UNIVERSITY OF WATERLOO
FINAL EXAMINATION
FALL TERM 2005

COURSE NUMBER	AMATH 250
COURSE TITLE	Introduction to Differential Equations
DATE OF EXAM	Tuesday December 20, 2005
TIME PERIOD	9:00 a.m. - 11:30 p.m.
DURATION OF EXAM	$2\ \frac{1}{2}$ hours
NUMBER OF EXAM PAGES (including this cover sheet)	10 pages
INSTRUCTOR	B. Marshman
EXAM TYPE	Closed book
ADDITIONAL MATERIALS ALLOWED	NO AIDS

Name ______________________________

ID No. ______________________________

Instructions:

1. It is important that your conclusions be justified and that your solutions be well-organized.
2. Use the reverse side of page if necessary.

Questions	**Mark**	**Out of**
1		15
2		13
3		12
4		20
5		14
6		14
7		12
Total		100

[15] 1. a) (i) Find the general solution of the DE $y' = 3x^2(1 - y)$.

(ii) Sketch the family of solutions, indicating any equilibrium solutions, and including the solutions for which $y(0) = 2,\ y(0) = 0$, and $y(0) = -2$.

b) (i) Solve the I.V.P. $y' = -\frac{2}{x}y + x, \quad y(1) = 0$, and state the domain of the solution.

[13] 2. At time $t = 0$, a group of N_0 students returns to campus from a tropical holiday, infected with a flu virus (to which none of the remaining students is immune). The number $N(t)$ of infected students grows rapidly according to the model

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{2000}\right) \quad , \quad N(0) = N_0$$

(a) Use **direction field analysis** to sketch typical solutions, indicating any equilibrium solutions. What happens eventually, for any N_0 in (0, 2000)?

(b) Suppose the campus health unit acts quickly and manages to isolate (i.e., quarantine) h infected students per day. The revised model is

$$\frac{dN}{dt} = 0.01N\left(1 - \frac{N}{2000}\right) - h, \quad N(0) = N_0.$$

(i) Show that, if we define $y = \frac{N}{2000}$ (the fraction of infected students) this model becomes

$$\frac{dy}{dt} = -0.01\left(y^2 - y + \lambda\right), \quad y(0) = y_0,$$

where y_0 and λ are constants you should determine.

(ii) Explain how you know that if $h > 5$ students per day, then no epidemic occurs (i.e., the fraction of infected students decreases steadily).

[12] 3. a) Assume that the frequency ν of resonance of an organ pipe depends on the speed c of sound in the air and the length l of the pipe. Use dimensional analysis to determine how ν depends on l and c.

b) Assume that the frequency ν_0 of the fundamental mode of vibration of a violin string depends on the length L of the string, the linear density σ (mass per unit length) of the string, and the tension (force) T on the string. Use dimensional analysis to determine how ν_0 depends on L, σ, and T.

c) Show that, in both cases, halving the length doubles the frequency.

d) How else could you double the frequency ν_0 of the violin string? Give a physical

[20] 4. A 1 kg mass on a spring with constant $k = 1$ Nm^{-1} is immersed in a viscous medium with damping constant $2c$ kg s^{-1}. When an external force $F(t) = \cos t$ N is applied, the displacement $y(t)$ from equilibrium satisfies

$$\frac{d^2y}{dt^2} + 2c\frac{dy}{dt} + y = \cos t.$$

a) Find the transient solution $y_h(t)$ for the case $c > 1$, and sketch the solution for (i) $y(0) = 0,\ y'(0) = 1$, and (ii) $y(0) = 1,\ y'(0) = 0$. Is the system under-, over-, or critically damped?

b) Find the steady-state solution $y_p(t)$, and its amplitude A. [Your answers will contain the constant c.]

c) Use your result from b) to determine the minimum positive value $c_{\min}$ of c for which the amplitude $A \leq \dfrac{1}{\sqrt{2}}$ m.

4. d) **Given** that, for $F(t) = \cos \omega t$, the steady-state amplitude is

$$A(\omega) = \frac{1}{\sqrt{(1-\omega^2)^2 + 4c^2\omega^2}} \quad ,$$

discuss the phenomenon of 'amplitude resonance'. In particular, explain why it cannot occur unless $0 \le c \le c_{\min}$.

e) Sketch typical curves $A(\omega)$ for different values of c, giving reasons.

f) The amplitude of the driving force $F(t) = \cos \omega t$ is 1 m. Find a condition on ω such that the steady-state amplitude $A(\omega) > 1$ (i.e., the input is amplified)

[14] 5. (a) Use the method of eigenvalues and eigenvectors to find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{x} \, .$$

(b) Find the fundamental matrix $\Phi(t)$ for the system in a), and hence state the solution for $\mathbf{x}(0) = \mathbf{a}$.

(c) Sketch the family of orbits of the system, showing the isoclines. Justify your sketch briefly.

[14] 6. (a) Define the Laplace tranform $\mathcal{L}[f(t)]$, and state sufficient conditions for its existence.

(b) Use the definition to find $\mathcal{L}[H(t-a)]$, where $H(t-a) = \begin{cases} 0 & \text{for} \quad 0 \leq t < a \\ 1 & \text{for} \quad t \geq a \end{cases}$

c) The current $y(t)$ in the electric circuit at left is governed by the DE

$$L\frac{dy}{dt} + Ry = V(t), \quad y(0) = 0.$$

where L is the inductance, R the resistance, and $V(t)$ the applied voltage.

Suppose that $L = 1$ henry, $\quad R = 2$ ohms, and $V(t) = \begin{cases} 12 \text{ volts} & 0 \leq t < 10 \\ 0 & t \geq 10 \end{cases}$.
Use Laplace transforms and the second shift theorem to find the current $y(t)$ for $t > 0$, and plot your solution on the given axes.

[12] 7. a) Using the value of $\mathcal{L}[f'(t)]$ given in the table, show that

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

b) Use a) to solve the IVP $y'' + y = \sin 2t$, $y(0) = 1, \quad y'(0) = 0,$

c) Give a possible physical interpretation for the problem in b), and state the period of the solution.

ROUGH WORK PAGE

(do not hand in)