# Graph traversals and visual ordering: Eulerians, Hamiltonians and pairwise comparisons

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# Graph traversals and visual ordering: Eulerians, Hamiltonians and pairwise comparisons

## Outline

- Comparison of treatment groups

  → A new multiple comparison display

- Visual ordering as graph traversal

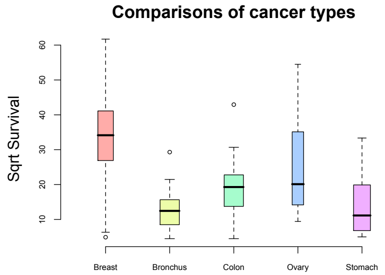
  → Eulerians and hamiltonians

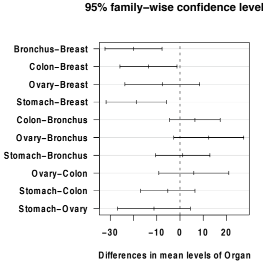
- Parallel coordinates

  → guided by scagnostics

# Comparison of treatment groups

## Vit. C treated cancer patients: Cameron and Pauling 1978



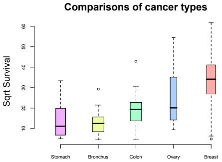



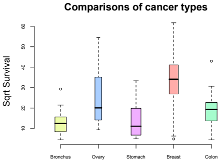

- Easy to visually compare adjacent groups
- not so easy for distant groups

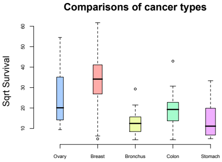
- Which pairs are significantly different?
- 95% Tukey HSD comparisons

Response is sq. root of survival times

# Comparison of treatment groups



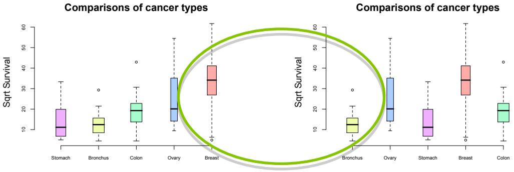



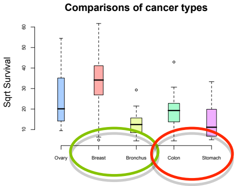

*Require 3 sequences for all pairwise comparisons.*

Note there is duplication: Breast-Ovary and Bronchus-Colon are in first and third plots

Theory

# Comparison of treatment groups



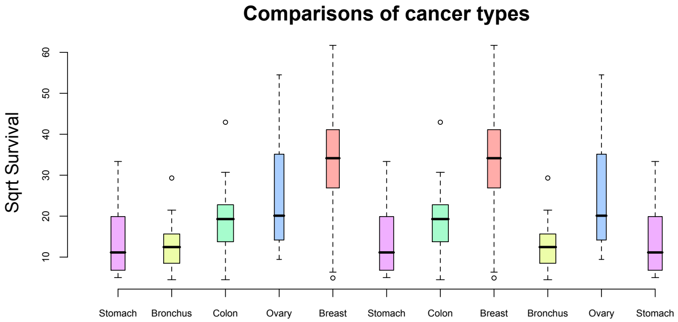




*Require 3 sequences for all pairwise comparisons.*

Note there is duplication: Breast-Ovary and Bronchus-Colon are in first and third plots

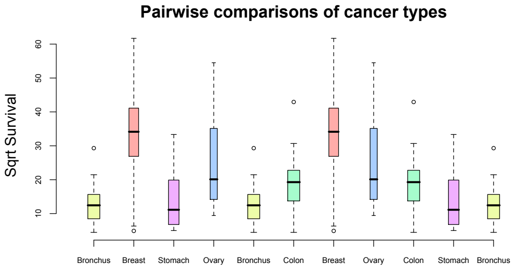
# Comparison of treatment groups

More compactly: Glue the sequences in the first two plots together, append an extra 'Stomach'.




Theory1

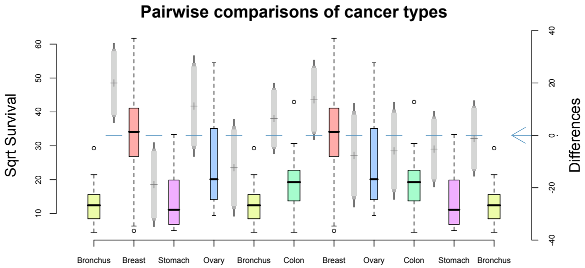
# New pairwise comparison display




- Rearrange boxplots so significantly different means appear on lhs.

