Clique types in Johnson graphs

Pavel Shuldiner
Department of Statistics & Actuarial Science, University of Waterloo

Wayne Oldford
Department of Statistics & Actuarial Science, University of Waterloo

July 14, 2020

2 Introduction

In graph theory, the Johnson graph is a class of undirected graphs defined over a family of sets where two sets of size m, which serve as nodes, are adjacent if they intersect in a $m-1$ elements. This class of graphs were named after American mathematician, Selmer Martin Johnson, who worked on bounds for codes on the now so-called Johnson scheme, which is the association scheme analogue of Johnson graphs.

Later, these graphs were generalized into a larger class of graphs known as the Generalized Johnson graphs $J_n(m,k)$ where nodes are $m-$subsets of an $n-$set and two nodes are adjacent if they intersect in k elements. These graphs were of particular interest to us because they serve as a tool for exploratory data analysis TODO: cite oldford hurley 2011.

In this paper, we make use of algebraic and combinatorial ideas in order to capture the structure of cliques in Johnson graphs. In particular, we characterize the total number of cliques in $J_n(m,m-1)$, define two natural types of cliques and provide a partial solution to the open problem of finding the coclique number of $J_n(m,m-1)$. Additionally, we extend our results to the more general case of generalized Johnson graphs $J_n(m,k)$. Finally, we establish a connection between Erdos Renyi graphs and navigation graphs and use our results to demonstrate how one may conduct hypothesis tests on cliques to investigate a dataset.

3 Clique types and clique number of a Johnson graph

Proposition 3.1. *The Johnson graph $J_n(m,m-1)$ is $m(n-m)-$regular. The number of edges in a Johnson graph $J_n(m,m-1)$ is given by*

$$\binom{n}{m}\frac{m(n-m)}{2}.$$

Proof. See [3]. □

Proposition 3.1 provides us with an enumeration of the number of K_2 copies present in the Johnson graph $J_n(m,m-1)$. We will soon enumerate copies of K_r for $r \geq 3$ in $J_n(m,m-1)$, which we recall are known as cliques.

Definition 3.2. Let G be a graph. We say that a clique H is a maximum clique if there is no clique with more vertices than H. Moreover, we let $\omega(G)$ denote the number of vertices in a maximum clique of G.

Proposition 3.3. *Let G be the Johnson graph $J_n(2,1)$. Then the intersection of a maximal clique in G is either 0 or 1. Moreover, for $n \geq 3$, the size of a maximal clique is either 3 or $n-1$ and hence*

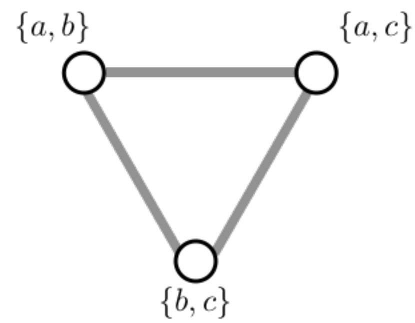
$$\omega(J_n(2,1)) = \max\{n-1,3\}.$$

Proof. Let H be a maximal clique in $G = J_n(2,1)$. Consider S the intersection of all the nodes in H

$$S = \bigcap_{v\in H}\nu(v).$$

Since every node has 2 elements and any two nodes intersect in exactly one element, we know that $|S| \leq 1$. Thus, $|S| \in \{0, 1\}$.

If $|S| = 0$, then one can show that H must be a triangle. Indeed, consider the configuration below.



Is it possible to find a vertex which is adjacent to all vertices in the triangle above? The answer is no because once a vertex is adjacent to any two of the vertices, such as $\{a, b\}, \{a, c\}$, it must be either of the form $\{a, d\}$ or $\{b, c\}$. In the first case, $\{a, d\}$ is not adjacent to the third member of the triangle and hence cannot be part of the clique. In the second case, it is already in the triangle.

Thus, a triangle is the largest possible clique in $J_n(2, 1)$ when we impose the restriction that it has an empty intersection.

Now, suppose that $|S| = 1$ and that $S = \{i\}$. Then we know that a maximum clique will have the form $H = \{i\} \times (\mathcal{N}_n \setminus \{i\})$ for some $i \in \mathcal{N}_n$. This is a clique of size $n - 1$ which cannot be any larger as we exhausted all nodes that contain i.

