Eikosograms:
The picture of probability

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Statistical Society of Canada

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Venn

"... symbolic and diagrammatic systems are to some extent artificial. And they ought therefore be so constructed as to work in perfect harmony together"

... Symbolic Logic, 1881 (p. 139, 2nd Ed. 1894)

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Diagrams give meaning to symbols.

Thursday, November 3, 2011

Common experience:

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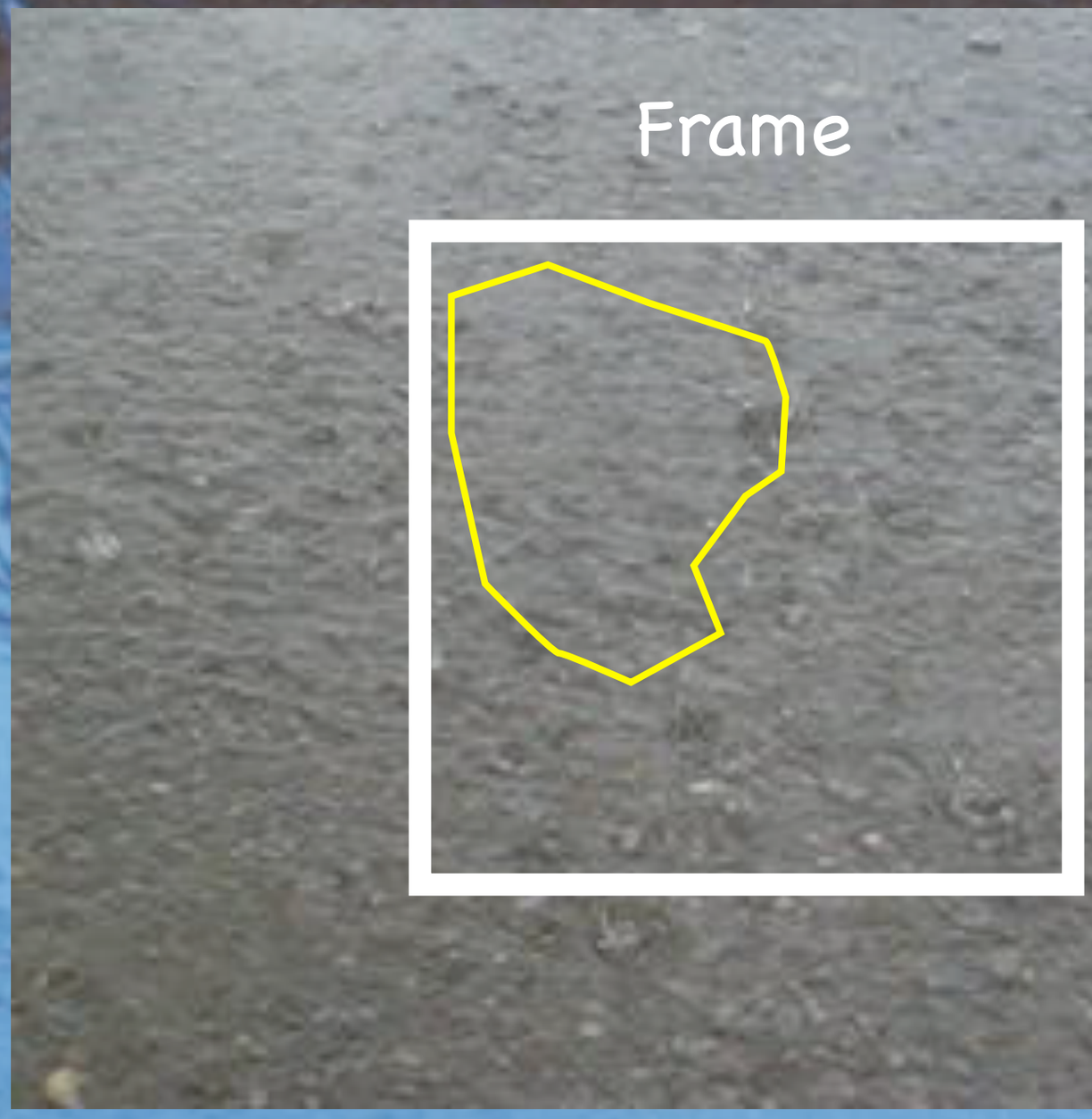
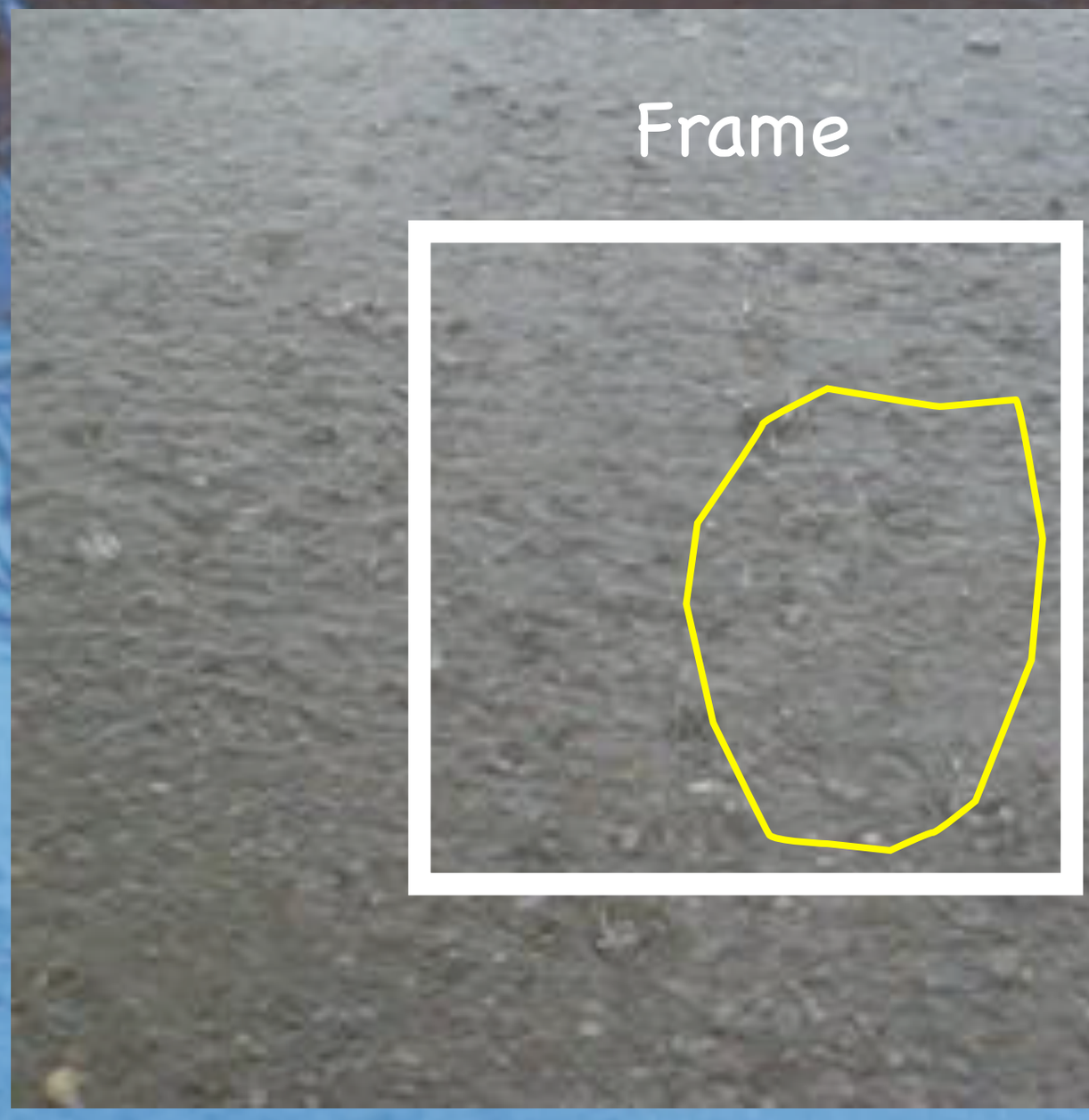


Of the raindrops which fall within the frame, what proportion fall within the yellow region?

Common experience:



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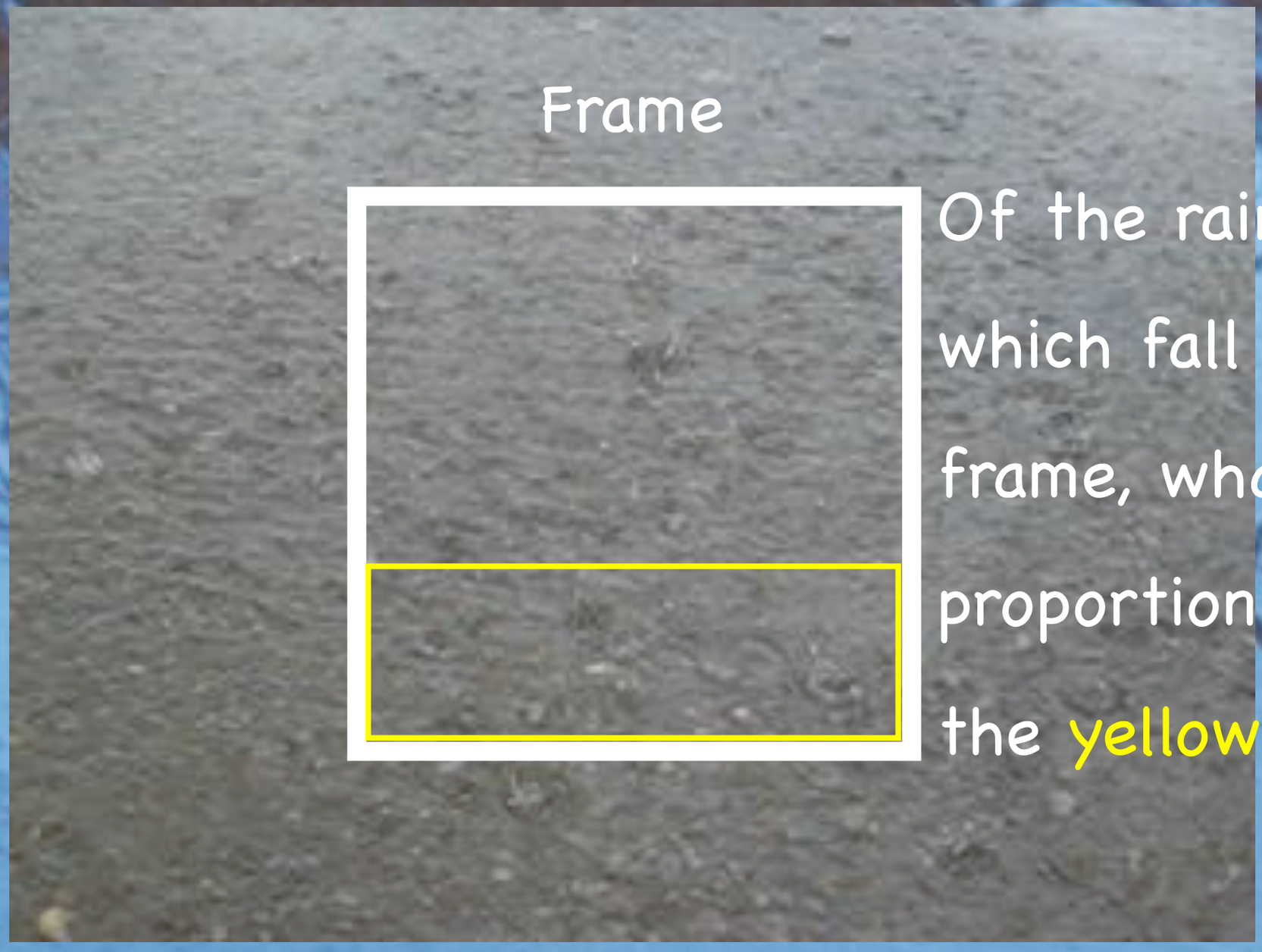

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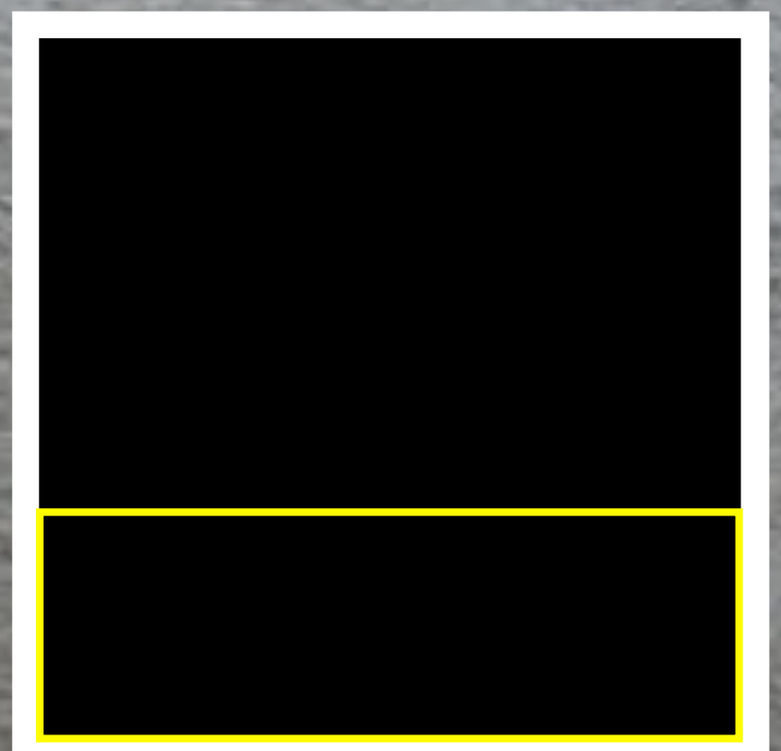

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Common experience:

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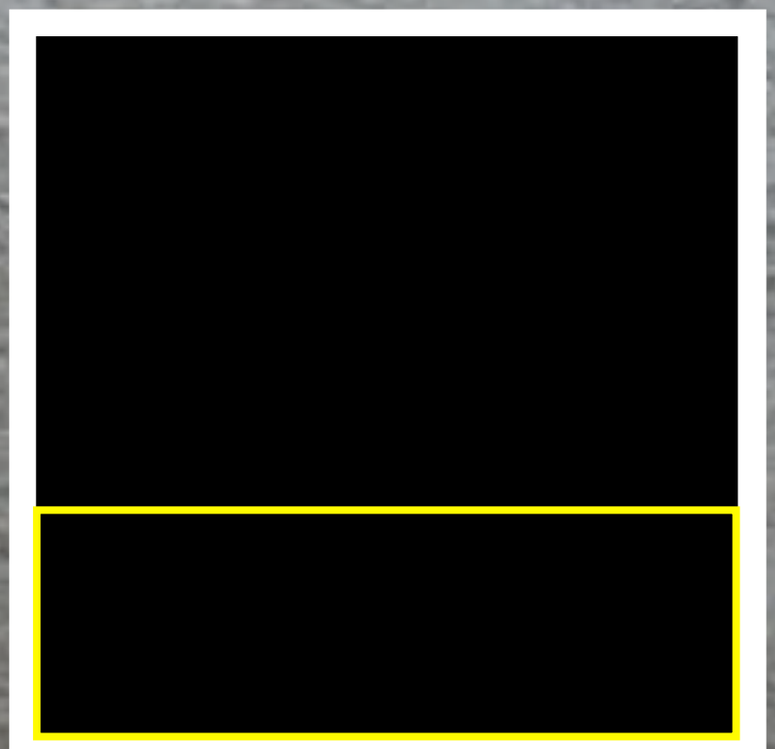
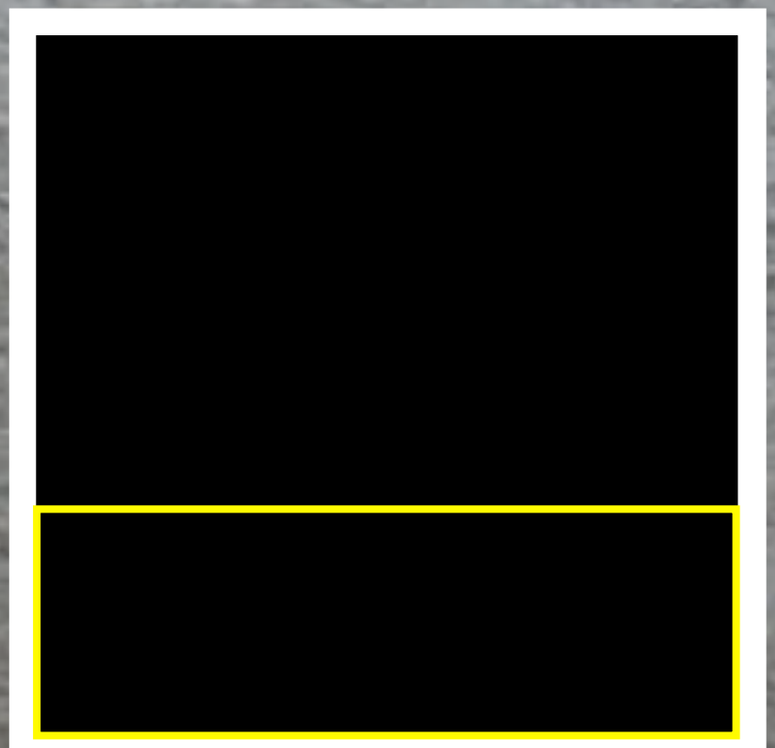


Of the raindrops which fall within the frame, what proportion fall within the yellow region?

Relative area of yellow region to that of the frame.

Common experience:

Frame

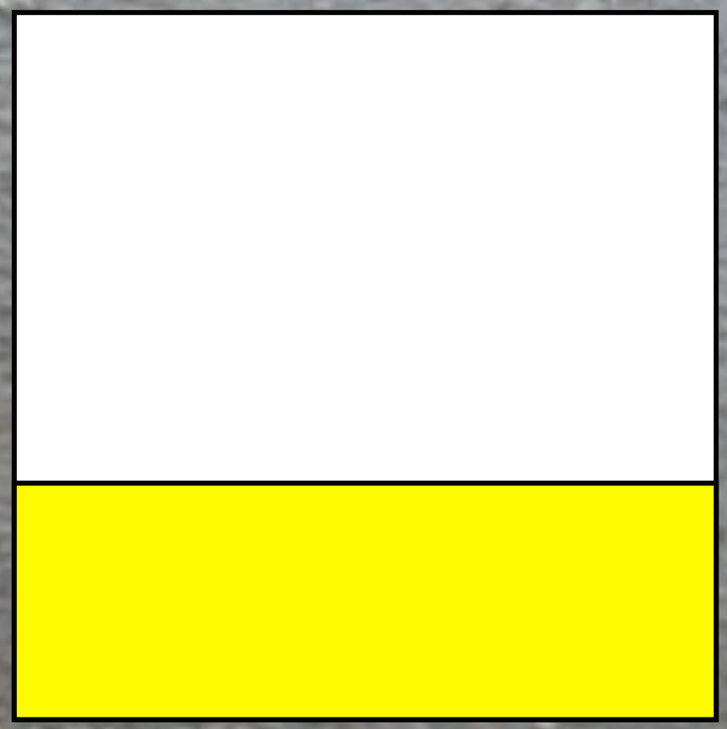


Of the raindrops which fall within the frame, what proportion fall within the yellow region?

Relative area of yellow region to that of the frame.

That's what we mean by probability.

Frame



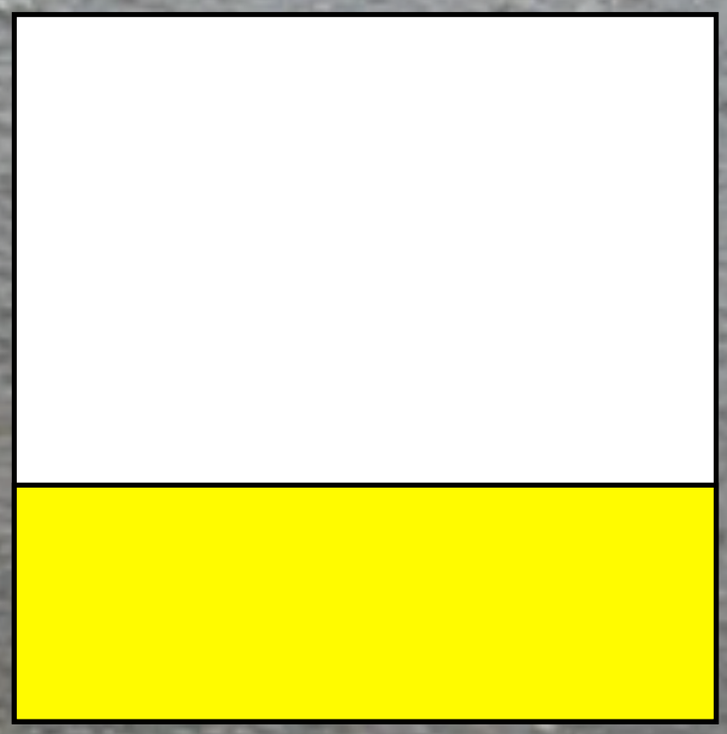
Frame

0

1

0

0 Frame 1



Region

1/3 <- Proportion

yellow

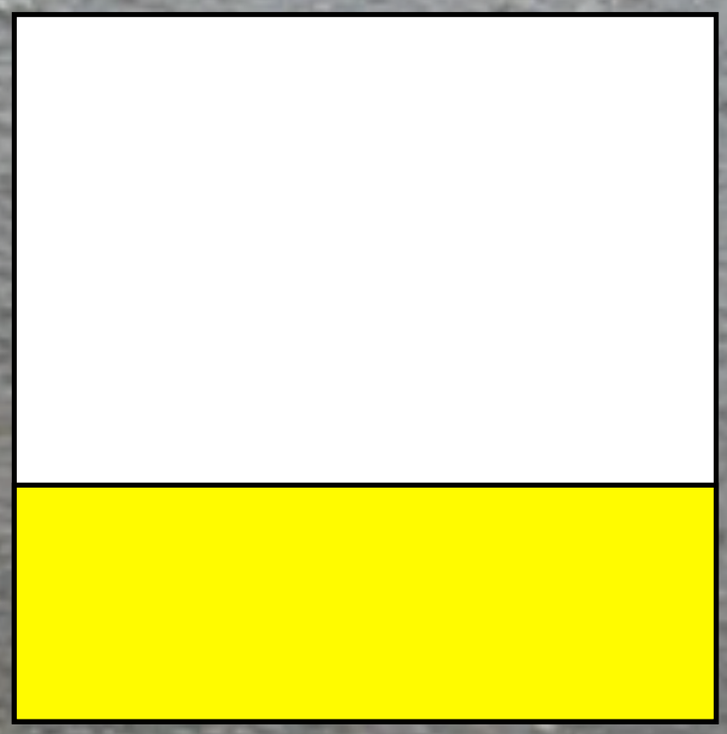
in this frame

0

Variable to label the partition

Y= white

Y= yellow



1/3 <- Proportion Y = yellow in this frame

0

Variable to label the partition

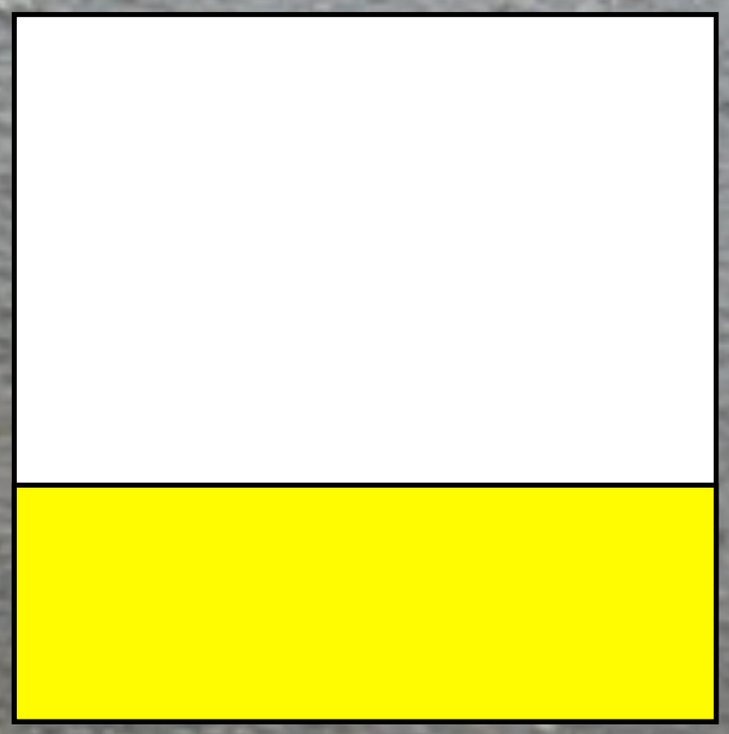
Y= b

Y= a

0

Frame

1



1/3 <- Proportion

Y = yellow

in this frame

0

Variable to label the partition

Y= y_2

Y= y_1

0

Frame

1

1/3 <- Proportion

Y = yellow

in this frame

0

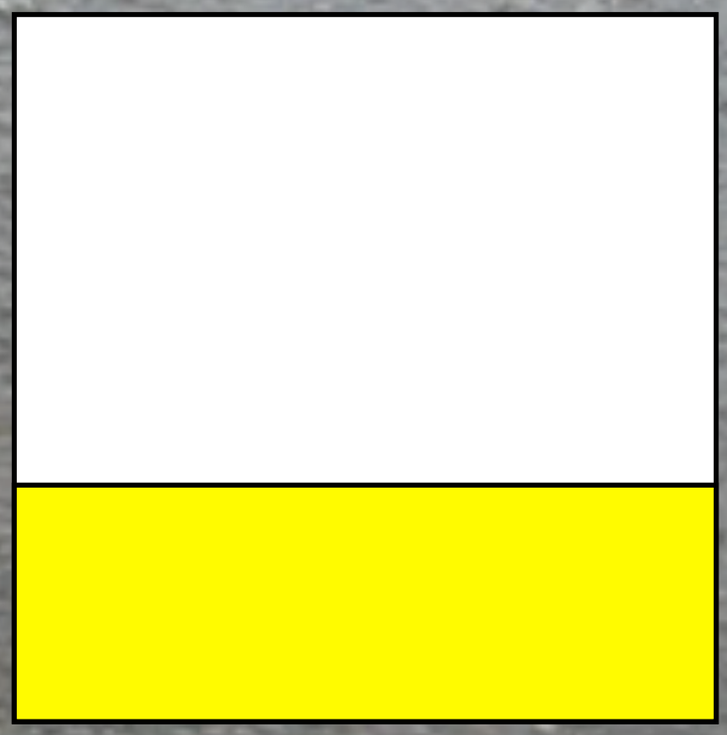
Variable to label the partition

Y= 2

0

Frame

1



Y= 1

1/3 <- Proportion

Y= 1

in this frame

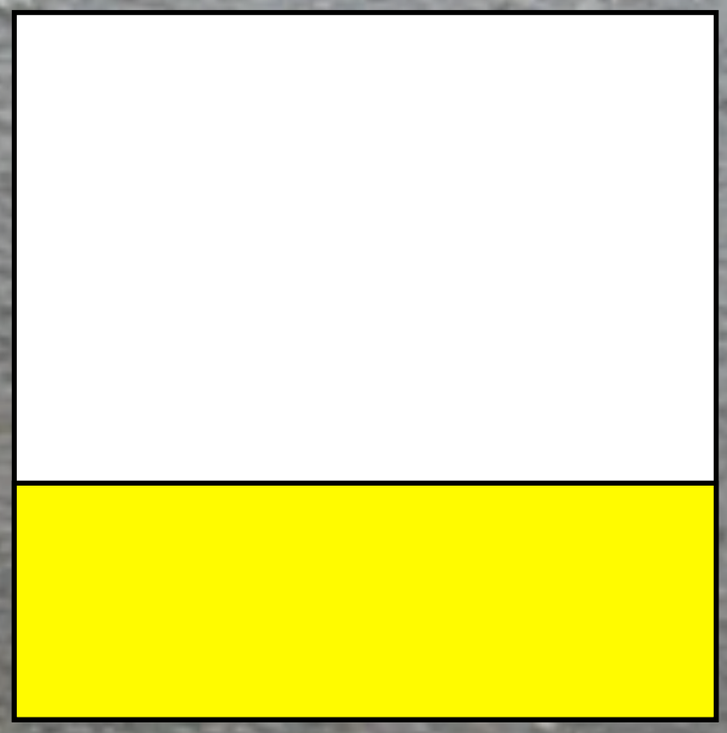
0

Frame

0

1

Y= 2



1/3 <- Proportion

Y= 1

Y= 1

in this frame

0

Symbolically: Pr(Y= 1 | Frame) = 1/3

and Pr(Y= 2 | Frame) = 2/3

Pr(Region of interest | Conditions which hold)

Eikosogram

eikos - like truth, likely, probable, reasonable, probability, likelihood *

gramma - that which is drawn, picture, piece of writing, ...

Symbolically: $Pr(Y= 1 \mid Frame) = 1/3$

and $Pr(Y= 2 \mid Frame) = 2/3$

* Thanks to Greg Bennett

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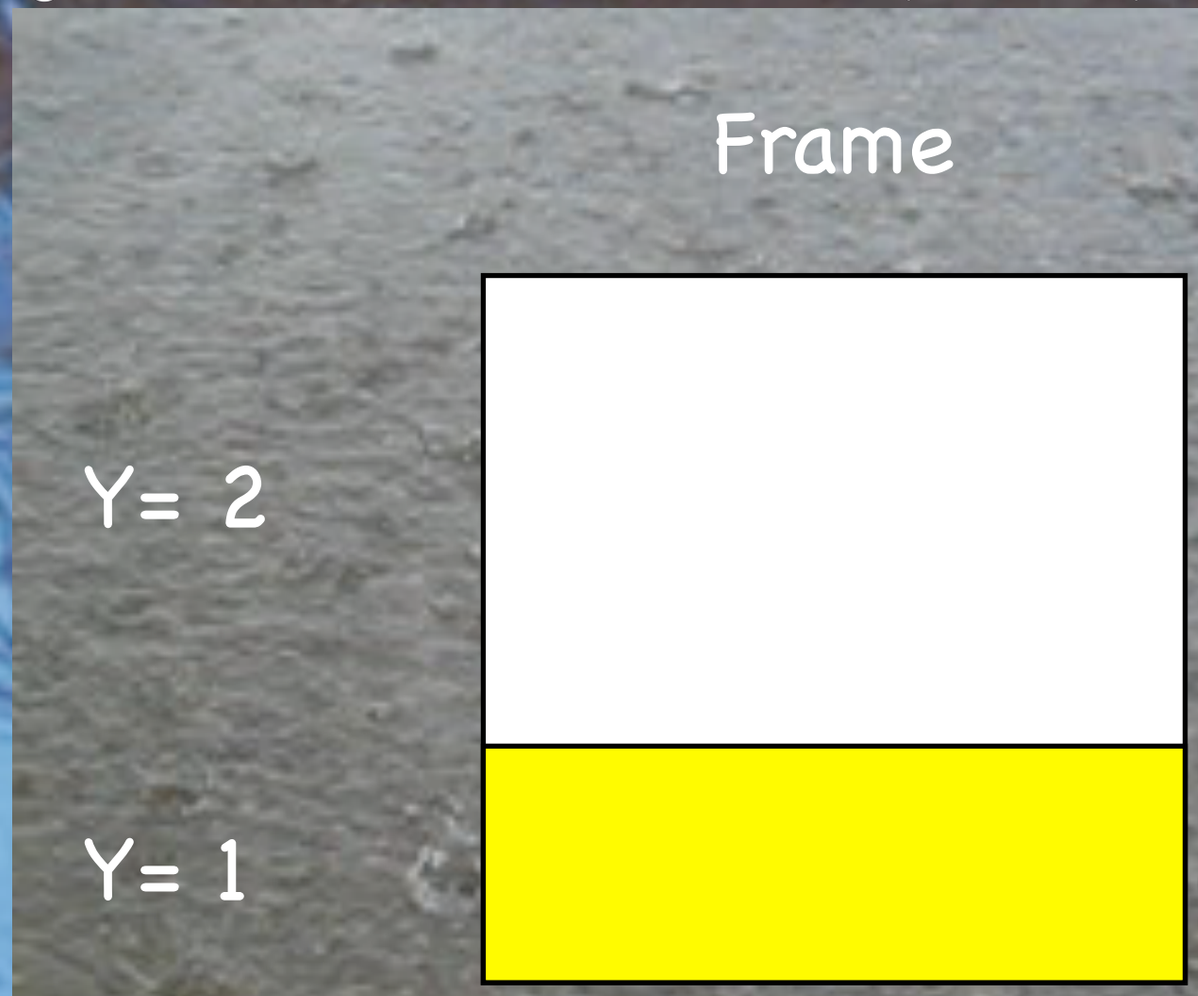

Symbolically: $Pr(Y= 1 \mid Frame) = 1/3$

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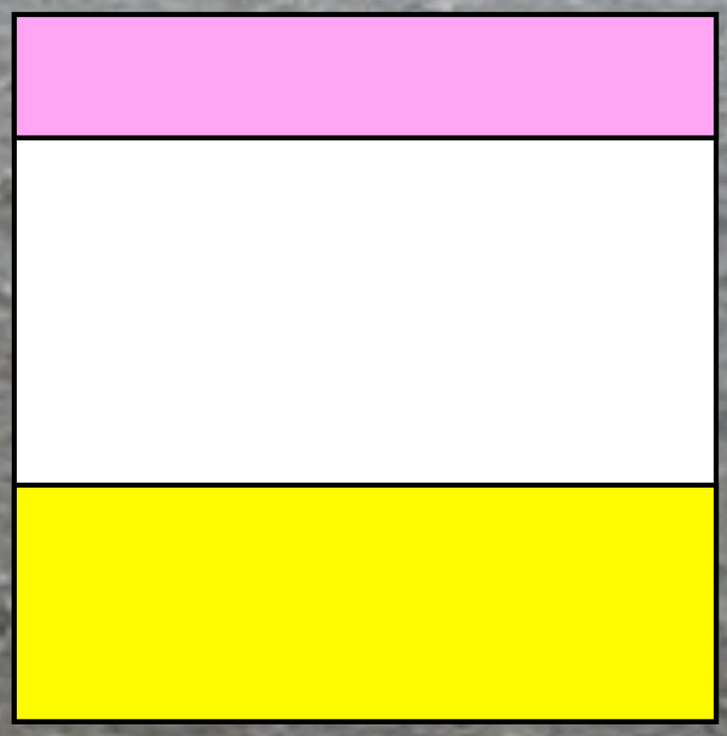

Symbolically: Pr(Y= 1 | Frame) = 1/3

and Pr(Y= 2 | Frame) = 5/6 - 1/3 = 1/2

and Pr(Y= 3 | Frame) = 1/6

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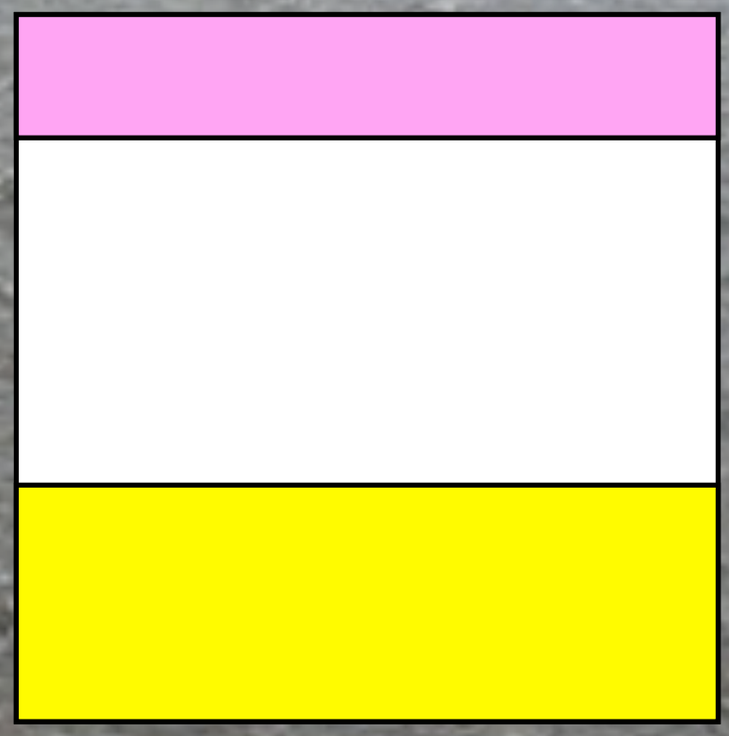
Frame



Symbolically: $\Pr(Y= 1 \mid \text{Frame}) = 1/3$

and $\Pr(Y= 2 \mid \text{Frame}) = 5/6 - 1/3 = 1/2$

and $\Pr(Y= 3 \mid \text{Frame}) = 1/6$

$\sum_y \Pr(Y= y \mid \text{Frame}) = 1$

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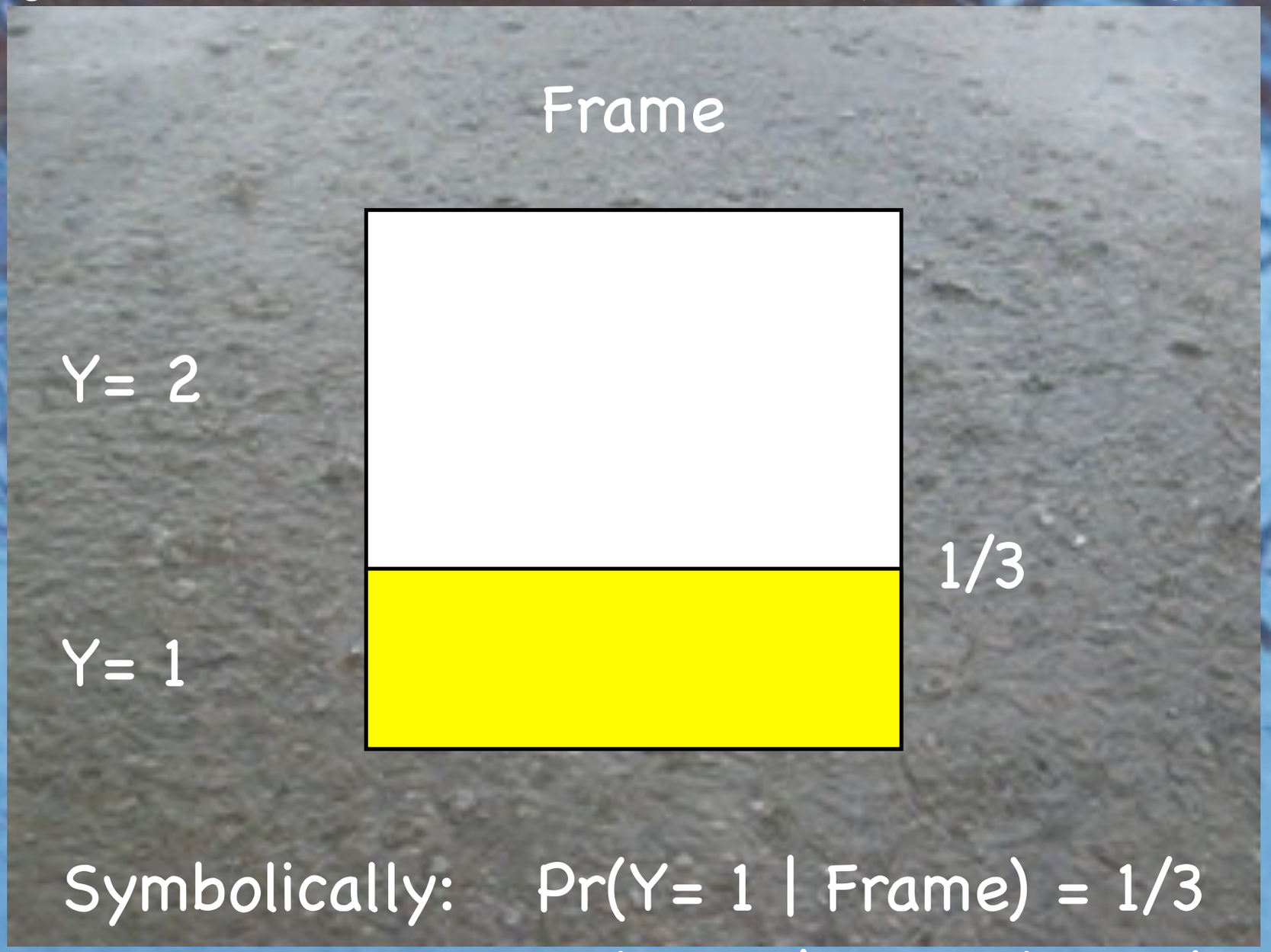

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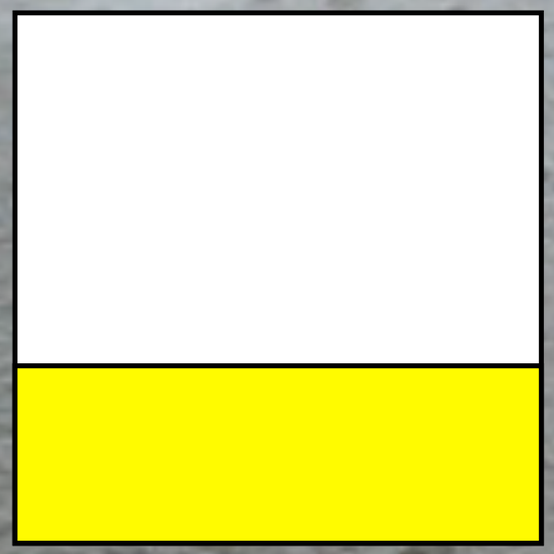

Frame

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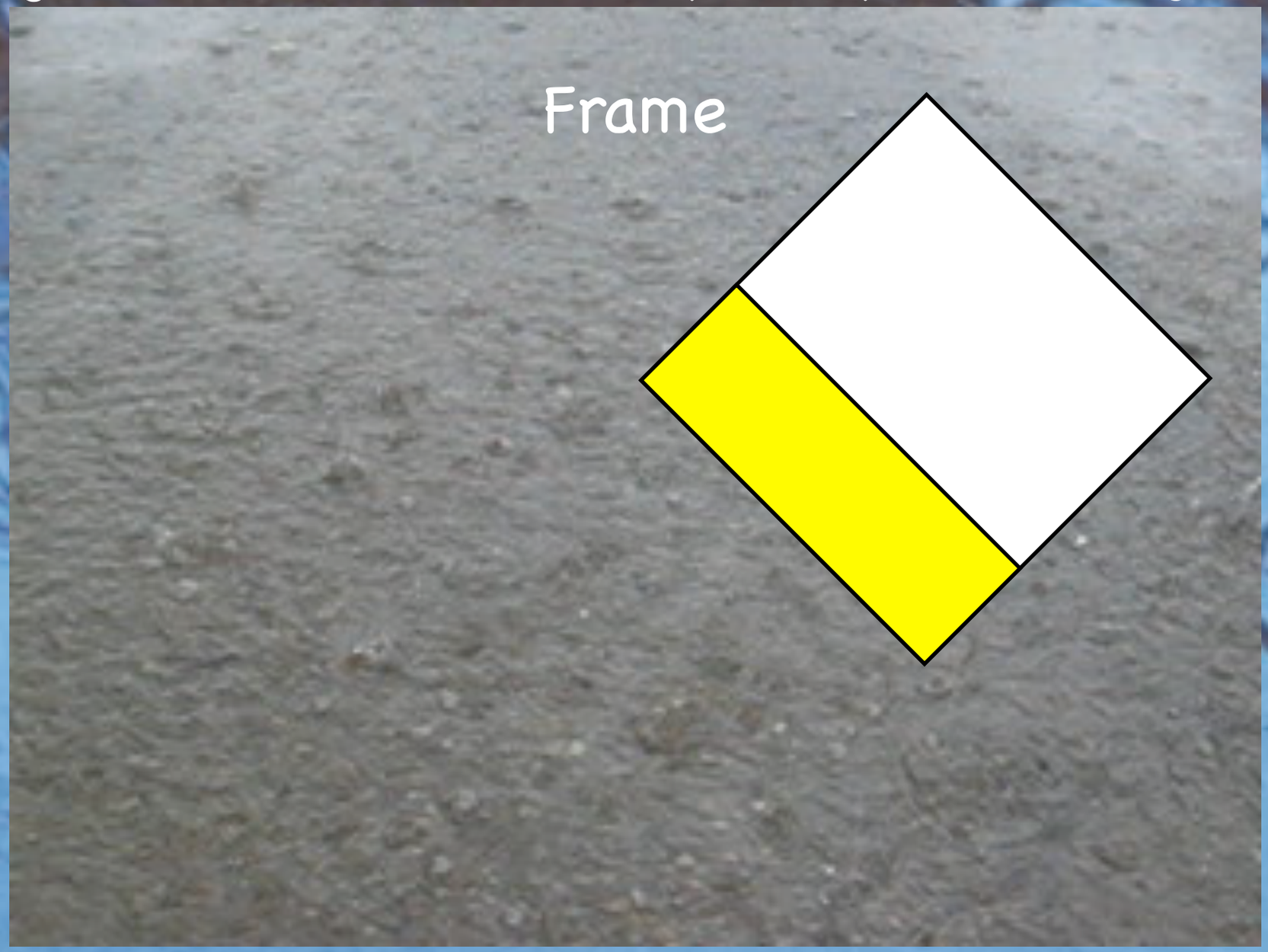

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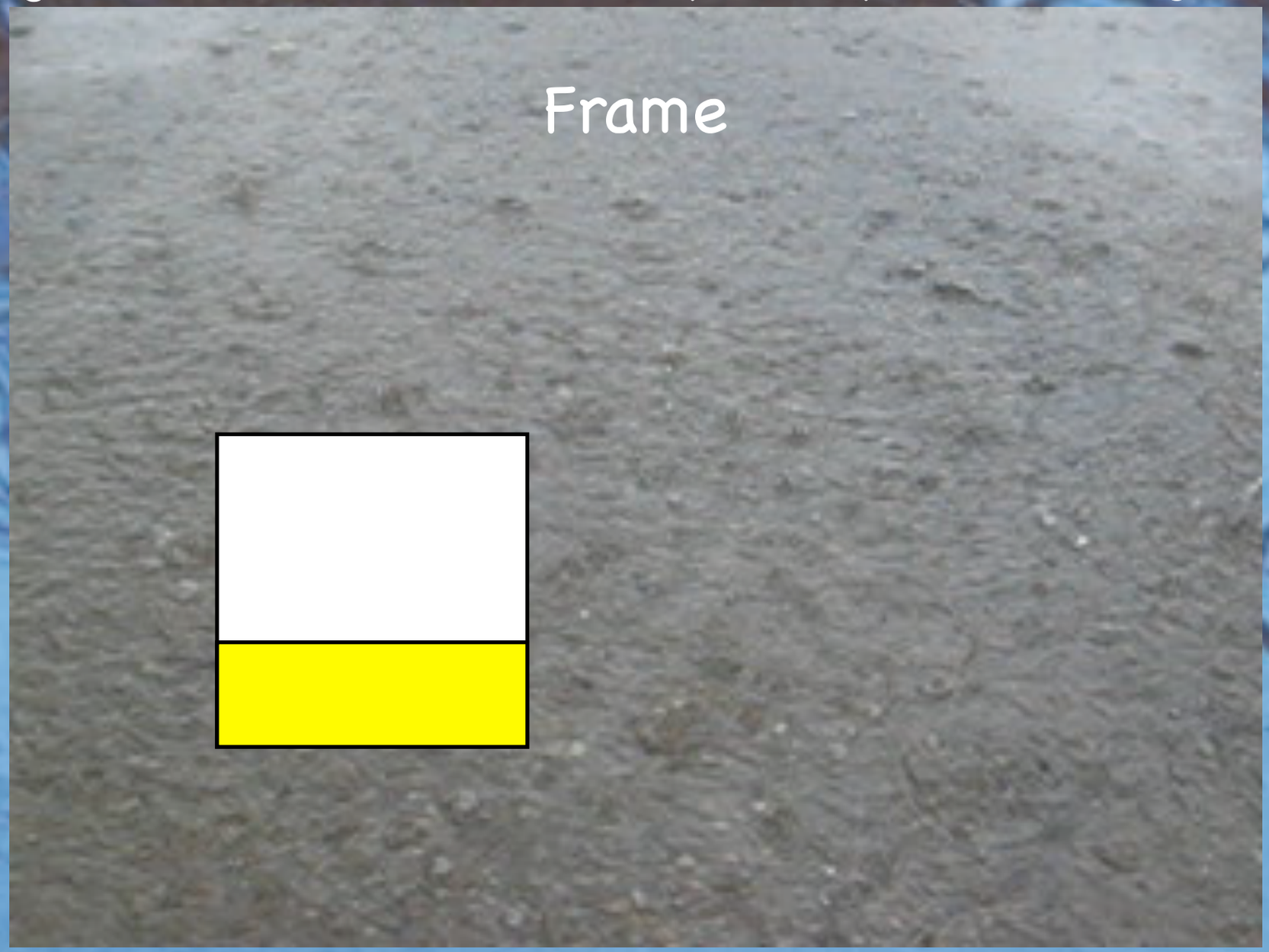