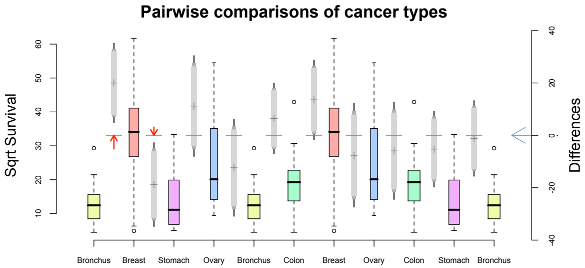
Theory2

# New pairwise comparison display




- Rearrange boxplots so significantly different means appear on lhs.
- Overlay 99% (HSD) CIs $(\mu_{left} - \mu_{right})$

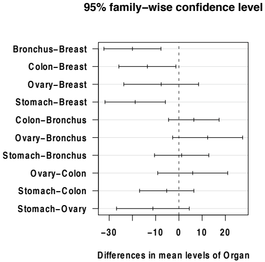
# New pairwise comparison display

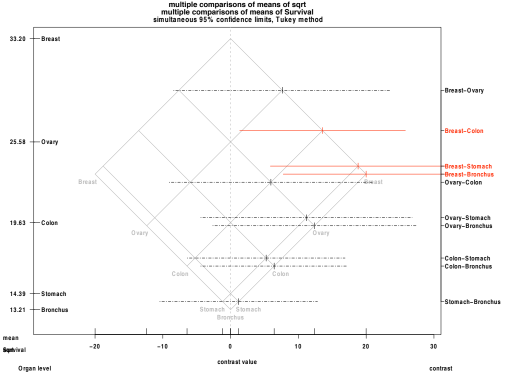



- Rearrange boxplots so significantly different means appear on lhs.
- Overlay 99% (HSD) CIs $(\mu_{left} - \mu_{right})$
- Red arrow: significantly different comparisons
- Simple yet informative

Theory2

# Improvement on..???

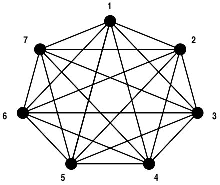



Hsu, Periggia (1994), Heiberger and Holland (2006)

# Graphs: nodes, edges and weights

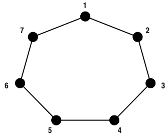
- $n$ variables, cases, factor levels, boxplots: identify with nodes of graph
- visualisation: requires graph traversal
- All possible pairings are of interest: place an undirected edge between each pair of nodes
- Graph is complete, $K_n$



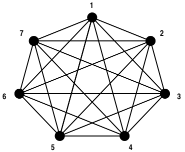

- Dissimilarity measure: edge weight

# Hamiltonian and Eulerian paths

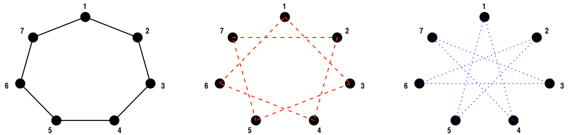
Hamiltonian path gives a permutation of vertices




Eulerian path visits all edges




Hamiltonian decomposition: an eulerian tour composed of edge-distinct hamiltonian cycles



# Classical results: Euler paths- existence

- Eulerian tour (closed path) exists when every vertex is even. ie for $K_{2m+1}$

▸ Example: $K_5$

- Eulerian path (open) exists when two vertices are odd. *Augment $K_{2m}$ with extra edges to achieve this.*

# Which eulerian?

- How many?
  - $K_7$: about 130 million choices
  - $K_{21}$ has more than $3.4 \times 10^{184}$ discounting cyclic permutations

Online Encyclopedia of Integer Sequences (Sloane 2004)

- Prefer eulerians where low-weight edges (interesting comparisons) occur early on.
- Standard algorithm follows unused edges until all are visited. Our version (GrEul) picks low-weight edges.

Example: GrEul

Hierholzer 1873

# Classical results: Hamiltonian Decompositions

$K_n$ can be decomposed as follows:

- For $n = 2m + 1$, into either
  - $m$ hamiltonian cycles, or
  - $m$ hamiltonian paths and an almost-one factor Example: $K_5$
- For $n = 2m$ into either
  - $m$ hamiltonian paths, or
  - $m - 1$ hamiltonian cycles and a 1-factor (or perfect matching).

Lucas-Walecki (1892) Alspach(1990)

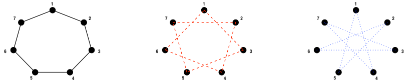
# Which hamiltonian?

- Depends on question of interest.
- Sort nodes, eg by median
- Find shortest or lowest-weight path: (TSP)
- Choice of weights?
  - How interesting is the comparison between treatements? or the relationship between variables?