Thus, we know that the maximal cliques in $J_n(2, 1)$ can only only be of either size 3 or $n - 1$ and hence $\omega(J_n(2, 1)) = \max\{n - 1, 3\}$. □

We saw above that there are only two possible sizes for the intersection of a clique in $J_n(2, 1)$ for $n \geq 3$. We will soon generalize this observation and demonstrate that there are only two possible clique intersection sizes. Now, we categorize of maximal cliques with empty total intersection.

Proposition 3.4. *Let G be the Johnson graphs $J_n(m, m-1)$ with $n \geq m + 1$ and let H be a maximal clique in G. If B denotes the union*

$$B := \bigcup_{v \in H} \nu(v),$$

then $\bigcap_{v \in H} \nu(v) = \varnothing$ if and only if $\nu(a) \cup \nu(b) = B$ for a, b distinct in H.

Proof. Suppose that $\bigcap_{v \in H} \nu(v) = \varnothing$. Fix a and b distinct vertices in H and suppose that $B \neq \nu(a) \cup \nu(b)$. We may assume that a and b have the form $\nu(a) = \{x\} \cup e, \nu(b) = \{y\} \cup e$ for some $(m-1)$-set of variables $e \subseteq \mathcal{N}_n$ and $x \neq y$ in $\mathcal{N}_n$.

Since $B \neq \nu(a) \cup \nu(b)$, there exists some $z_1 \in B \setminus (\nu(a) \bigcup \nu(b))$. Let c_1 be a vertex in H for which $z_1 \in \nu(c_1)$. Since $c_1, a, b \in H$ and H is a clique it must be that c_1 is adjacent to a and b.

Now, since $z_1 \notin \nu(a) \cup \nu(b)$, it must be that c_1 has the form $\nu(c_1) = \{z_1\} \cup f$ where f is a $(m-1)$-subset of $\mathcal{N}_n$. Moreover, since $x \neq z_1$, we see that in order for c_1 to be adjacent to a, f must be e. If $f = e$, then we also then guarantee that c_1 is adjacent to b.

We now inductively extend our construction into a sequence $z_1, \ldots, z_l$ of variables and nodes $c_1, \ldots, c_l$ where

1. $z_i \in \nu(c_j)$ if and only if $i = j$ for all $i = 1, \ldots, l$,
2. $z_i \notin \nu(a) \cup \nu(b) \bigcup (\cup_{s=1}^{i-1} \nu(c_s))$,
3. $B = \nu(a) \cup \nu(b) \bigcup (\cup_{s=1}^{i-1} \nu(c_s))$ and
4. a, b and all c_i are pairwise adjacent.

As before, we see that since c_s has the form $\nu(c_s) = \{z_s\} \cup e_s$ where $z_s \notin e_s$ and H is a clique, it must be that $e_s = e$.

Now, we demonstrate that this collection must include all vertices from H. Fix $u \in H$ and suppose that $u \notin \{a, b\} \bigcup (\cup_{i=1}^{l} \{c_i\})$.

We now consider two cases regarding the structure of u: either $\nu(u)$ is a subset of $\nu(a) \bigcup \nu(b)$ or it is not.

Case 1: $u = \{z_s\} \cup e_u$ for some $1 \leq s \leq l$.

Since H is a clique, we know that u must be adjacent to $c_s = \{z_s\} \cup e$ and u must be adjacent to a. The former implies that $|e_u \cap e| = m - 1$ as $z_s \in \nu(c_s) \cap \nu(u)$. The latter implies that $|e_u \cap e| = m$ as $z_s \notin \nu(a)$. This is impossible.

Case 2: $\nu(u) \subseteq \nu(a) \cup \nu(b)$.

Suppose that $\nu(u) \subseteq \nu(a) \cup \nu(b)$. Then it must be that u is of the form $\nu(u) = \{x, y\} \cup e_u$ for some $(m-2)$ subset e_u of e to ensure the adjacency to a and b. However, now u is not adjacent to c_i for all $1 \leq i \leq l$ as $x, y \notin \{z_i : 1 \leq i \leq l\}$ and $|e_u \cap e| = m - 2$. This contradicts that H is a clique.