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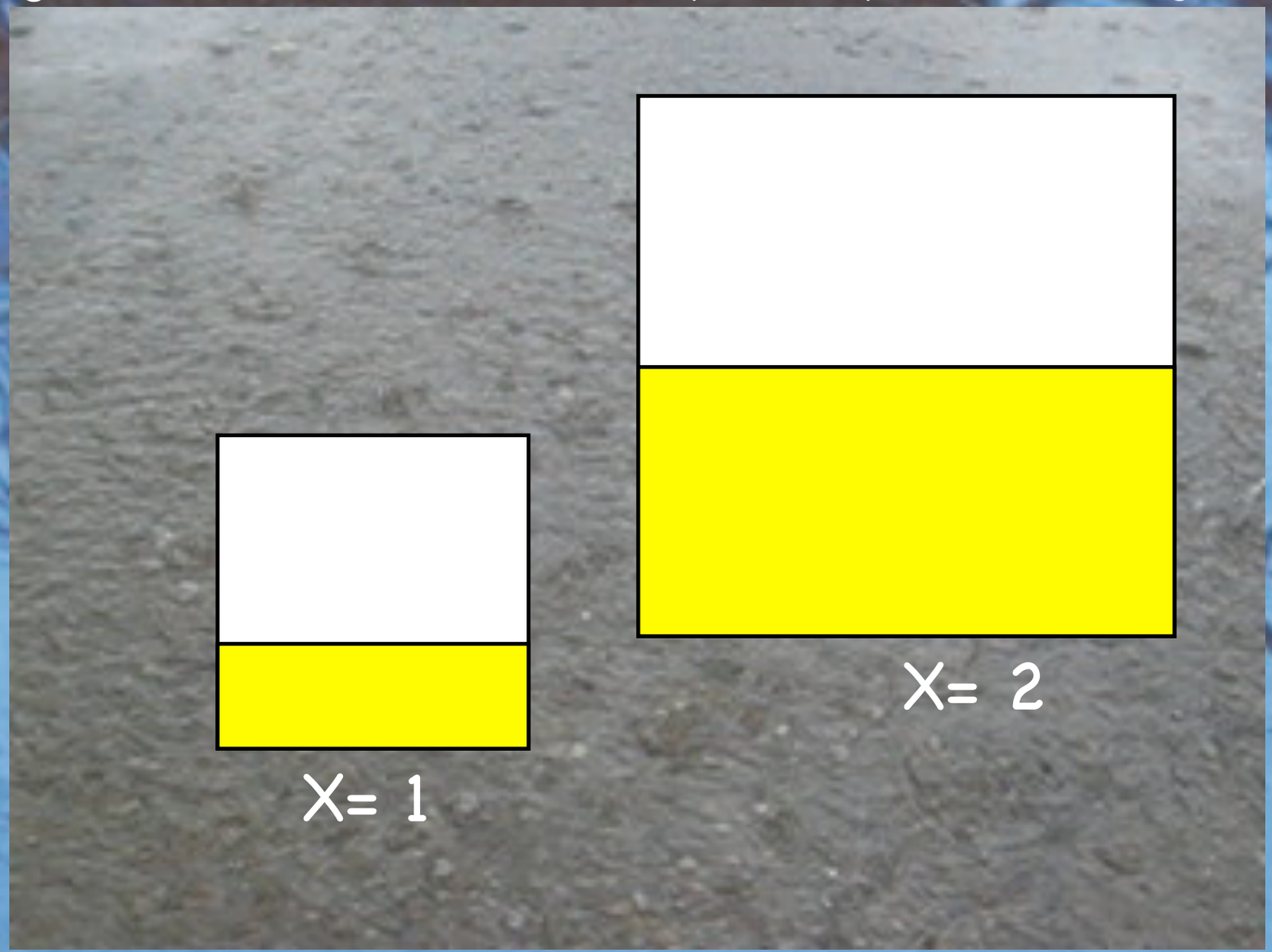

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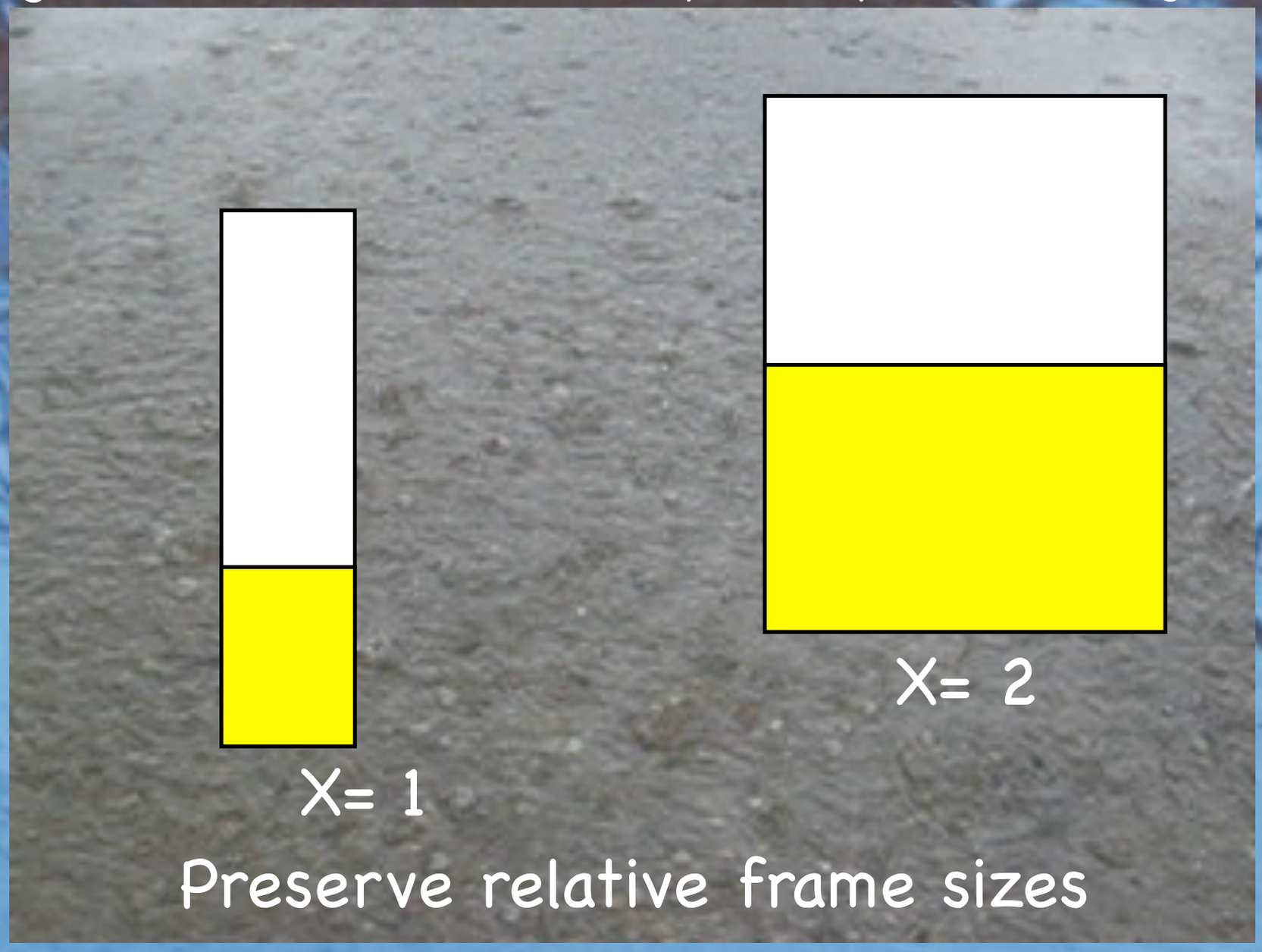

Preserve relative frame sizes

Eikosogram

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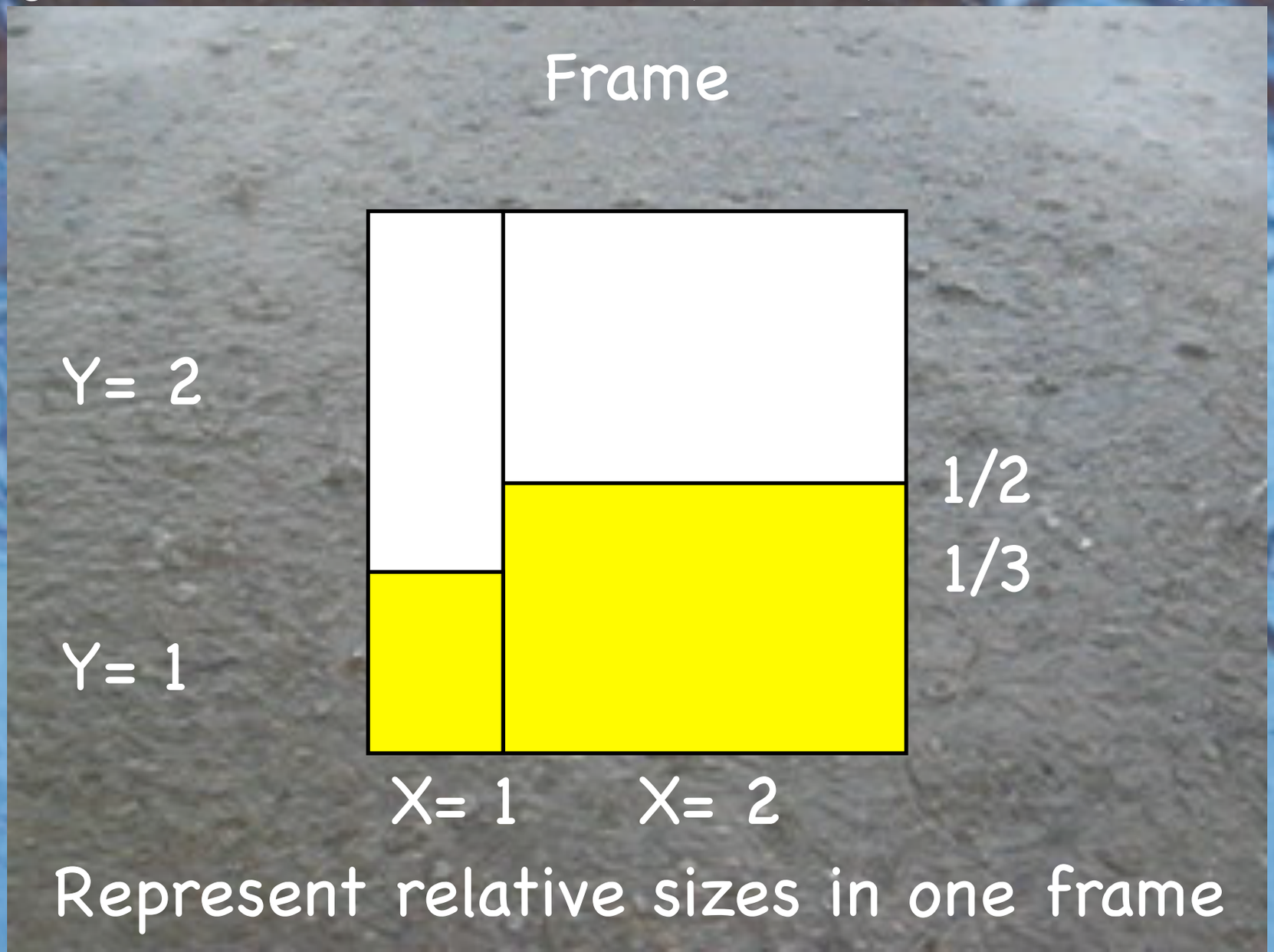

Represent relative sizes in one frame

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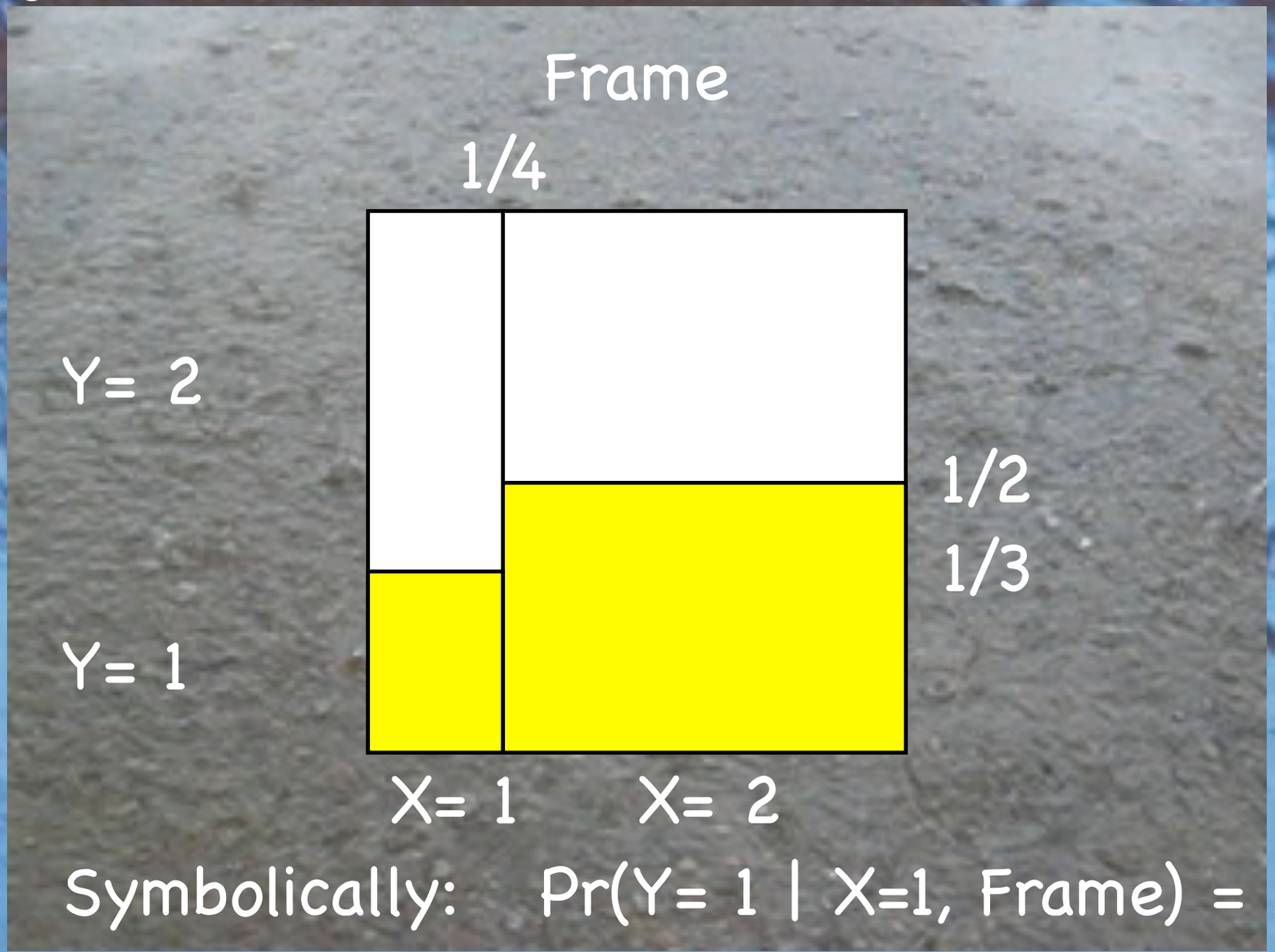

Symbolically: Pr(Y= 1 | X=1, Frame) = 1/3

and Pr(Y= 1 | X=2, Frame) = 1/2

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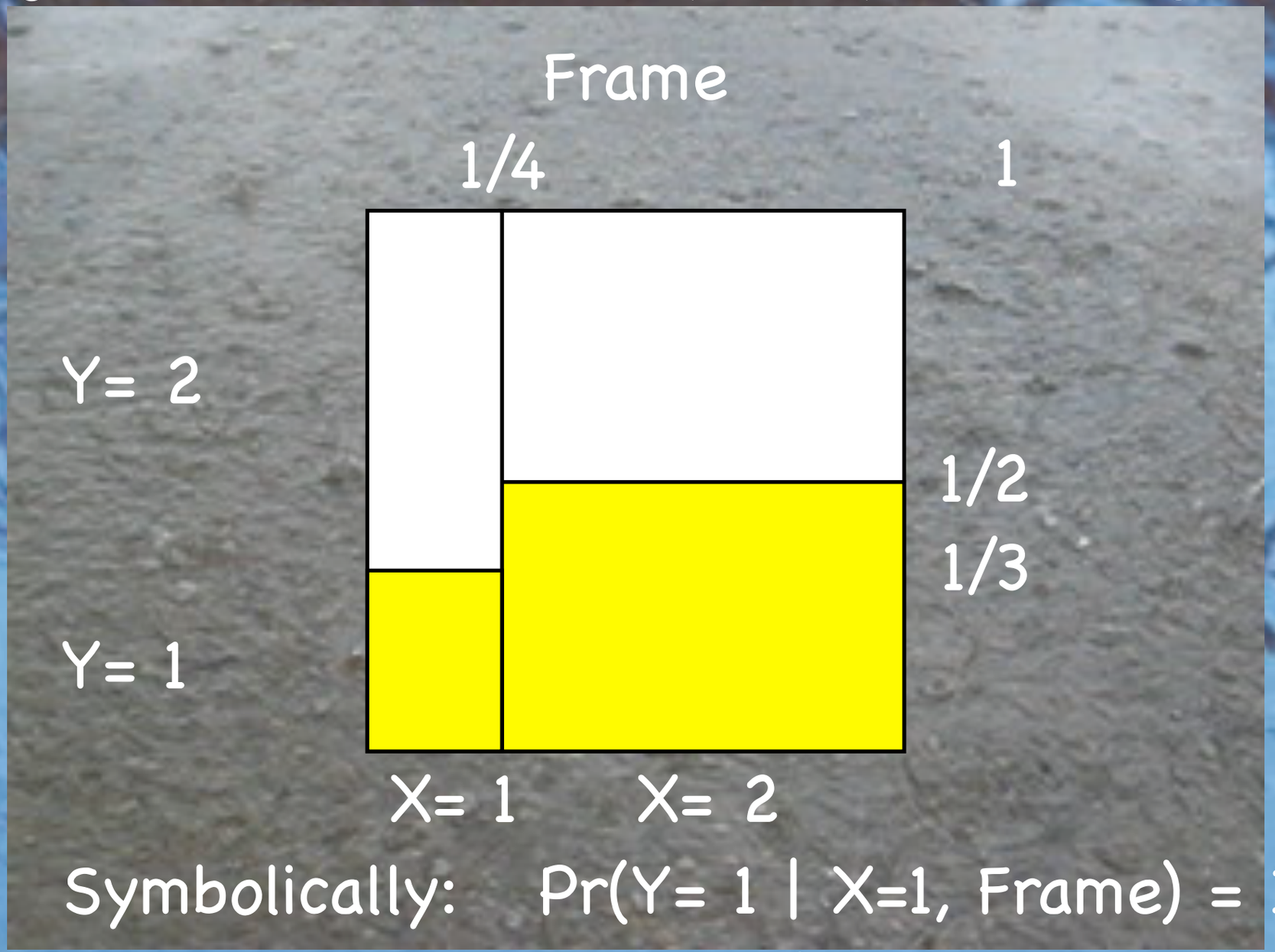

Symbolically: $Pr(Y= 1 \mid X=1, Frame) = 1/3$

and $Pr(Y= 1 \mid X=2, Frame) = 1/2$

$Pr(X= 1 \mid Frame) = 1/4$

$Pr(X= 2 \mid Frame) = 3/4$

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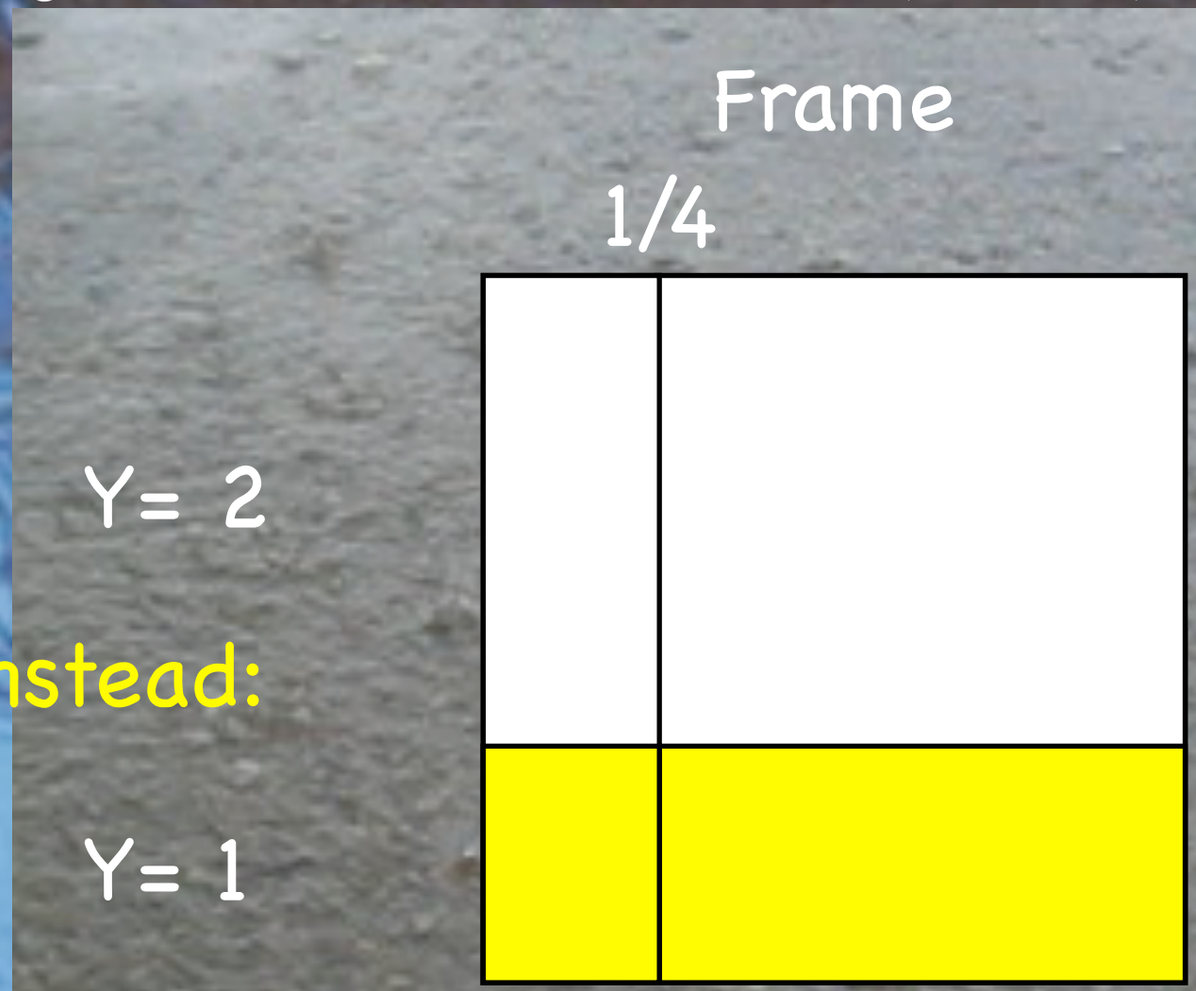

Suppose instead:

Symbolically: $\Pr(Y=1 \mid X=1, \text{Frame}) = 1/3$

$= \Pr(Y=1 \mid X=2, \text{Frame}) = 1/3$

and $\Pr(Y=2 \mid X=1, \text{Frame}) = \Pr(Y=2 \mid X=2, \text{Frame}) = 2/3$

Values of X and Y are independent

Eikosogram

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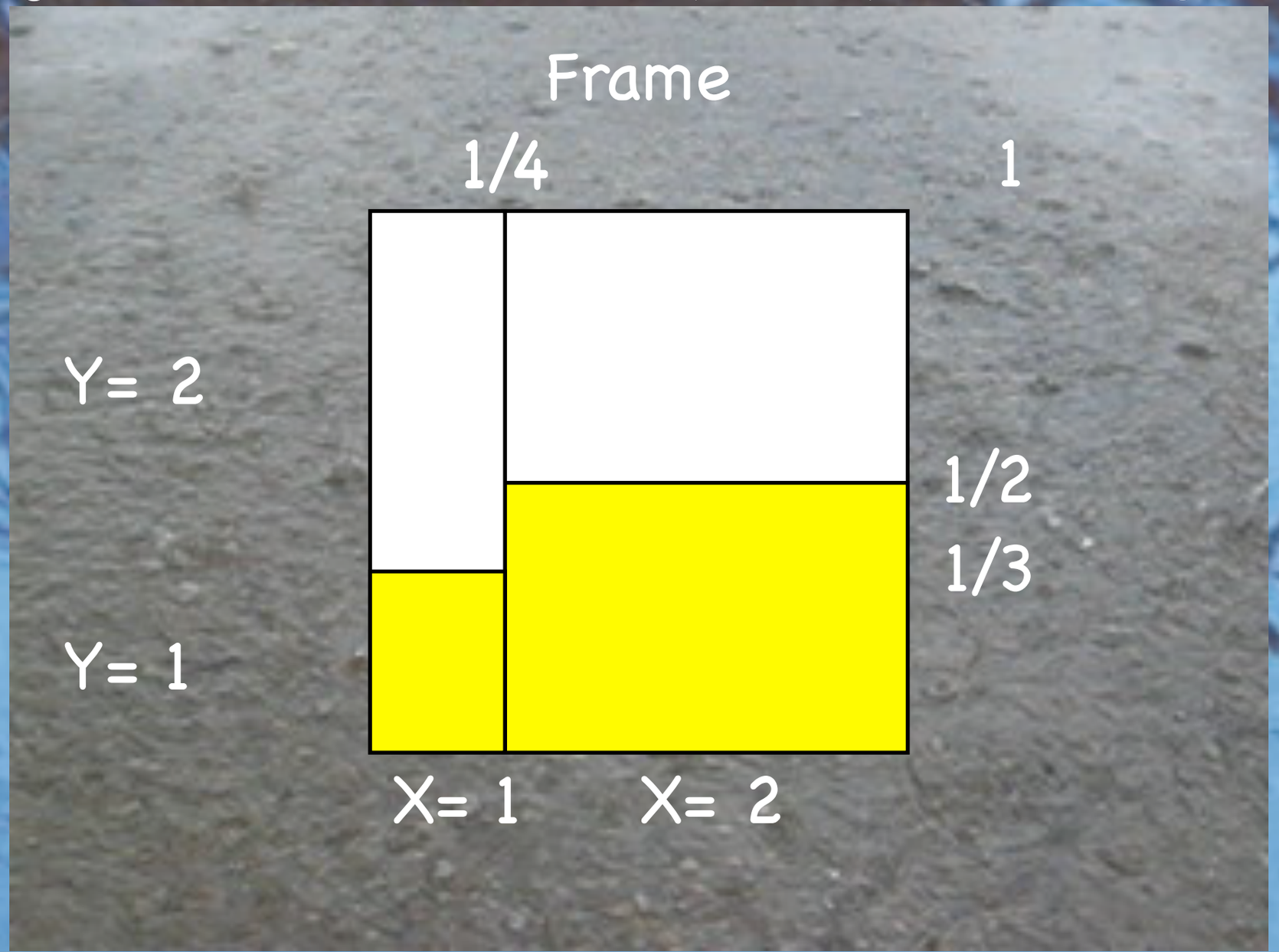

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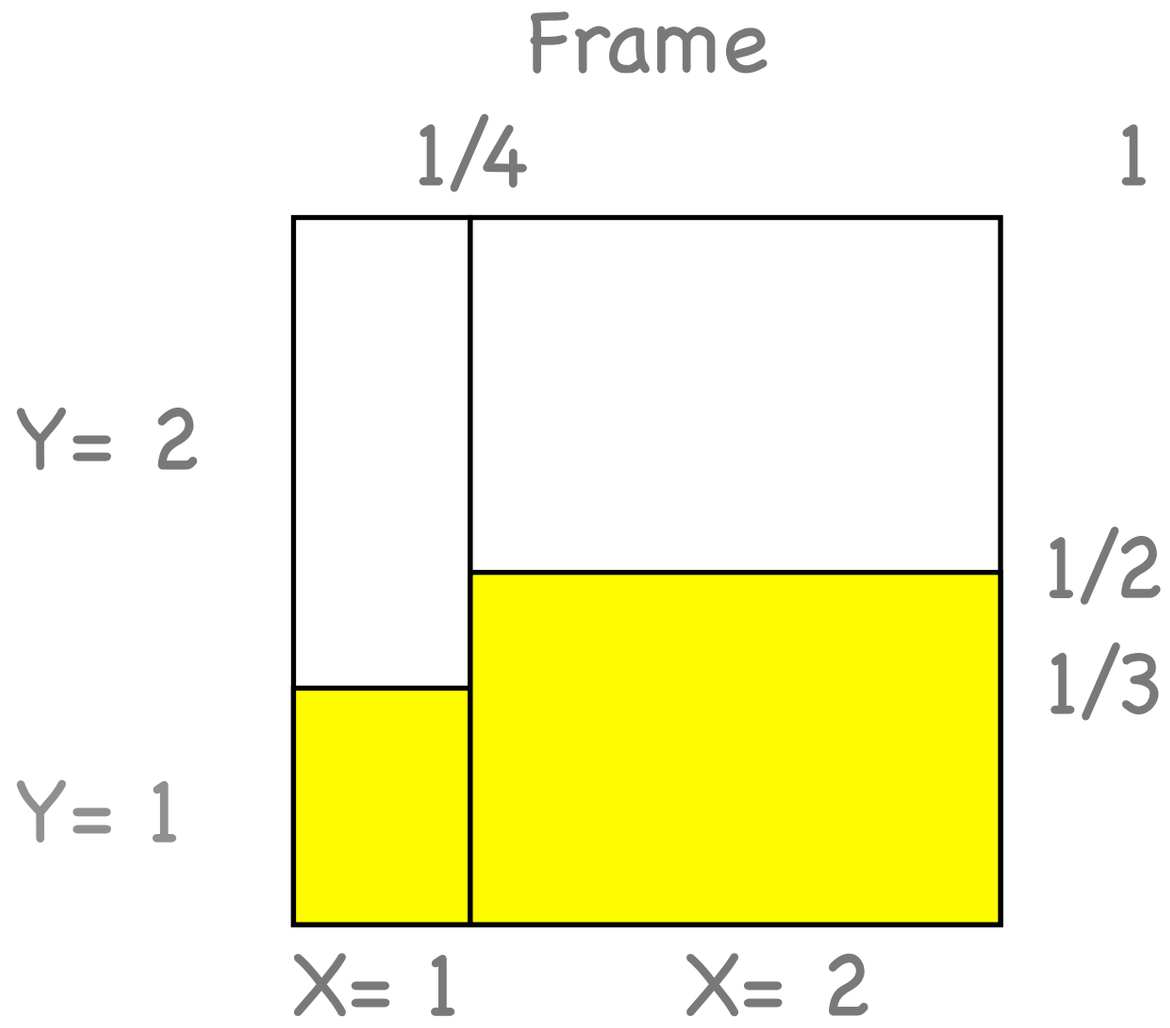

Symbolically: $Pr(Y= 1 \mid X=1, Frame) = 1/3$

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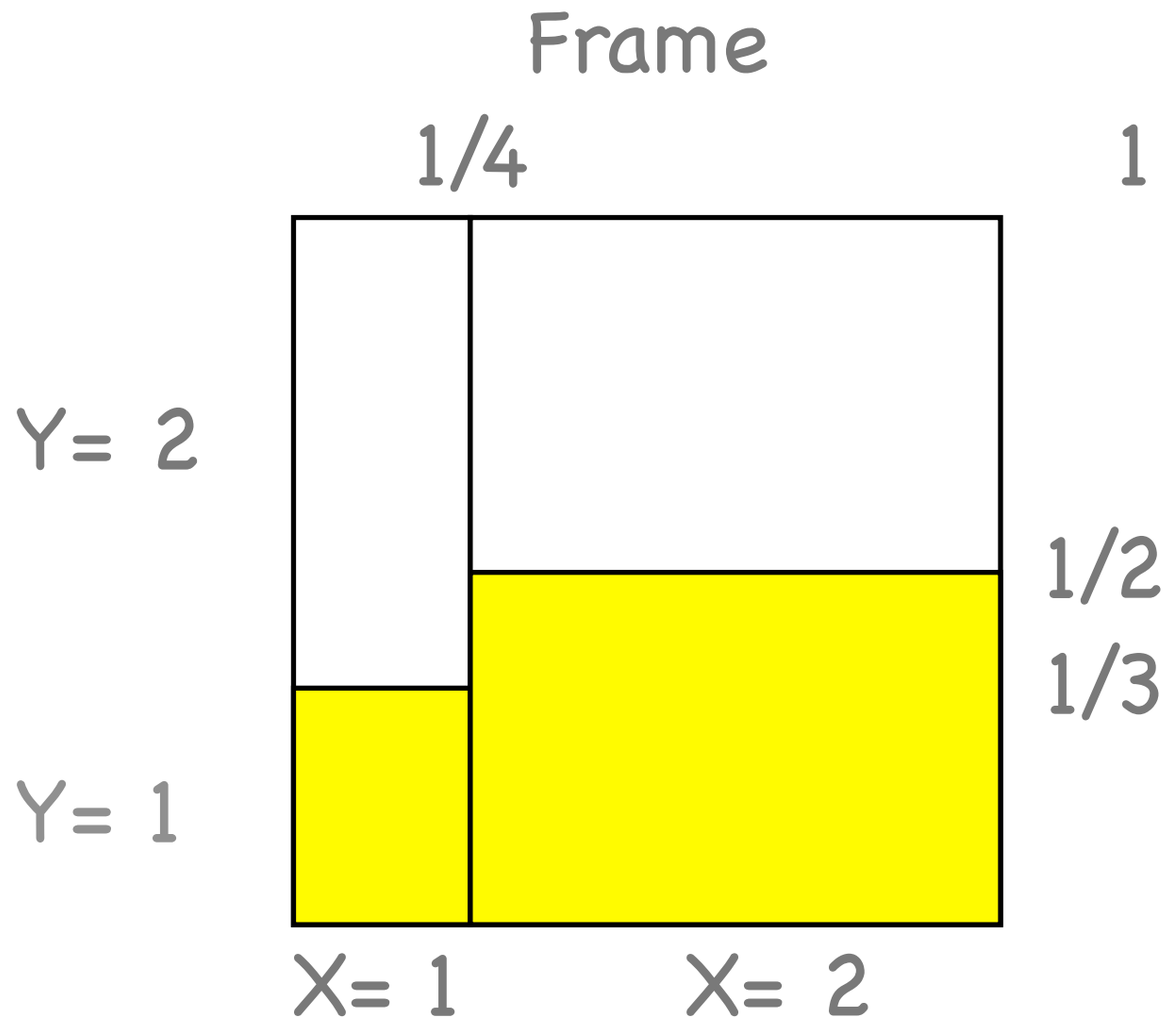

Symbolically: $Pr(Y= 1 \mid X=1, Frame) = 1/3$

or sometimes more simply as:

$Pr(Y=1 \mid X=1)$ ... common Frame

Eikosogram

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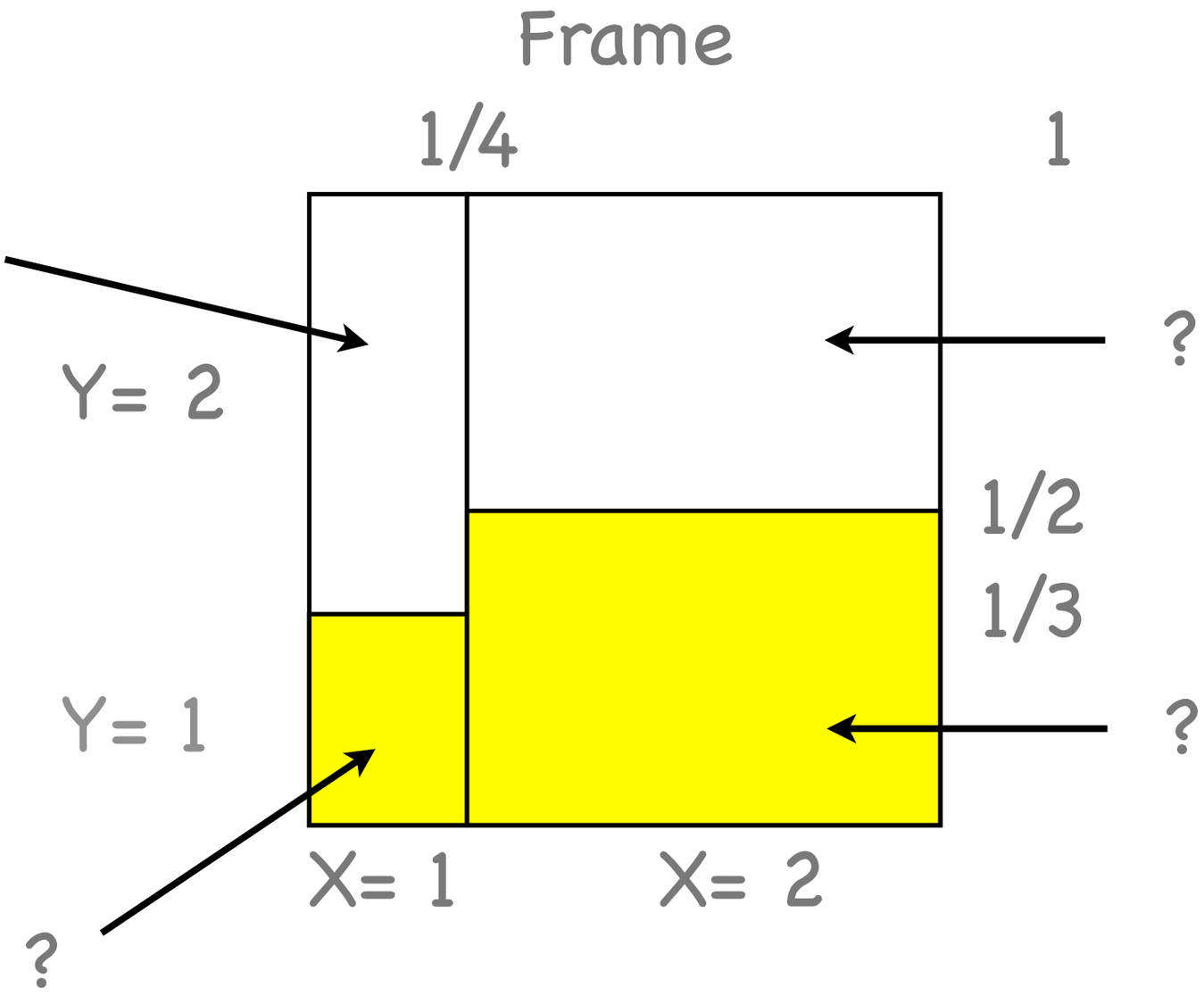

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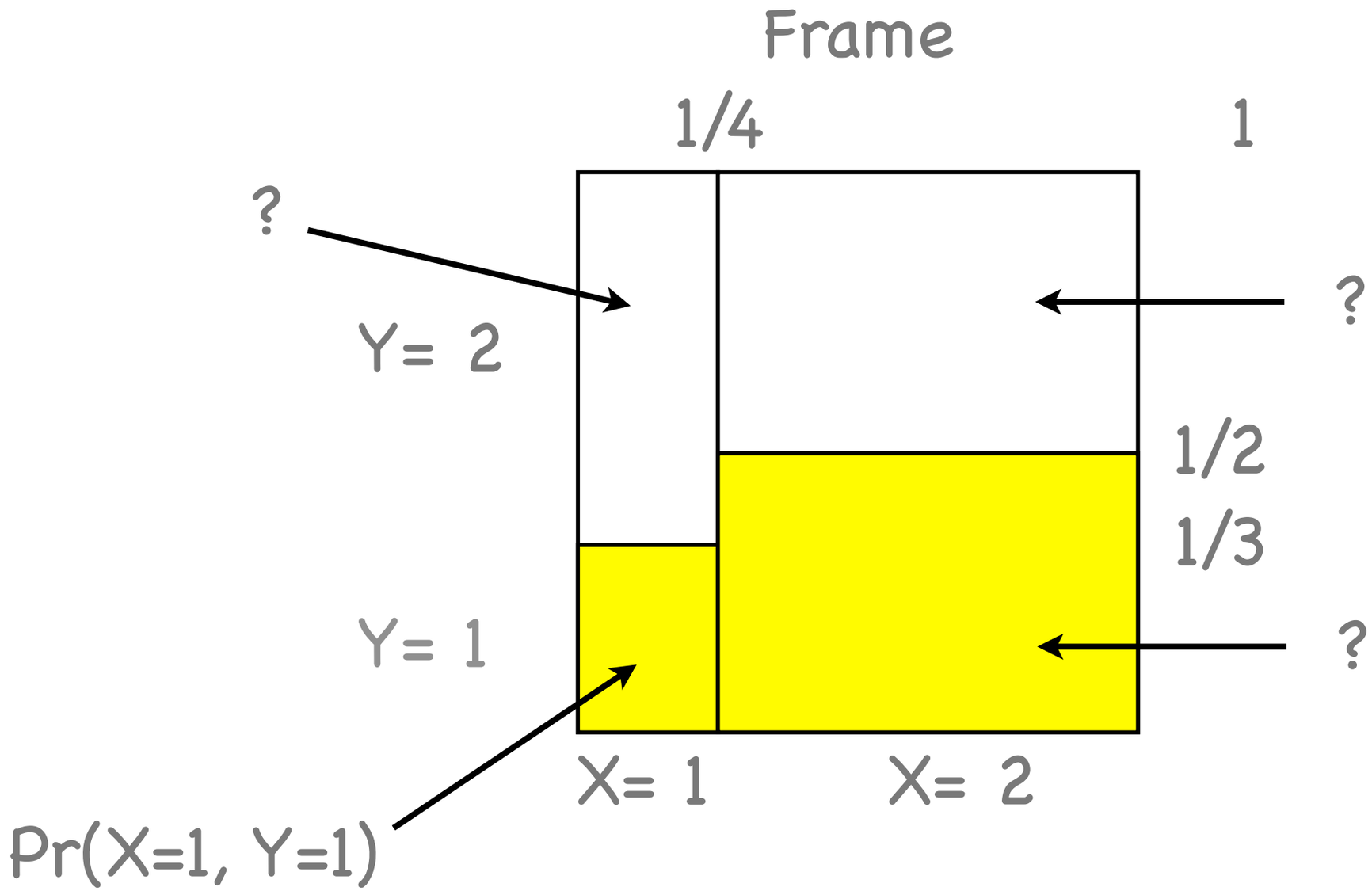

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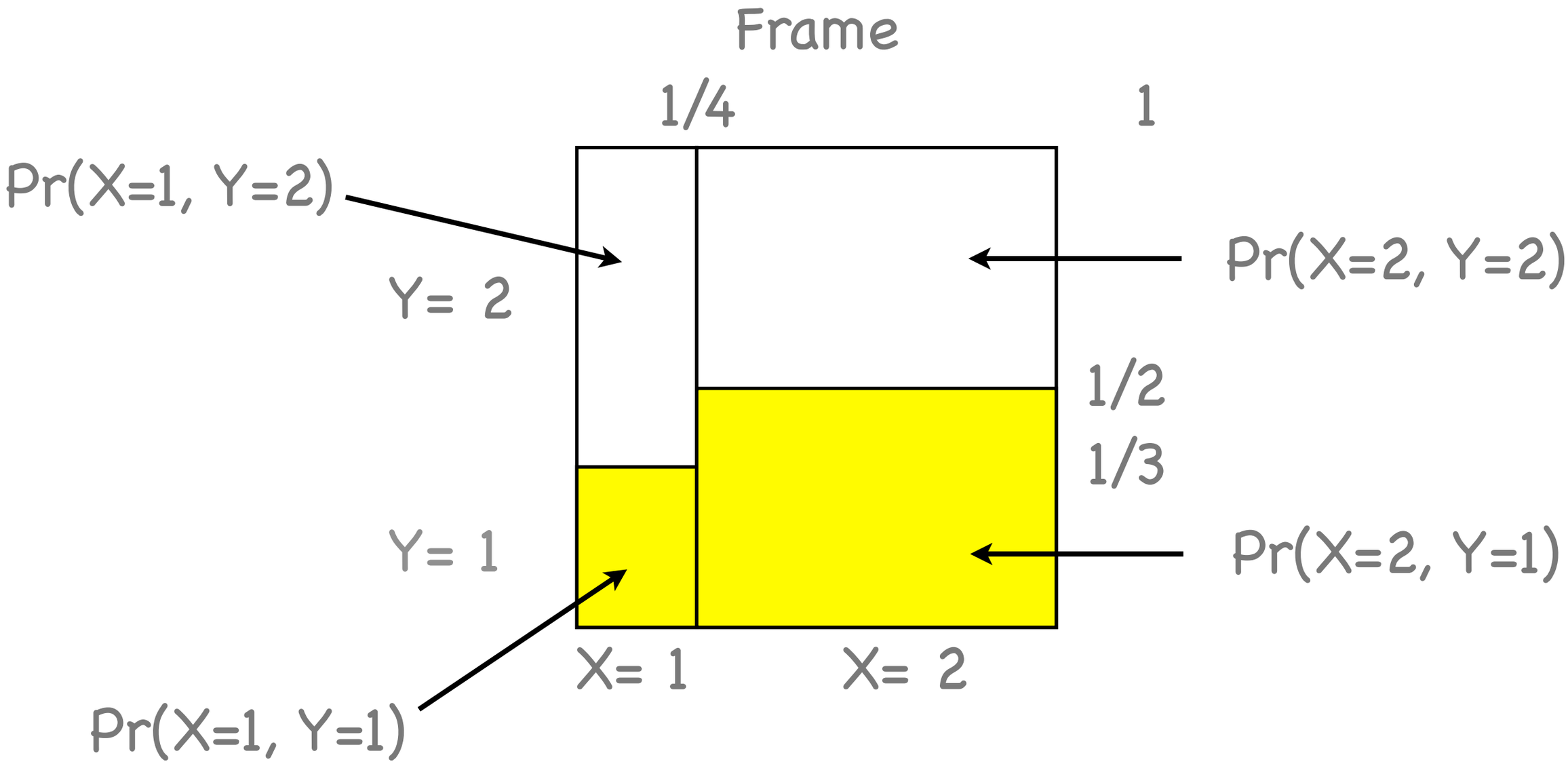

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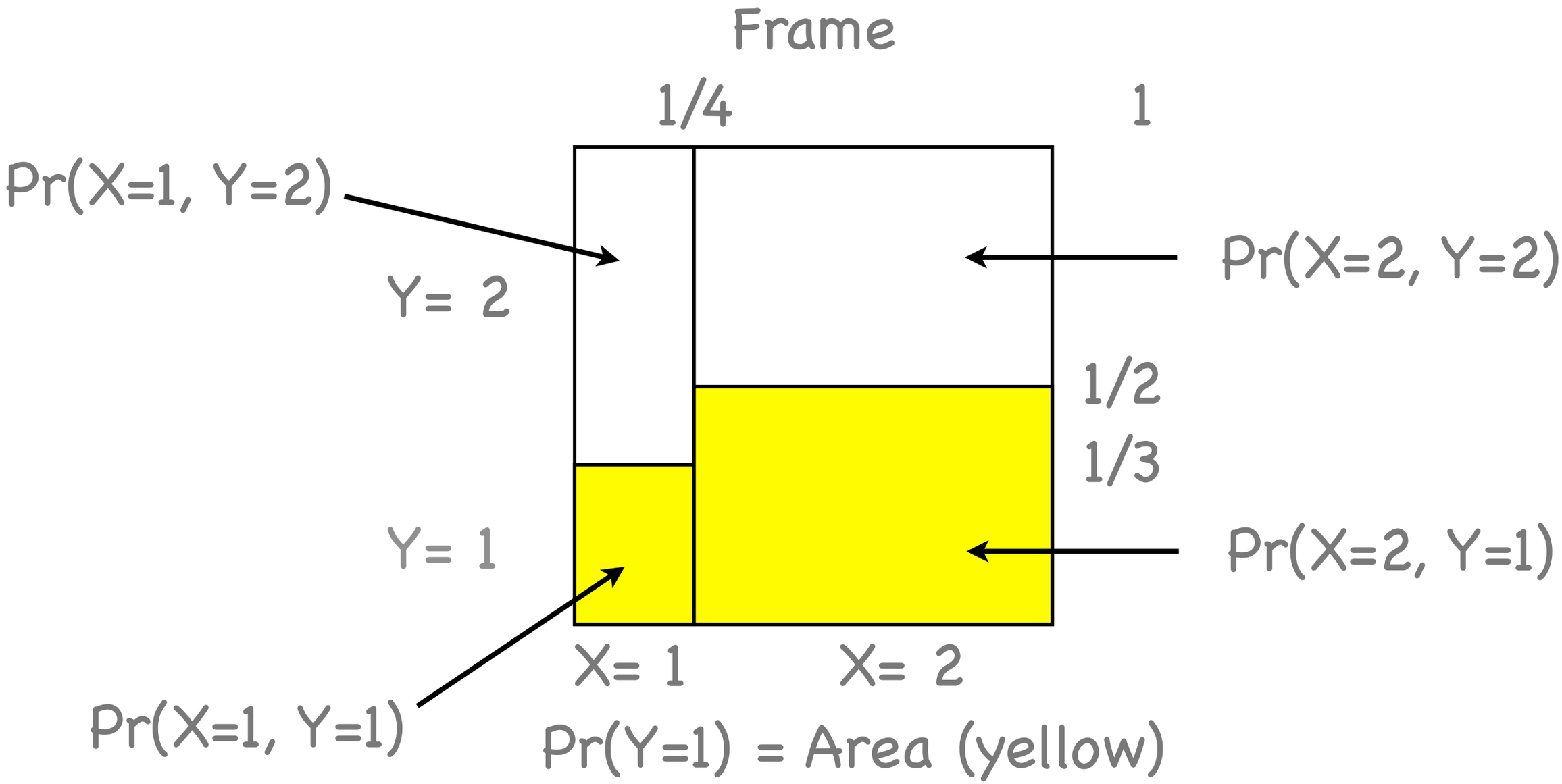

