[10] 1. Solve the following initial-value problem:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \ , \ y(0) = -1,$$

to get an explicit expression for $y(x)$.

[15] 2. (a) Use the integrating-factor method to solve the initial-value problem:

$$xy' + 2y = 4x^2 \quad , \quad y(1) = 2.$$

(b) Give a qualitatively correct sketch of the 1-parameter family of solution curves representing the general solution found in (a). Highlight the solution satisfying the initial conditions.

[15] 3. Assume that the electrical power P generated by a wind turbine depends upon the following physical quantities:

$$\begin{aligned} \rho &= \text{density of air (mass per unit volume)} \\ v &= \text{air speed} \\ A &= \text{area swept out by turbine blades.} \end{aligned}$$

[Power is work done per unit time.]

(a) Construct the dimensional matrix for the problem, deduce the number of dimensionless variables available, and find them.

(b) Hence deduce how P depends functionally on ρ, ν, and A.

(c) If the wind speed doubles, deduce the factor by which P changes.

[20] 4. Consider the following initial-value problem:

$$\begin{aligned} &\ddot{y} + 2\dot{y} + y = 4e^{-t}, \\ &y(0) = 2 \ , \quad \dot{y}(0) = -1 \end{aligned}$$

where $\dot{} = \dfrac{d}{dt}$.

(a) **State** the formula for the Laplace transform of the second derivative $\ddot{y}$.

(b) Find the Laplace transform of the given initial-value problem, and solve it algebraically for $Y(s)$.

(c) **Prove** the s-differentiation formula: $\mathcal{L}\{-ty(t)\} = \dfrac{dY(s)}{ds}$.

(d) Hence deduce the solution to the given initial-value problem.

[20] 5. Consider the double-loop circuit shown below:

It can be shown that the current $I_2(t)$ in the second loop is related to the applied voltage $E(t)$ by the DE:

$$R_1 L C \ddot{I}_2 + (R_1 R_2 C + L)\dot{I}_2 + (R_1 + R_2) I_2 = E(t).$$

Let $R_1 = R_2 = 1\,\text{ohm}$, $C = 1\,\text{farad}$, and $L = 1$ henry in all the following calculations.

(a) Is the circuit under-, critically-, or overdamped? Find the damping ratio ζ. Explain your reasoning.

(b) Suppose the initial conditions are zero, and that at time $t = 0$ a 1-volt battery is applied: $E(t) = H(t)$. Solve this initial-value problem to find $I_2(t)$ **without using Laplace transforms**.

(c) Sketch $I_2(t)$ making sure that your sketch is qualitatively correct.

[20] 6. Solve the vector DE initial value problem:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t)$$

with

$$A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}, \mathbf{x}(0) = \mathbf{0}$$

$$\mathbf{f}(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as follows:

(a) Find the fundamental matrix $\Phi(t)$ for this problem. You may use the eigenvalue/eigenvector method or the Laplace transform method.

(b) Now use the method of **variation of parameters** to complete the solution.