Thus, we have just demonstrated that the collection $\{a, b\} \cup \{c_i : 1 \leq i \leq l\}$ must be all of H where for any vertex $u \in H$, $\nu(u)$ contains the set e and a singleton outside of e. This contradicts our assumption that $\cap_{v \in H} \nu(v) = \varnothing$ and hence it must be that $B = \nu(a) \cup \nu(b)$ for any two distinct a, b vertices in H.

Conversely, suppose that $B = \nu(a) \cup \nu(b)$ for any two distinct a, b in H. Then since $|\nu(a) \cup \nu(b)| = m + 1$, we have that $|B| = m + 1$. Furthermore, since any $a \in H$ is an $m-$subset of B, we find that $H \subseteq J$ where $J := \{A \subseteq B : |A| = m\}$.

We now demonstrate that $H = J$. Suppose that there is some $v \in J$ for which $v \notin H$. Since v satisfies $\nu(v) \subseteq B$ and has cardinality m, there is some unique $i_v \in B$ for which $\nu(v) = B \setminus \{i_v\}$. Fix a vertex $u \in H$ and similarly, note that that u must have the form $\nu(u) = B \setminus \{i_u\}$, for some $i_u \in B$. Now, we have that

$$\nu(v) \cap \nu(u) = B \setminus \{i_u\} \bigcap B \setminus \{i_v\} = B \setminus \{i_u, i_v\},$$

which is a set of size $m - 1$ as $i_u \neq i_v$. By maximality of H, it must be that $v \in H$ and hence $H = J$.

Now, we demonstrate that the intersection $\bigcap_{v\in J}\nu(v)$ is empty. Since $\bigcap_{v\in J}\nu(v)\subseteq B$, it suffices to show that for all $x\in B$, $x\notin\bigcap_{v\in J}\nu(v)$. Fix $x\in B$ and consider the vertex v_x where $\nu(v_x)=B\setminus\{x\}$. Since v_x is a node in J and $x\notin B$, we have that $x\notin\bigcap_{v\in J}\nu(v)$ and hence

$$\bigcap_{v\in J}\nu(v)=\varnothing.$$

□

Remark 3.5. Proposition 3.4 indicates that $J_n(m,m-1)$ has a family of maximal cliques that have the form

$$J_B:=\{v:|\nu(v)|=m,\nu(v)\subset B\},$$

where B is some $(m+1)-$subset of $\mathcal{N}_n$. We refer to maximal cliques of this form and their subcliques by D_{max}. Another family which we encountered in Theorem 3.3 is the one which takes on the form

$$J_A:=\{u:|\nu(u)|=m,A\subset\nu(u)\},$$

where A is some fixed $(m-1)-$subset of $\mathcal{N}_n$. We refer to maximal cliques of this form and their subcliques by D_{min}.

The following proposition asserts that only maximal cliques of type D_{max} and type D_{min} exist in $J_n(m,m-1)$ for $m\geq n+1$.

Theorem 3.6. *Let H be a maximal clique in the Johnson graph $J_n(m,m-1)$. Then the intersection of H is either 0 or $m-1$.*

Proof. We proceed inductively on m. Consider the base case $m=2$.

Let H be a maximal clique in $J_n(2,1)$. Suppose $s=\bigcap_{v\in H}\nu(v)$. Then since the intersection of any two nodes is of size 1 and the intersection of the whole clique is bounded below by 0, we find that

$$0\leq|\bigcap_{v\in J}\nu(v)|\leq 1=2-1.$$

Now, suppose for some $m_0\geq 2$, any $m\leq m_0$ and $n\geq m+1$ satisfy that all maximal cliques H in $J_n(m,m-1)$ have an intersection with

$$\left|\bigcap_{v\in H}\nu(v)\right|\in\{0,m-1\}.$$

Fix $n'\geq m_0+1$ and let I be a maximal clique in $J_{n'}(m_0+1,m_0)$. Suppose that I is such that

$$m_0>\left|\bigcap_{v\in I}\nu(v)\right|=s>0.$$

Let $A:=\left|\bigcap_{v\in I}\nu(v)\right|$ denote the intersection of the clique and without loss of generality, suppose that $A:=\mathcal{N}_s$ and let I_{-A} denote the collection