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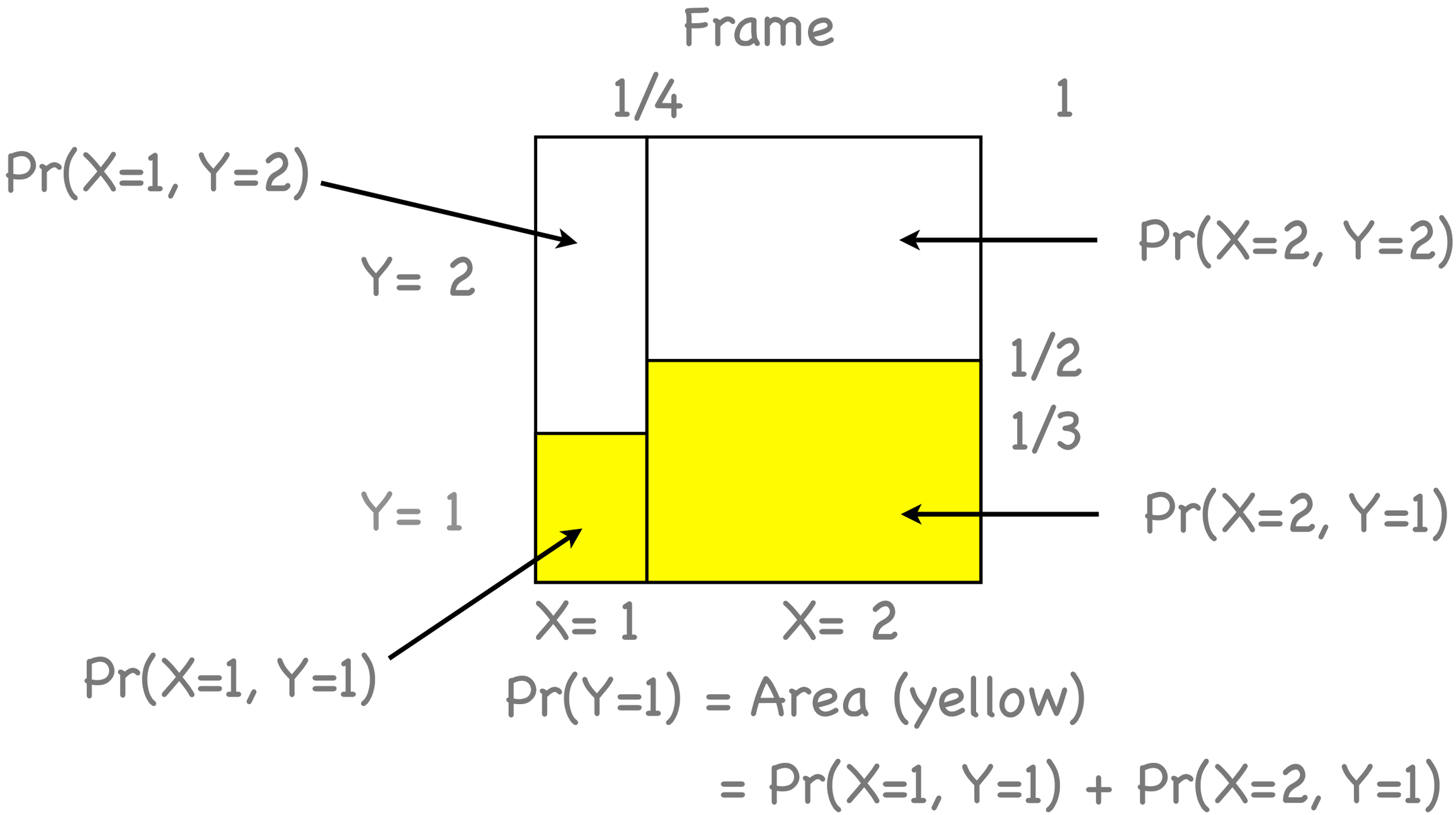

Pr(Y=1) = Area (yellow)

= Pr(X=1, Y=1) + Pr(X=2, Y=1)

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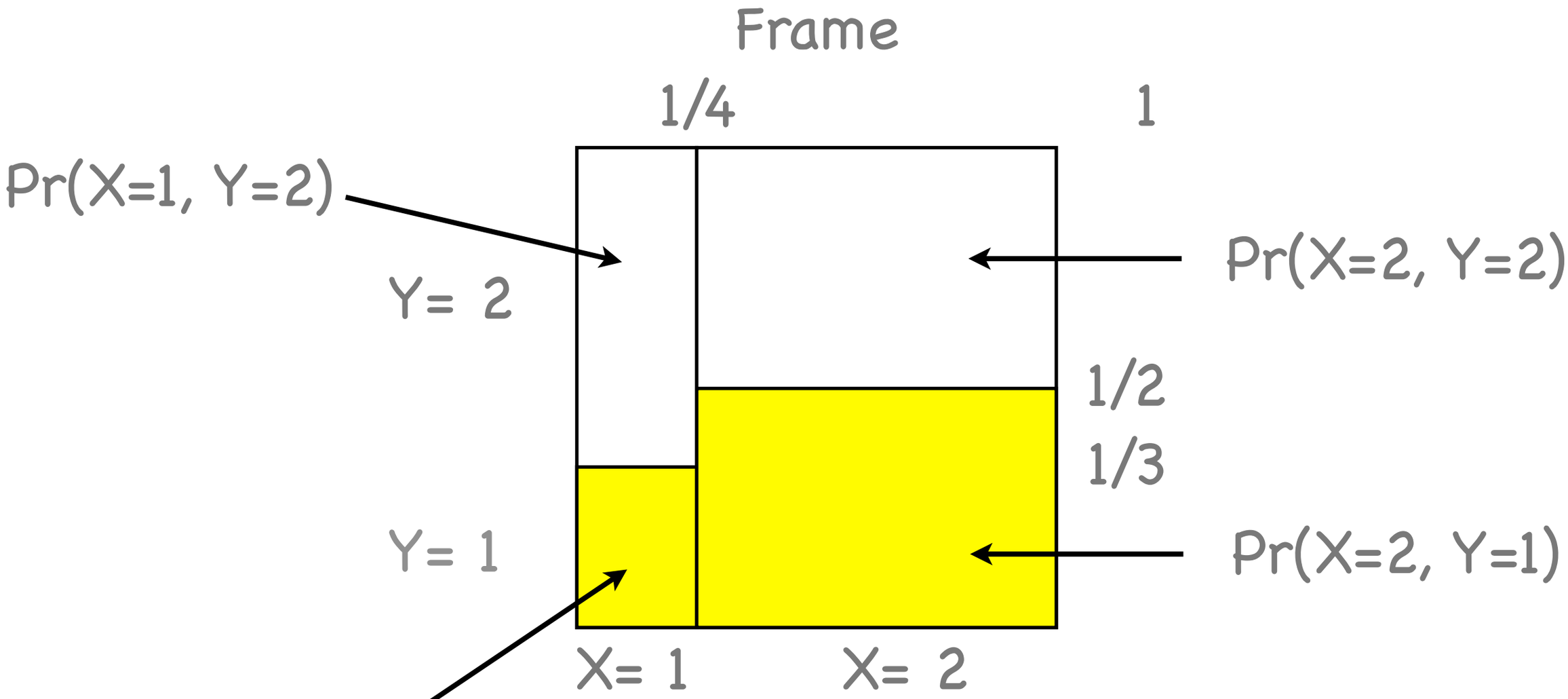

Pr(Y=1) = Area (yellow)

= Pr(X=1, Y=1) + Pr(X=2, Y=1)

Pr(Y=2) = Pr(X=1, Y=2) + Pr(X=2, Y=2)

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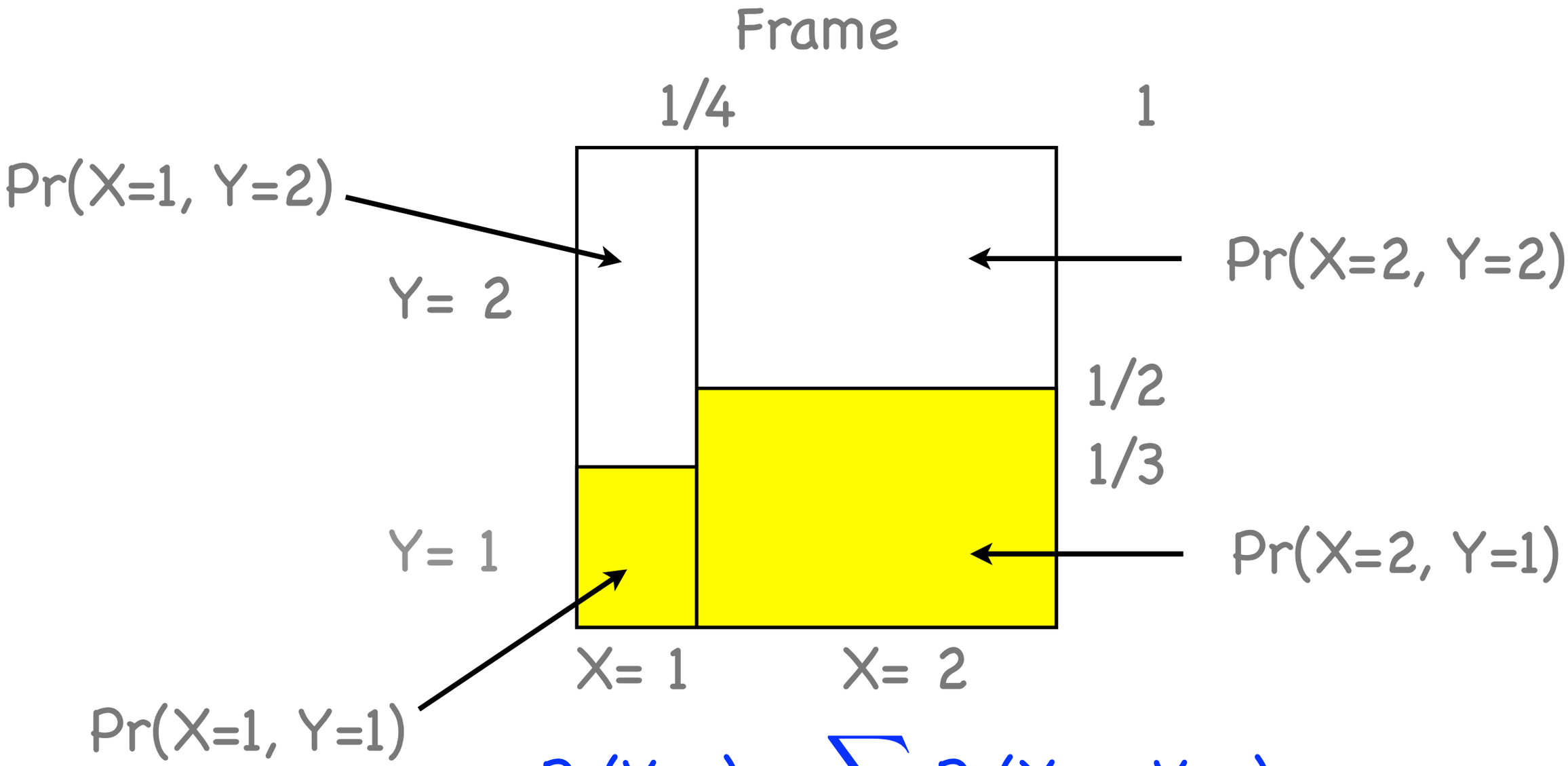

$$Pr(Y=y) = \sum_{all\ x} Pr(X=x, Y=y)$$

$$Pr(Y=y \mid Frame) = \sum_{all\ x} Pr(X=x, Y=y \mid Frame)$$

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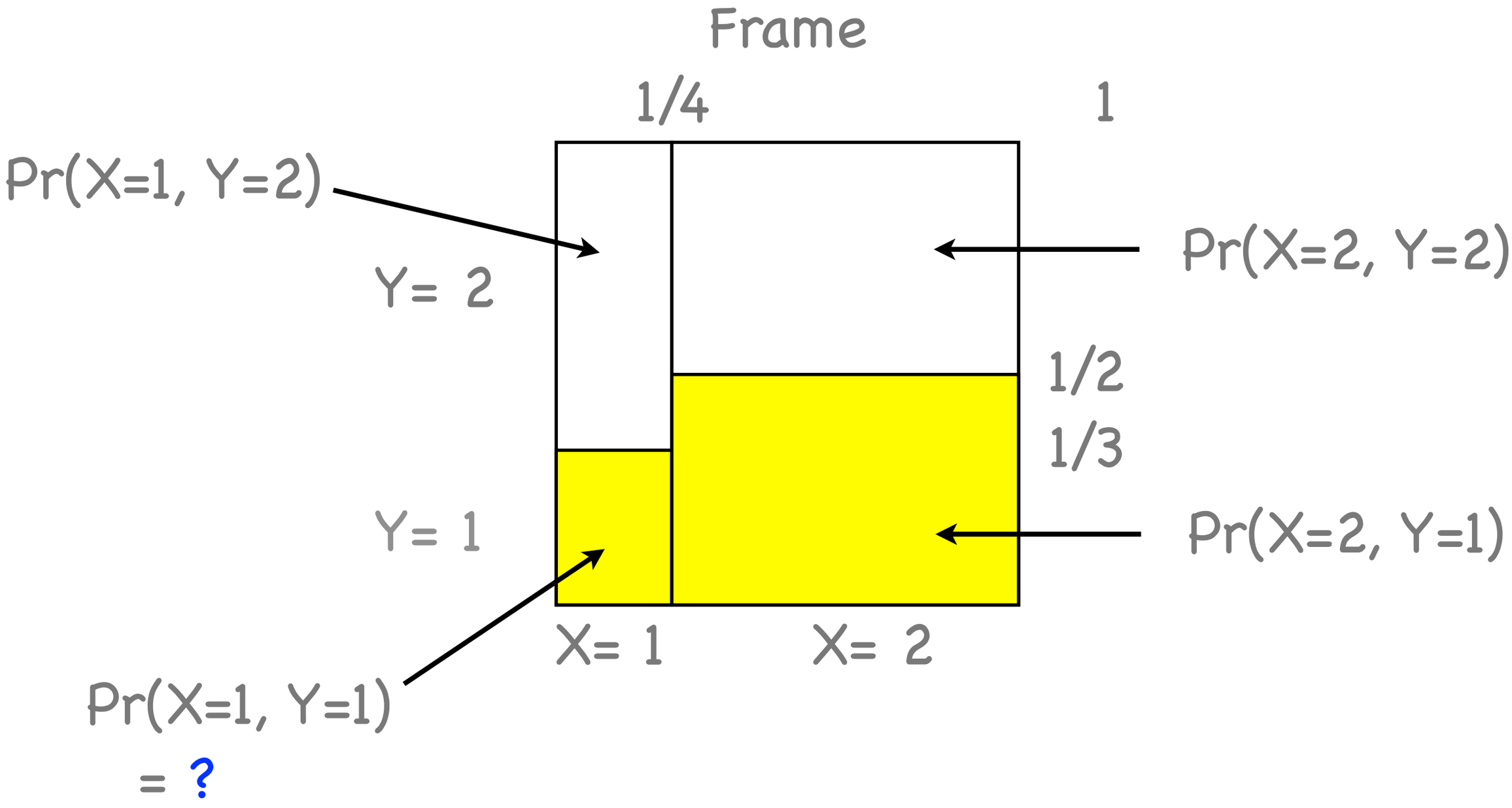

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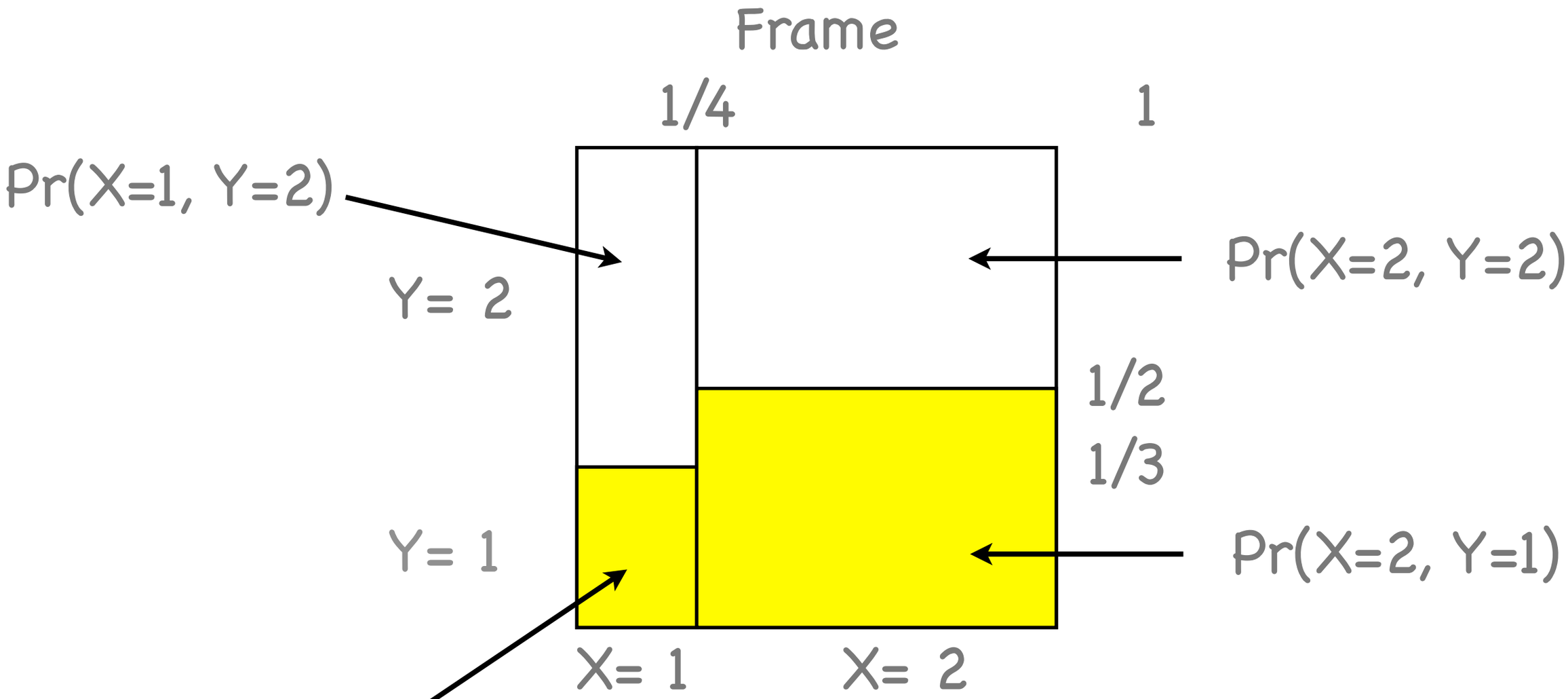

Pr(X=1, Y=1)

= Area(rectangle: X=1, Y=1)

= height x width

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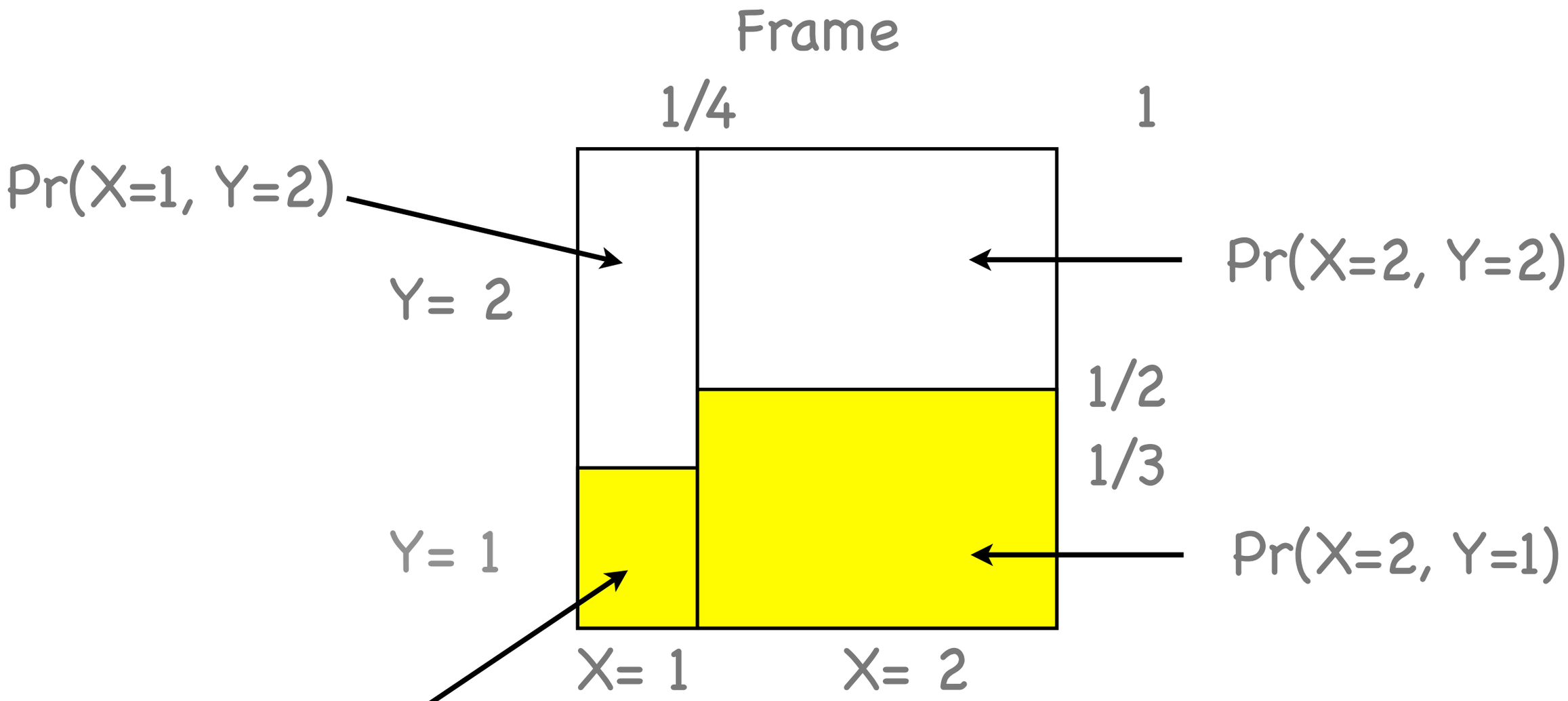

Pr(X=1, Y=1)

= Area(rectangle: X=1, Y=1)

= height x width

= Pr(Y=1 | X=1) x Pr(X=1)

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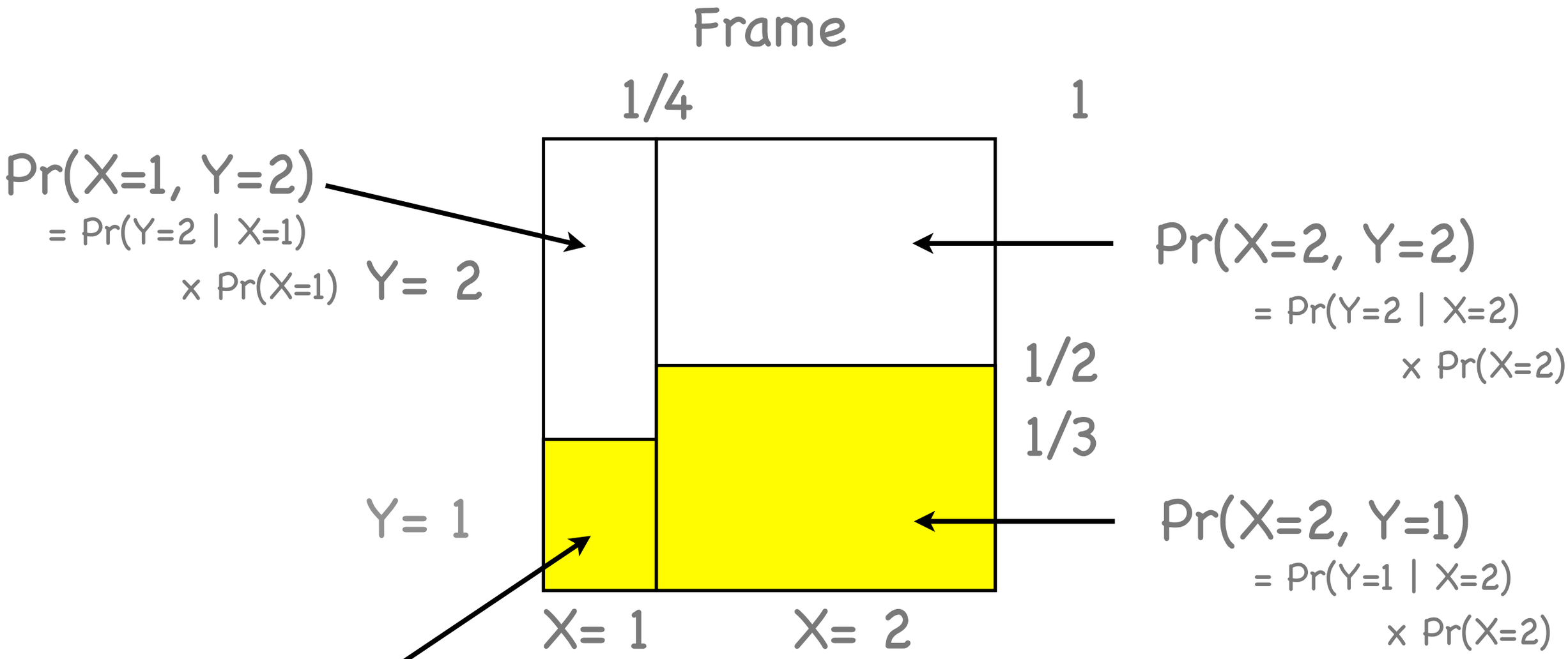


= Area(rectangle: X=1, Y=1)

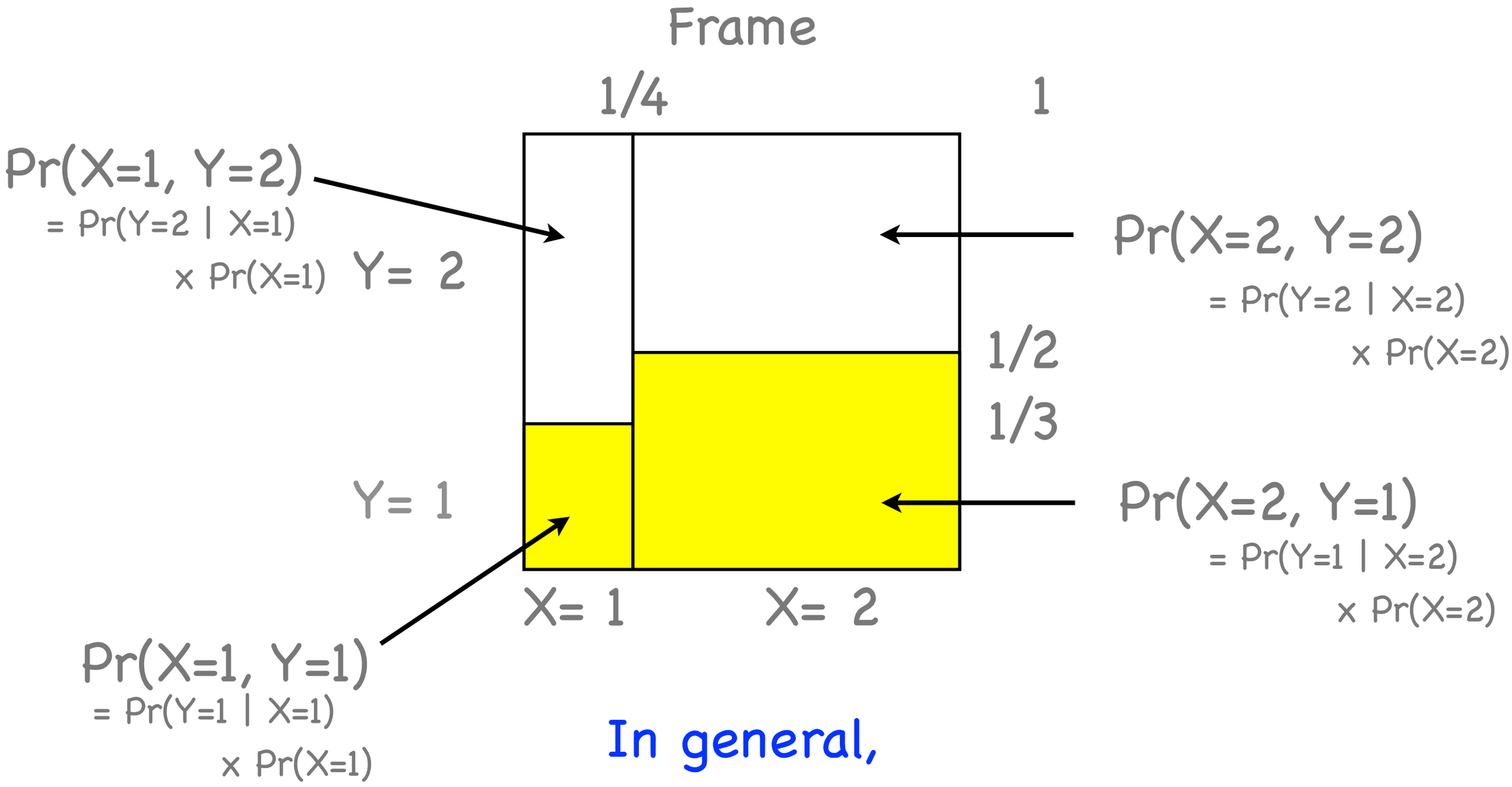
= height x width

= Pr(Y=1 | X=1) x Pr(X=1)

Eikosogram

eikos - like truth, likely, probable, reasonable, probability, likelihood *

gramma - that which is drawn, picture, piece of writing, ...



In general,

$$P(X,Y) = Pr(Y \mid X) \times Pr(X)$$

Eikosogram

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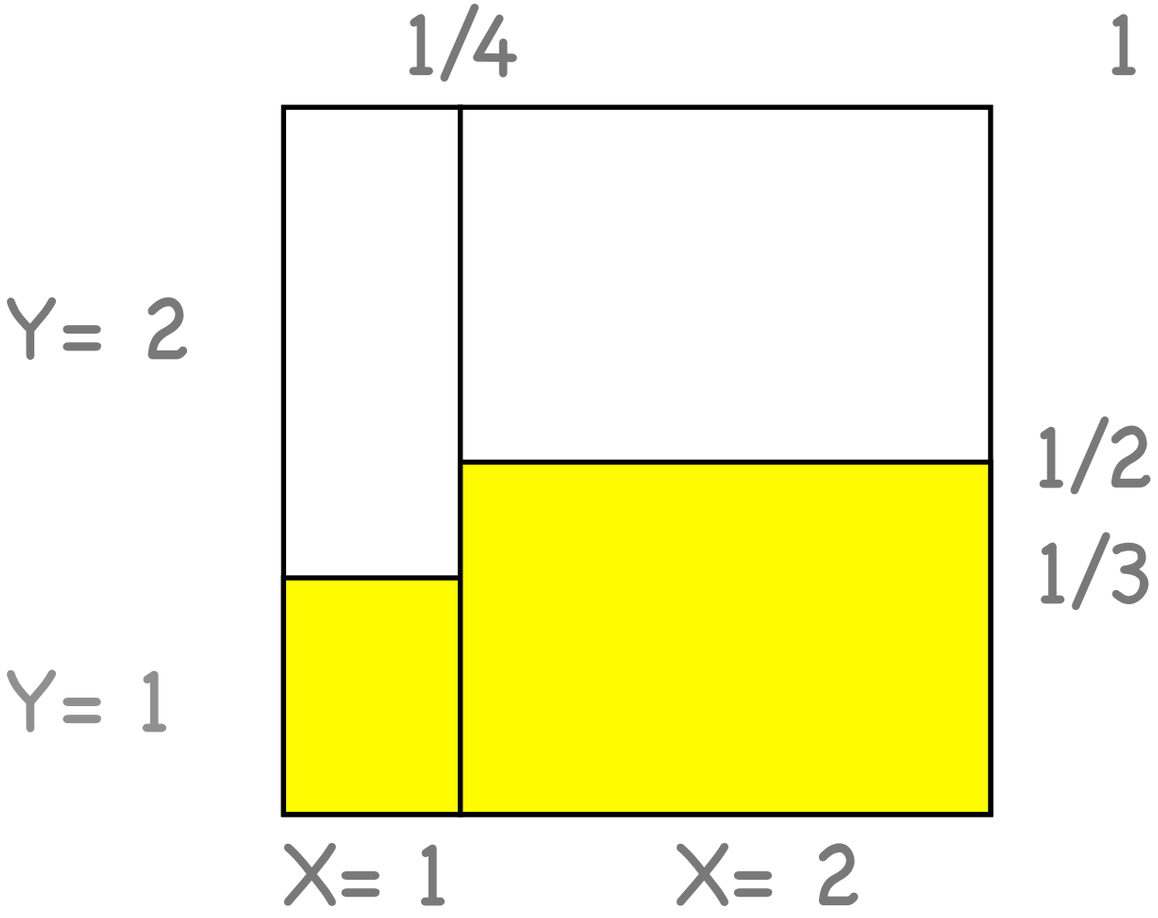
Frame



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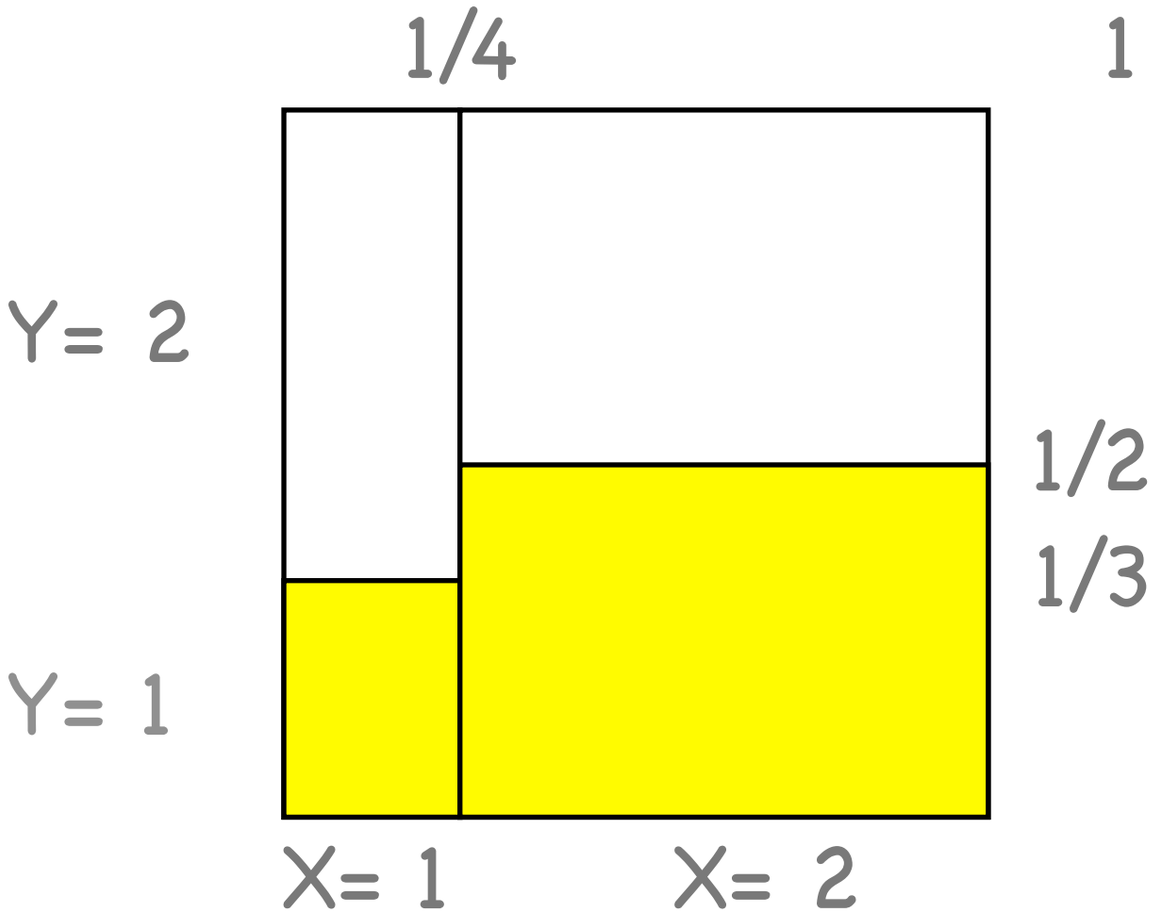
Reorder X and Y:



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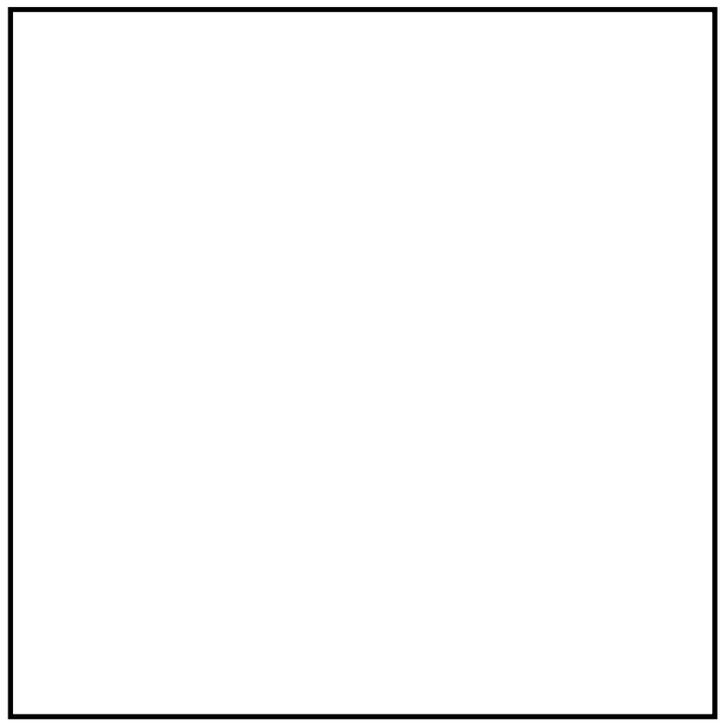

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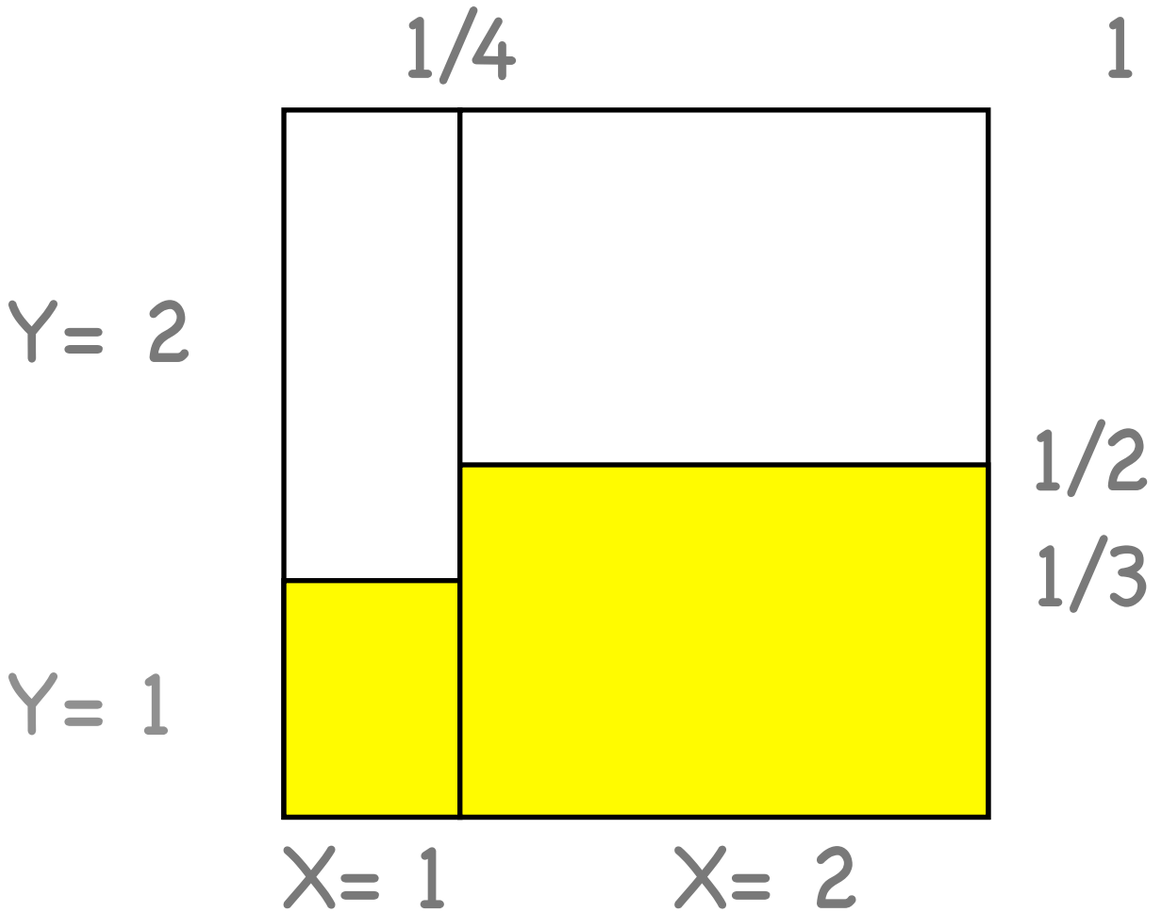

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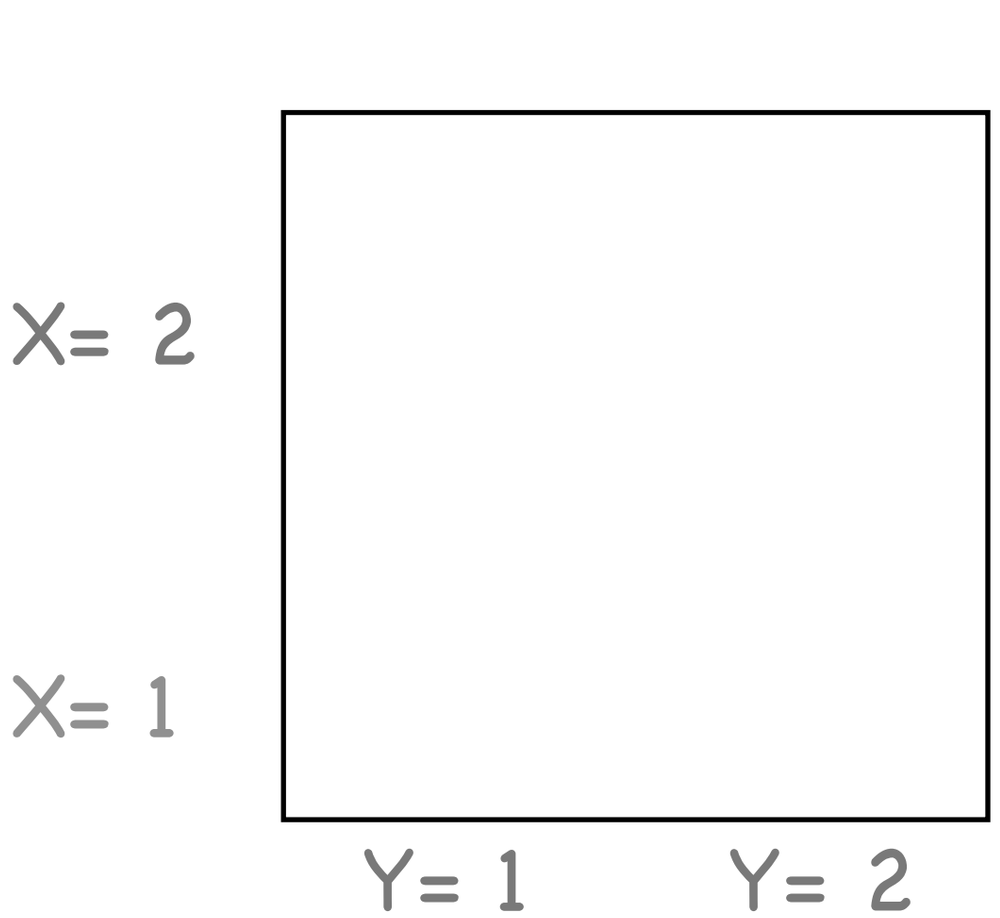

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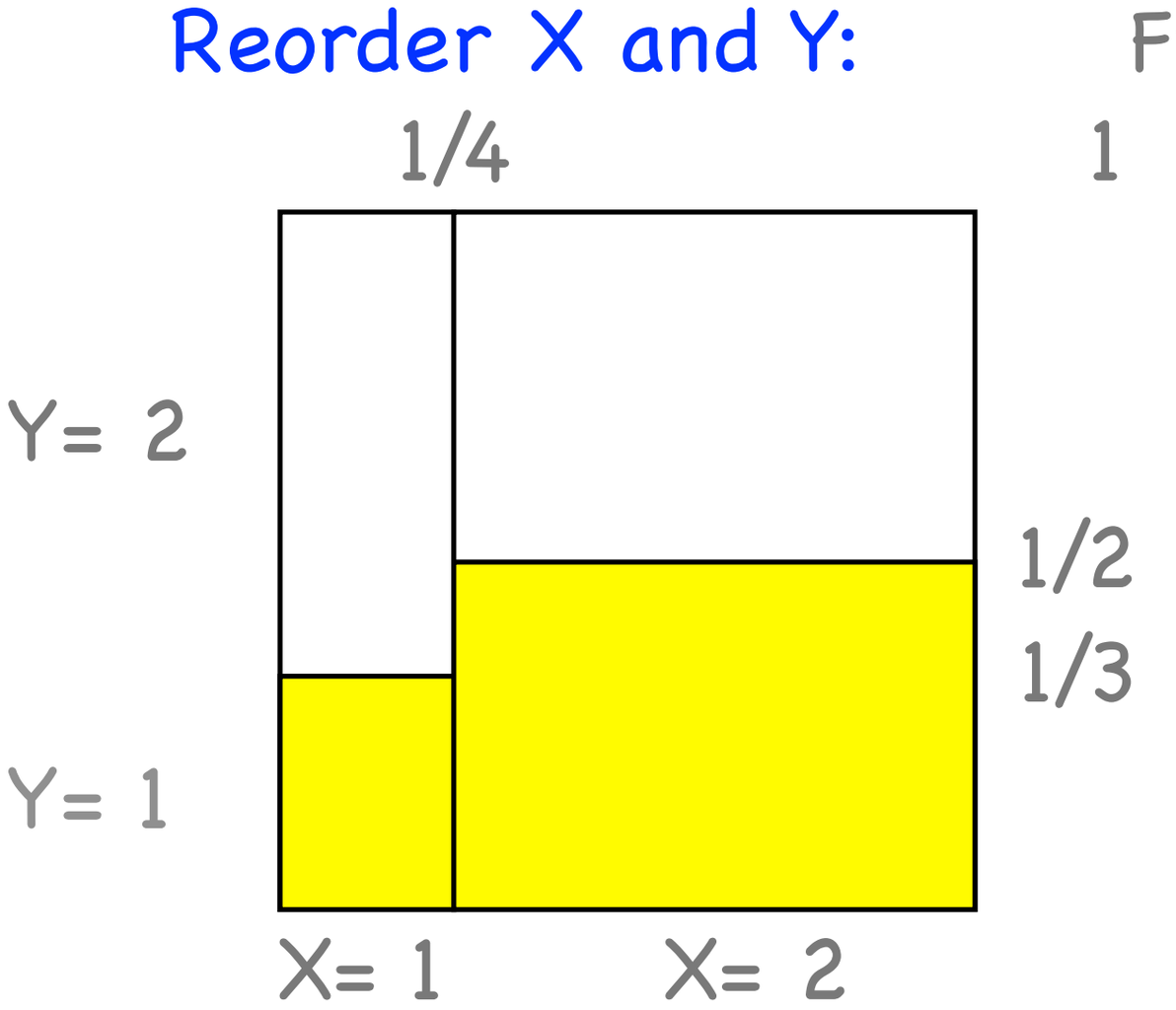