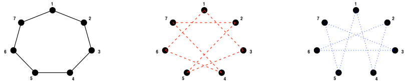
# Which hamiltonian decomposition?

How many?

- $K_7$: 2 canonical forms



120 like this



and 840 like this

- $K_{11}$: 45,000+ canonical forms

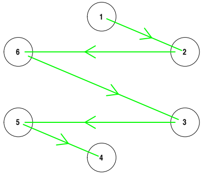
Colburn (1982)

Lucas-Walecki construction: gives one canonical form SkipLW

# Hamiltonian decomposition algorithm

– for decomposition into hamiltonian cycles

- When $n$ is even $n/2-1$ edges must be visited twice
- Lucas-Walecki construction (1892)
- Construction: $n$ even




1 2 6 3 5 4

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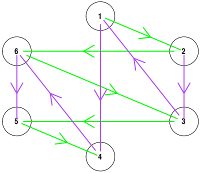


1 2 6 3 5 4

2 3 1 4 6 5

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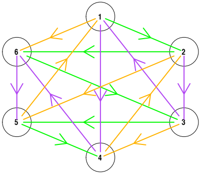
same as zig-zag method used in Wegman (1990)

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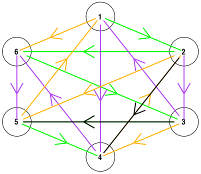


1 2 6 3 5 4

2 3 1 4 6 5

3 4 2 5 1 6

black edges- visited twice

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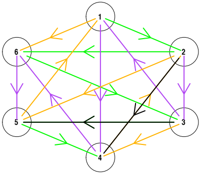



black edges- visited twice

1 2 6 3 5 4
2 3 1 4 6 5
3 4 2 5 1 6

- Construction: $n$ odd

7 1 2 6 3 5 4
7 2 3 1 4 6 5
7 3 4 2 5 1 6 7

# Hamiltonian decomposition algorithm

– for weighted graphs

Goal: a decomposition where weights increase: first hamiltonian has lowest weights, 2nd has next lowest weights etc.

- Greedy algorithm:
  - Start with Lucas-Walecki construction
  - WHam: use TSP for first hamiltonian, using weights, vary cycle order, direction and contact point in others.

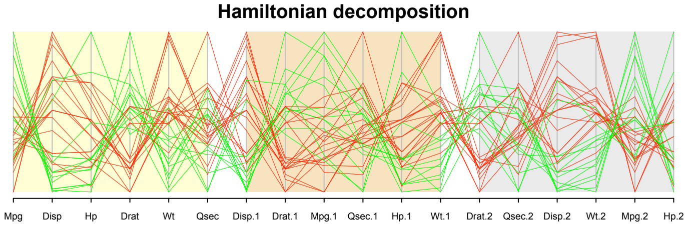
- Or:
  - Or, peripatetic TSP: k-best edge-disjoint hamiltonians
  - use other seriation as alternatives to TSP

# Applications

- Pairwise comparison of treatments
- Parallel coordinates
- Interaction plots
- Star glyphs of multivariate data

# Parallel coordinates

## mtcars data from R: 6 variables

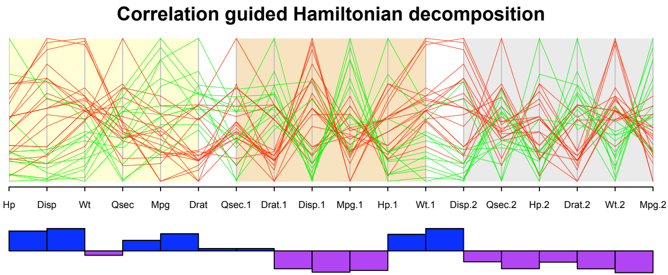



- Shows all pairs of variables adjacently.

Panel colors - three hamiltonian paths. Line color -transmission type.

# Parallel coordinates

## mtcars data from R: 6 variables

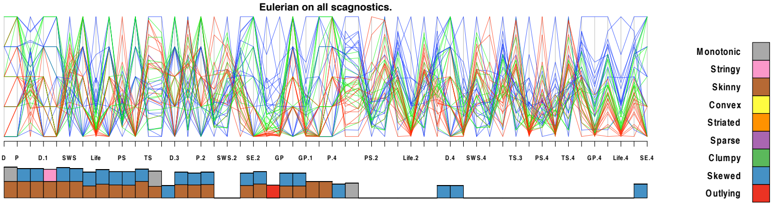



- Shows all pairs of variables adjacently.
- WHam: use correlation to choose decomposition
- Add correlation guide.

Panel colors - three hamiltonian paths. Line color -transmission type.