$$I_{-A}:=\{\nu(v)\setminus A:v\in I\}.$$

For every vertex $v \in I$, the removal of A results in a node of size $m_0 + 1 - s$. We now demonstrate that I_{-A} forms a maximal clique in $J_{n'-s}(m_0+1-s, m_0-s)$. First, suppose that a is adjacent to b in $J_{n'}(m_0+1, m_0)$. So, we have that $|\nu(a) \cap \nu(b)| = m_0$. Moreover, since $A \subseteq \nu(a) \bigcap \nu(b)$, we see that

$$\left|(\nu(a) \setminus A) \bigcap (\nu(b) \setminus A)\right| = m_0 - s.$$

Therefore, the collection I_{-A} forms a clique in $J_{n'-s}(m_0+1-s, m_0-s)$.

Next, we show that I_{-A} is a maximal clique. Suppose there is some $v \in J_{n'-s}(m_0 + 1-s, m_0-s)$ for which v is adjacent to all of I_{-A} but is not in I_{-A}. Consider the vertex v' with variable set $\nu(v') = \nu(v) \cup A$. By construction this is a unique node in $J_{n'}(m_0+1, m_0)$ and it is adjacent to all $u \in I$. As I was assumed to be maximal, it must be that $v' \in I$ and we have a contradiction as then $v \in I_{-A}$.

Since I_{-A} is a maximal clique in $J_{n'-s}(m_0+1-s, m_0-s)$, by the inductive hypothesis, $i := |\bigcap_{v \in I_{-A}} \nu(v)| \in \{0, m_0 - s\}$. We note that i must be 0 as otherwise removing the intersection of a collection of sets would yield a collection of sets with nontrivial intersection, which is impossible.

Now, since I_{-A} is a maximal clique with intersection 0, Proposition 3.4 implies that

$$\left|\bigcup_{v \in I_{-A}} \nu(v)\right| = (m_0 + 1 - s) + 1.$$

Since we have only removed the variable set A from I in our construction of I_{-A}, this implies that

$$\left|\bigcup_{v \in I} \nu(v)\right| = \left|\bigcup_{v \in I_{-A}} (\nu(v) \dot\cup A)\right| = s + (m_0 + 1 - s) + 1 = (m_0 + 1) + 1.$$

Now, we demonstrate that for any two distinct vertices $a, b \in I$, $\nu(a) \cup \nu(b) = \bigcup_{v \in I} \nu(v)$. First, we note that $\nu(a) \cup \nu(b) \subseteq \bigcup_{v \in I} \nu(v)$ as $a, b \in I$. On the other hand, since $|\nu(a) \cup \nu(b)| = m_0 + 2$ and $|\bigcup_{v \in I} \nu(v)| = m_0 + 2$, we have that

$$\nu(a) \cup \nu(b) = \bigcup_{v \in I} \nu(v).$$

By Proposition 3.4, I must be a clique for which

$$\bigcap_{v \in I} \nu(v) = \varnothing,$$

contradicting our assumption that $|A| = s > 0$. The claim then holds by the principle of strong induction on m_0. □

As a consequence of the above result, we can obtain the clique number of $J_n(m, m-1)$ for all $n \geq m + 1$:

Theorem 3.7. *The clique number $\omega(J_n(m, m-1))$ of the Johnson graph $J_n(m, m-1)$ is given by*

$$\max(m+1, n-m+1),$$

whenever $n \geq m+1$.

Proof. Since there are only two types of maximal cliques, in order to determine the clique number it suffices to compare their sizes. For maximally intersecting maximal cliques, the size of such a clique is $n-m+1$. On the other hand, an empty intersecting clique has size $m+1$. Therefore, the clique number of $J_n(m, m-1)$ is given by $\omega(J_n(m, m-1)) = \max(m+1, n-m+1)$. □

It is easy to show that for $n \geq 2m, \omega(J_n(m, m-1)) = n-m+1$. Indeed, we have that

$$n-m+1 \geq m+1 \iff n+1 \geq 2m+1 \iff n \geq 2m.$$

Thus, we can conclude that $\omega(J_n(m, m-1)) = m+1$ for $m+1 \leq n \leq 2m$ and $\omega(J_n(m, m-1)) = n-m+1$ for $n \geq 2m$.