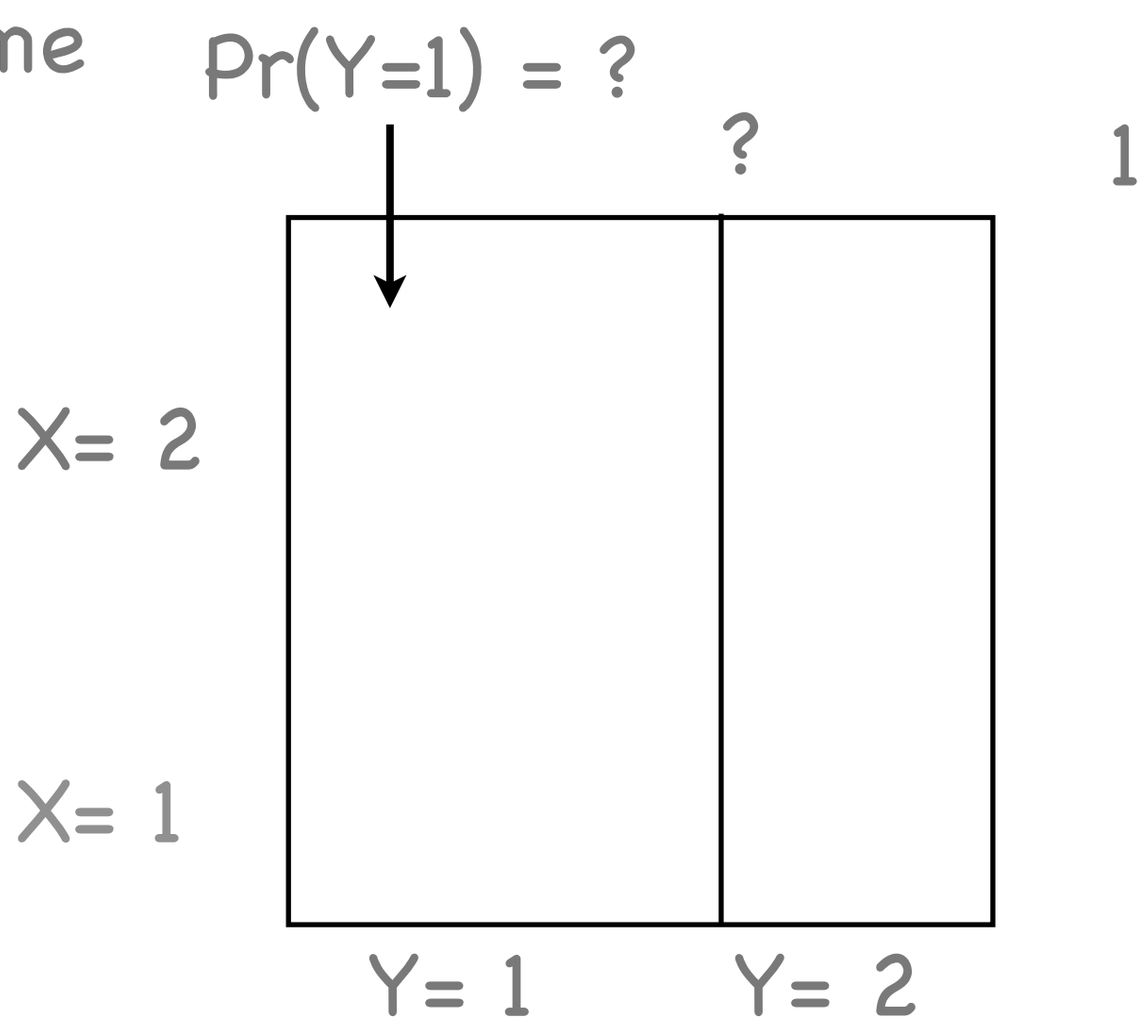

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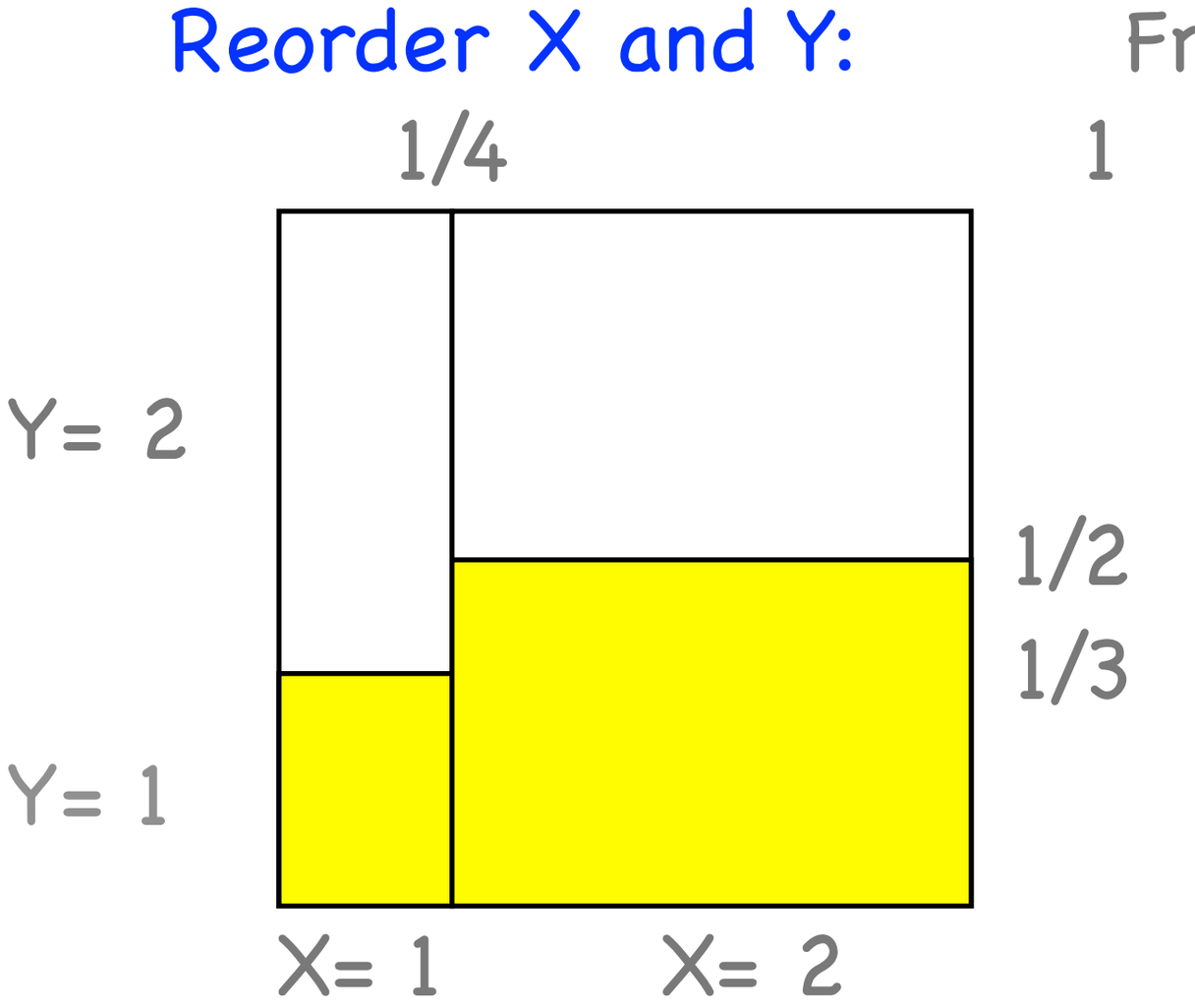

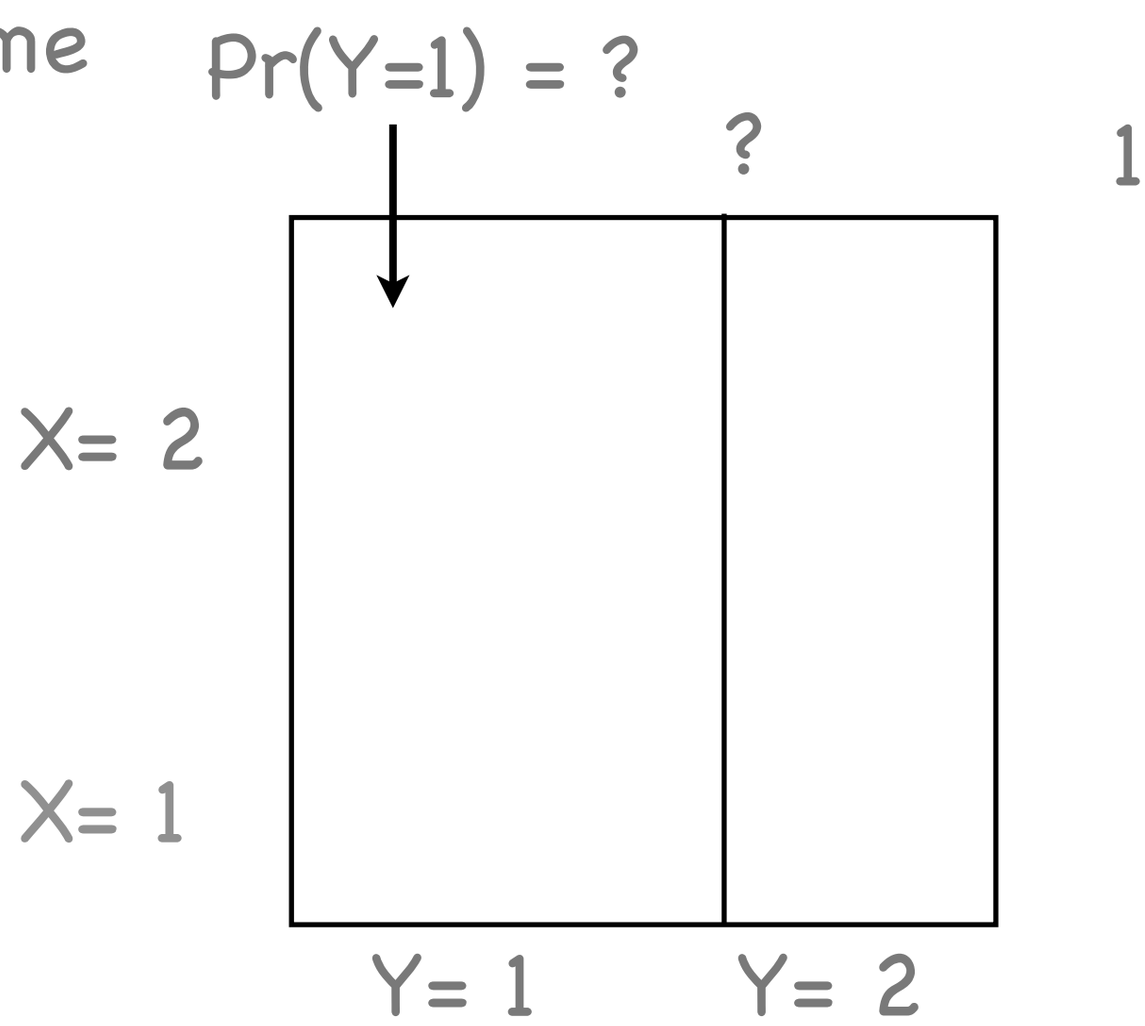

$$Pr(Y= 1) = Pr(X=1, Y=1) + Pr(X=2, Y=1)$$

Eikosogram

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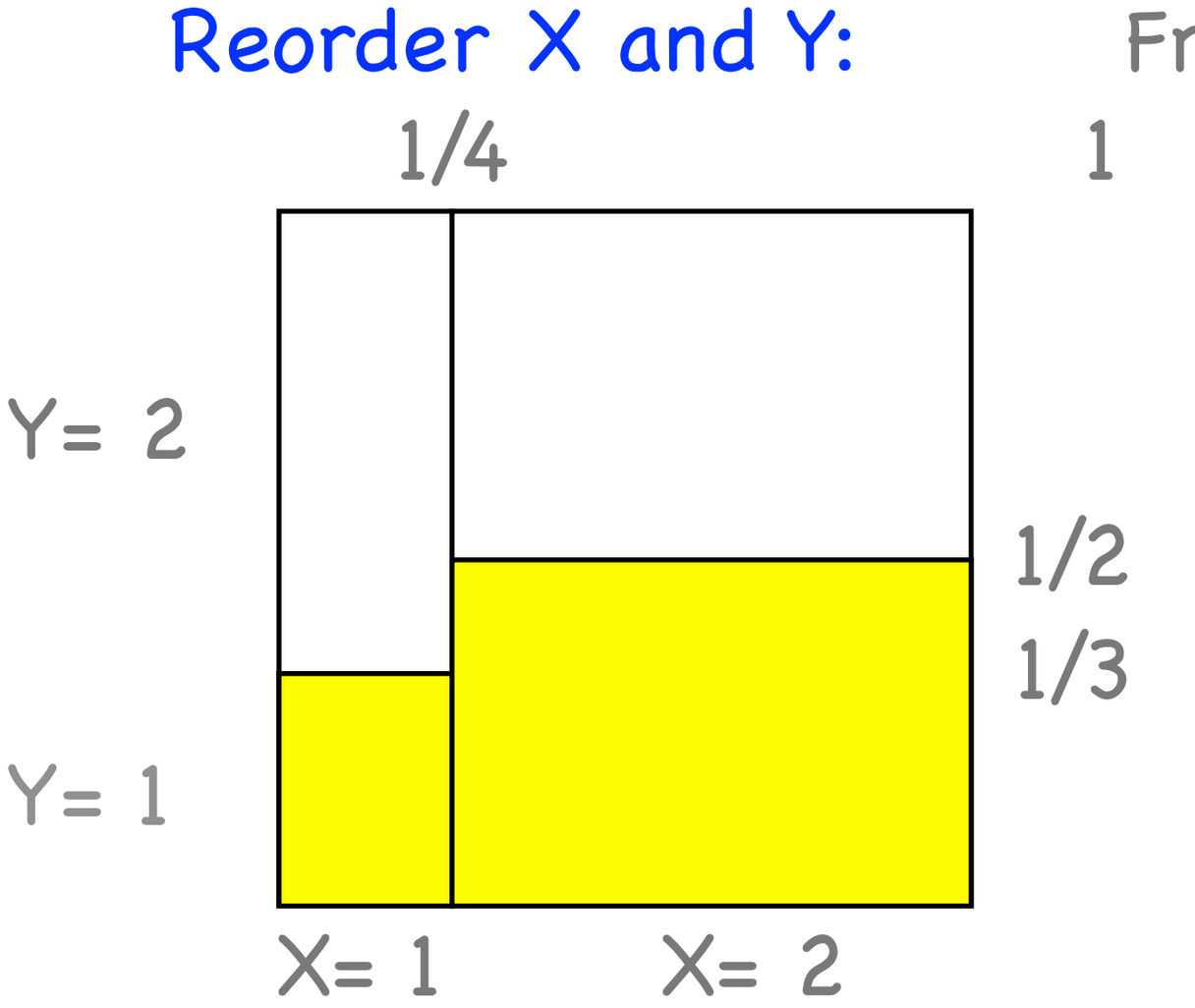
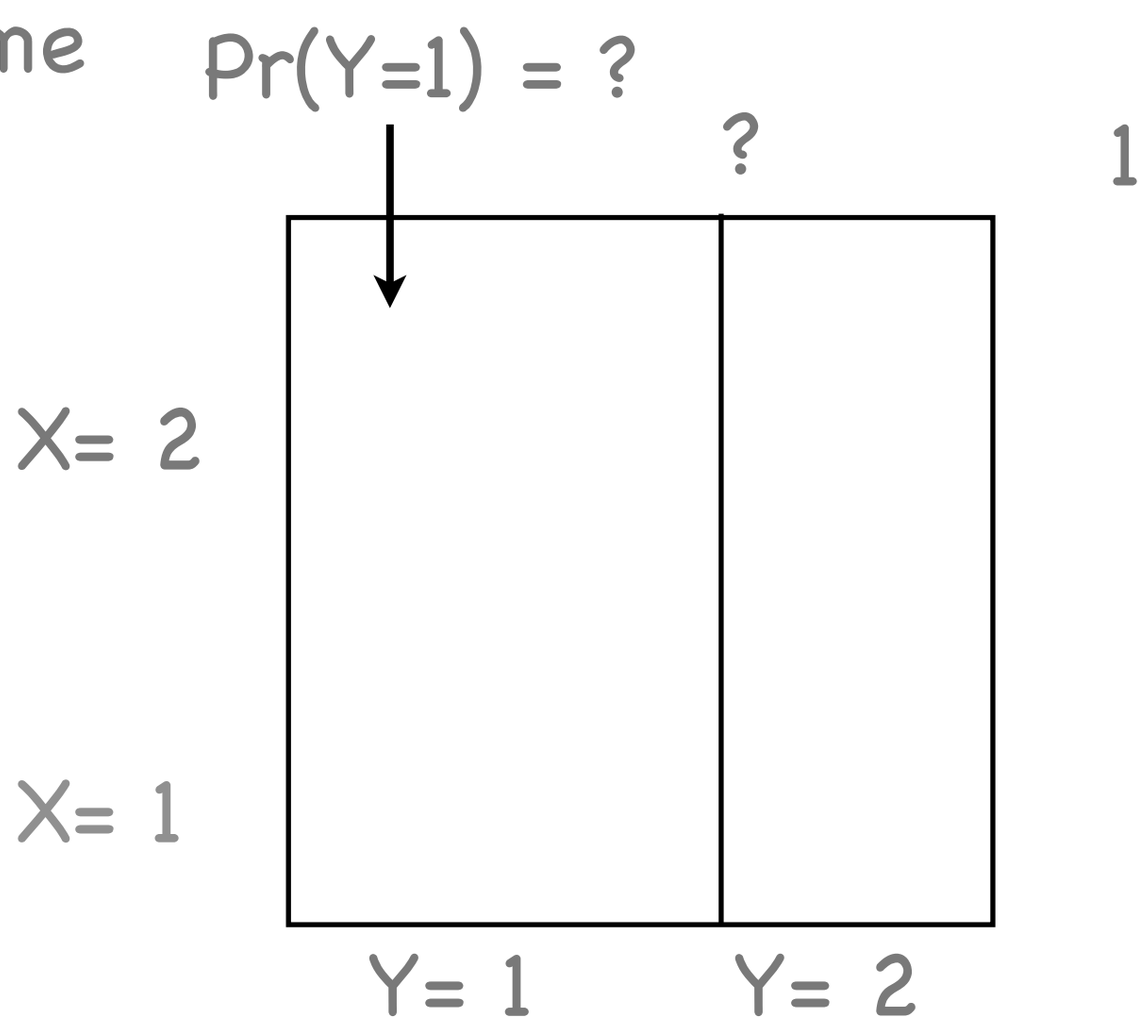
Reorder X and Y:





$$\Pr(Y=1) = \Pr(X=1, Y=1) + \Pr(X=2, Y=1)$$
$$= (1/3 \times 1/4) + (1/2 \times 3/4)$$
$$= 11/24$$

Eikosogram

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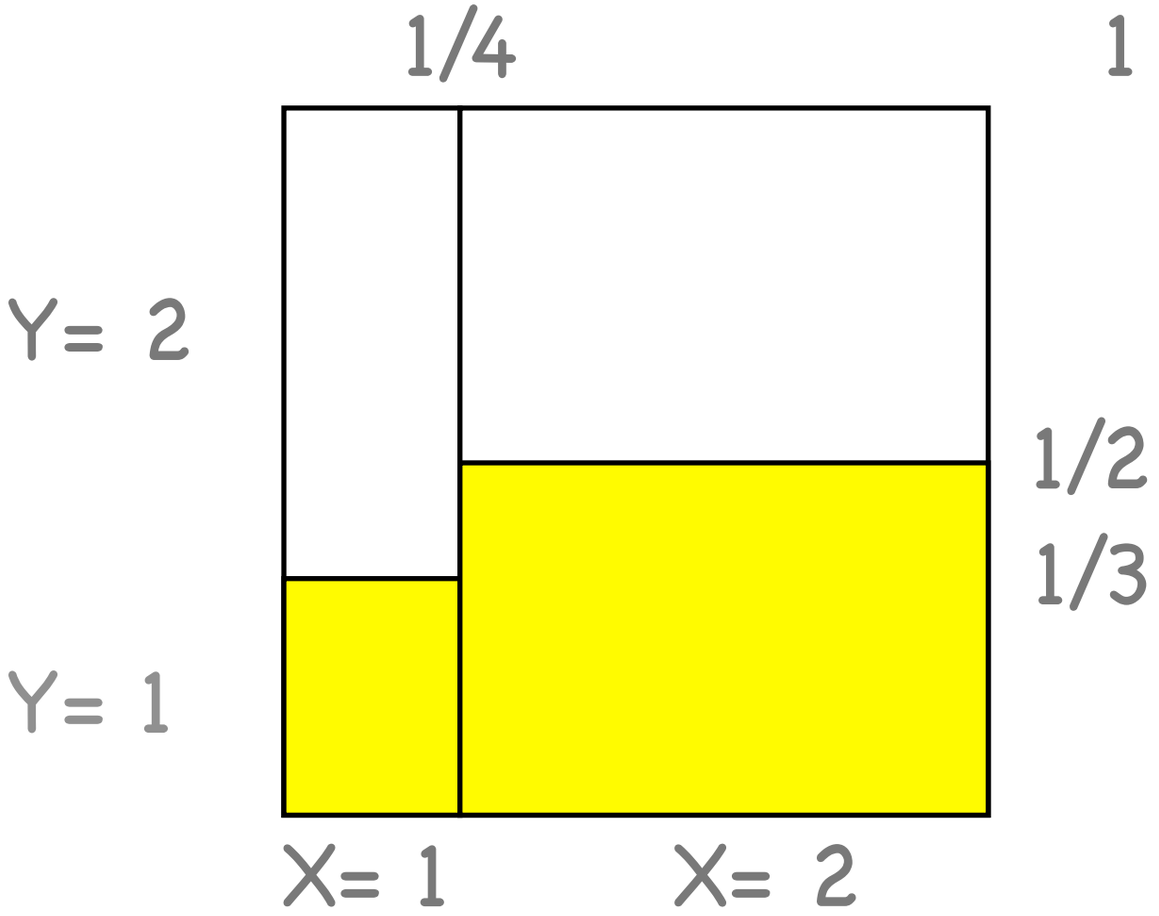
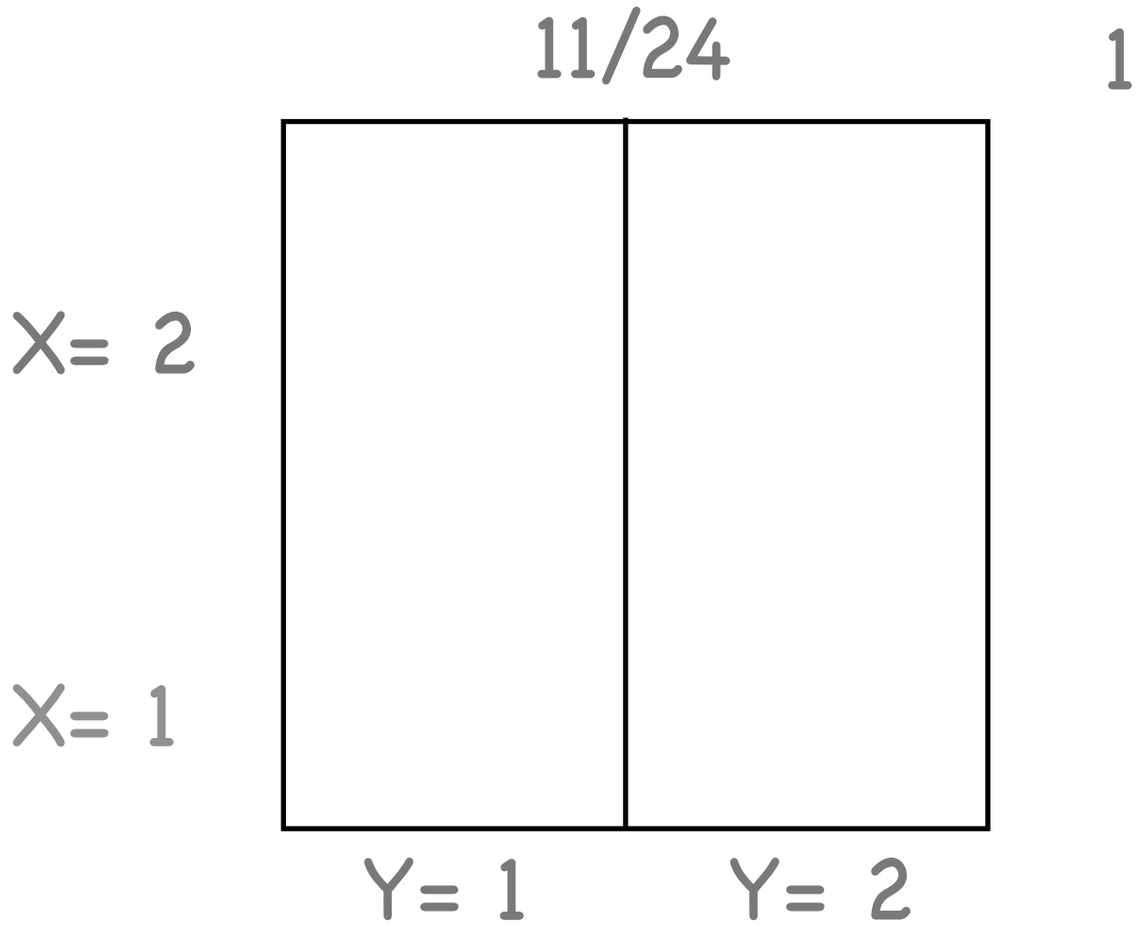
Reorder X and Y:



Frame



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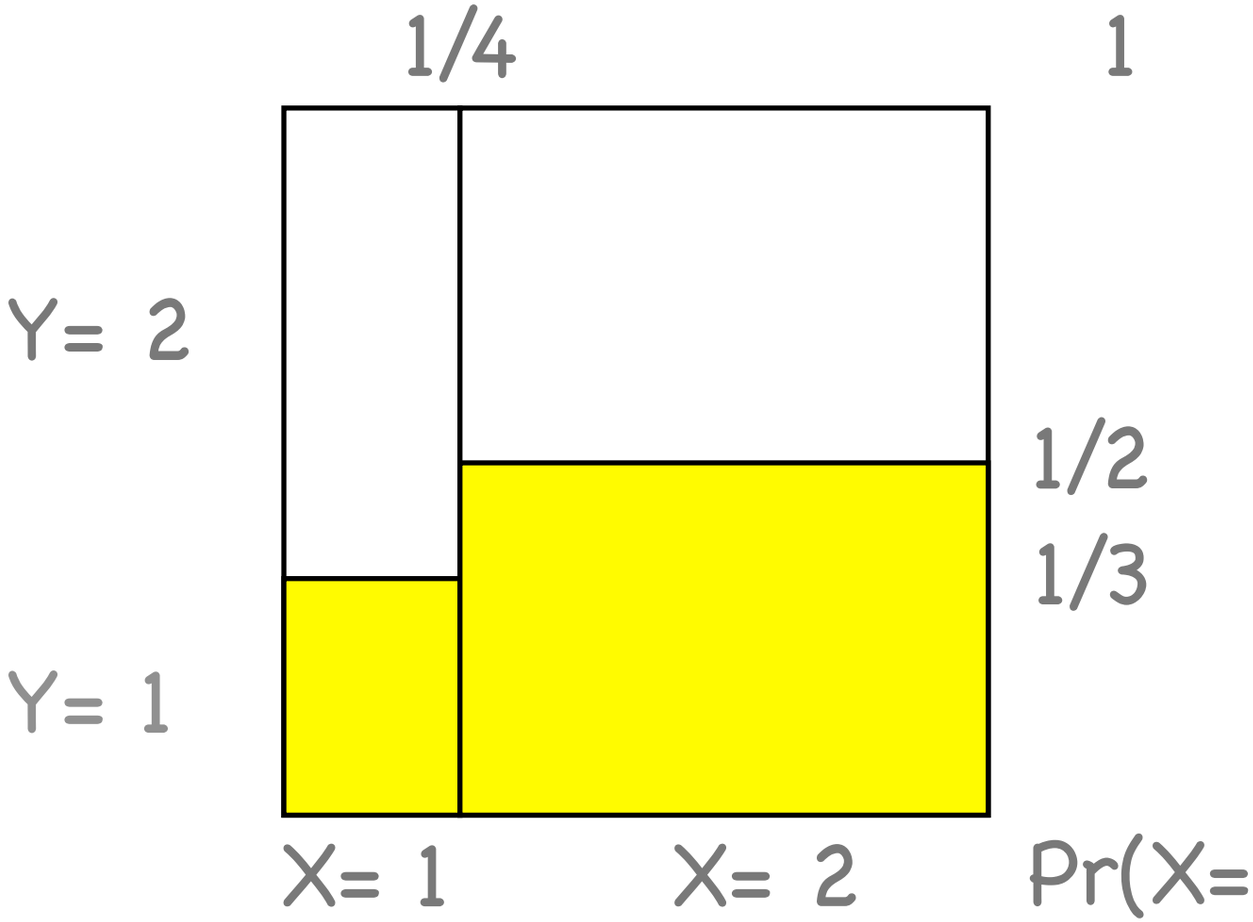

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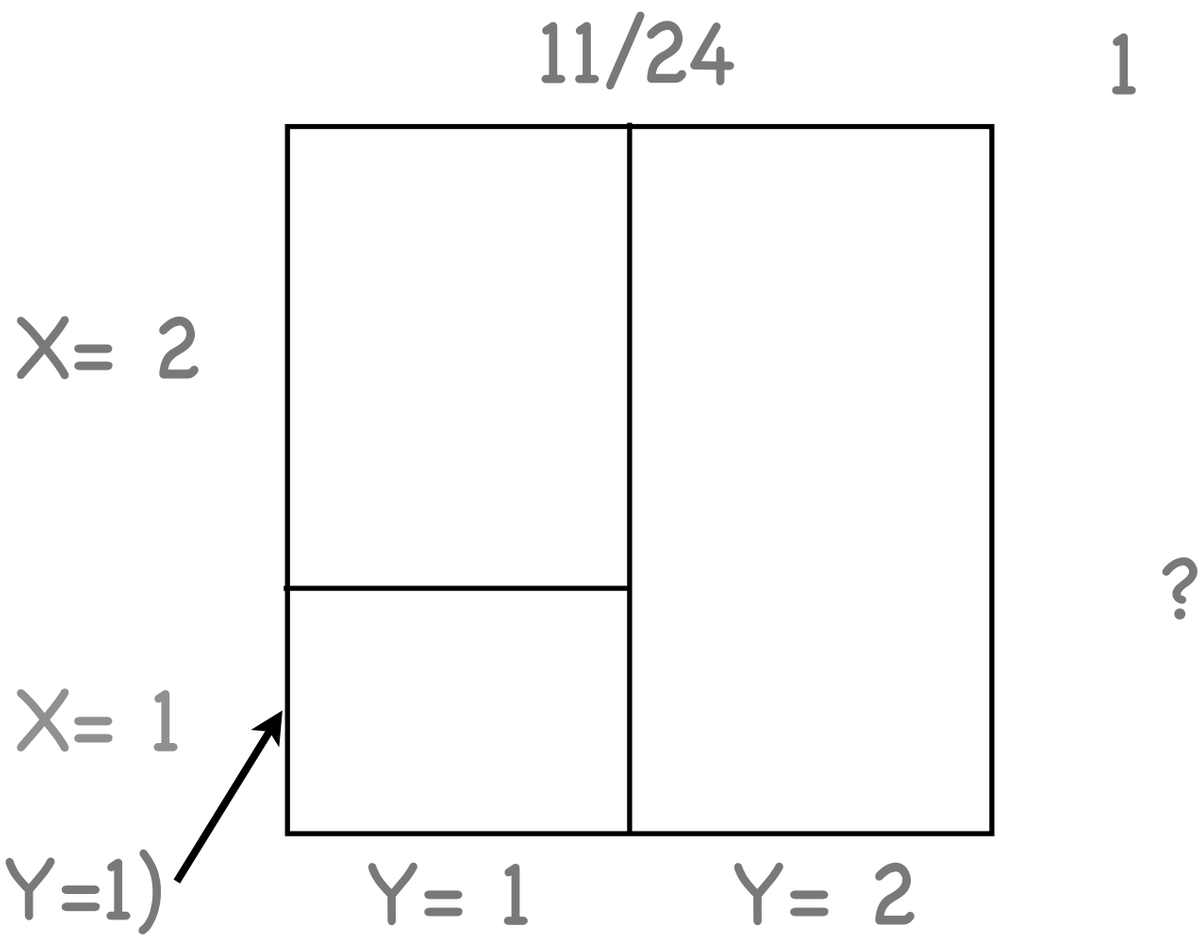

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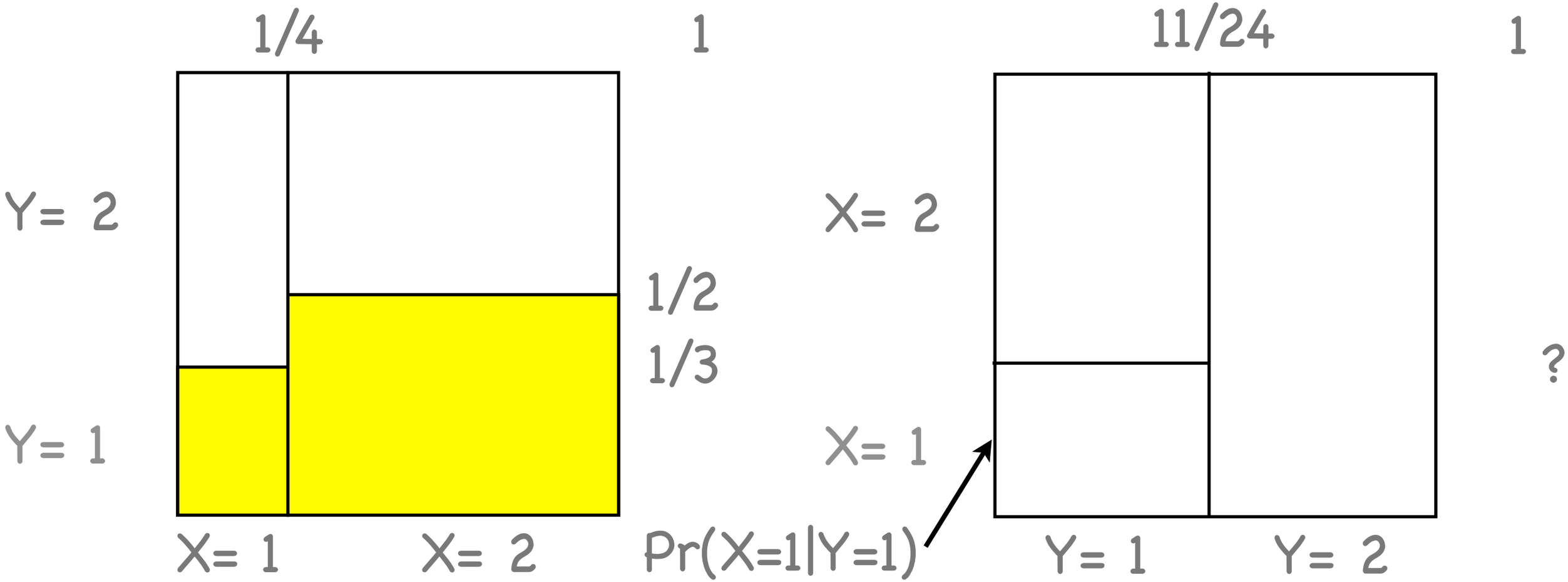
Reorder X and Y:

Frame



$$Pr(X=1 \mid Y=1) = Pr(X=1, Y=1) / Pr(Y=1)$$

Eikosogram

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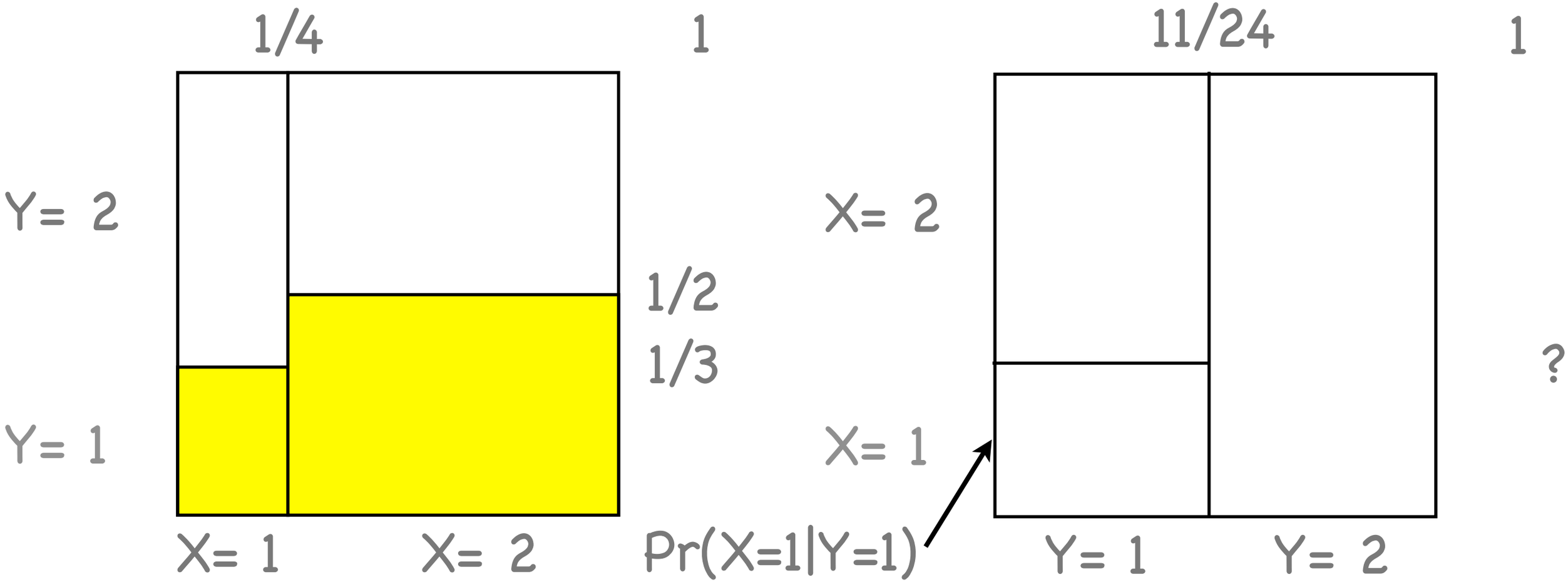
gramma - that which is drawn, picture, piece of writing, ...

Reorder X and Y:

Frame



$$\Pr(X=1 \mid Y=1) = \Pr(X=1, Y=1) / \Pr(Y=1)$$

$$= (1/3 \times 1/4) / (11/24)$$

$$= 2/11$$

Eikosogram

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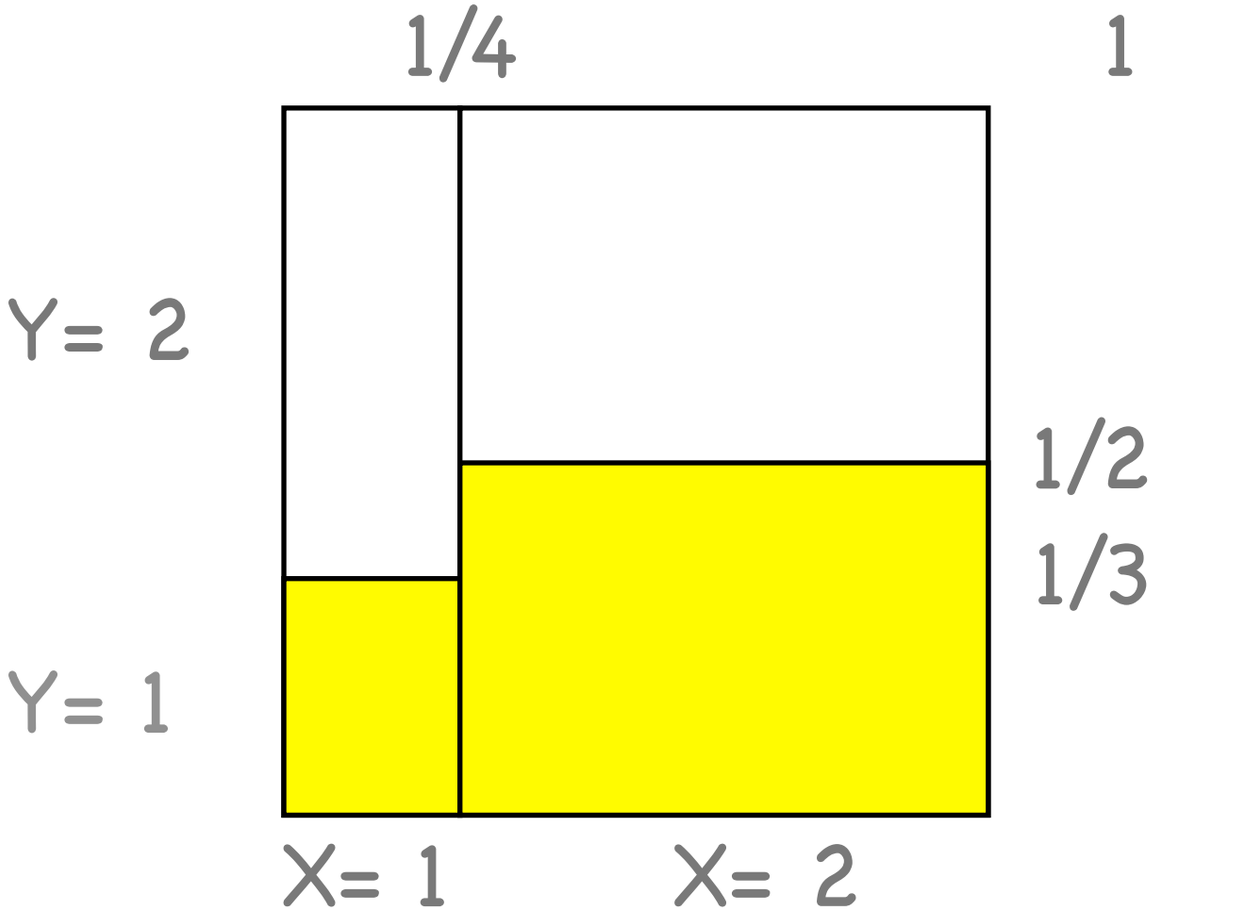
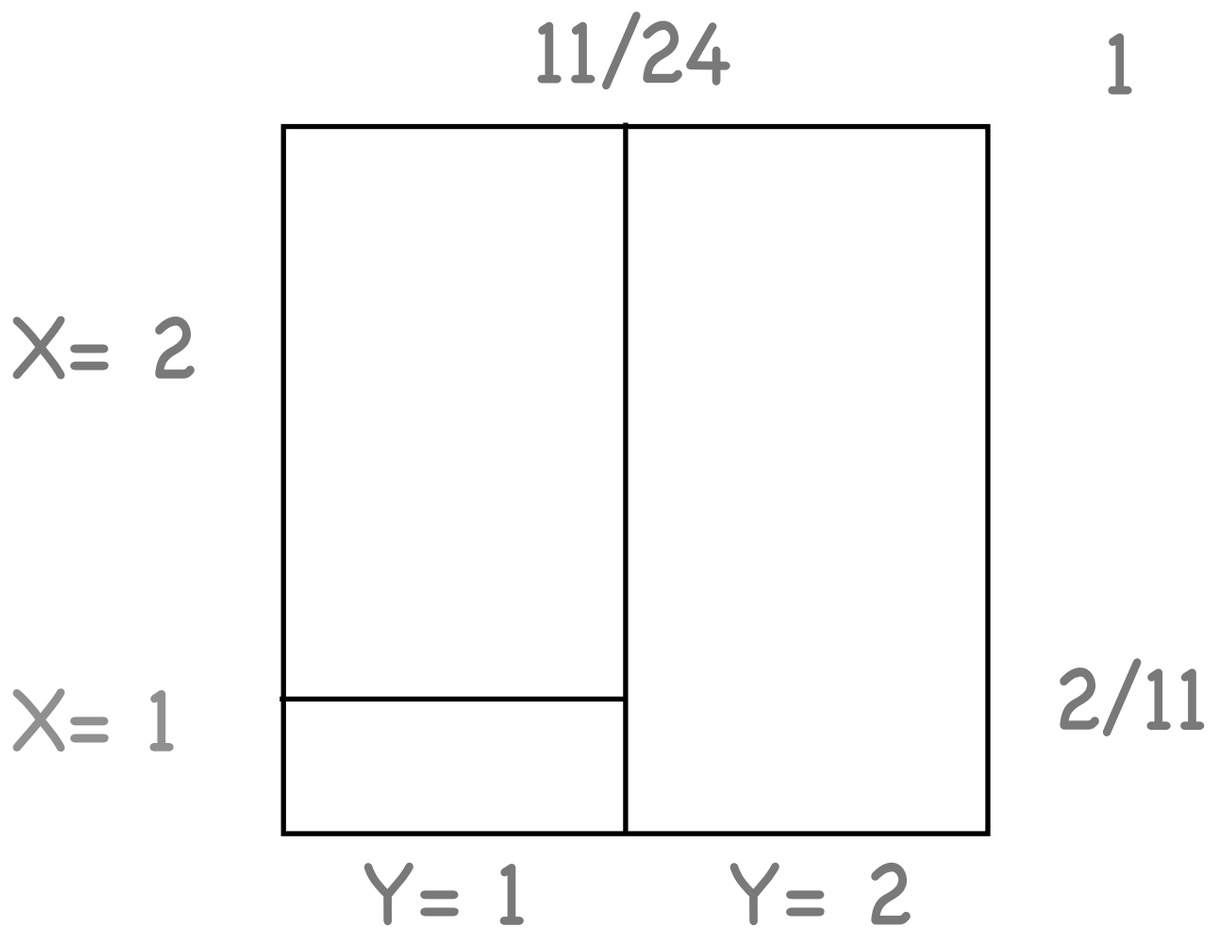
Reorder X and Y:

Frame





$$\Pr(X=1 \mid Y=1) = \Pr(X=1, Y=1) / \Pr(Y=1)$$
$$= (1/3 \times 1/4) / (11/24)$$
$$= 2/11$$

Eikosogram

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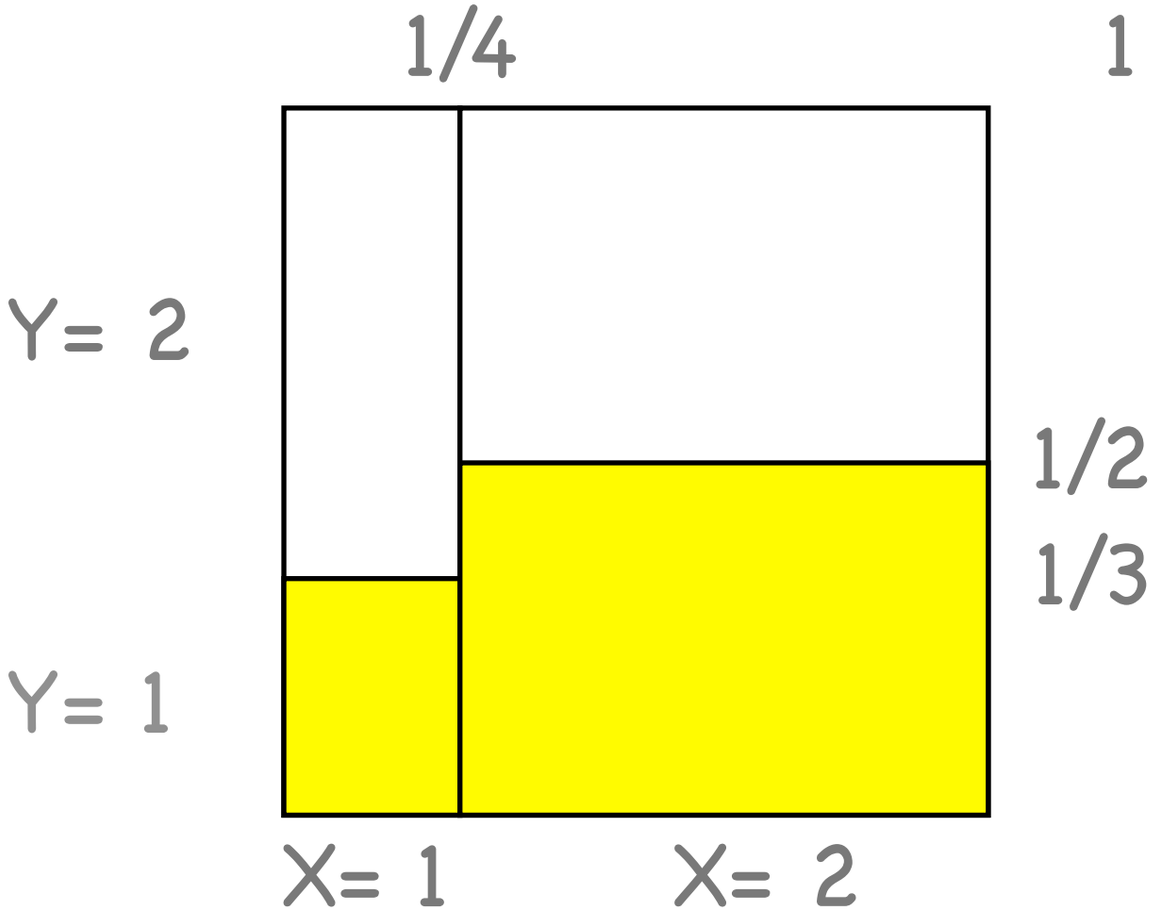
Reorder X and Y:



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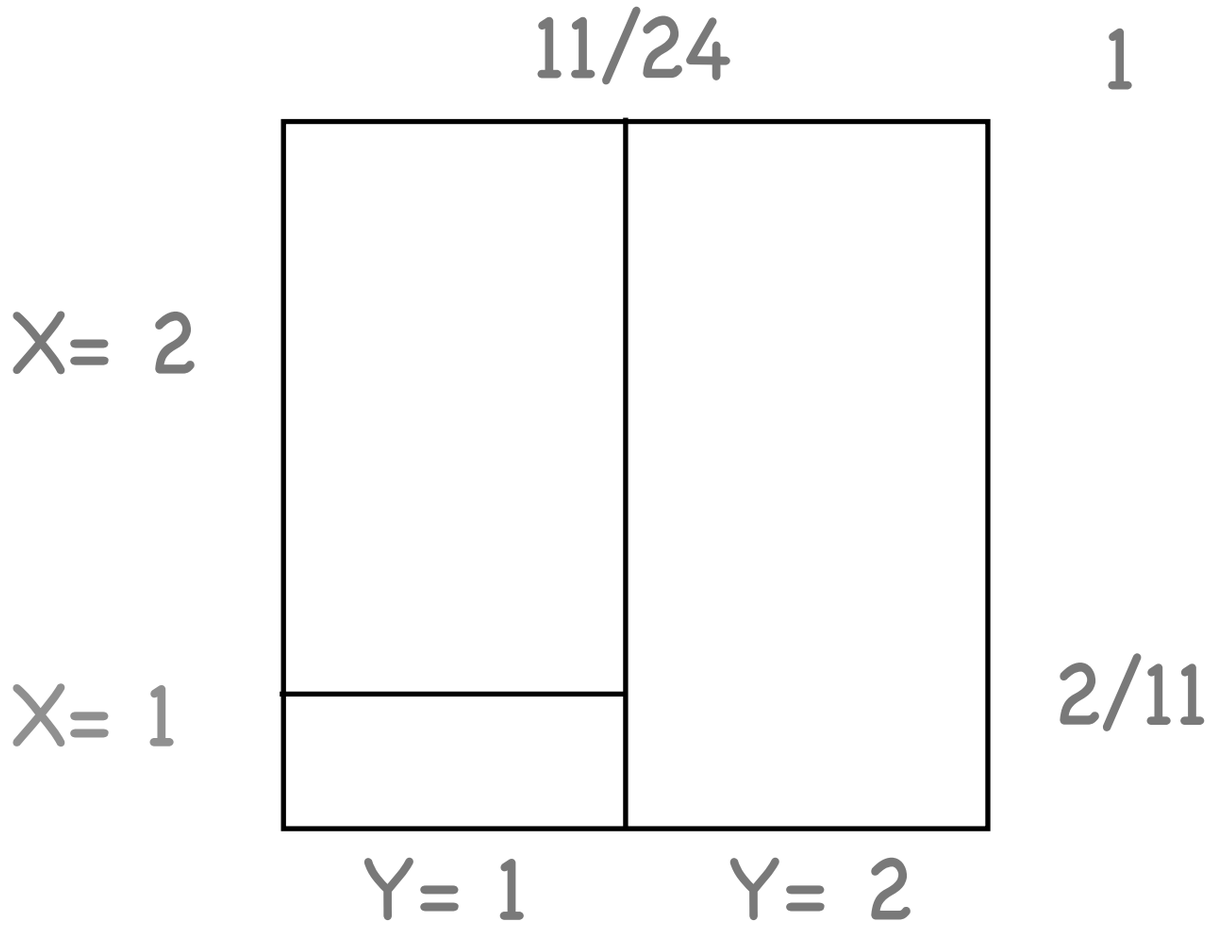

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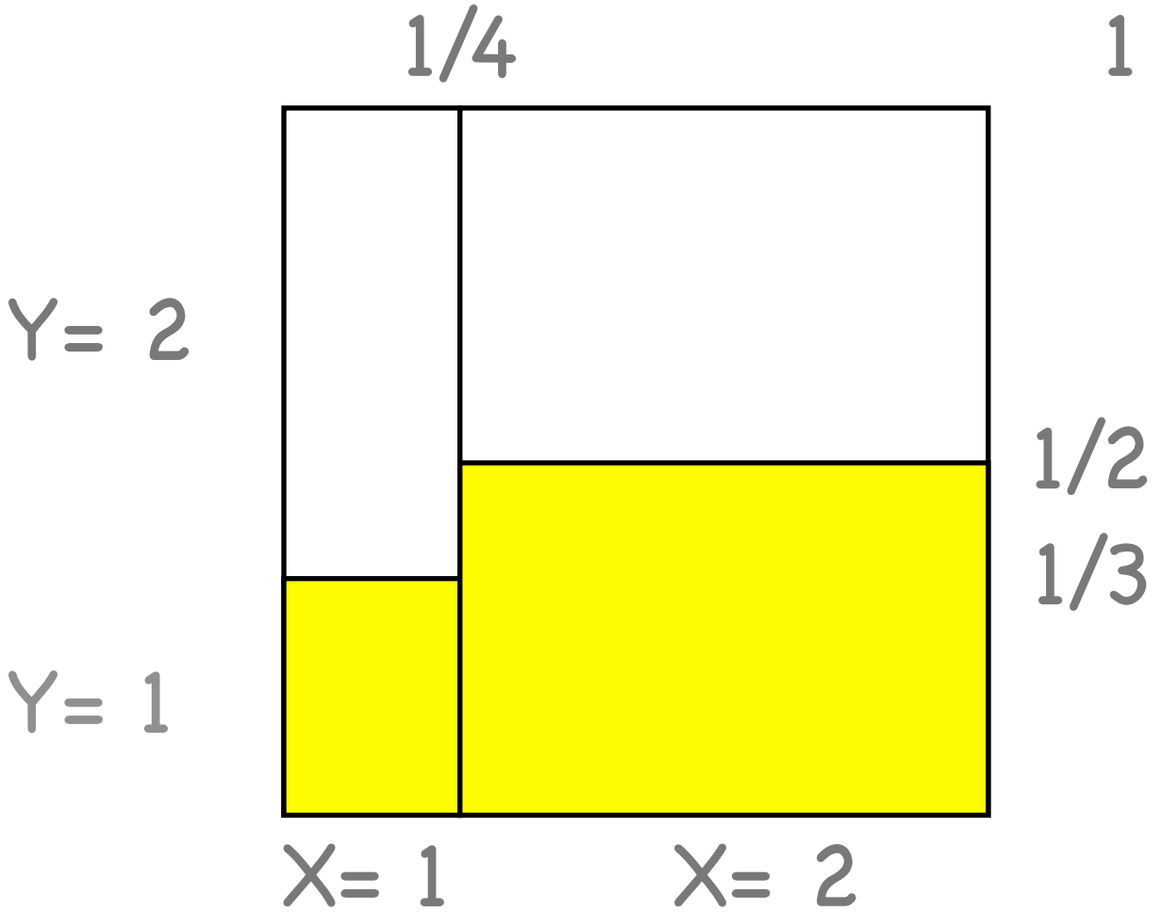
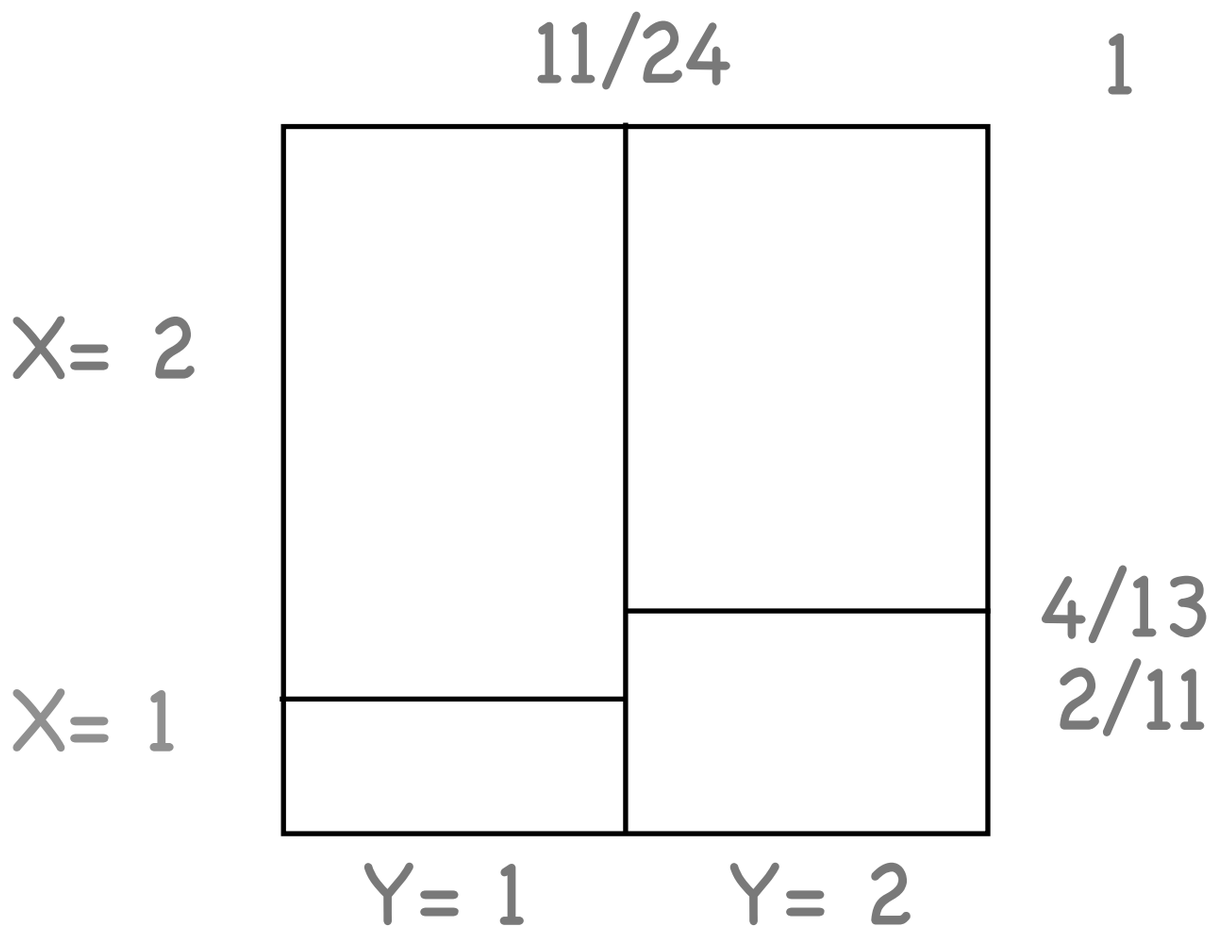
Reorder X and Y:

Frame





$$\Pr(X=1 \mid Y=2) = \Pr(X=1, Y=2) / \Pr(Y=2)$$
$$= (1/4 \times 2/3) / (13/24)$$
$$= 4/13$$

Eikosogram

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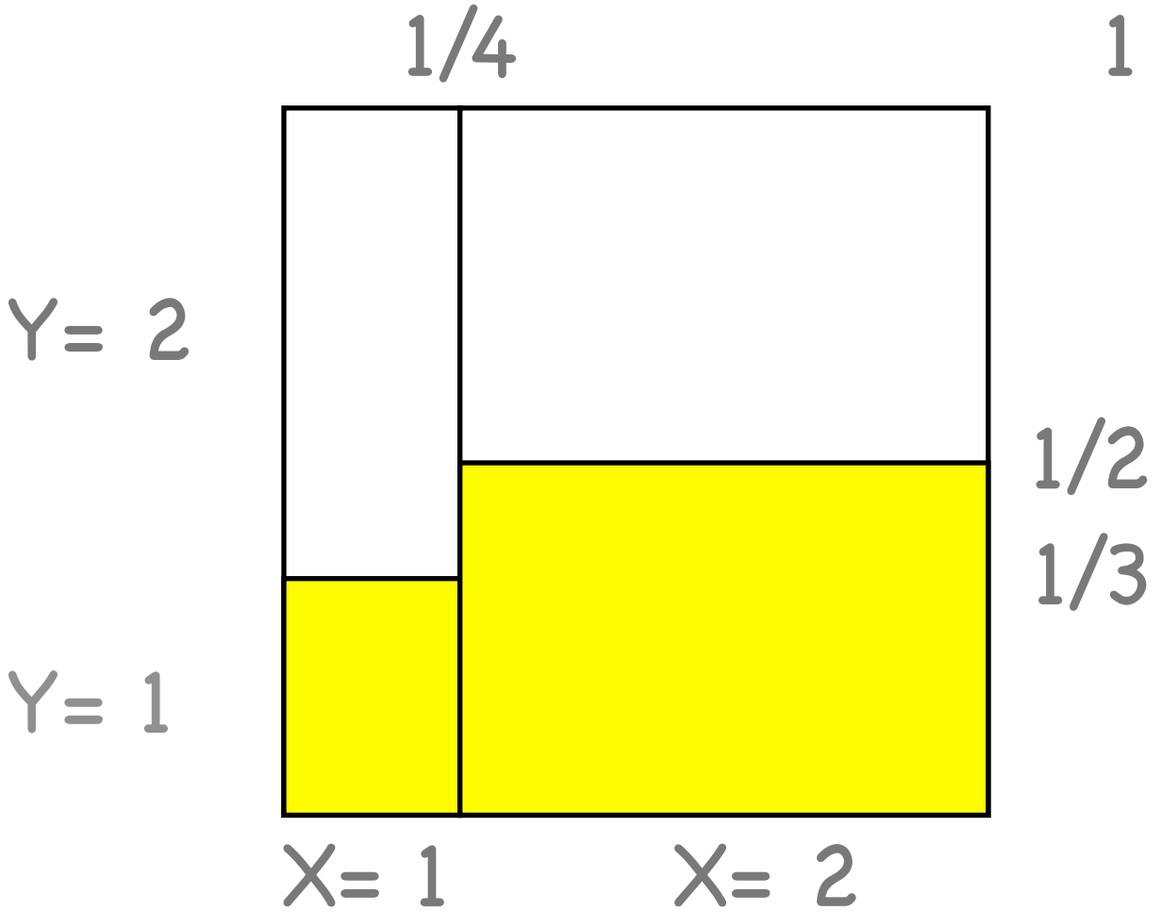
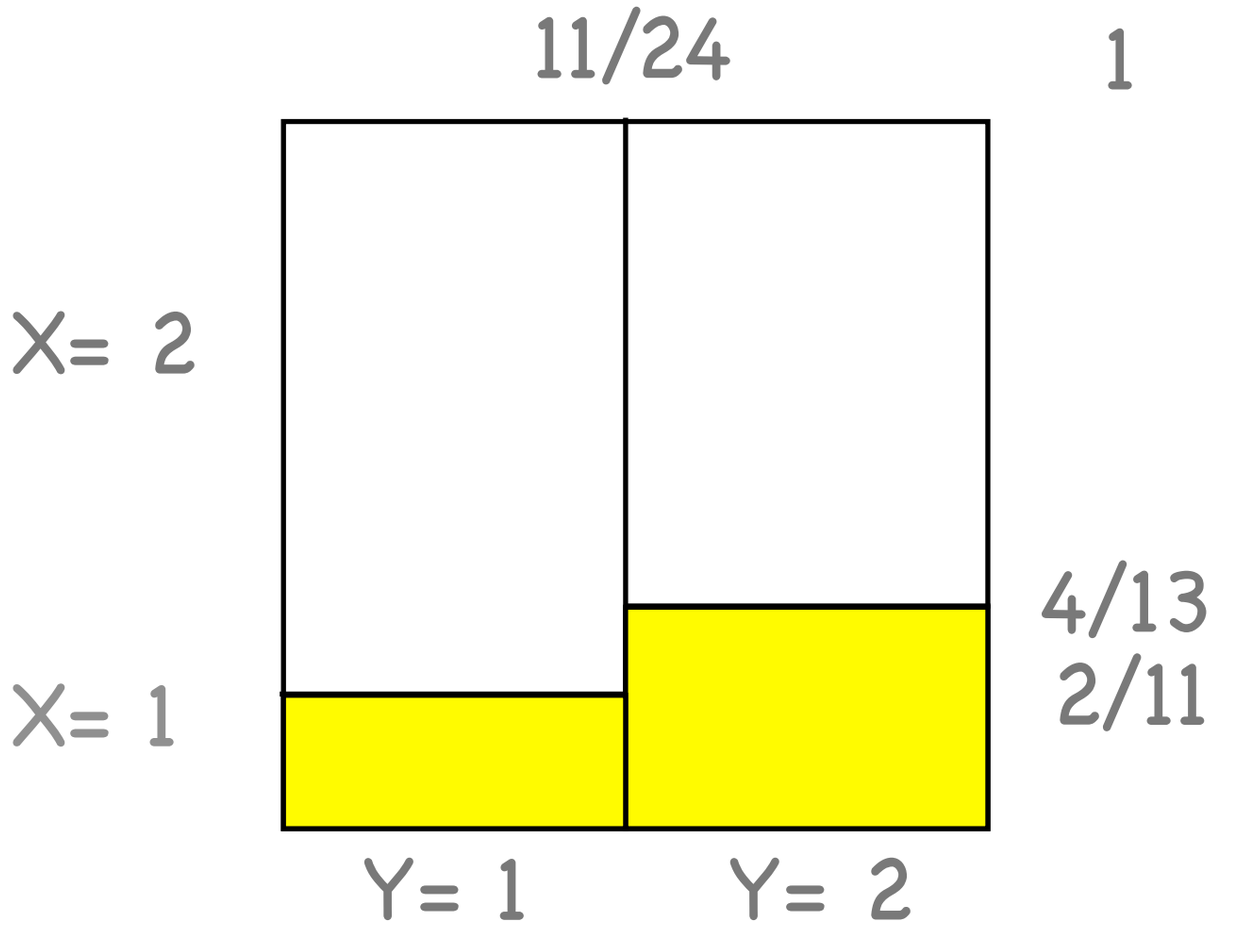
Reorder X and Y:



Frame



Eikosogram

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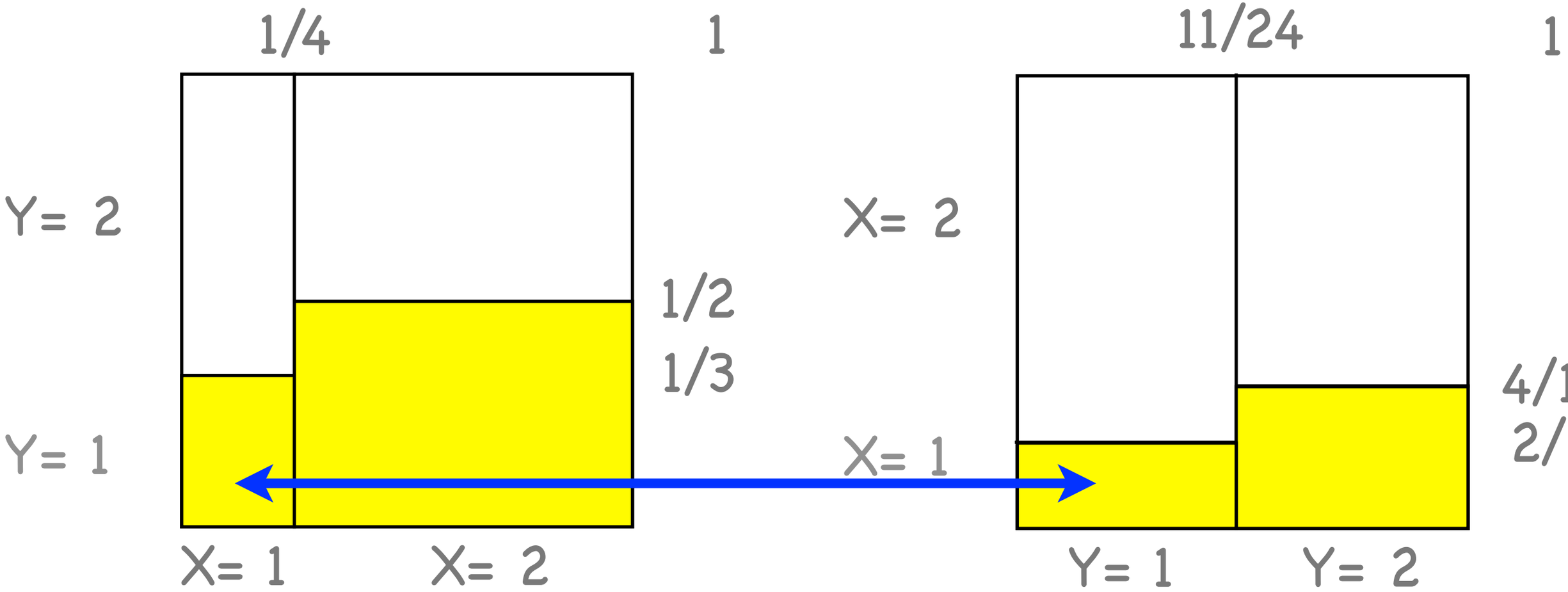
Matching areas:



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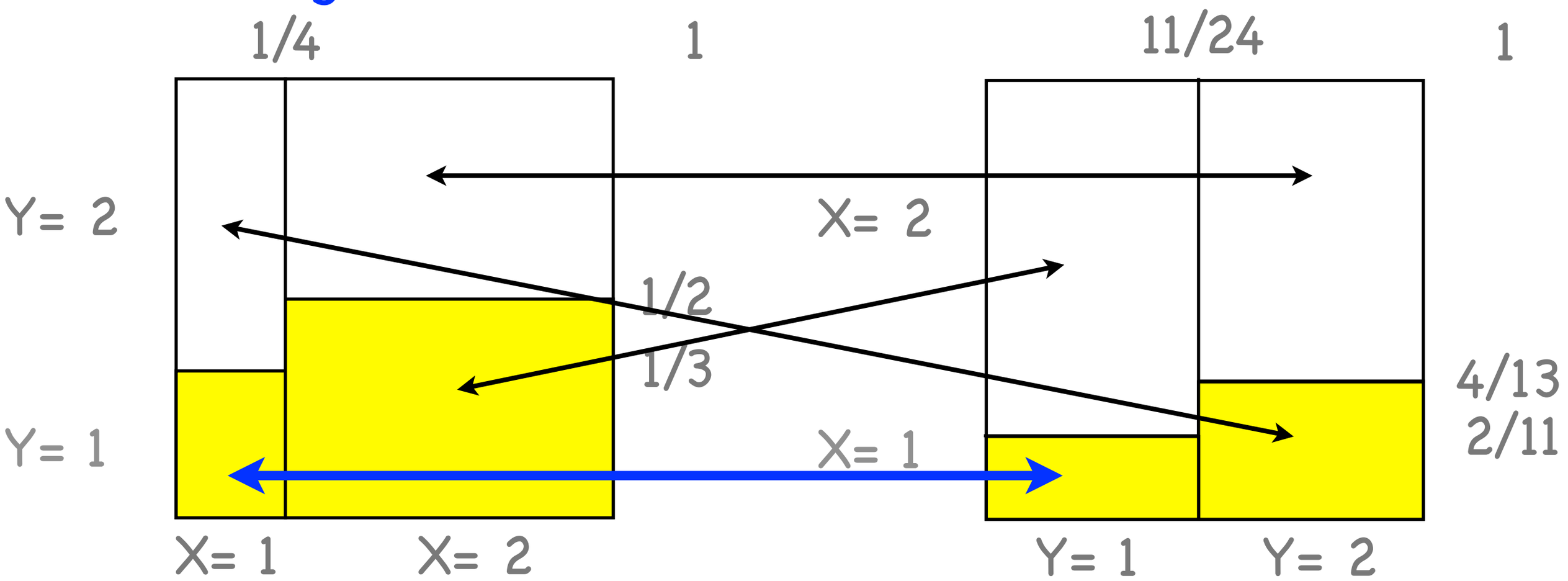

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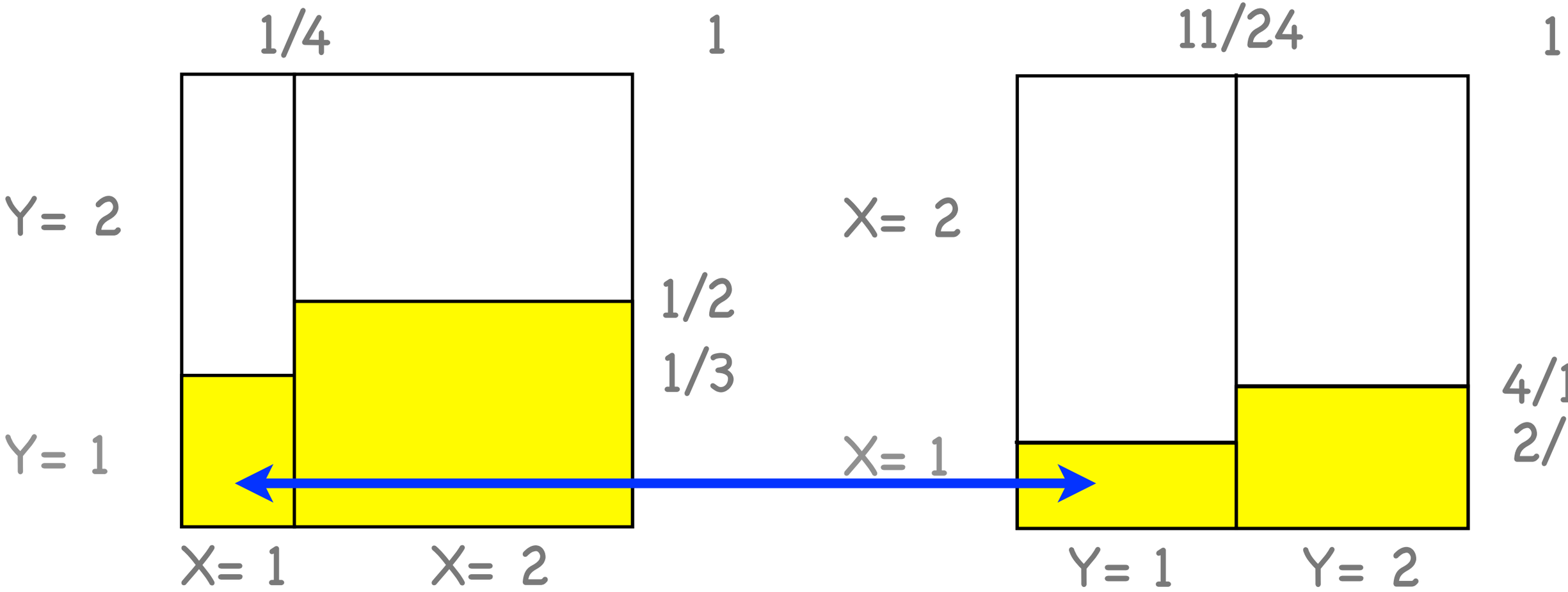

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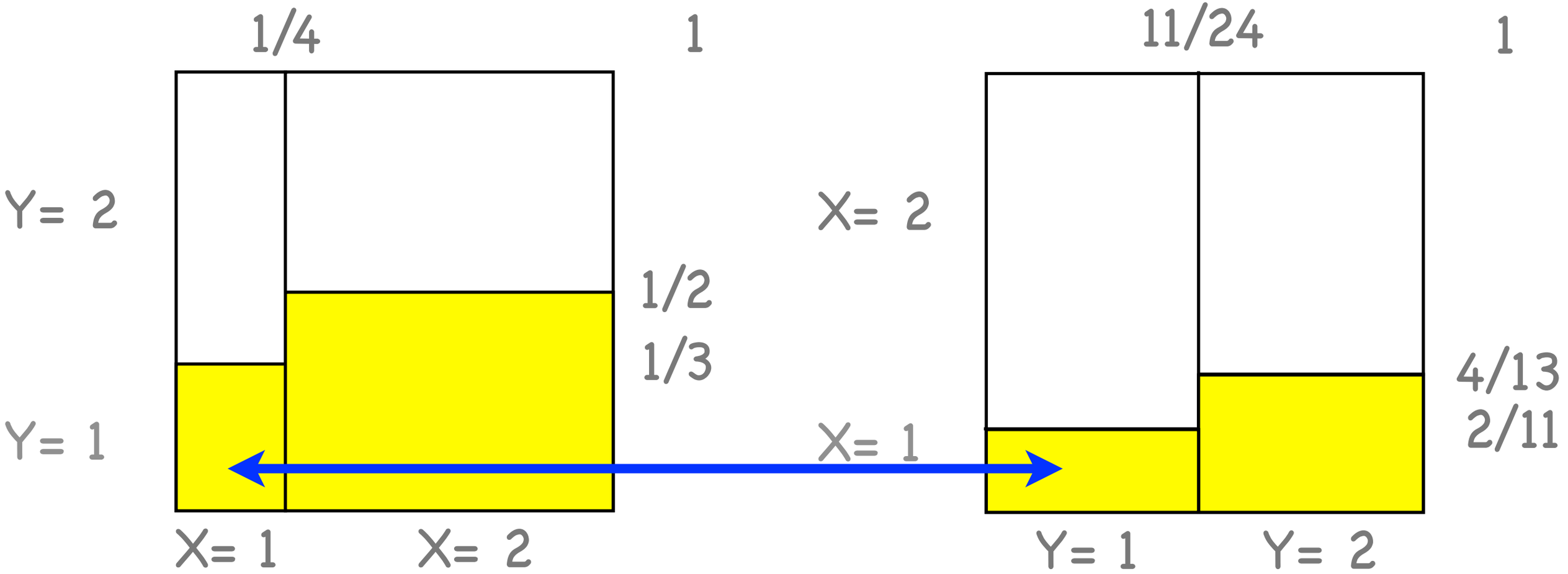

$$\Pr(Y=1 \mid X=1)\,\Pr(X=1) = \Pr(X=1, Y=1) = \Pr(X=1 \mid Y=1)\,\Pr(Y=1)$$

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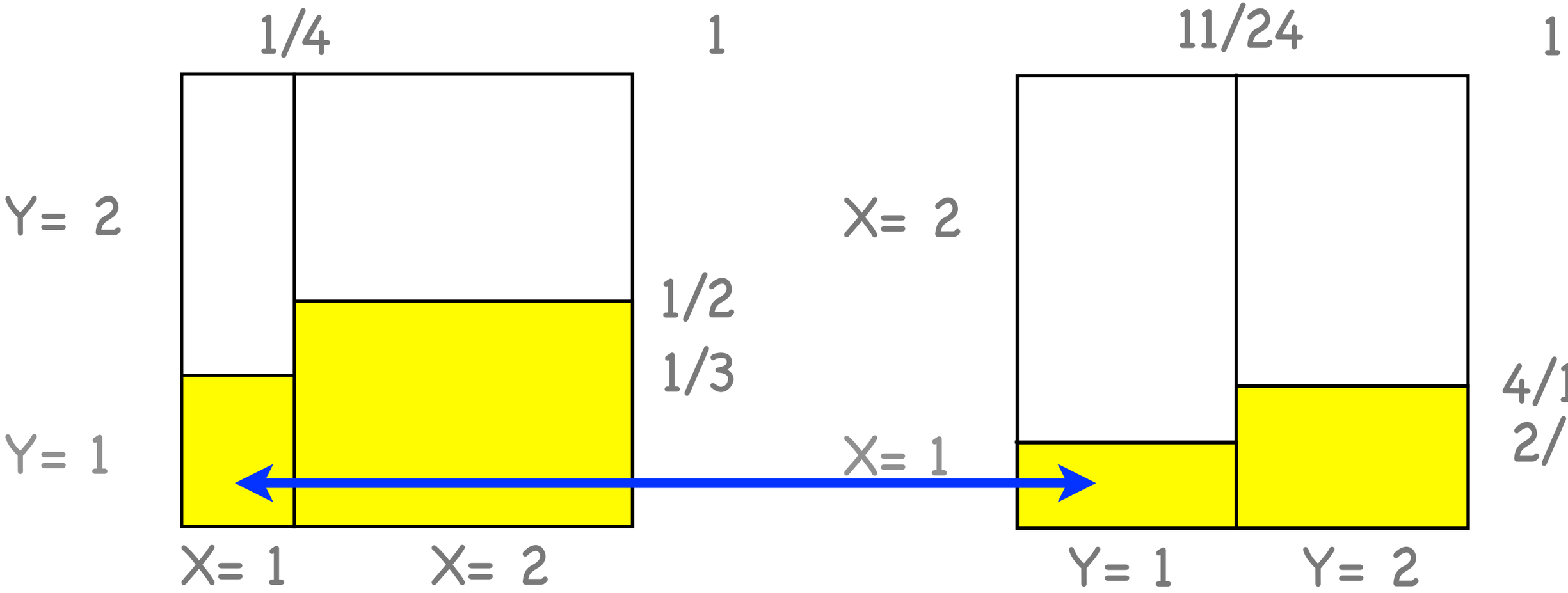
Matching areas:

Frame



$$\Pr(Y \mid X)\,\Pr(X) = \Pr(X, Y) = \Pr(X \mid Y)\,\Pr(Y)$$

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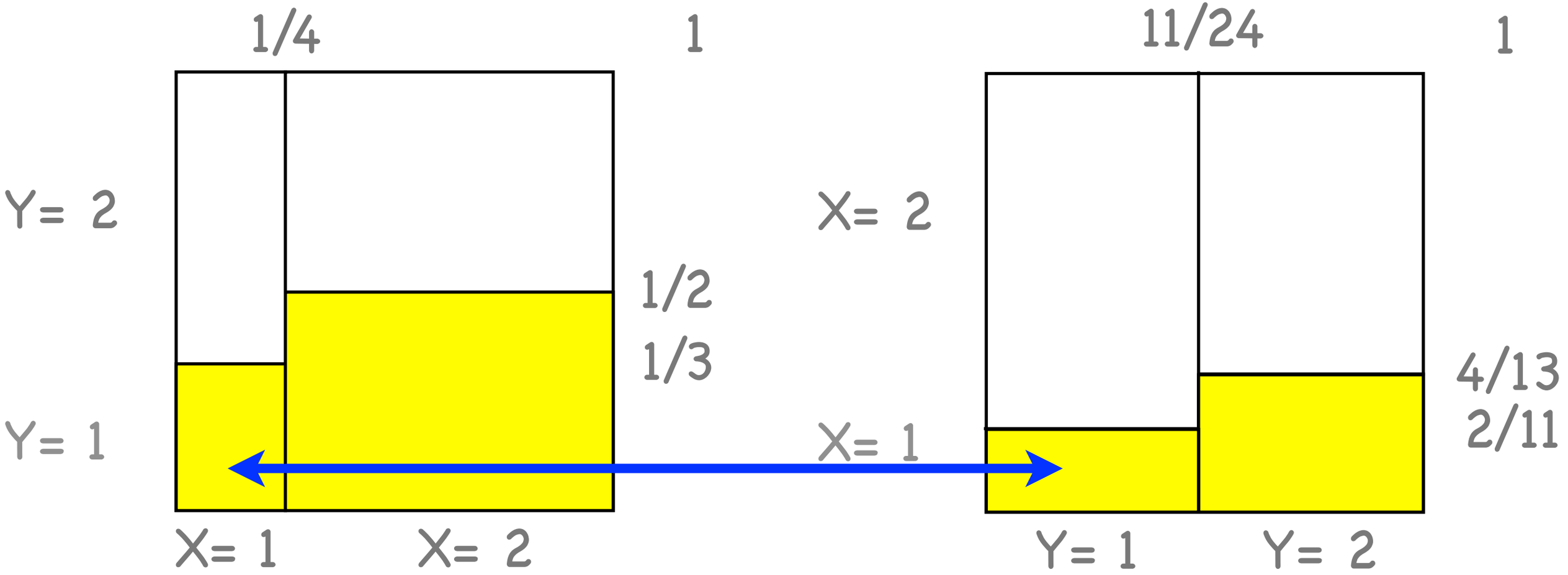

$$\Pr(Y \mid X)\ \Pr(X) = \Pr(X \mid Y)\ \Pr(Y)$$

"Bayes Theorem"

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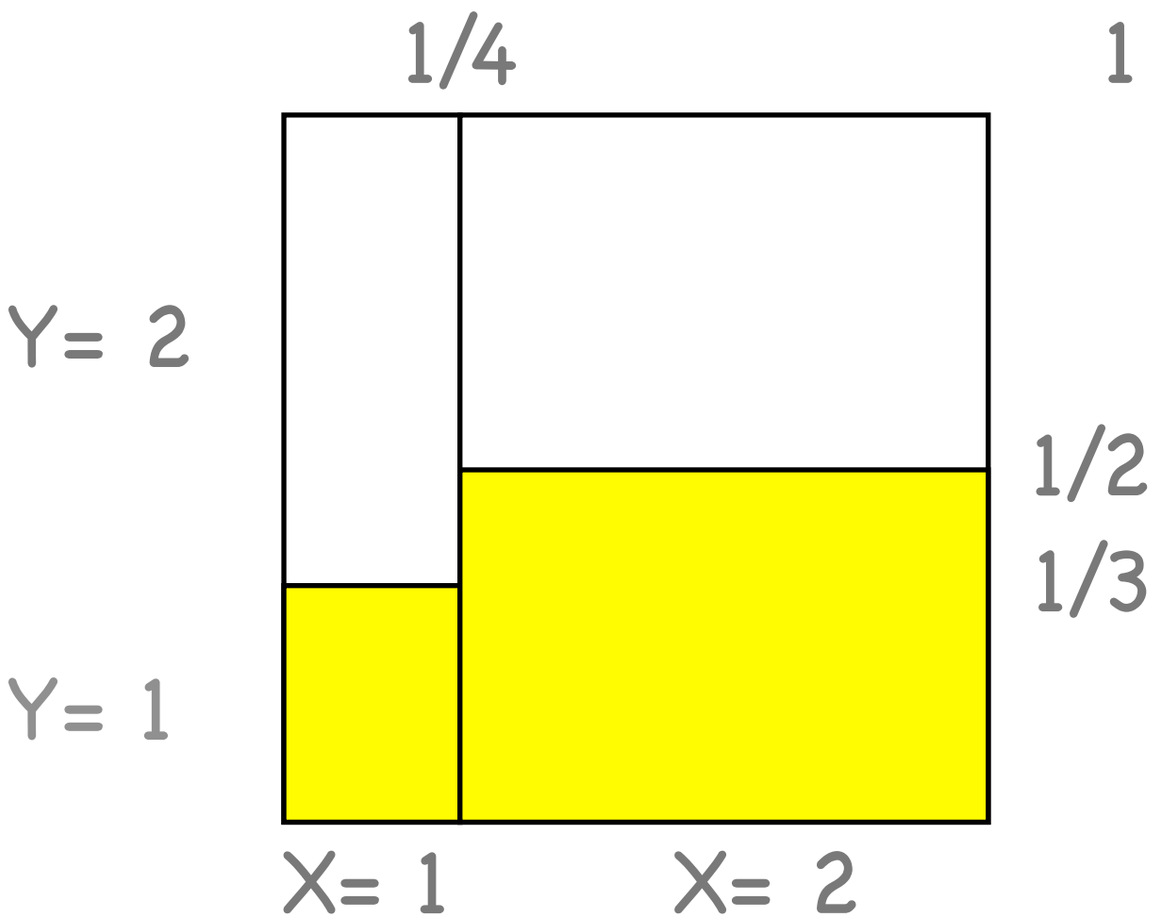

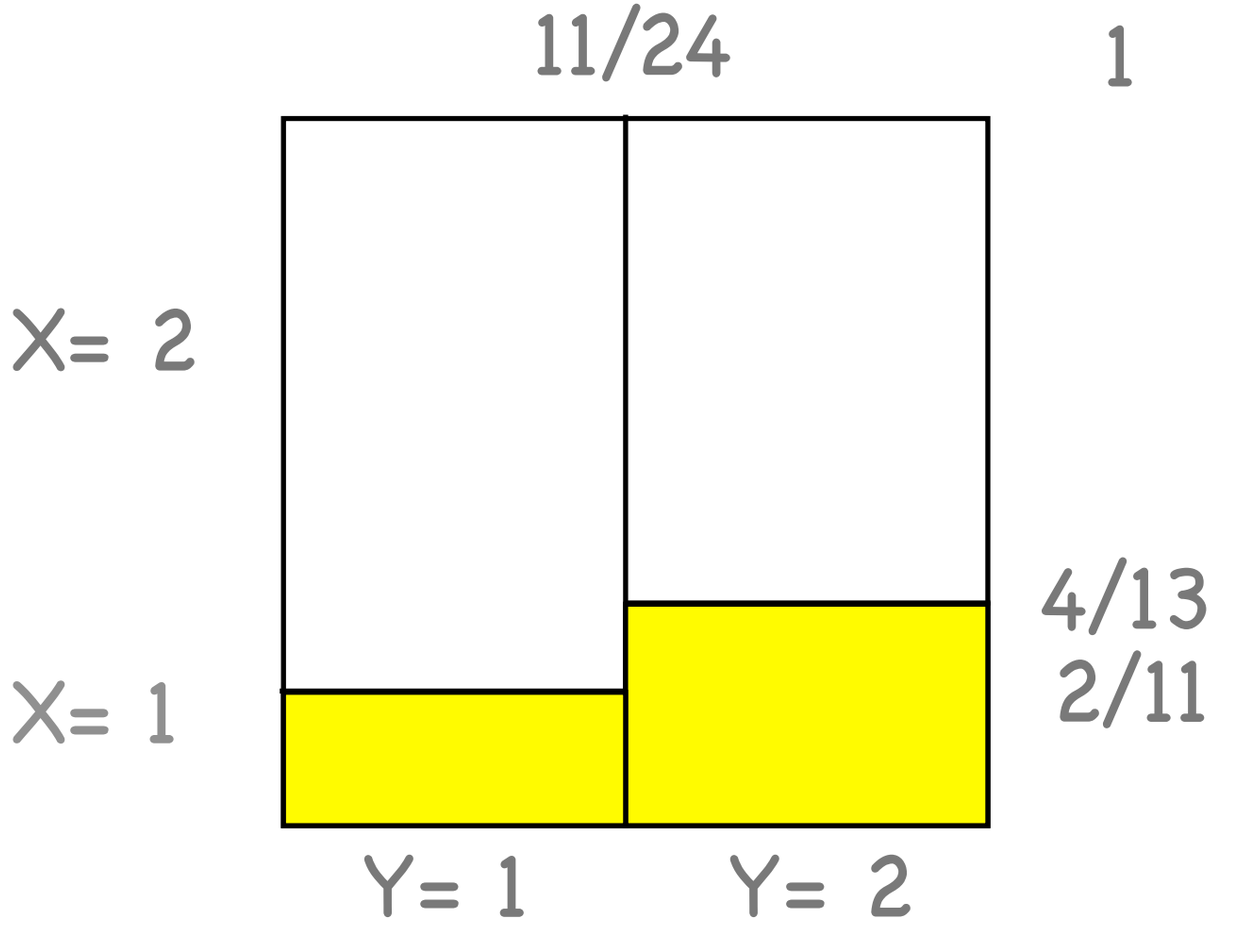

Why keep the colours layered?

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Water container metaphor and marginal probabilities.

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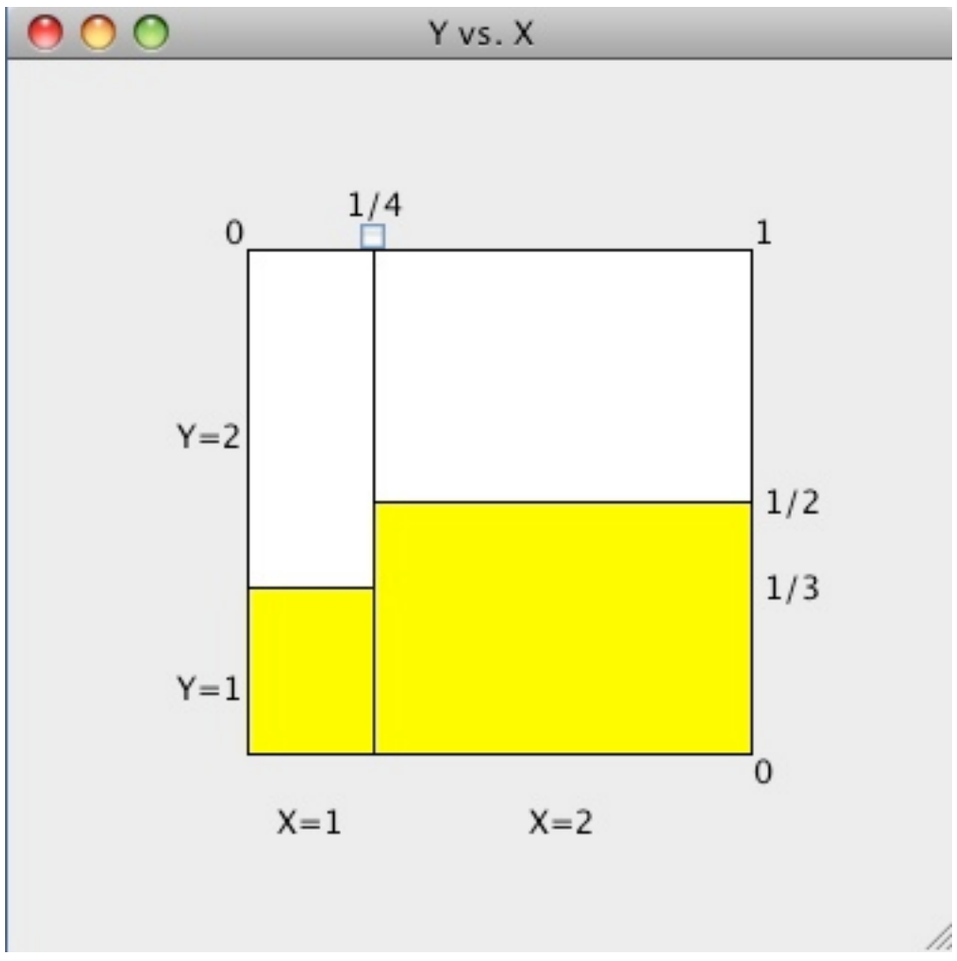

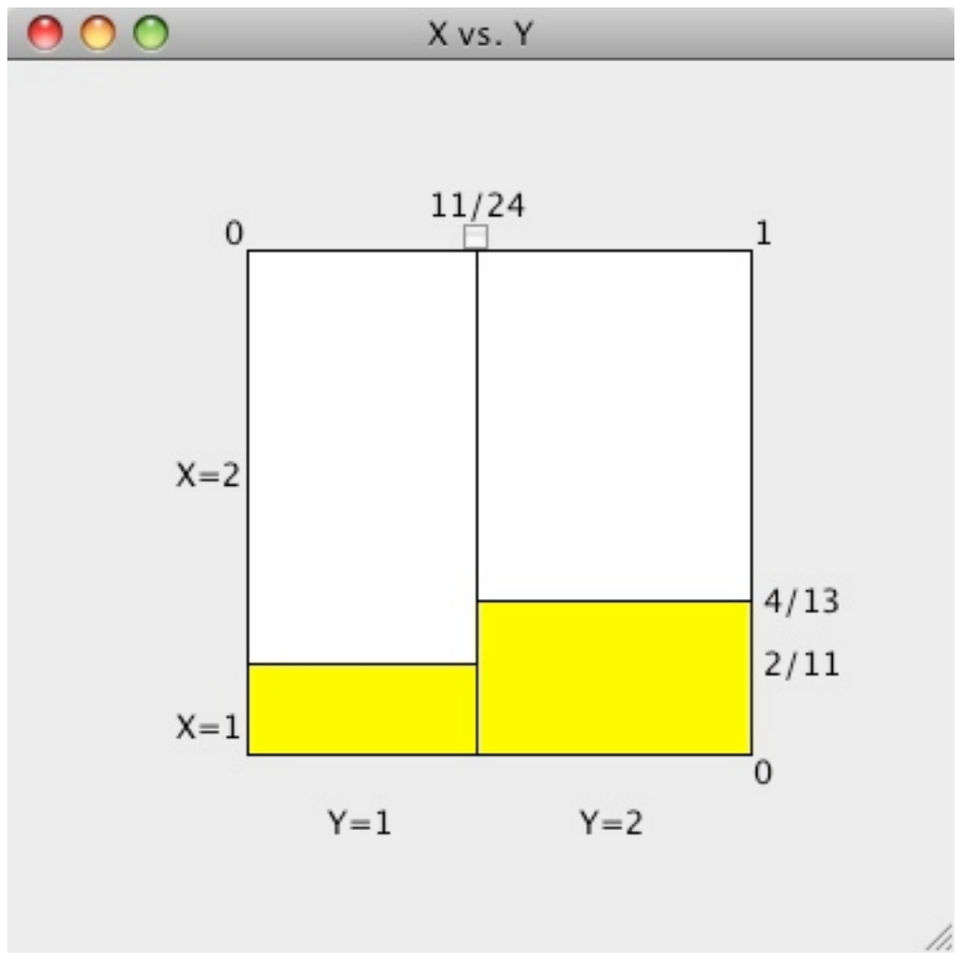

Remove the barrier, water level adjusts.

Get the marginal distribution.