# Parallel coordinates- more variables?

sleep data- 10 variables, 62 species




- Eulerian has 49 edges - use GrEul to follow interesting edges first.
- Barchart shows panel scagnostics

  scagnostics package, Hofmann et al.
- Lots of skinniness, skewness

Brain and body weight log transformed, colour by life expectancy
Use index values of 0.7 or more.

# Parallel coordinates- more variables?

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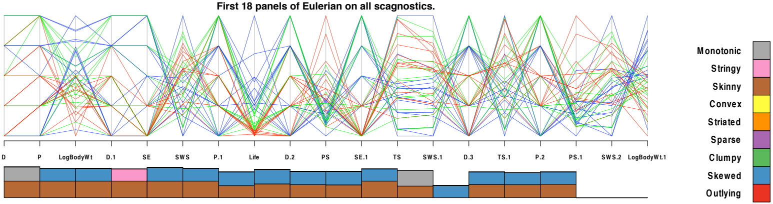
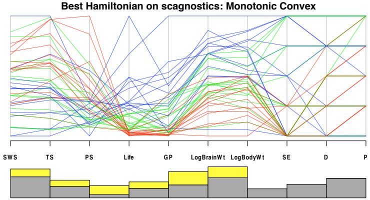
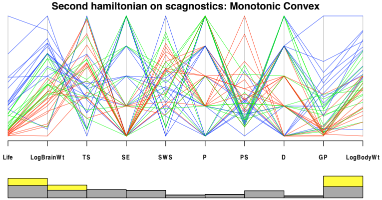



- Zoom on first 18 panels- captures ‘interesting” relationships
- Lots of skinniness, skewness

# Parallel coordinates- hamiltonian decomposition



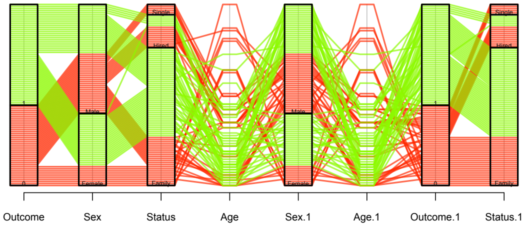




- Hamiltonians that chase "interesting" relationships-here correlational structure
- WHam: first two (of 5) hamiltonians

Monotone (grey) + convexity (yellow)

# Categorical data

## The Donner Party- 1846-47, Sierra Nevada




- Categorical variables: spread out uniformly within bars, along axis
- Double axis
- All pairwise relationships, and p(survival | x,y)

# Concluding remarks

- Other applications: PCP-categorical, star glyphs, interaction plots
- Wegman(1990) - LW hamiltonian path algorithm in parallel coordinate displays
- Bailey et al (2003)- Hamiltonian cycles, in DOE

★★★★

- Software EulerViz R-package
- Uses TSP(Hahsler et al), scagnostics (Hofmann et al)

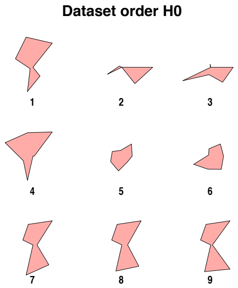
★★★★

- Further work... better algorithms?
- other types of graphs eg bipartite?
- Next talk....

# Cars data

- Task: visually cluster cases

Default ordering of variables.




789 look similar, and to 1?
Other groups: 23, 56
4 on its own

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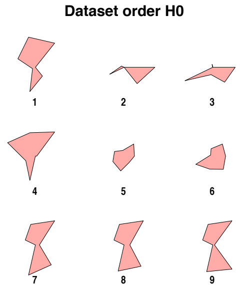


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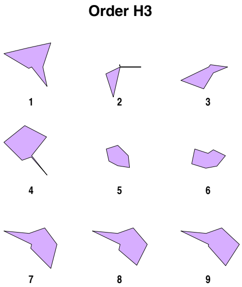
Another hamiltonian

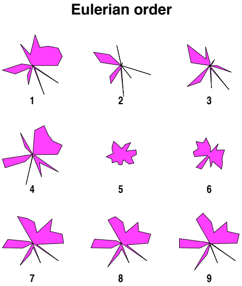



14 look similar
23 look different

Conclusions are order dependent!

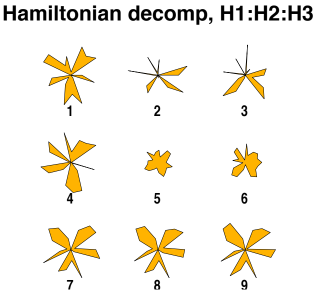
# Cars data

Eulerian order




Groups: 789,23,56,14

Another hamiltonian




Less shape variation between orderings.
Conclusions are less order dependent!