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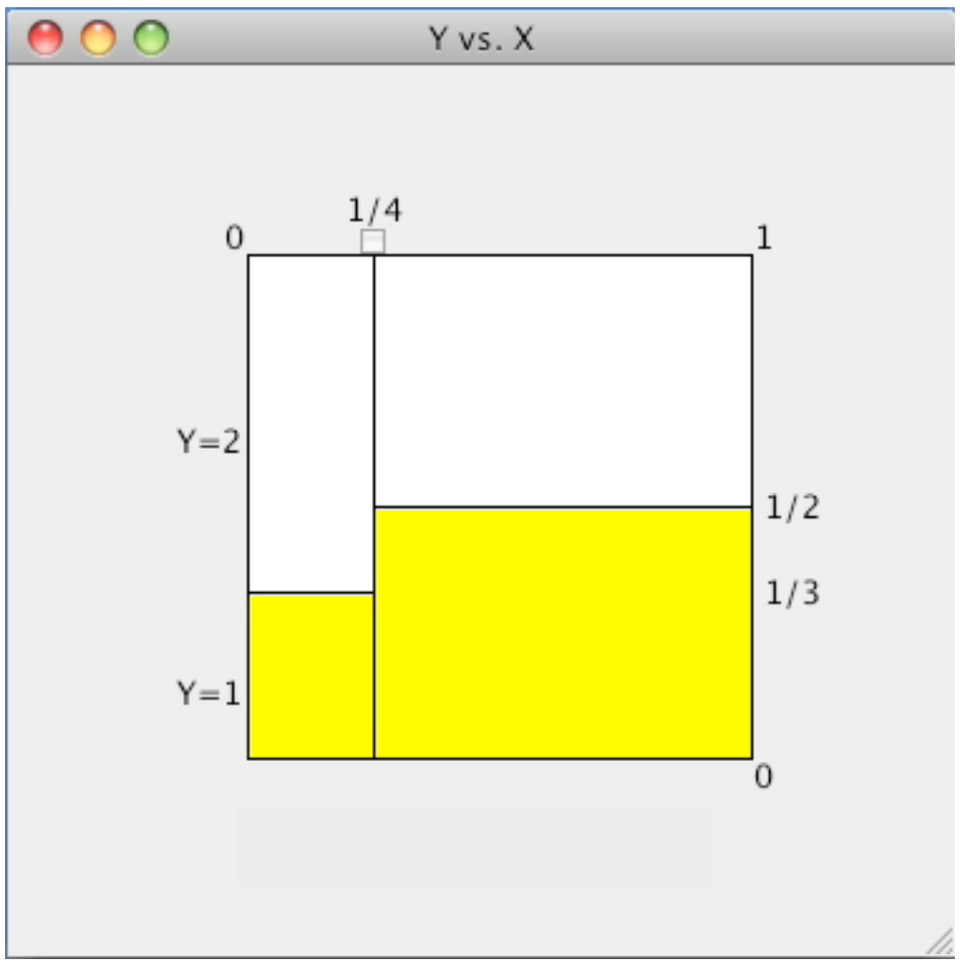
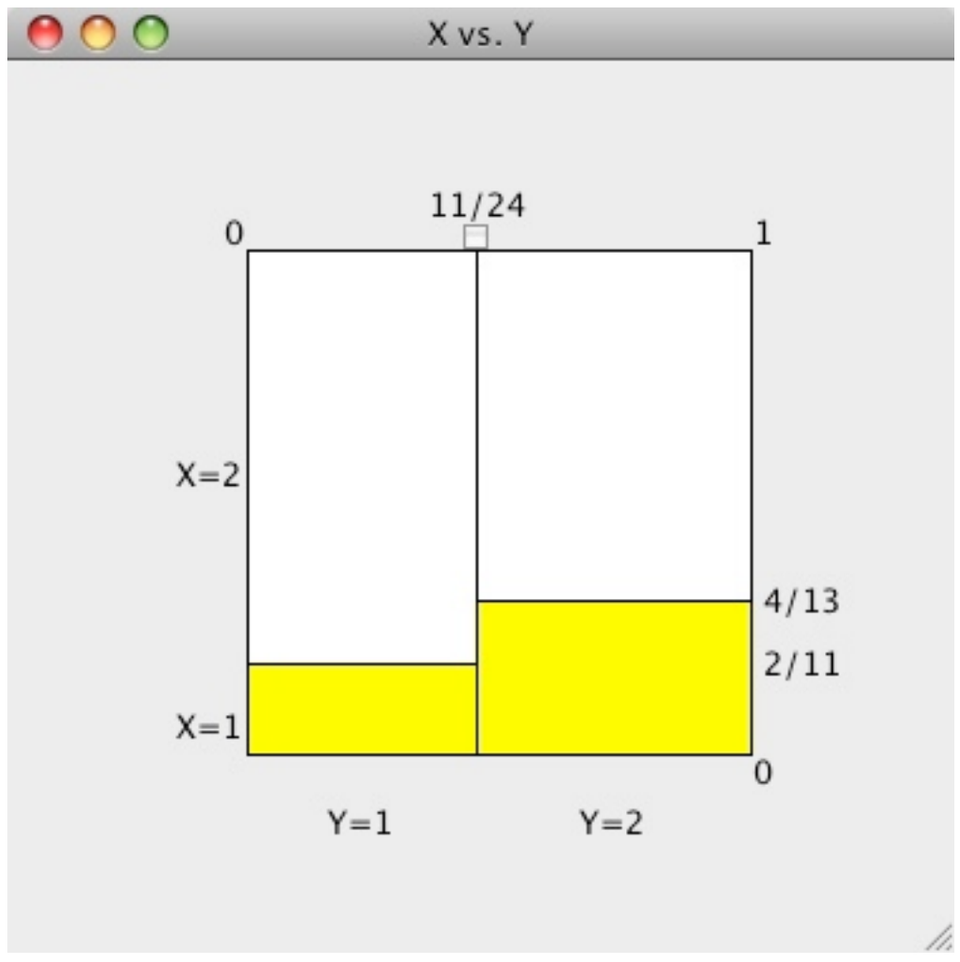
Water container metaphor and marginal probabilities.





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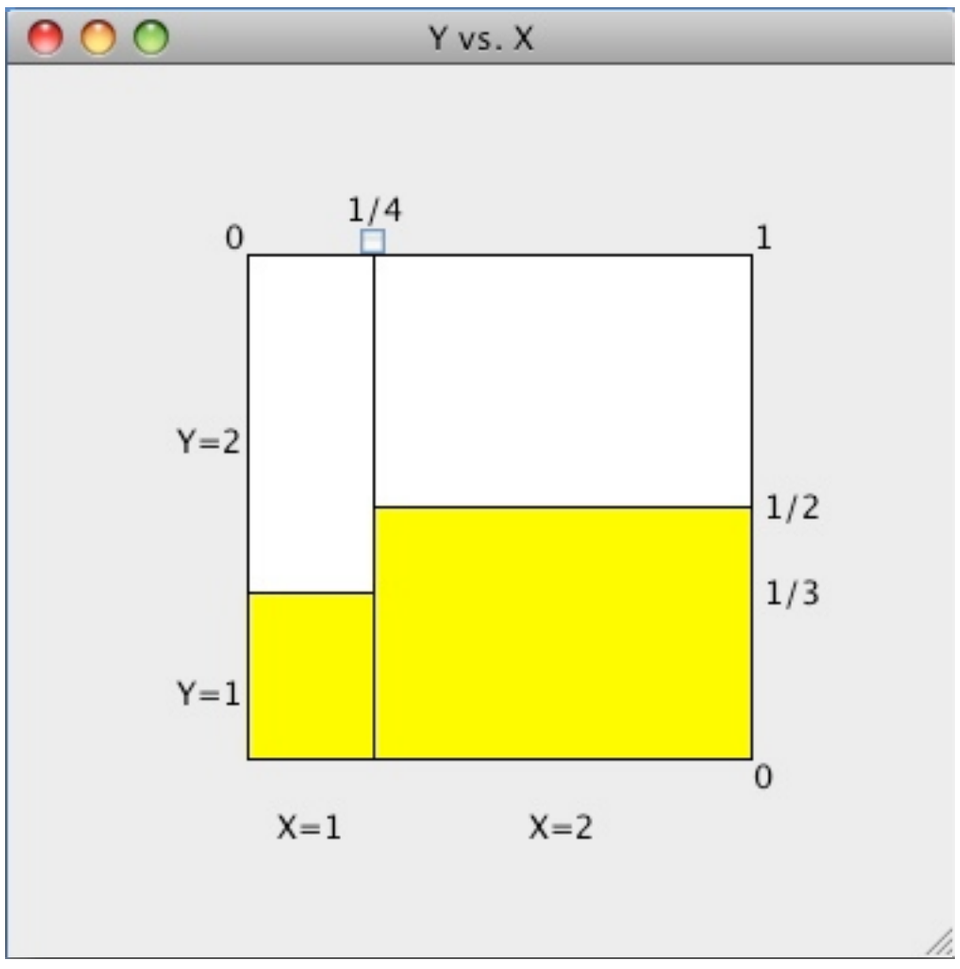

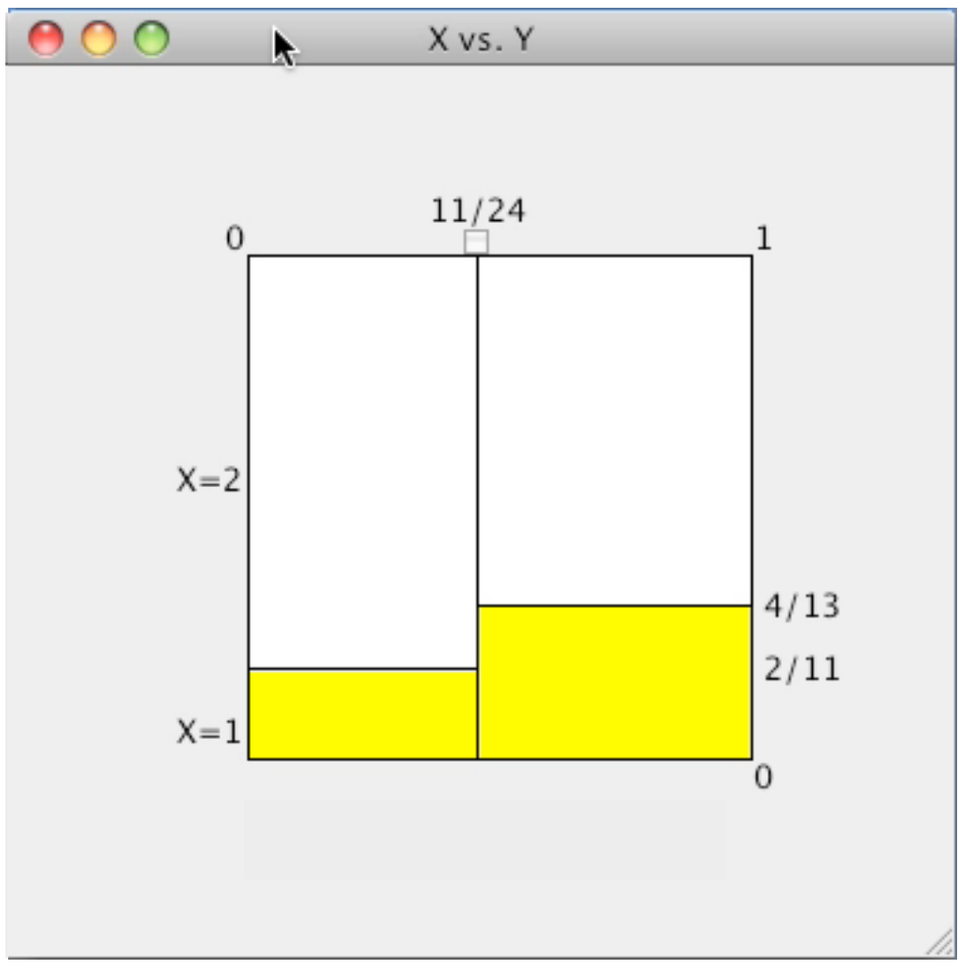

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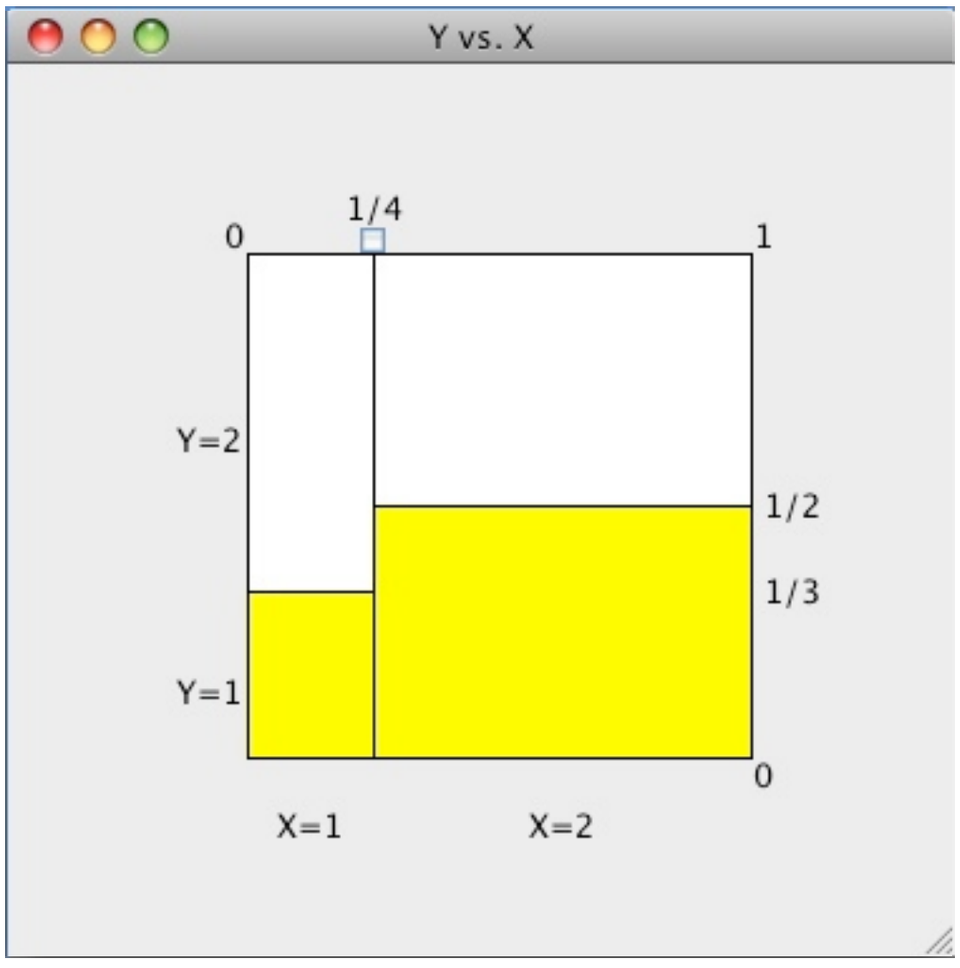

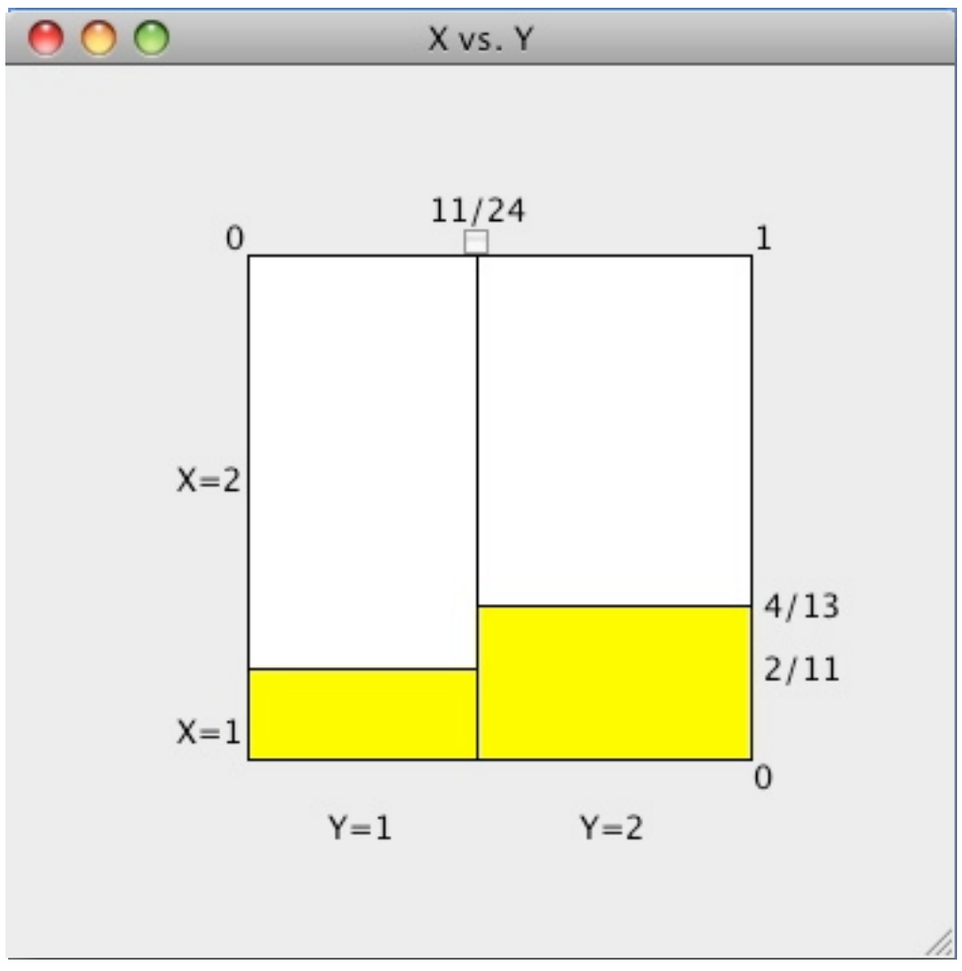

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Get the marginal distribution.

Example problem

Three gas stations: G1, G2, G3

Example problem

Three gas stations: G1, G2, G3

They always have different prices for their gas.

Example problem

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They always have different prices for their gas.

On any given day, any one of the three might have the cheapest gas, with equal probability 1/3 .

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All three are on route, but once you pass a gas station there is no turning back.

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You must buy gas at one, and only one, of the stations.

As you reach each station, you can see its price advertised but not the price of the next.

Example problem

Three gas stations: G1, G2, G3

They always have different prices for their gas.

On any given day, any one of the three might have the cheapest gas, with equal probability 1/3 .

All three are on route, but once you pass a gas station there is no turning back.

You must buy gas at one, and only one, of the stations.

As you reach each station, you can see its price advertised but not the price of the next.

You want the cheapest gas with high probability.

Example problem

Three gas stations: G1, G2, G3

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You must buy gas at one, and only one, of the stations.

As you reach each station, you can see its price advertised but not the price of the next.

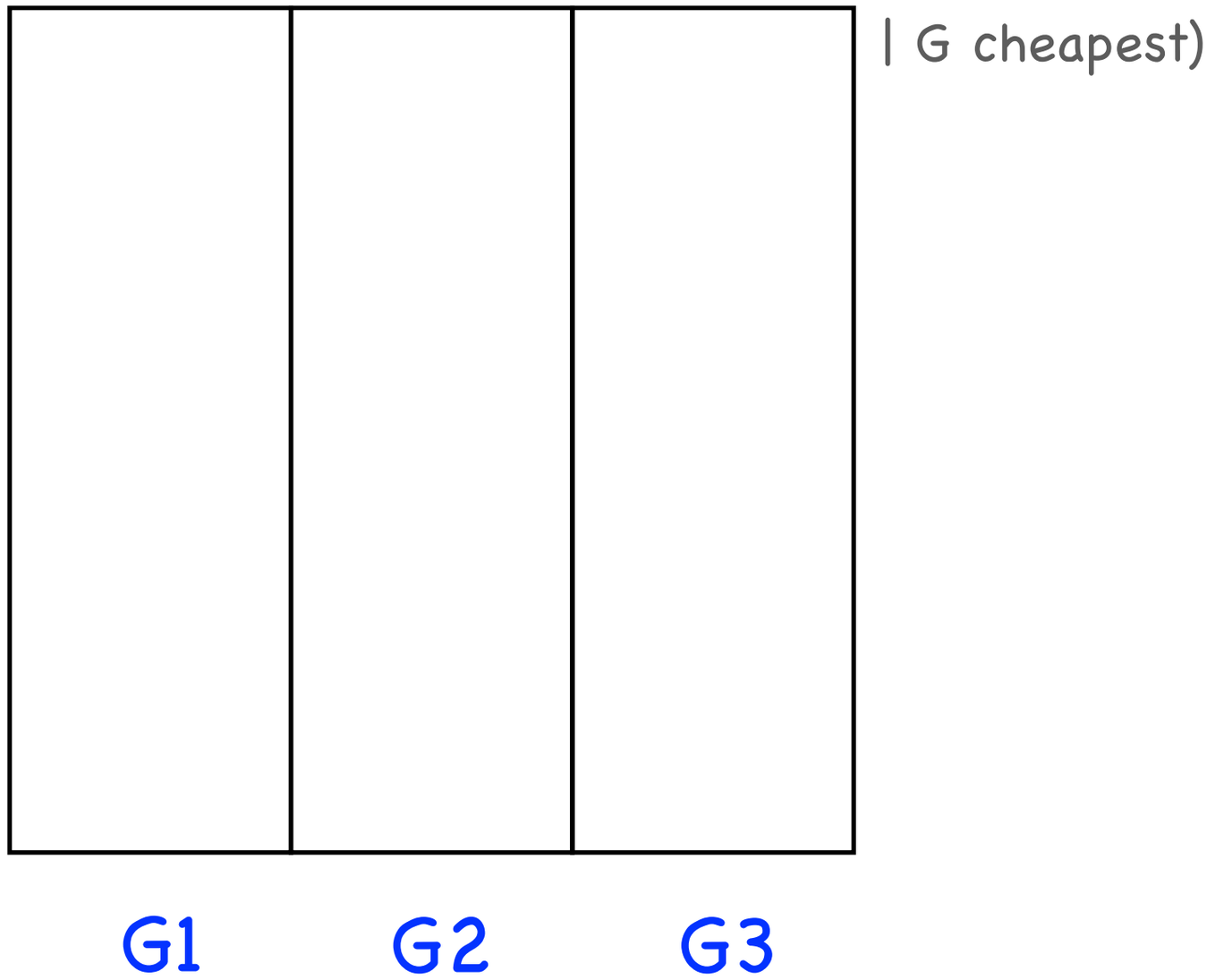
You want the cheapest gas with high probability.

Does it matter what you do?

Example problem

Three gas stations: G1, G2, G3

Each station has equal probability of being the cheapest.



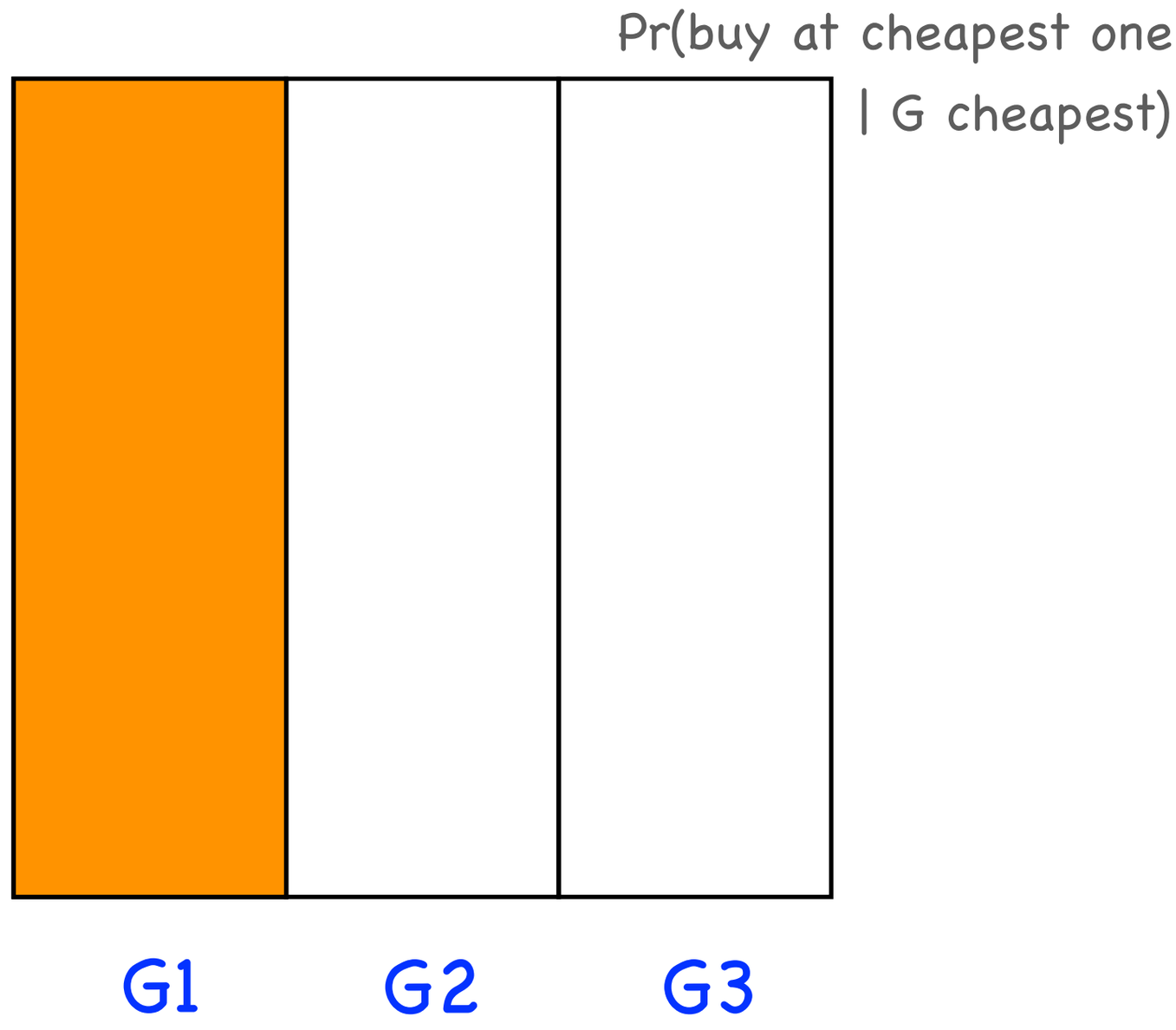
Example problem

Strategy

Always buy gas at the first station.

Pr(buy cheapest gas)

= Total Orange Area



Example problem

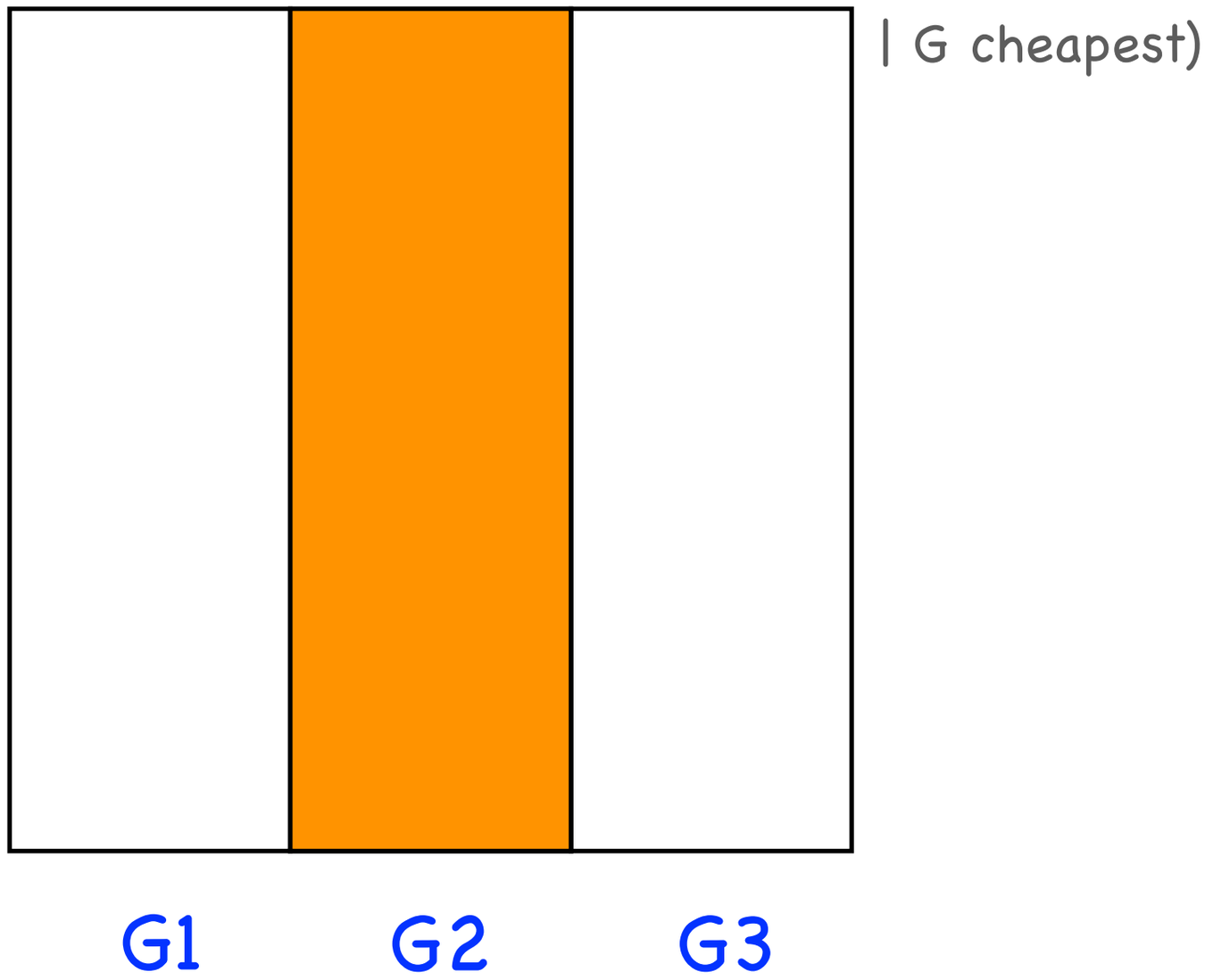
Strategy

Always buy gas at the second station.

Pr(buy cheapest gas)

= Total Orange Area

Same area!



Location of the cheapest gas

Example problem

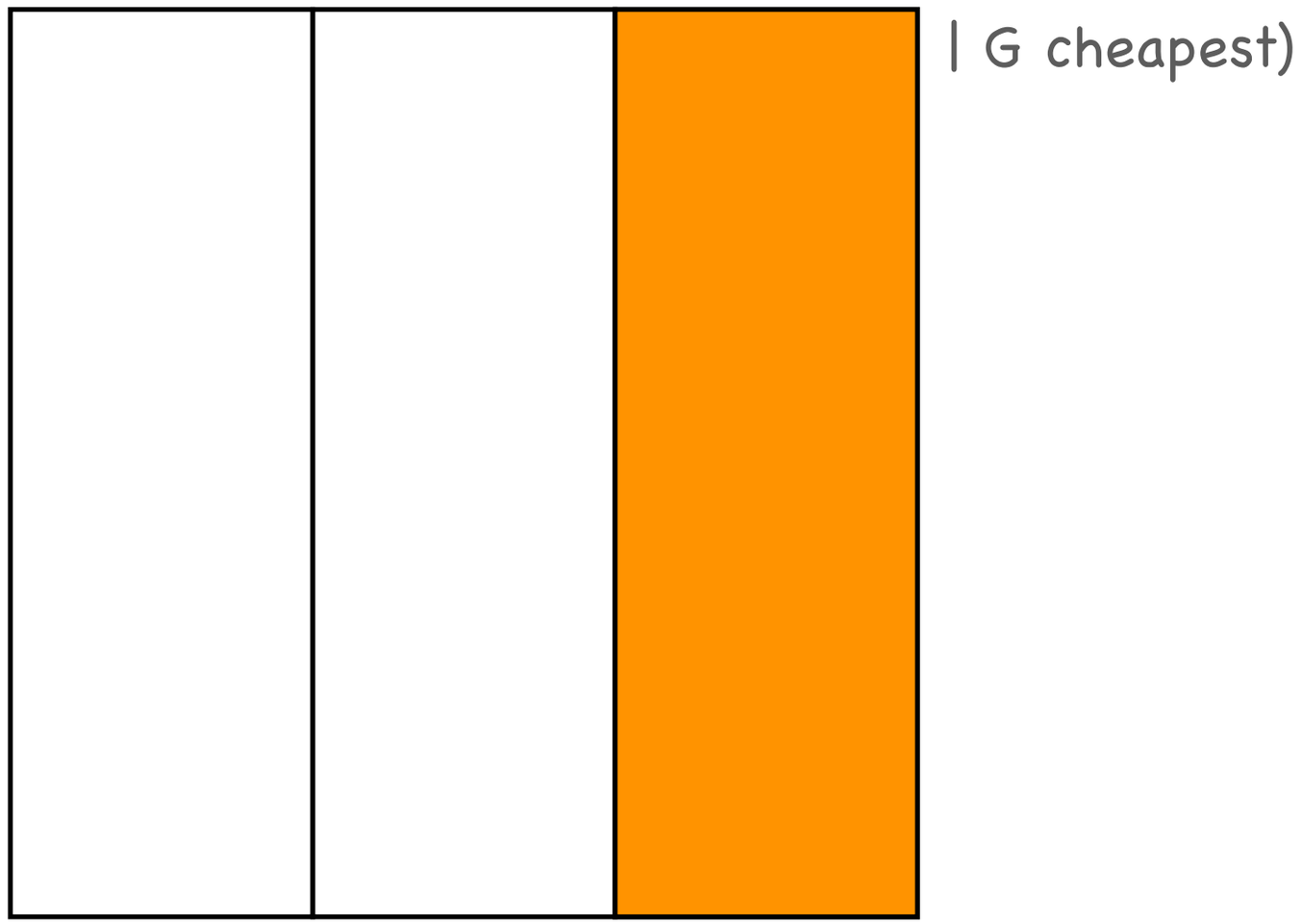
Strategy

Always buy gas at the last station.

Pr(buy cheapest gas)

= Total Orange Area

Same area!



Location of the cheapest gas

Example problem

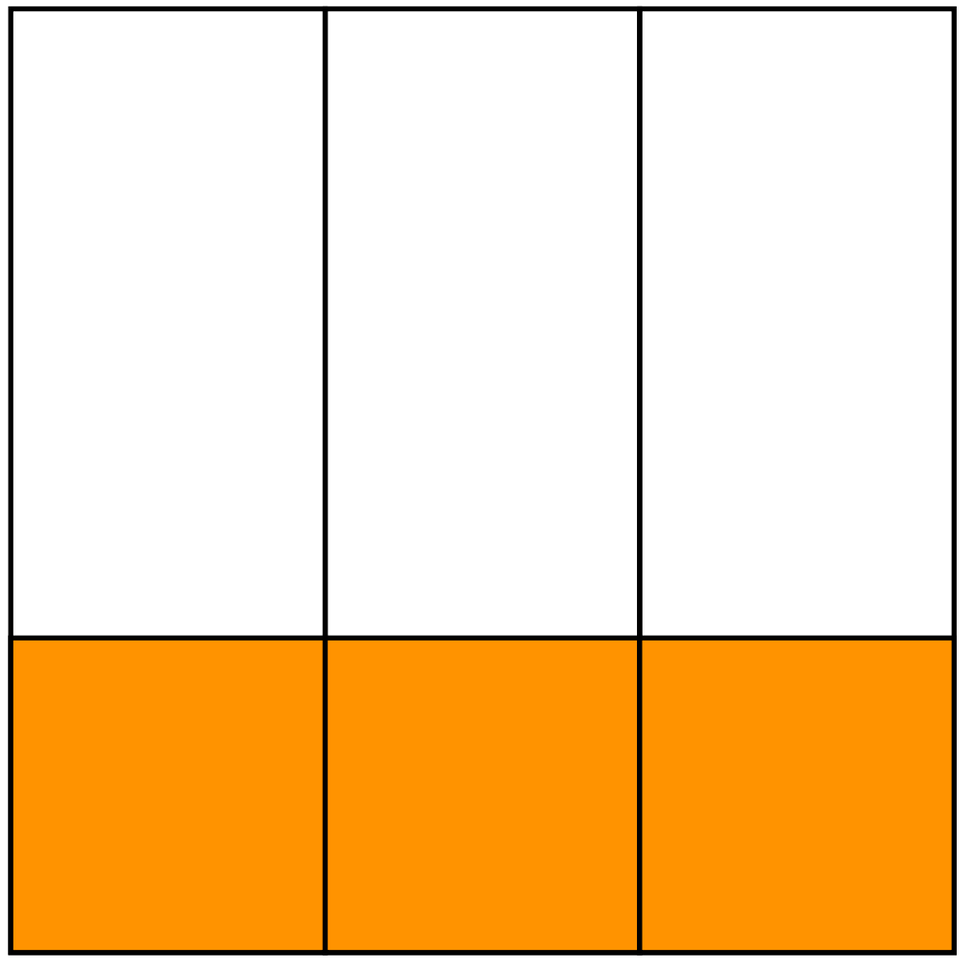
Strategy

Buy gas at each station with equal probability.

Pr(buy cheapest gas)

= Total Orange Area

Same area!



Example problem

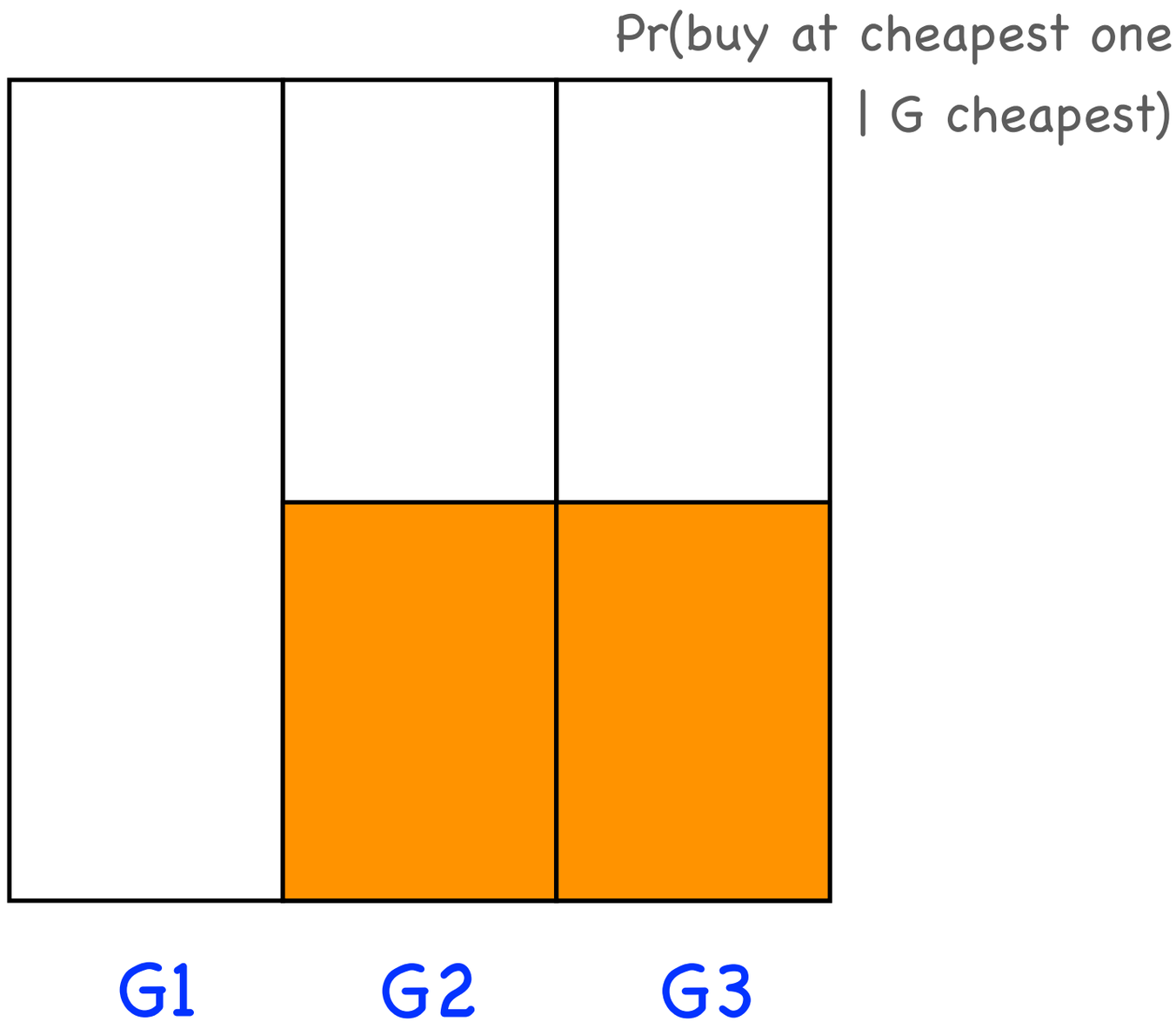
Strategy

Never buy gas at the first station, half the time at G2, half at G3.

Pr(buy cheapest gas)

= Total Orange Area

Same area!



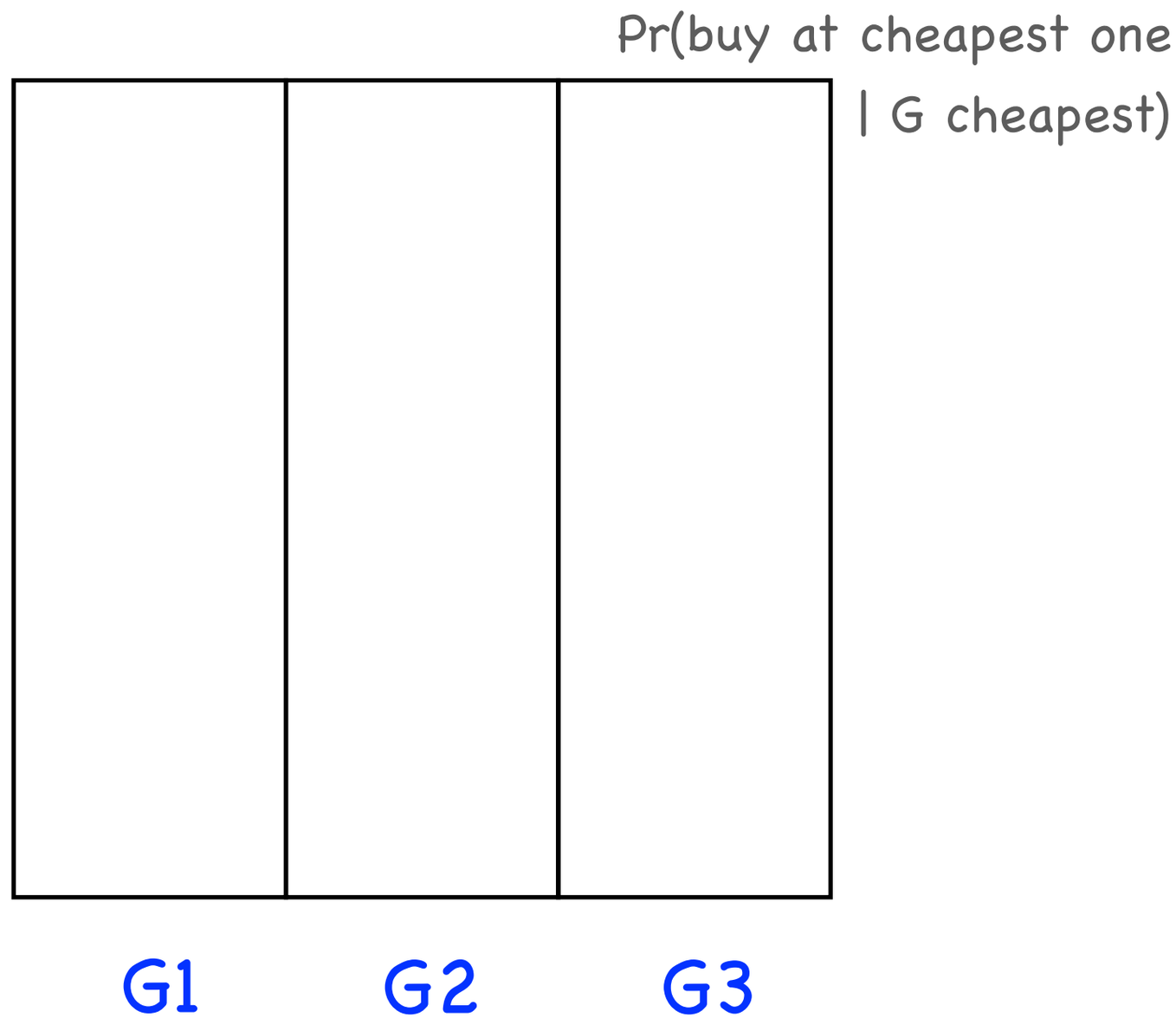
Example problem

Strategy

Never buy gas at the first station, BUT at the second if it's cheaper, otherwise the last.

Pr(buy cheapest gas)

= Total Orange Area



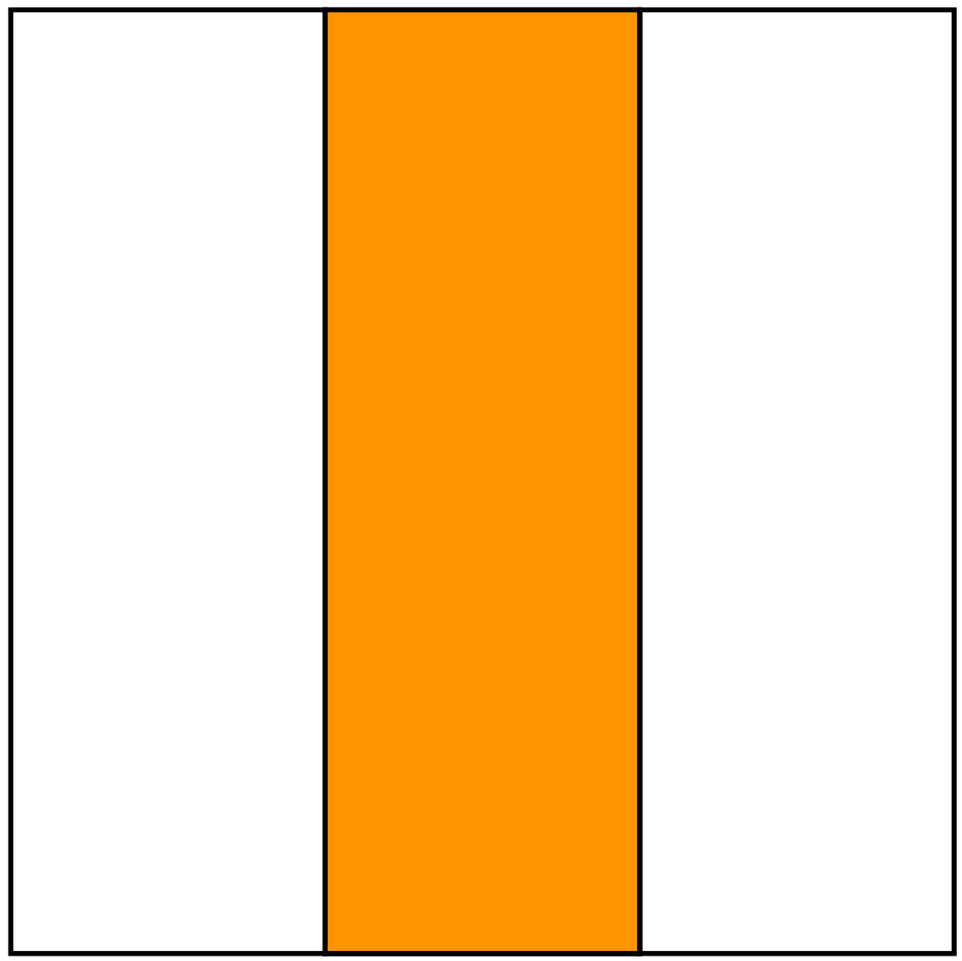
Example problem

Strategy

Never buy gas at the first station, BUT at the second if it's cheaper, otherwise the last.

Pr(buy cheapest gas)

= Total Orange Area



Example problem

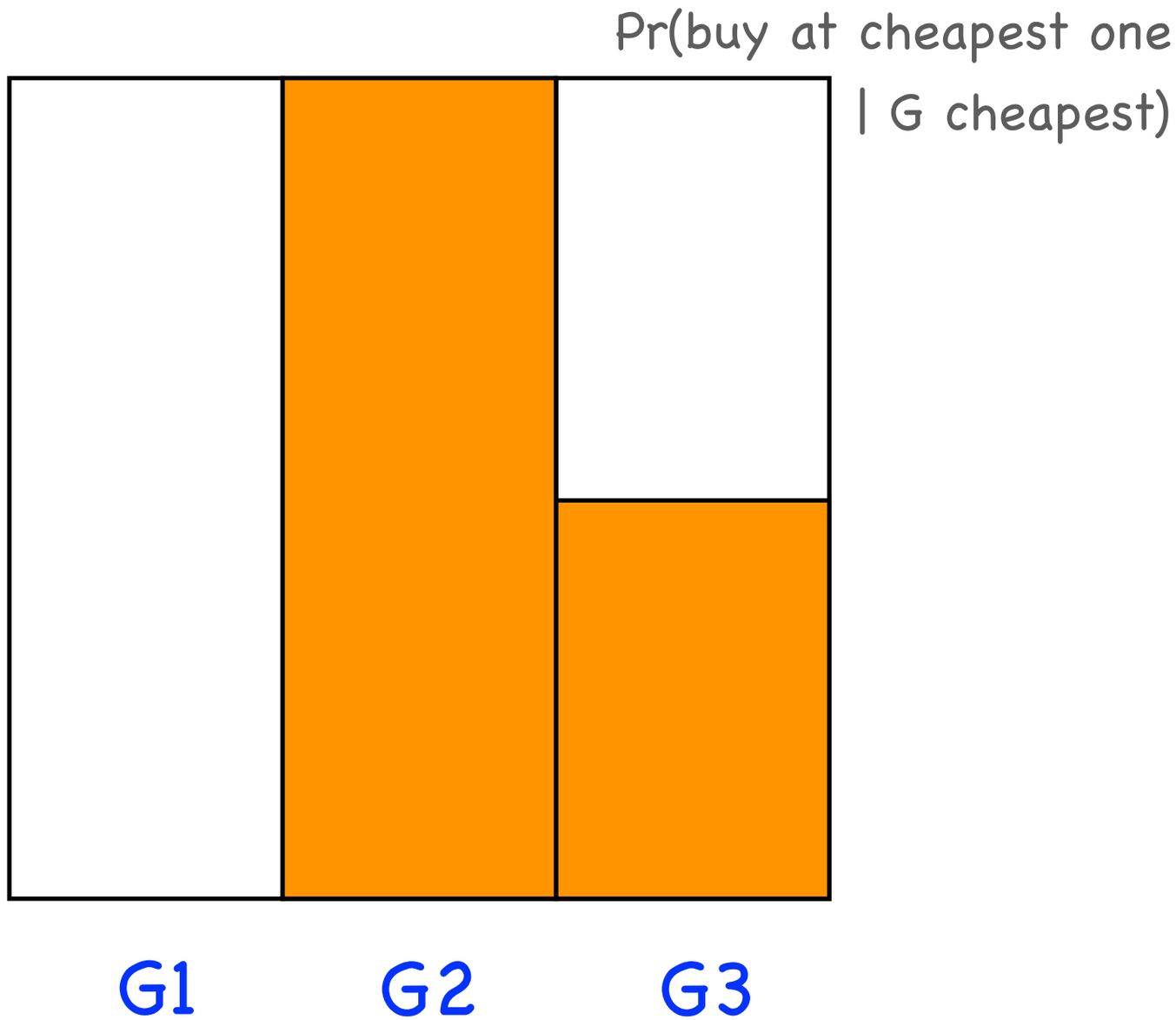
Strategy

Never buy gas at the first station, BUT at the second if it's cheaper, otherwise the last.

Pr(buy cheapest gas)

= Total Orange Area

Winner!



More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

One variable

on vertical axis

Z = *no*

Z = *yes*

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

One variable Z = *no*

on vertical axis

Z = *yes*

Nest rest of variables

along horizontal axis

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

One variable Z = *no*

on vertical axis

Z = *yes*

Nest rest of variables

along horizontal axis

X = *yes* X = *no*

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

One variable Z = *no*

on vertical axis

Z = *yes*

Nest rest of variables

along horizontal axis

Y=*yes* Y=*no*

X = *yes* X = *no*

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

One variable Z = *no*

on vertical axis

Z = *yes*

Nest rest of variables

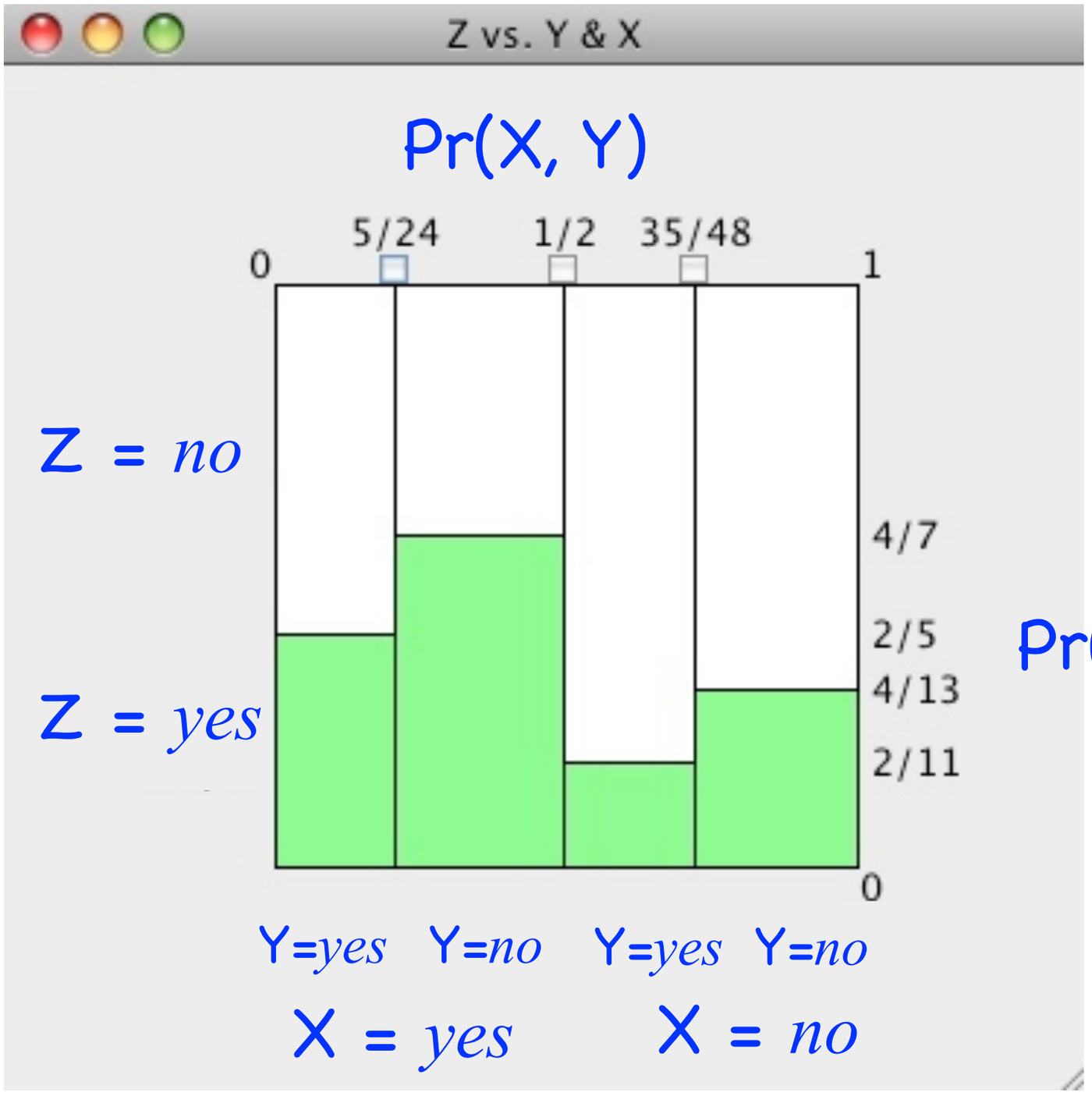
along horizontal axis

Y=*yes* Y=*no* Y=*yes* Y=*no*

X = *yes* X = *no*

More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

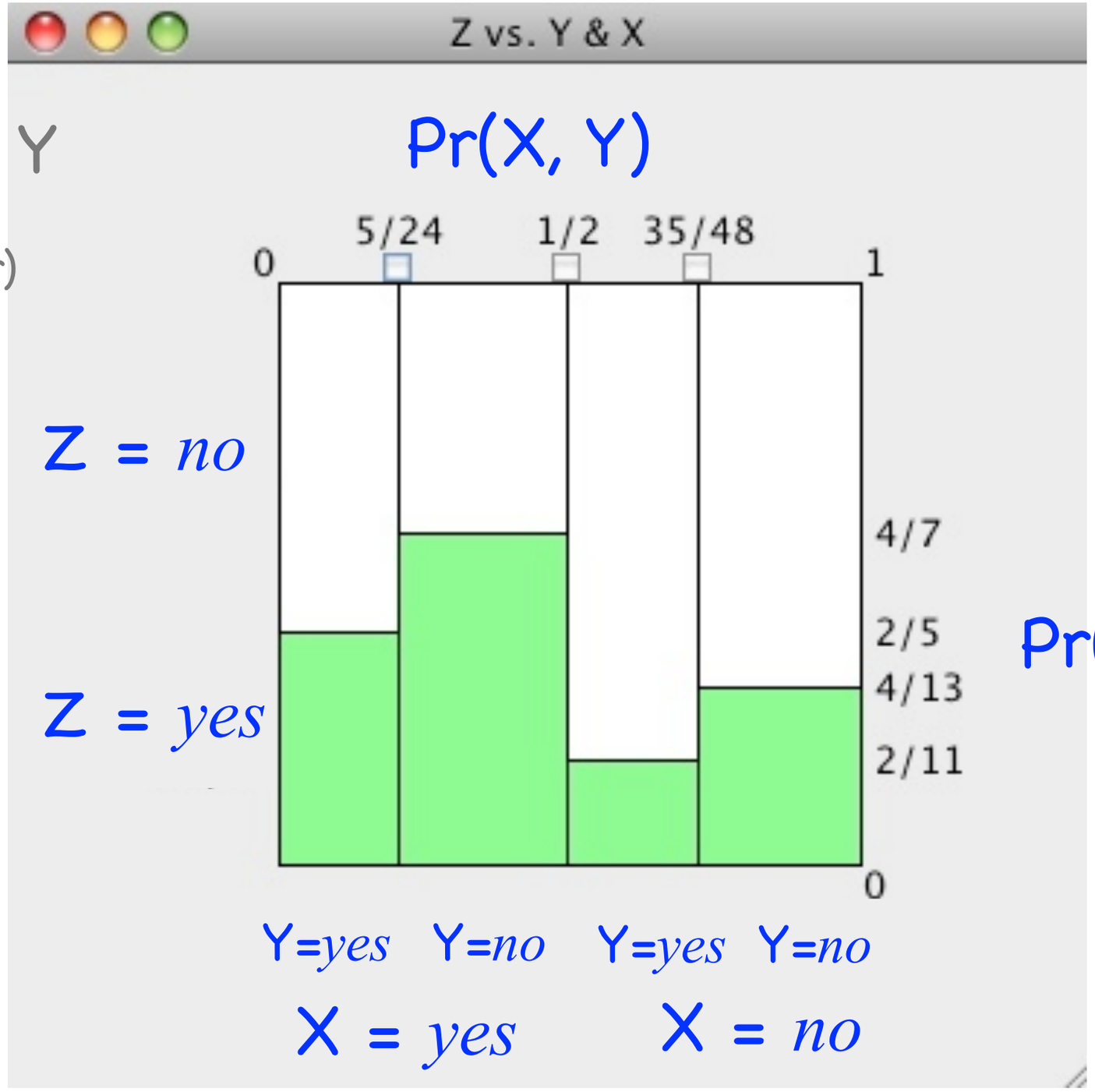


More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

Marginalize over Y

(water container metaphor)

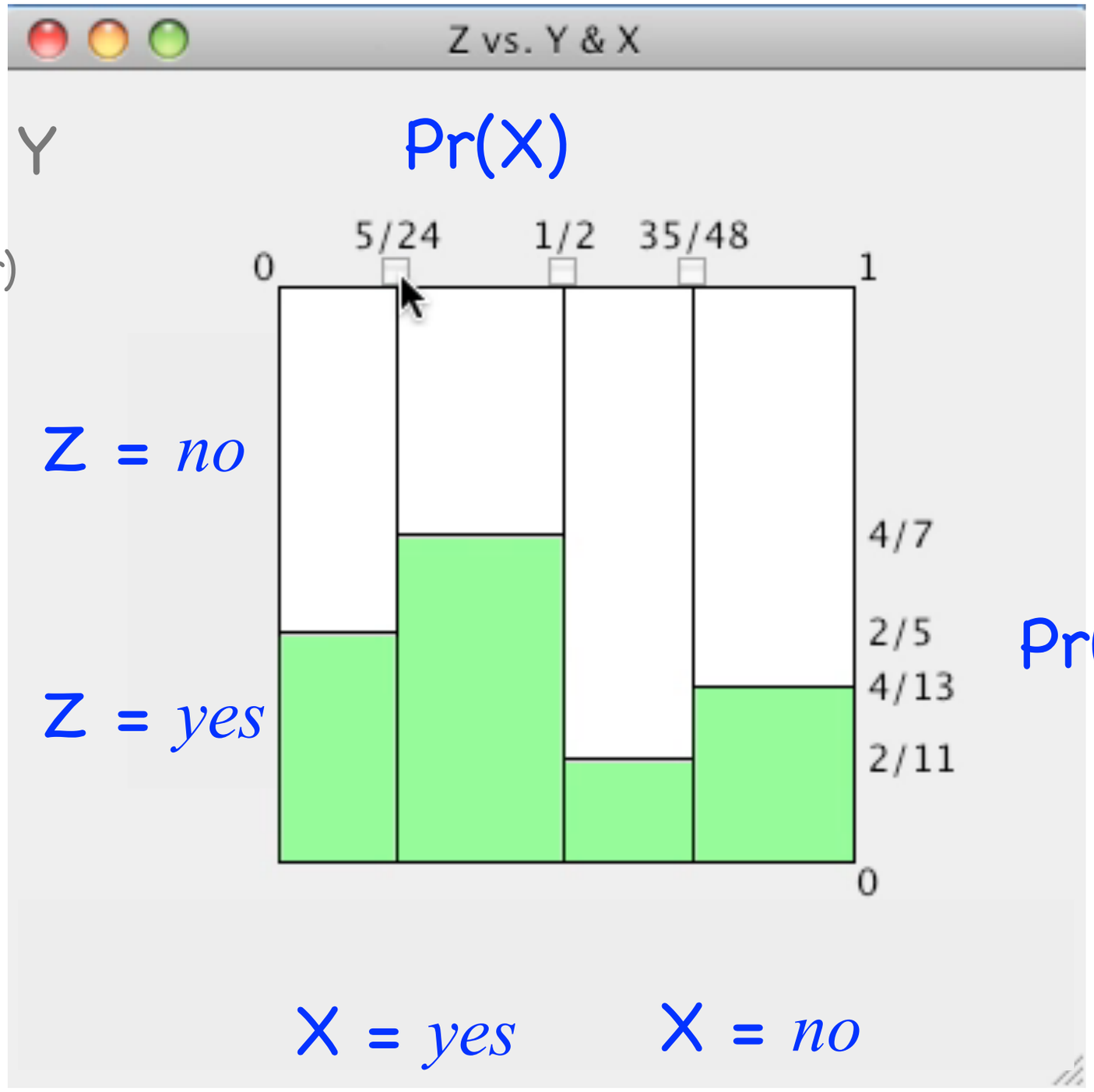


More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

Marginalize over Y

(water container metaphor)



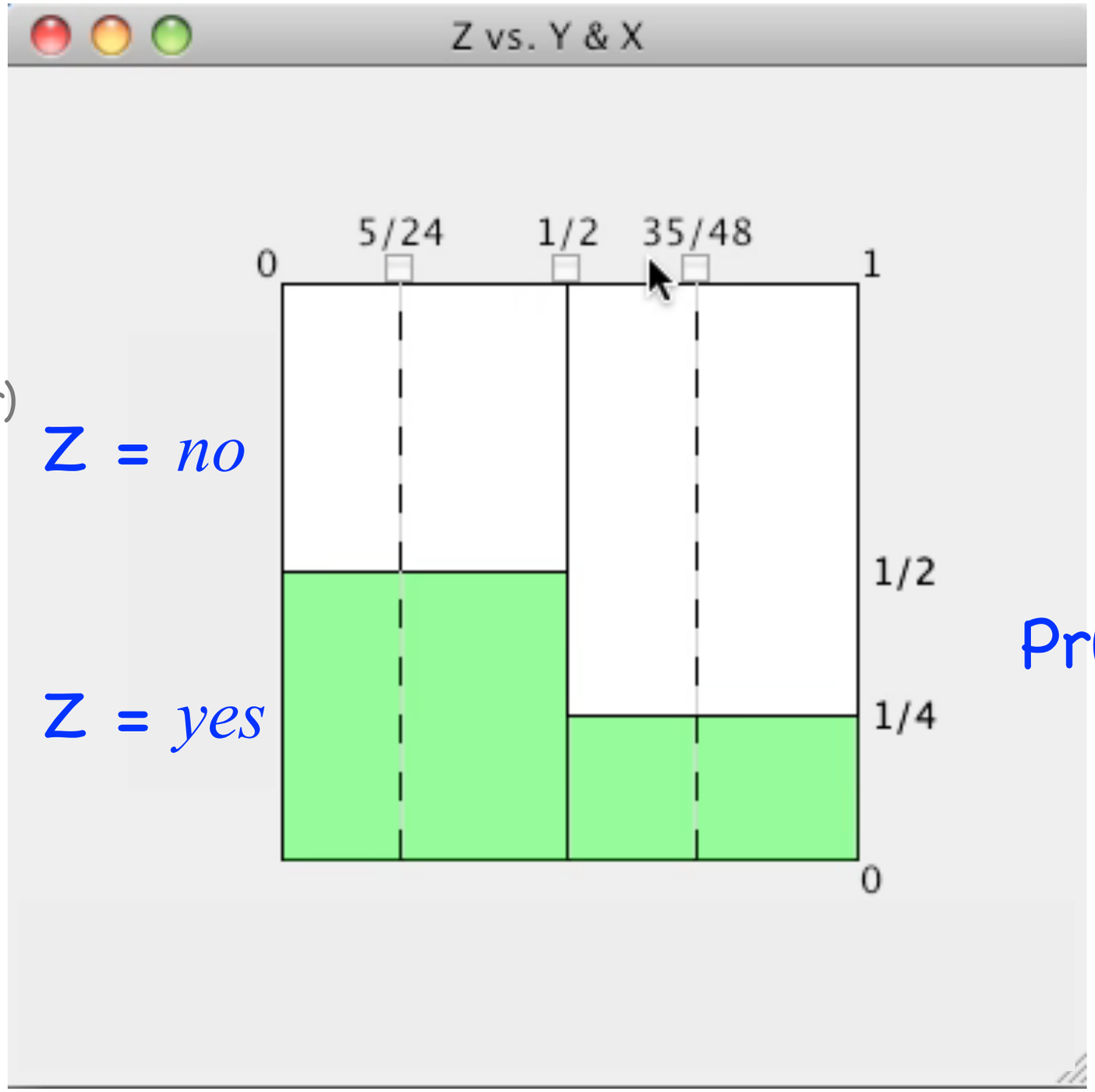
More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

Marginalize over

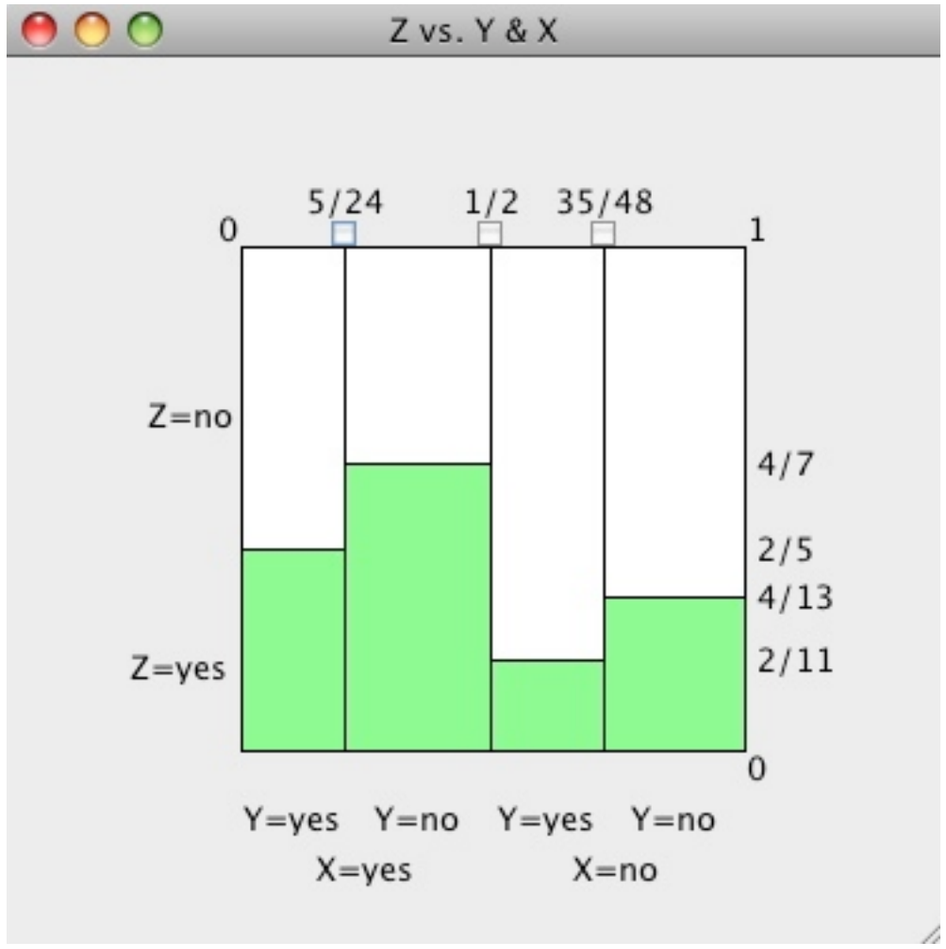
Y and X

(water container metaphor)



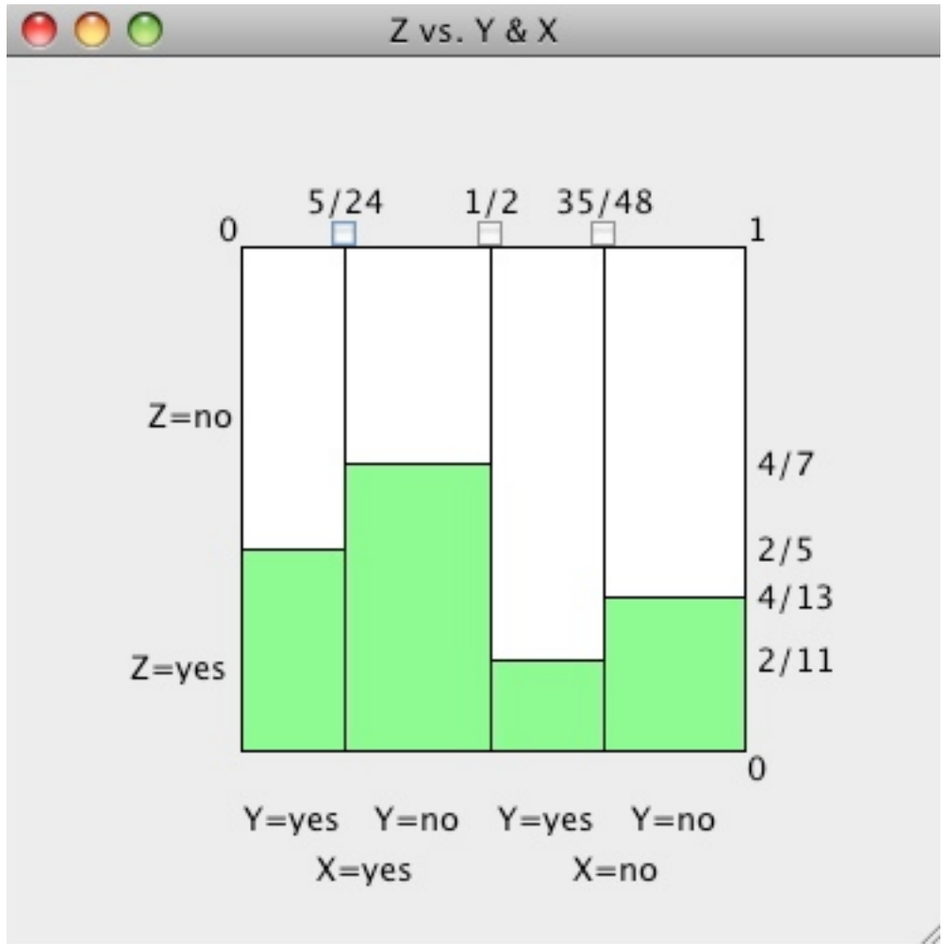
More variables

Three binary variables X, Y, Z $\in \{yes, no\}$



More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

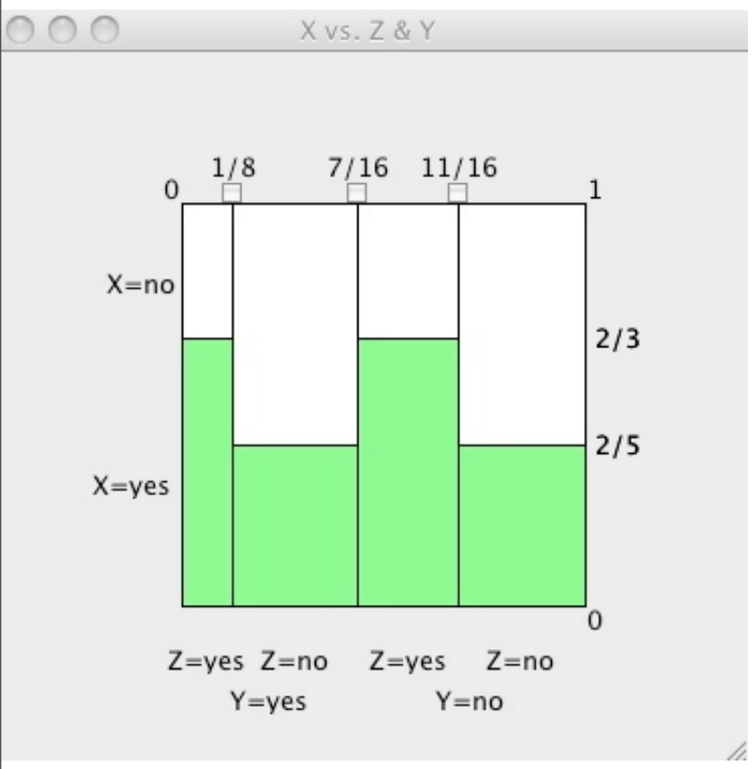


But there are two other choices for vertical axis.

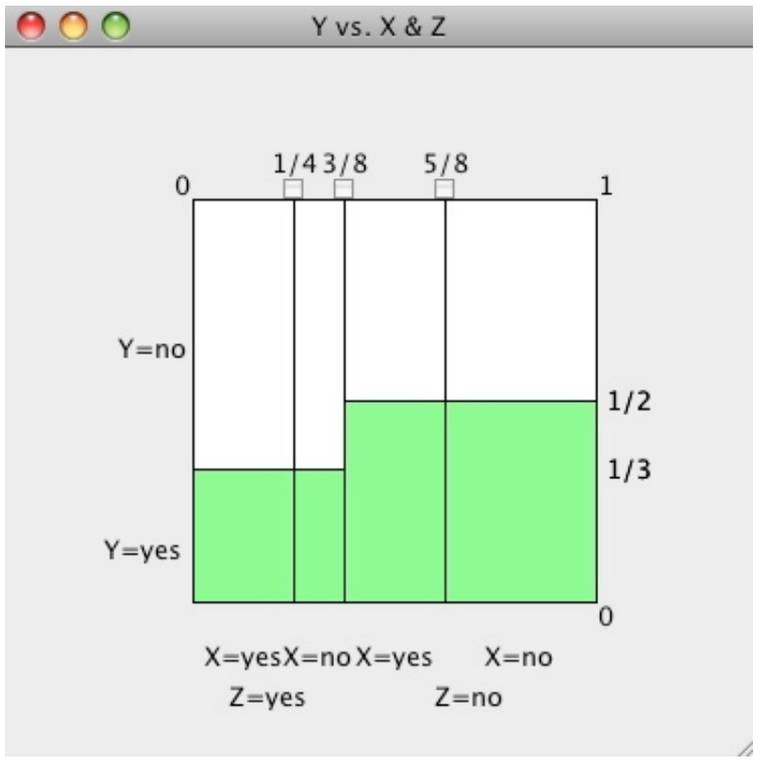
More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

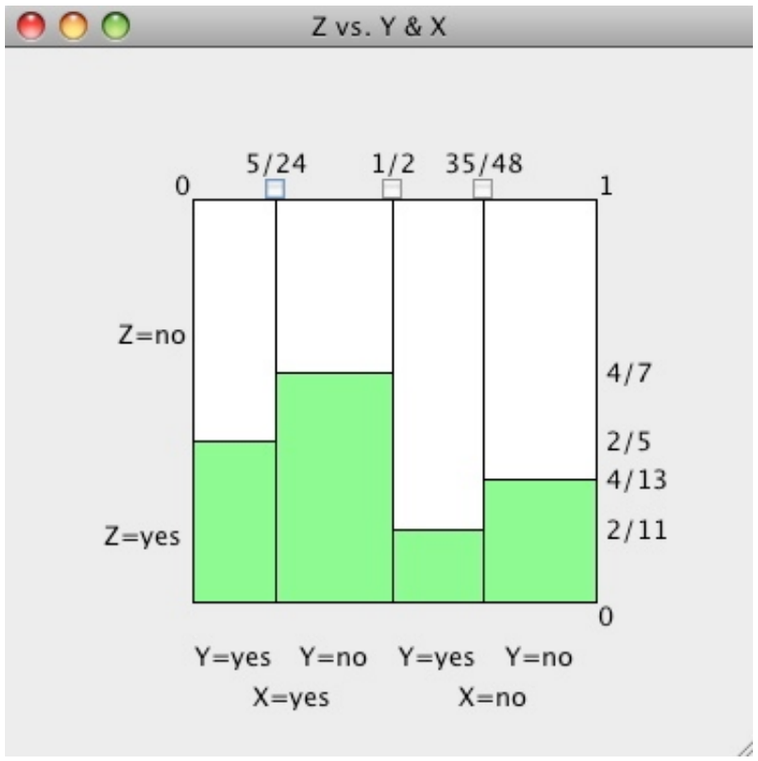
X



Y



Z

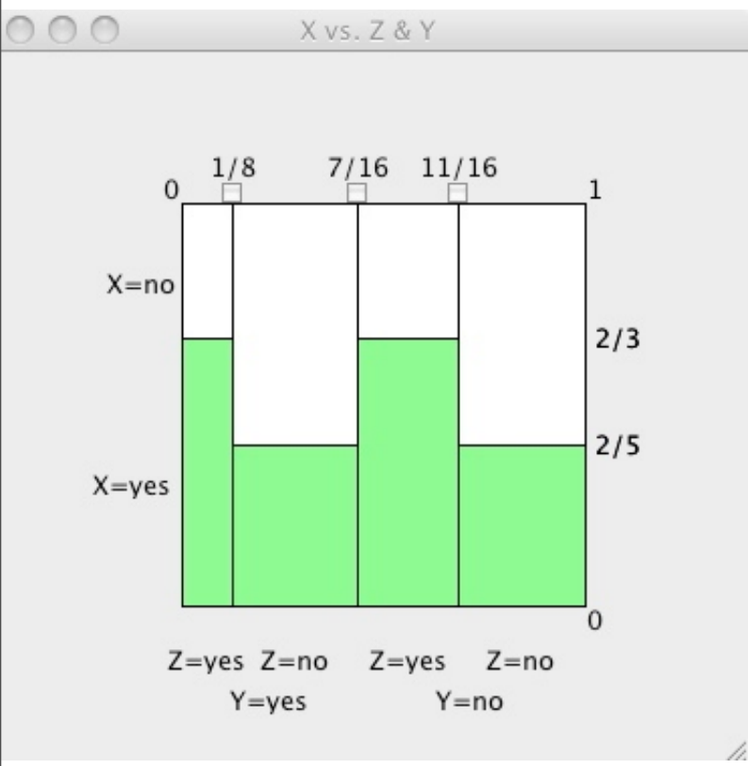


Note flat areas

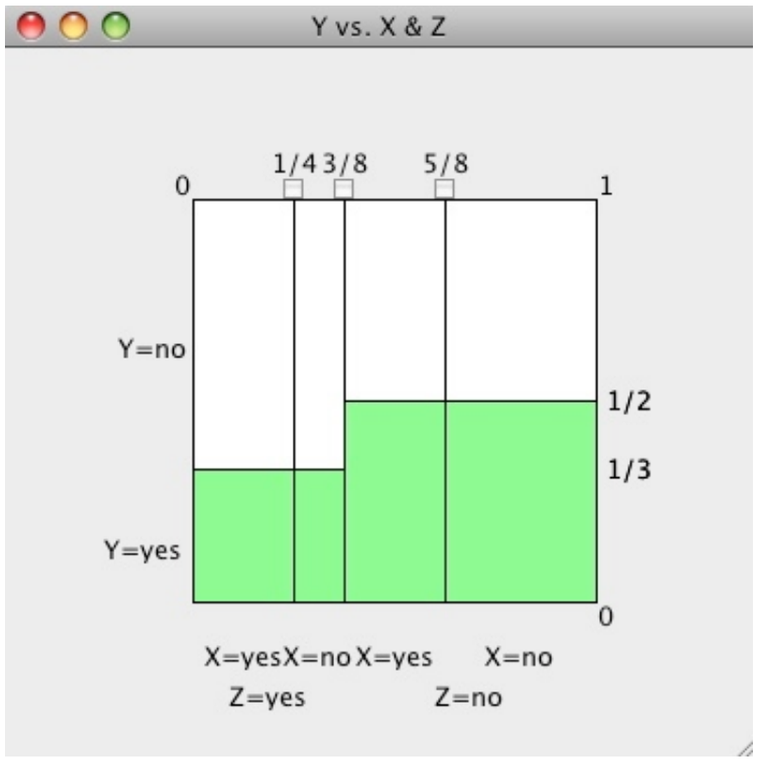
More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

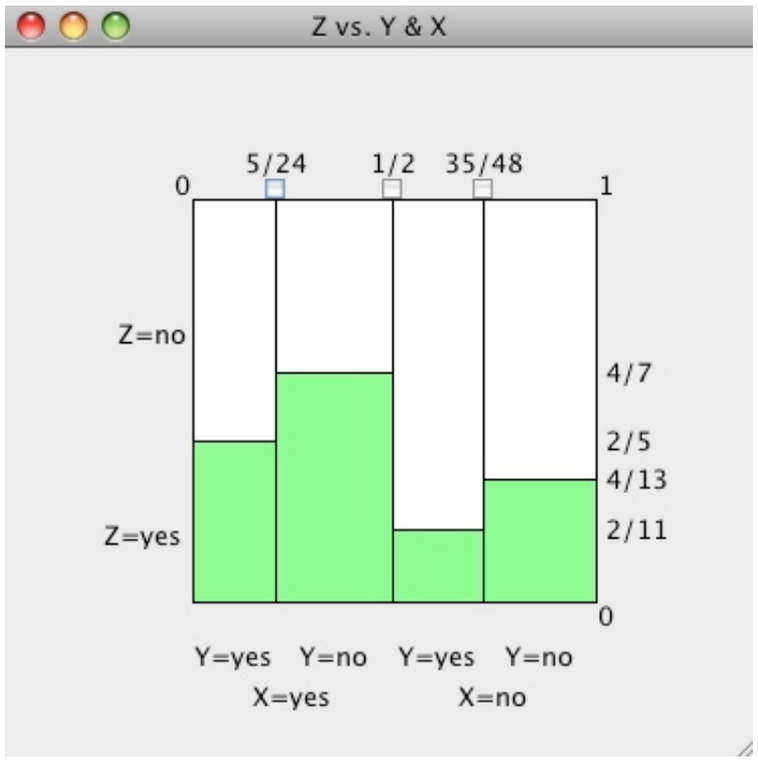
X



Y



Z



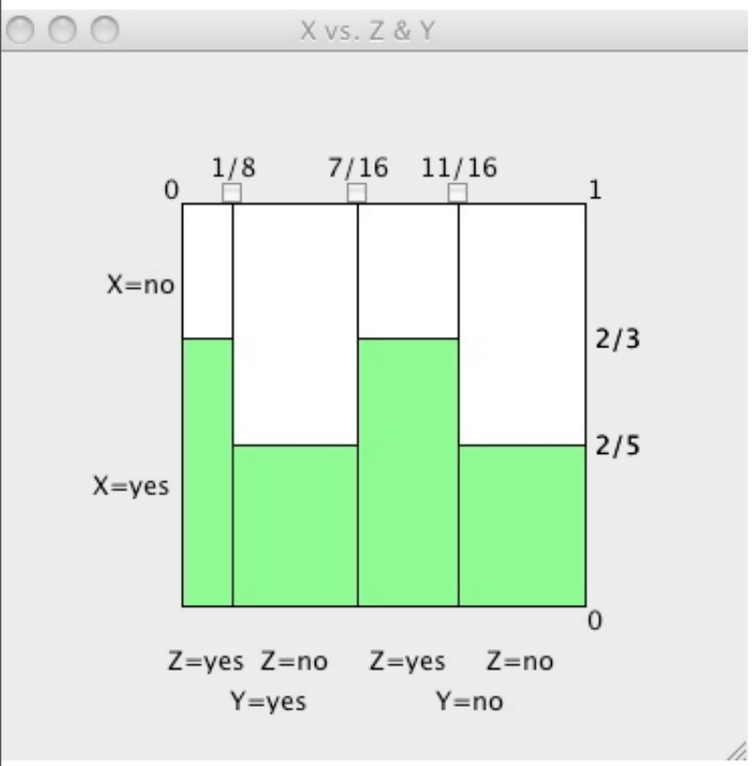
Note flat areas

Y independent of X given Z

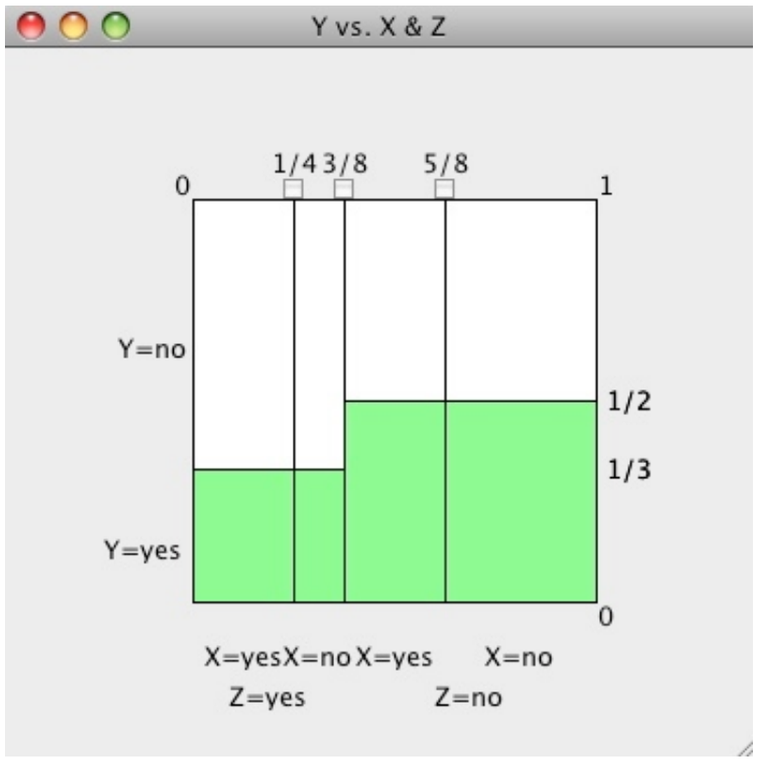
More variables

Three binary variables X, Y, Z $\in \{yes, no\}$

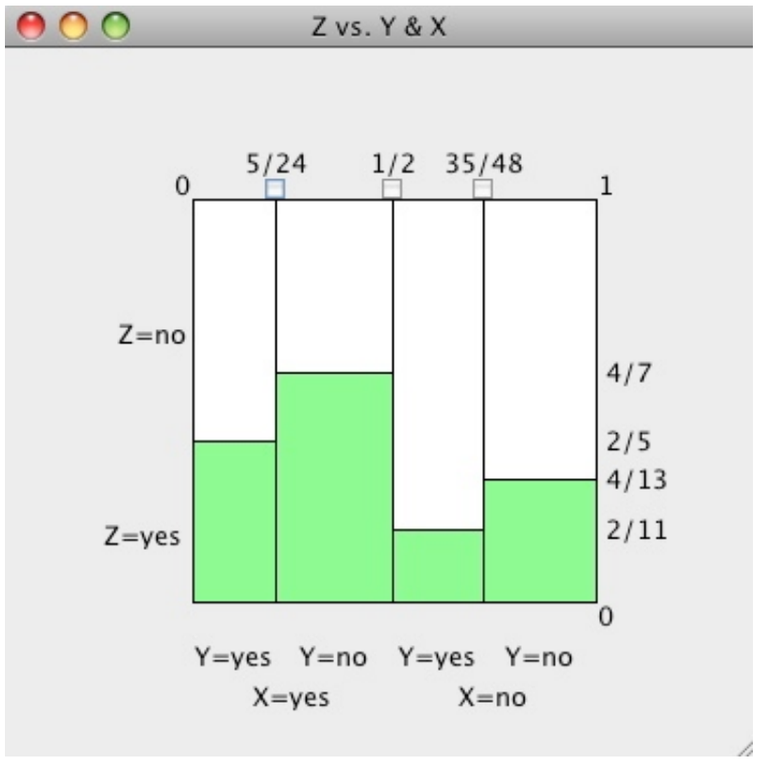
X



Y



Z



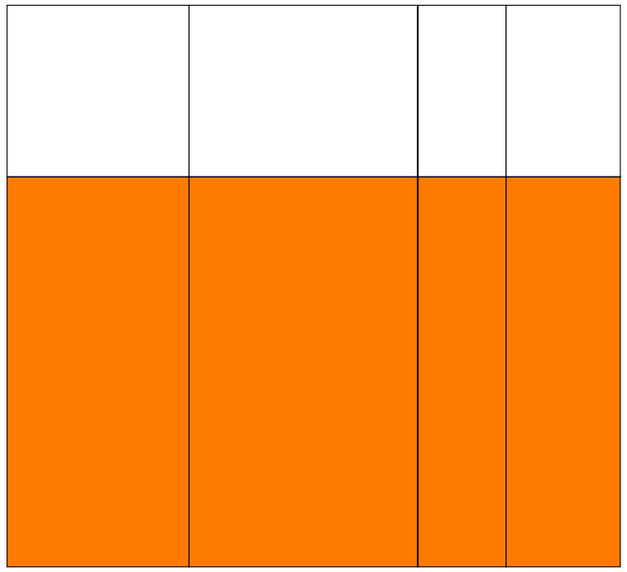
Flat areas across all values of some variable

<=>

conditional independence

Complete independence

Y

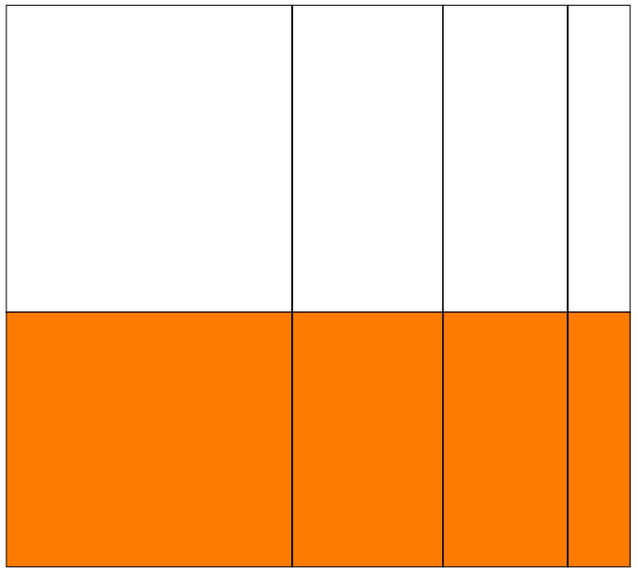


X & Z

Z

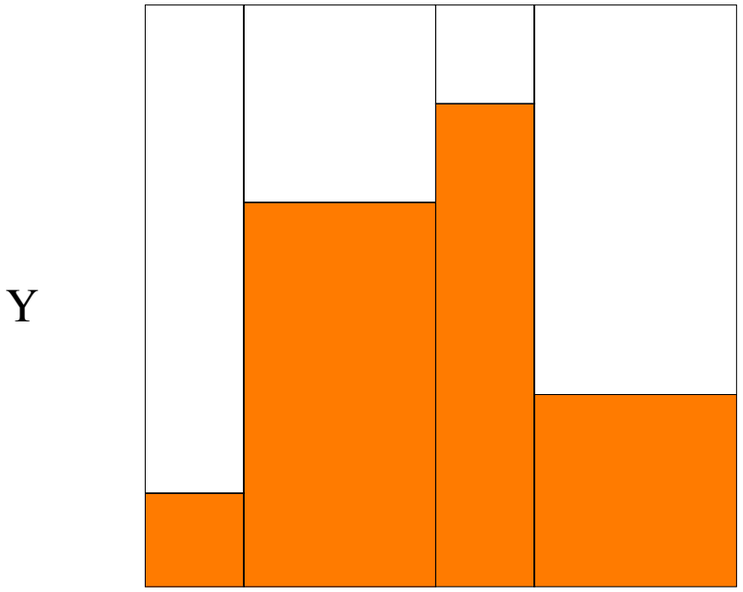
X & Y

X

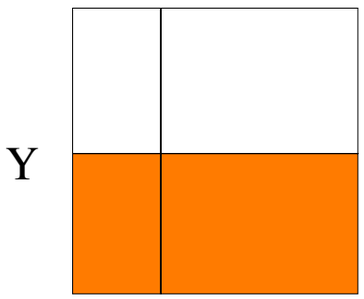


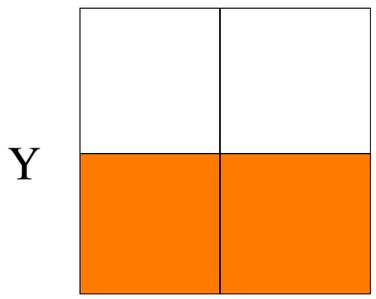
Z & Y

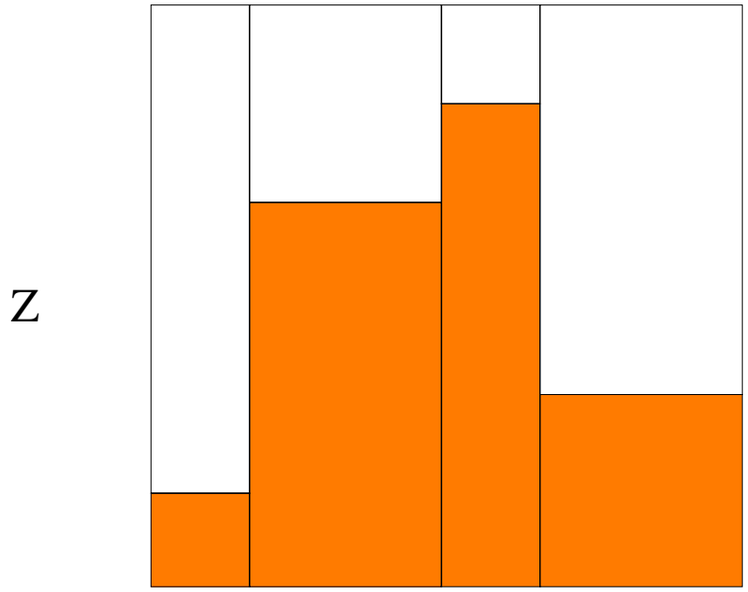
All pairwise, no conditional indep.



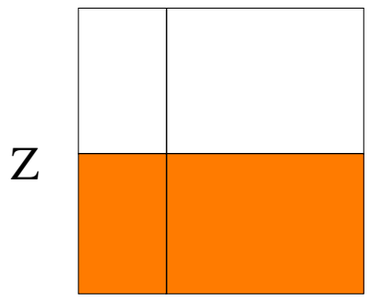
X & Z

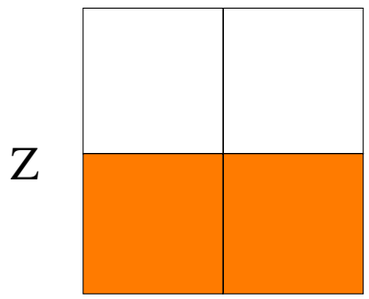


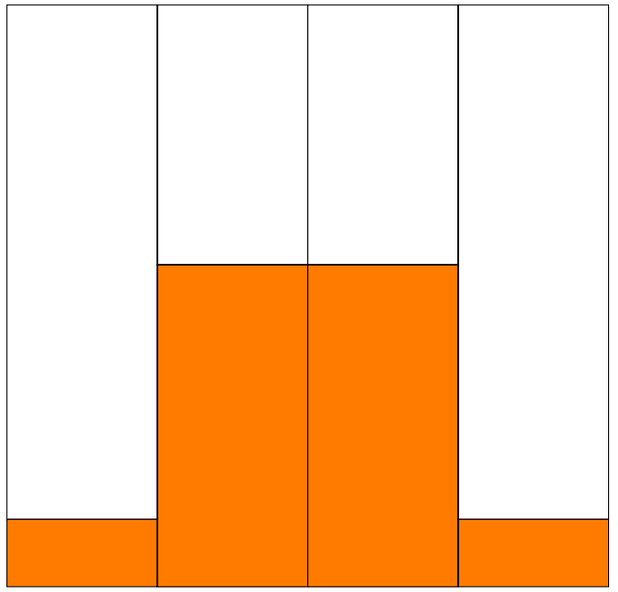




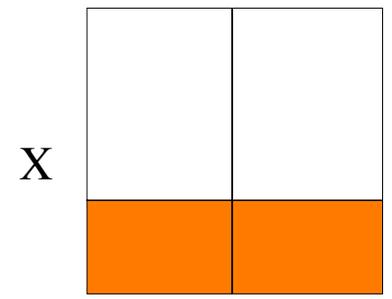
X & Y





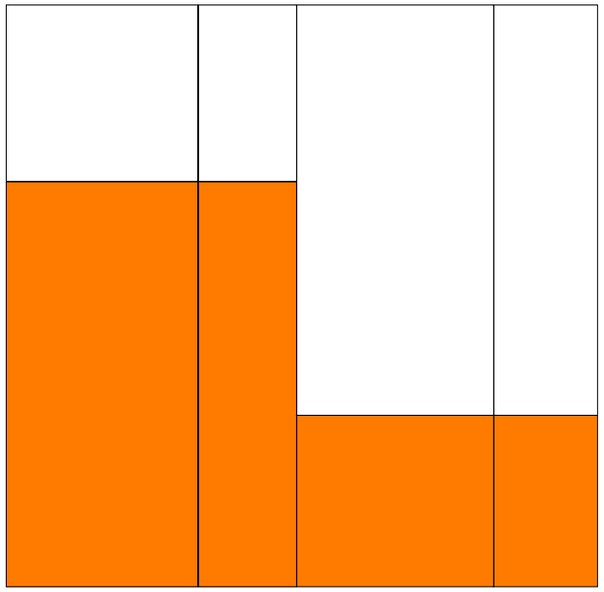


Z & Y



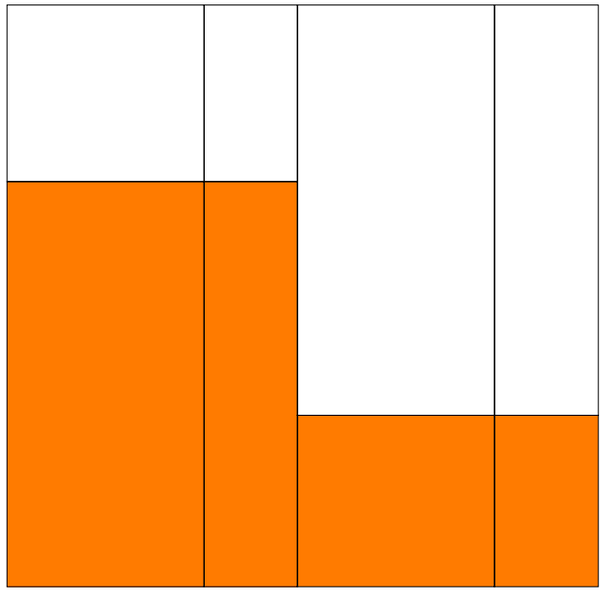
Some of each

Y



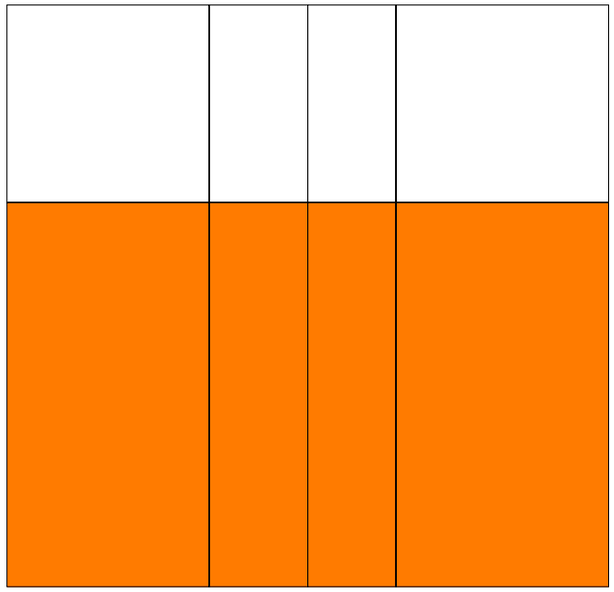
X & Z

Z



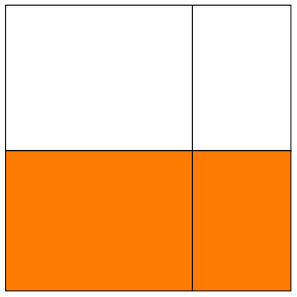
X & Y

X



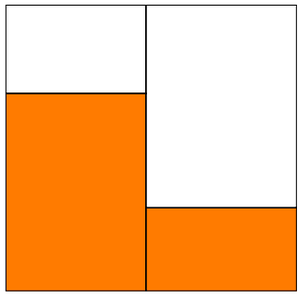
Z & Y

Y



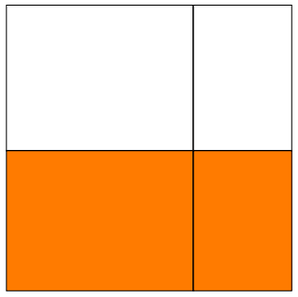
X

Y



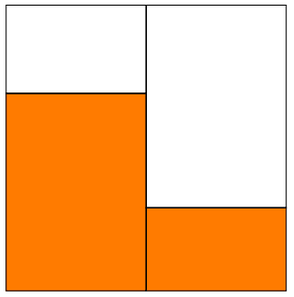
Z

Z



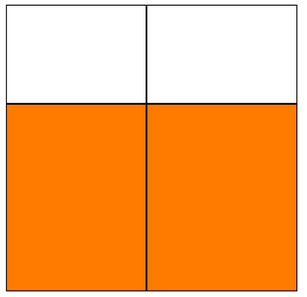
X

Z



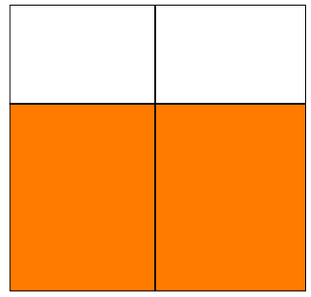
Y

X



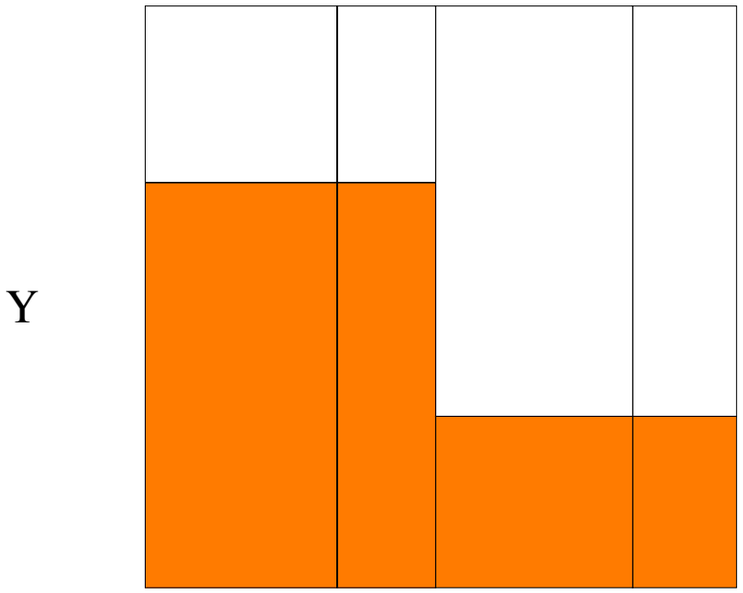
Y

X

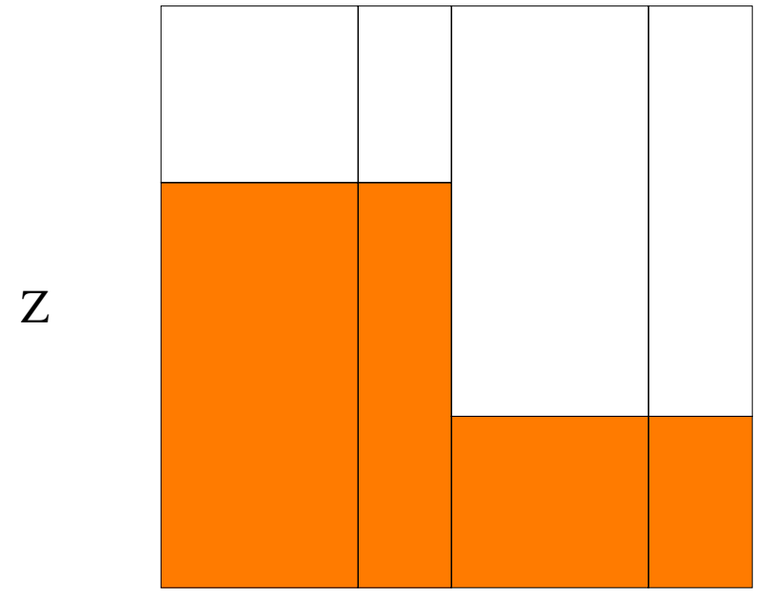


Z

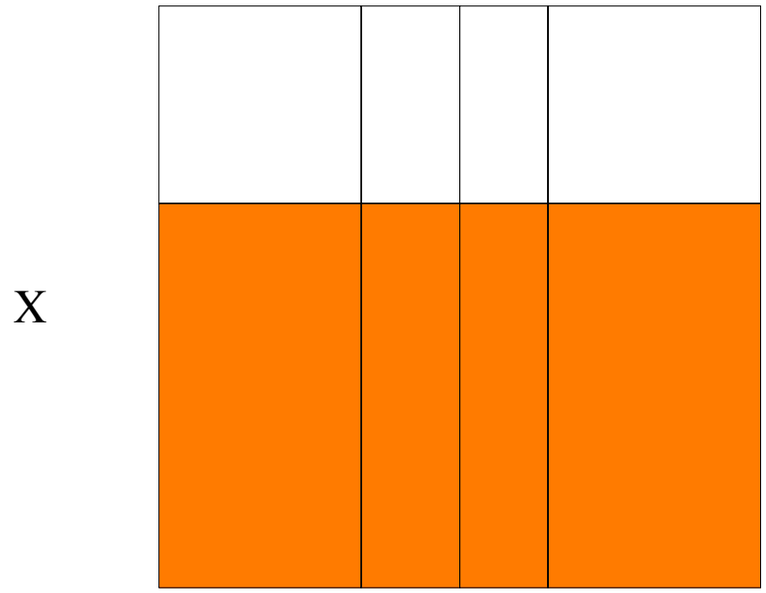
Some of each



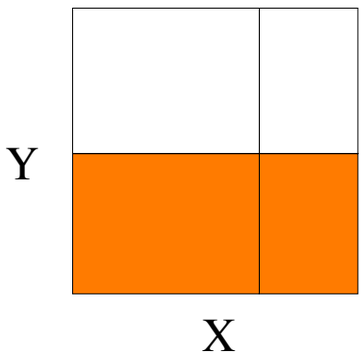
X & Z

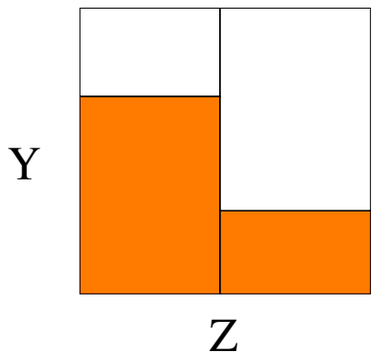


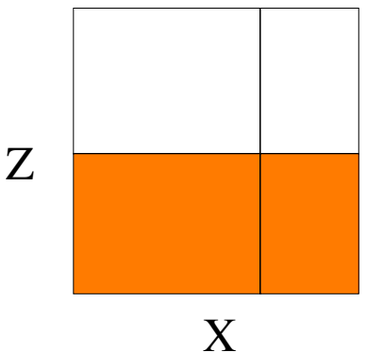
X & Y

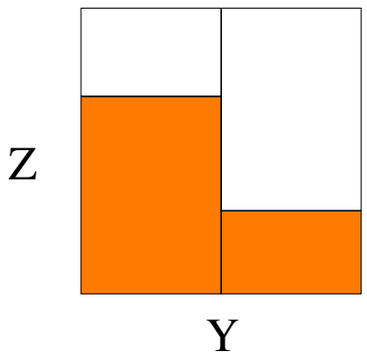


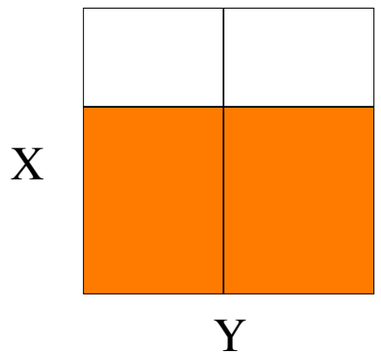
Z & Y

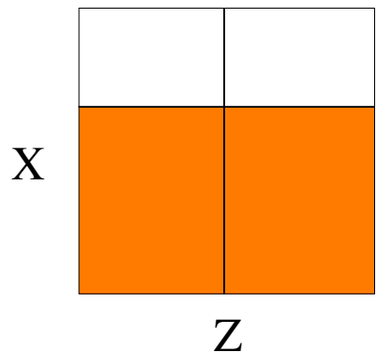






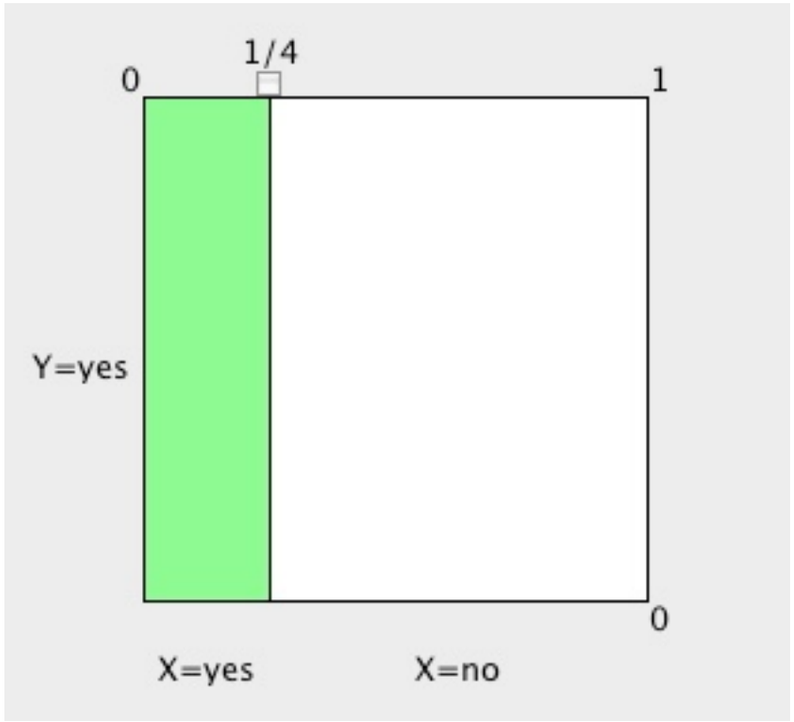






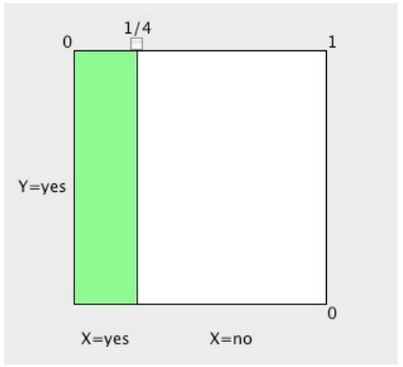
There are many more examples

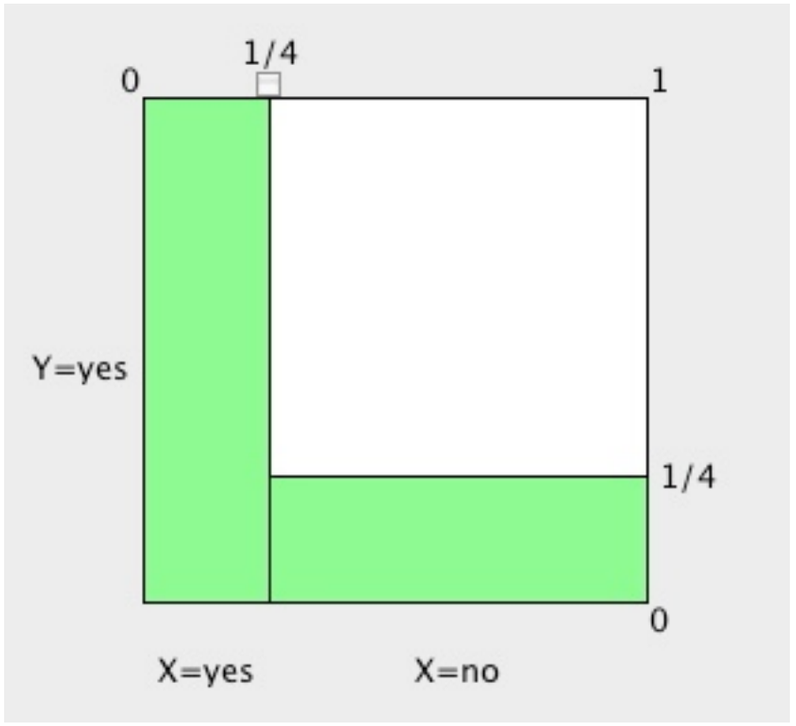
Binary Association



Positive

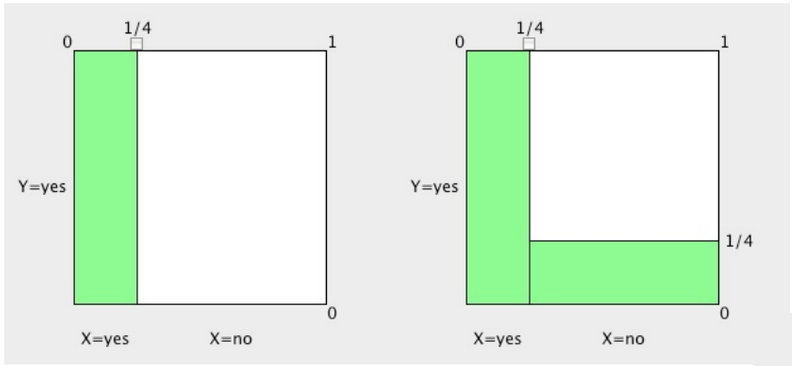
Binary Association

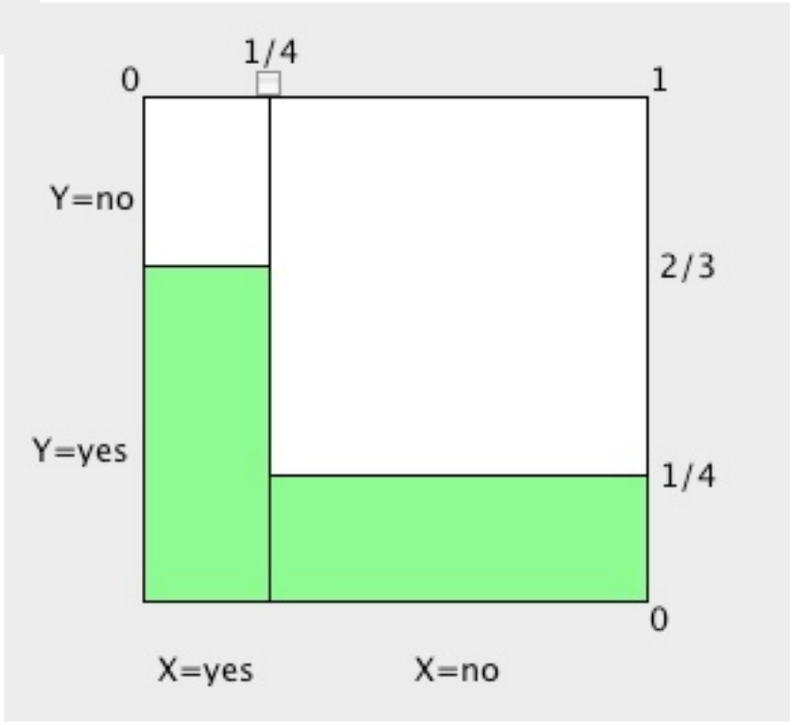




Positive

Binary Association





Positive

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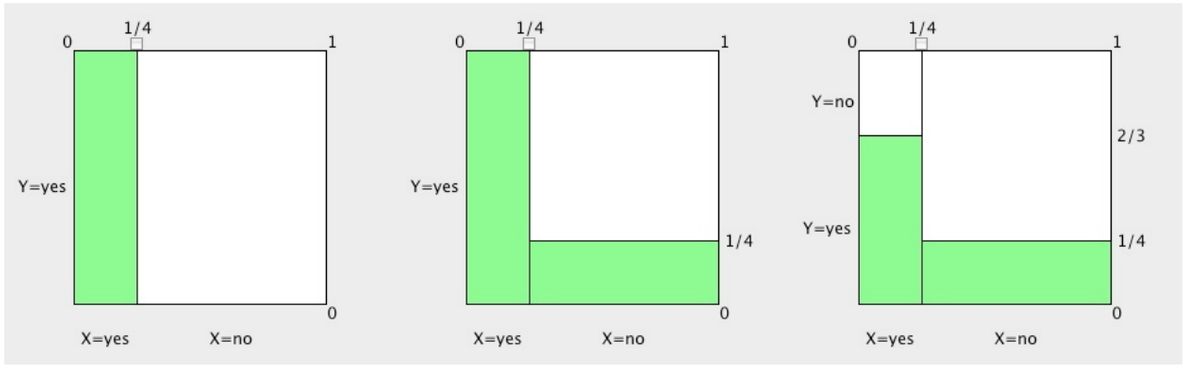

Positive

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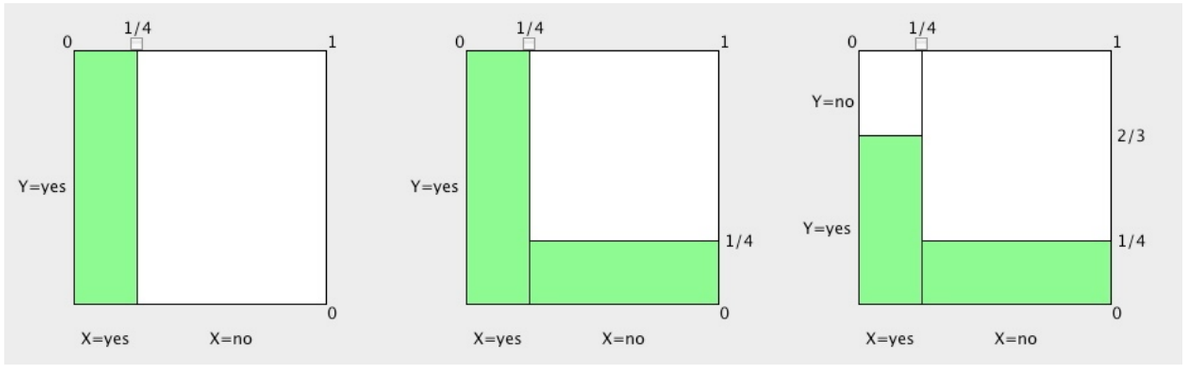
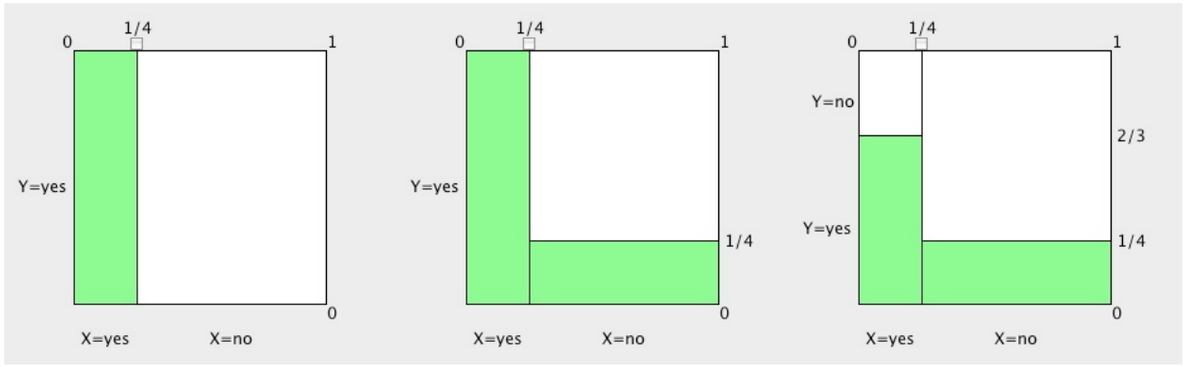


Positive

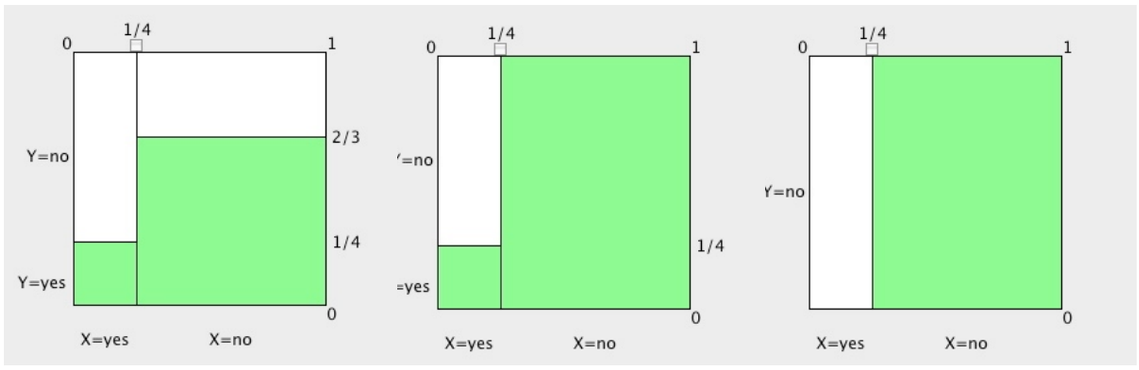
Negative

Binary Association



Positive

Negative



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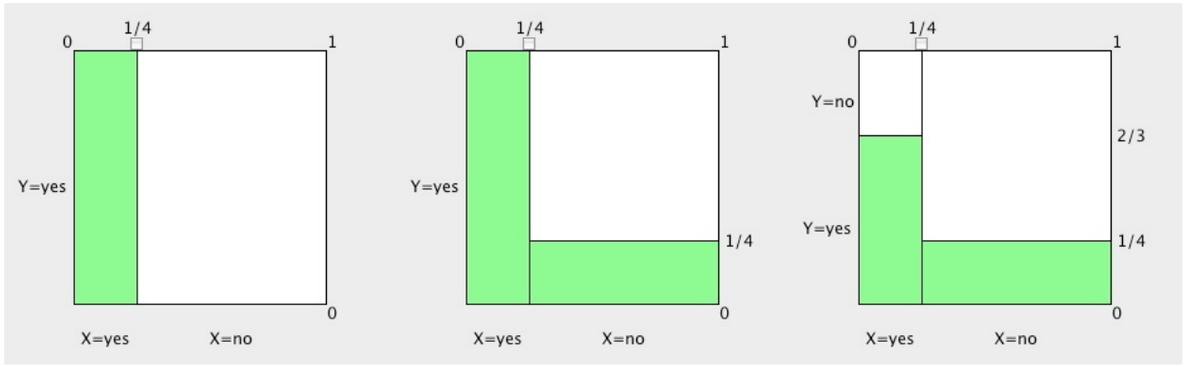

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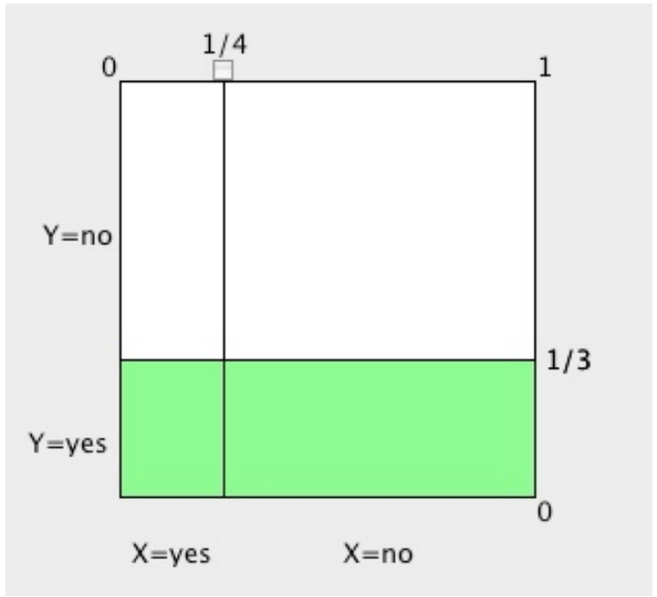

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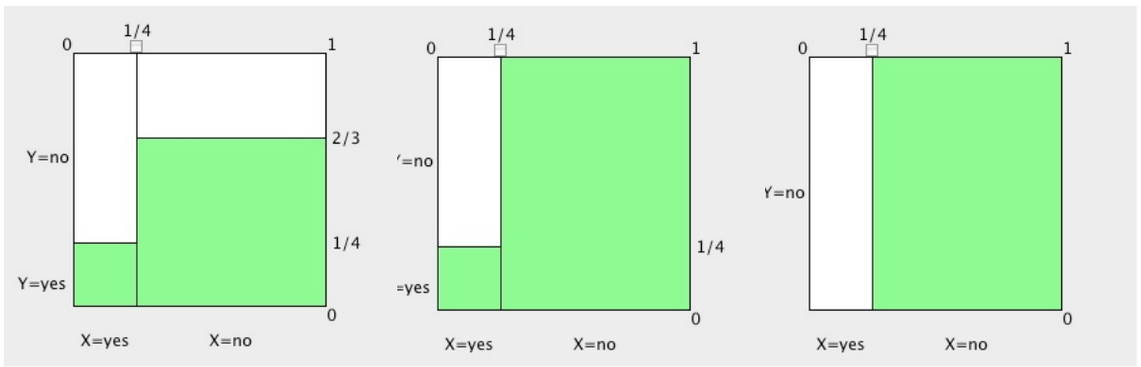

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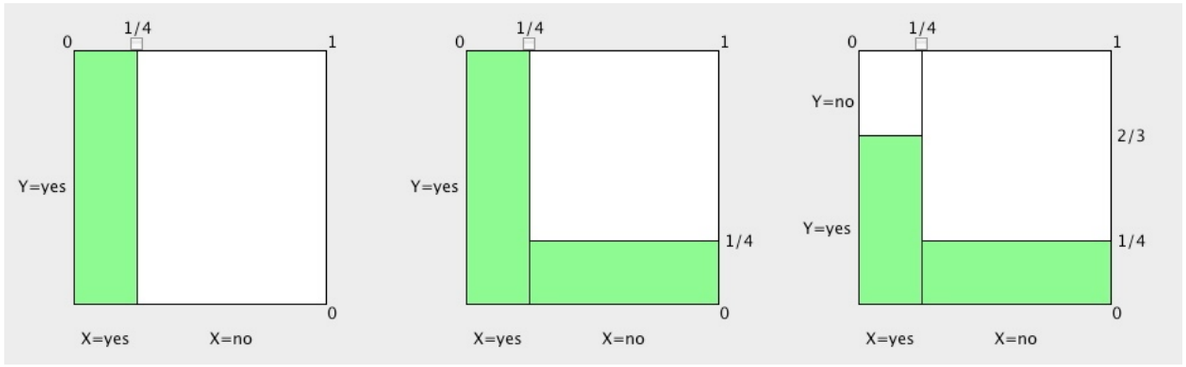
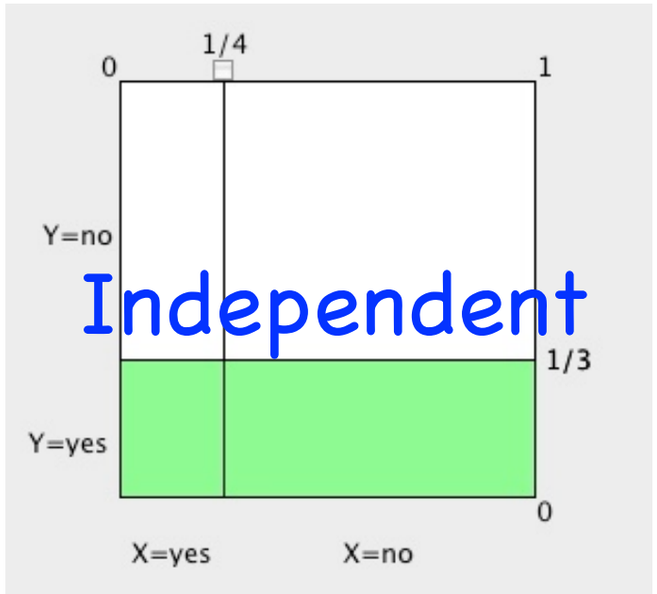


Positive



Independent

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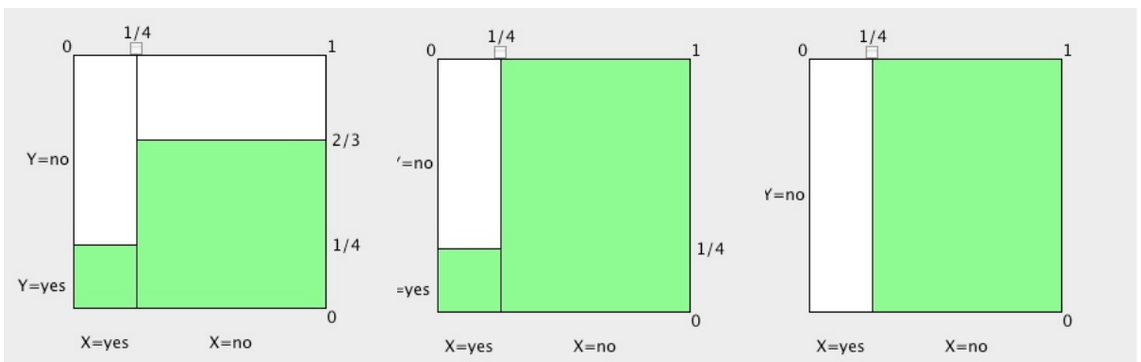
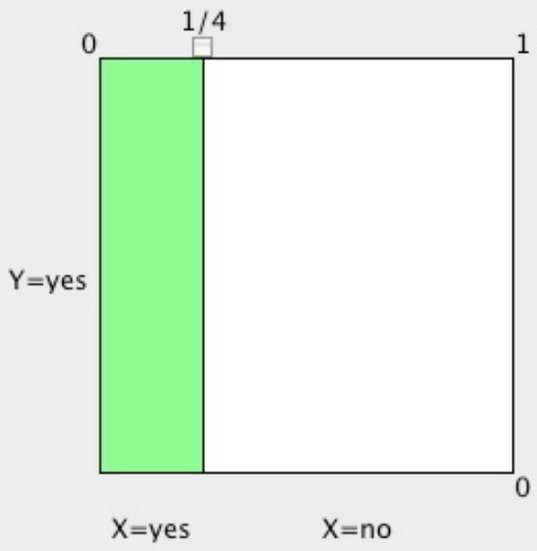


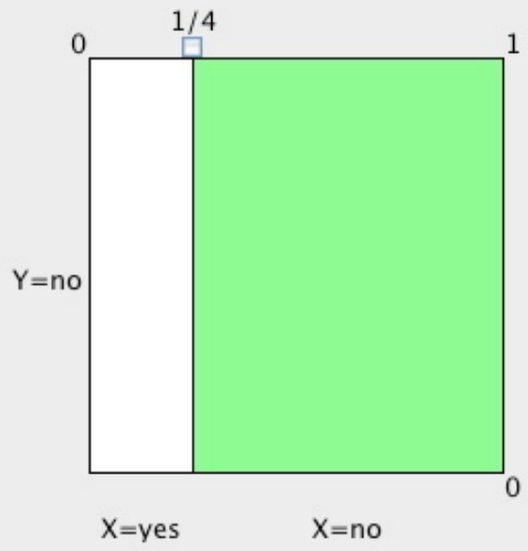
Some standard event properties taught

Events X and Y, "occur" with value "yes" and do not with value "no"

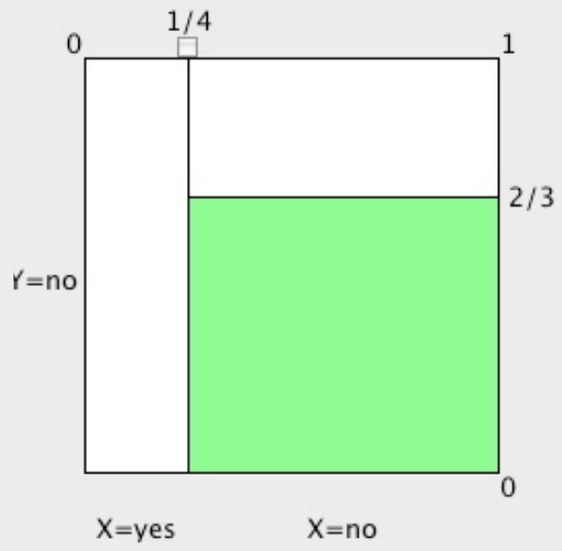
Coincident



Complementary



Mutually exclusive



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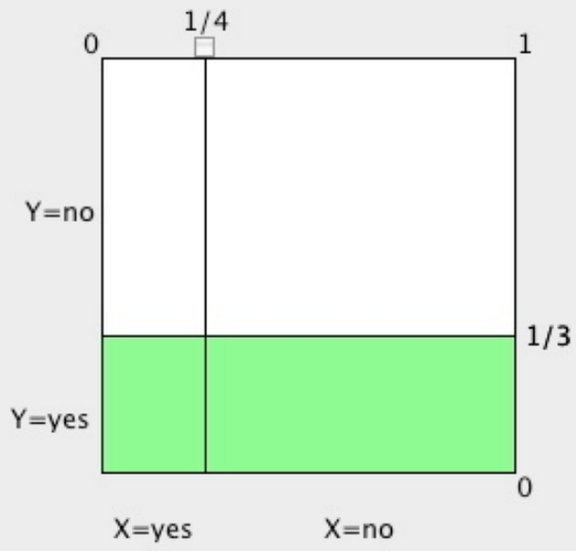
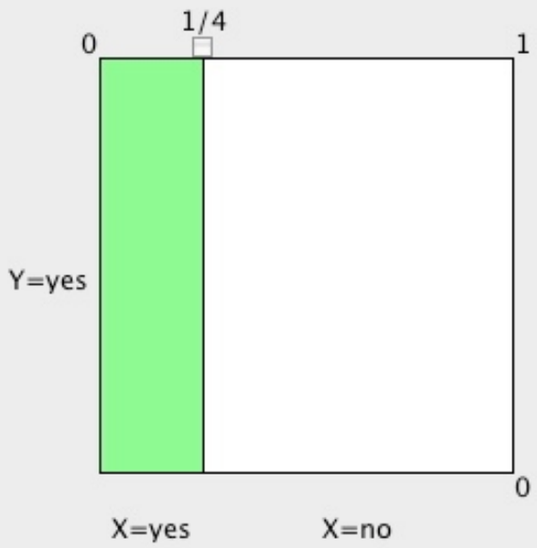


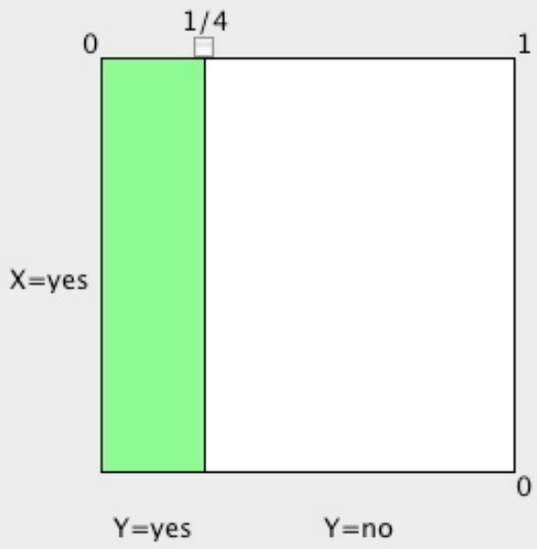
Visually distinct and easily reasoned about

Some standard event properties taught

Events X and Y, "occur" with value "yes" and do not with value "no"

Coincident





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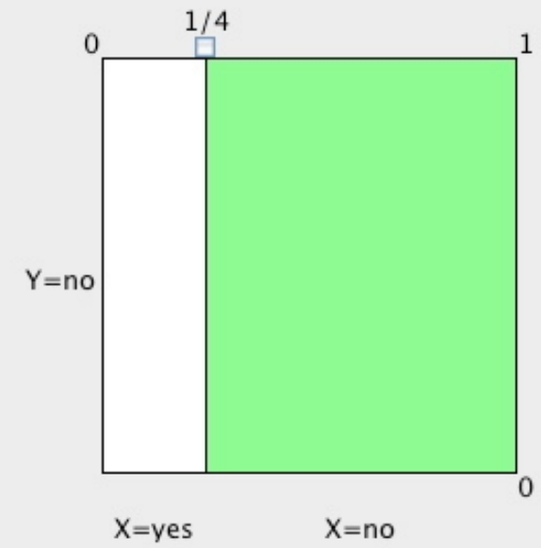
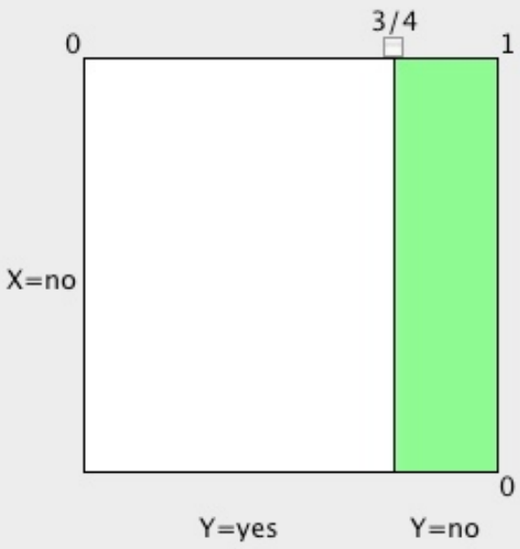
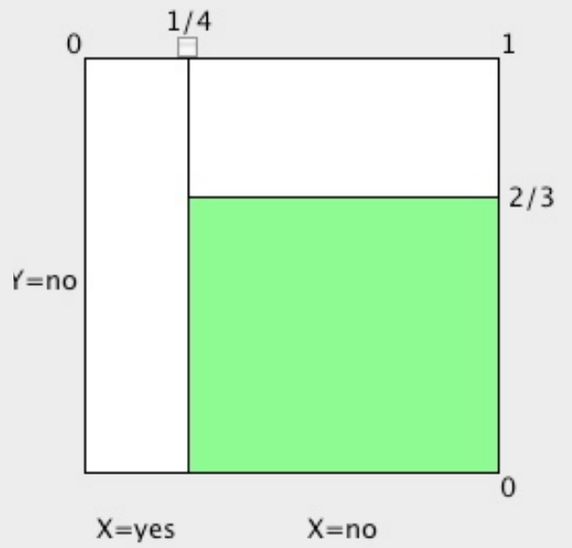
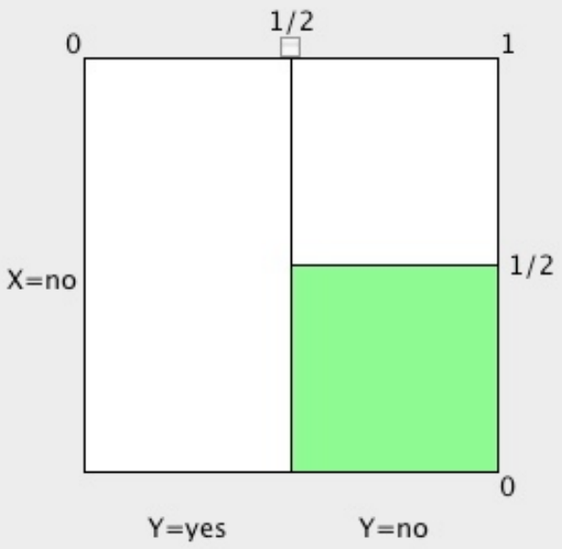




Mutually exclusive





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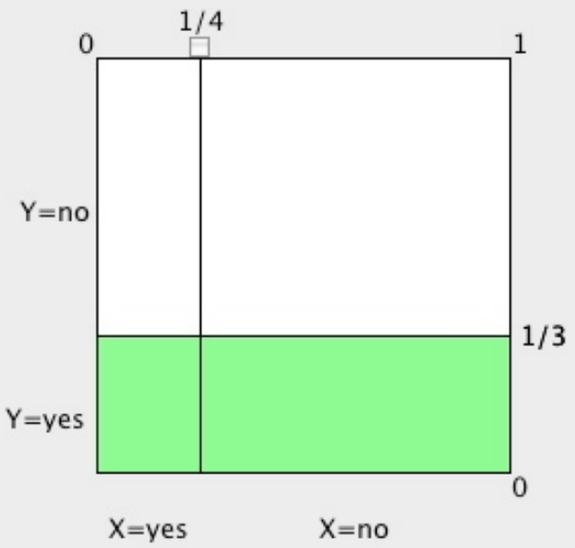
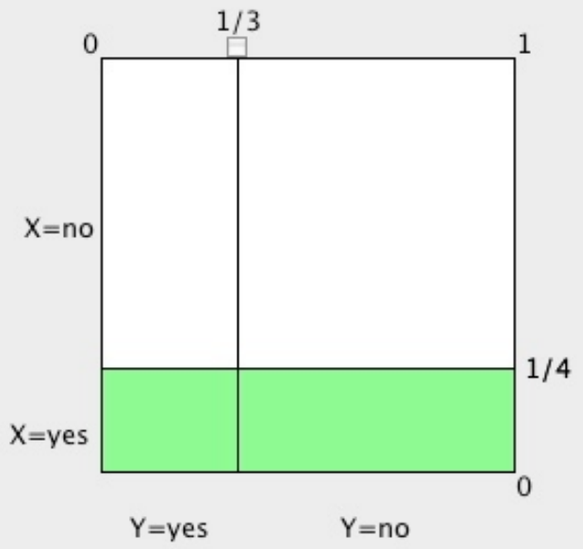




Visually distinct and easily reasoned about

Axioms

(motivated only by conditional proportions/probabilities, within a frame)

C1 $Pr(Y|Z) \geq 0$

C2 $Pr(Y|Y) = 1$

C3 If Y takes on only two distinct values, say $\mathtt{y}$ and $\mathtt{n}$, then

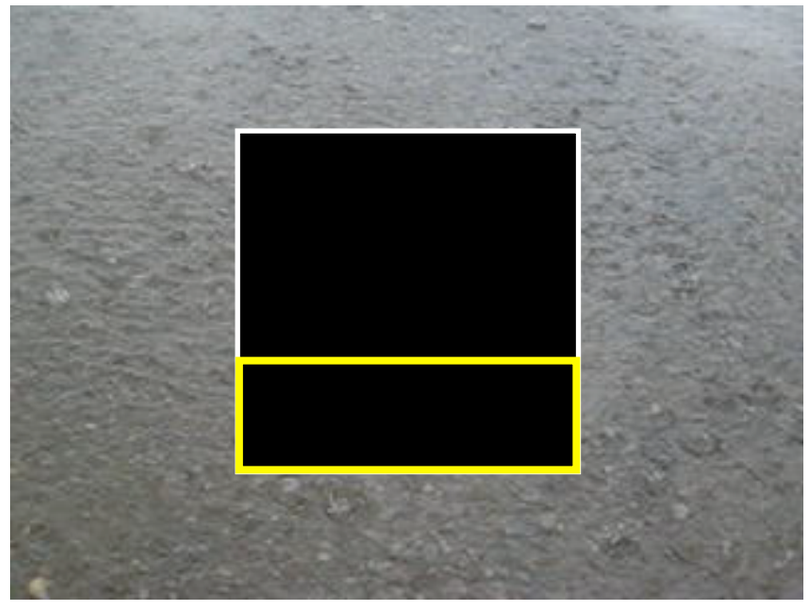
$$Pr(Y = \mathtt{y}|Z) + Pr(Y = \mathtt{n}|Z) = 1$$

C4 $Pr(Y, X|Z) = Pr(Y|X, Z)Pr(X|Z)$

Conditional, Jaynes - Renyi like axioms

Summary

- Ostensive definition of probability
  - Simple calculation
  - Visual reasoning
  - Distinct concepts = distinct visual
  - Sophisticated and important concepts
  - In harmony with notation
  - Lead to axioms for conditional probability



- Suggests good exercises, puzzles, etc.
- Eikosogram – old idea, new name, design, uses …
  – try it, practice with it, use it …
  – avoid Venn diagrams, …

Argument, applet, details, examples

Oldford (2003-8): www.stats.uwaterloo.ca/~rwoldfor

- Papers:
  - "Picturing probability: the poverty of Venn diagrams, the richness of Eikosograms" ... with W. Cherry
  - "Understanding probabilistic independence and its modelling via eikosograms and graphs"
  - "Probability, problems, and paradoxes pictured by eikosograms"
- Software: Java applet ... see my homepage and/or
  - "Eikosogram java application" ... with Glenn Lee (Master's essay/project) http://www.stats.uwaterloo.ca/~rwoldfor/software/eikosograms/