There are several consequences to the two propositions above. First, we may use our results to count cliques within Johnson graphs.

Theorem 3.8. *For $r \geq 3$, the number of $r-$cliques in $J_n(m, m-1)$ is given by*

$$\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}.$$

Proof. Let H be an $r-$clique for $r \geq 3$. Since every clique can be extended into a maximal clique, we know by Theorem 3.6 that H could be extended into either a clique with maximum intersection $m-1$ or minimum intersection 0. We shall demonstrate that H is a subclique of exactly one of these maximal clique families.

We begin by demonstrating that an $r-$clique H can be solely identified to be a member of one of the families by considering the total intersection or union of the clique H. To be thorough, we show how both intersection and union can be used to identify the clique type.

In proposition 3.4, we saw that for a maximal clique J with empty total intersection, the union $\nu(a) \cup \nu(b)$ of any two distinct $a, b \in J$ is equal to the union of the whole clique and has cardinality $m+1$. If H is a subgraph of such a clique J, then clearly the union of any two distinct vertices in H would satisfy the same two properties. That is, we find that in such case

$$\left| \bigcup_{v \in H} \nu(v) \right| = m+1.$$

Moreover, we note that since H has the form $H = \{v : \nu(v) := B \setminus \{x\}, \text{for some } x \notin \text{B}\}$, where B is some fixed $(m+1)-$subset of $\mathcal{N}_n$, we have that

$$\begin{aligned}\left|\bigcap_{v\in H}\nu(v)\right| &= \left|\bigcap_{i=1}^{r}(B\setminus\{x_i\})\right| \\ &= \left|B\setminus(\bigcup_{i=1}^{r}x_i)\right| \\ &= m+1-r.\end{aligned}$$

On the other hand, if H were a subclique from the family of maximally intersecting cliques, say for instance J_A as in Remark 3.5, then we know that H has the form $H=\{v:\nu(v):=A\dot{\cup}\{x_i\}\}$. Therefore, we find that

$$\begin{aligned}\left|\bigcup_{v\in H}\nu(v)\right| &= \left|\bigcup_{i=1}^{r}(A\dot{\cup}\{x_i\})\right| \\ &= \left|A\cup(\bigcup_{i=1}^{r}x_i)\right| \\ &= m-1+r.\end{aligned}$$

This is already different from the union we would find if H were a subclique from a clique of form J_B since $m-1+r\neq m+1$ for $r\geq 3$. Moreover, the intersection of the H in such case will be given by

$$\begin{aligned}\left|\bigcap_{v\in H}\nu(v)\right| &= \left|\bigcap_{i=1}^{r}(A\dot{\cup}\{x_i\})\right| \\ &= |A| \\ &= m-1,\end{aligned}$$

which does not equal to $m+1-r$ for $r\geq 3$.

Therefore, there are only two possible types of $r-$cliques in $J_n(m,m-1)$: those that are subsets of some empty intersecting maximal clique or some maximum intersecting maximal clique.

In the former case, there are

$$\binom{n}{m+1}\binom{m+1}{r}$$

such cliques. In the latter case, there are

$$\binom{n}{m-1}\binom{n-m+1}{r}$$

such cliques. □

Proposition 3.9. *For all integers $n>m>1$, the edges of $J_n(m,m-1)$ partition according to their membership in the set of all maximal D_{max} cliques.*

Proof. Fix an edge e in $J_n(m, m-1)$. Then for some vertices A and B, $\nu(e) = \nu(A) \cap \nu(B)$ and e can be identified with a set of size $m-1$. This set corresponds to a unique, maximal D_{max} clique in $J_n(m, m-1)$. In particular, the clique $\{v : \nu(v) = \nu(e) \cup x, x \in \mathcal{N}_n \setminus \nu(e)\}$ is a maximal D_{max} clique containing e. It is clear that it is unique and all maximal D_{max} cliques have the same size, since they are relabellings of one another.

As an additional verification, we note that according to Proposition 3.1, $J_n(m, m-1)$ has

$$\frac{m(n-m)}{2}\binom{n}{m}$$

edges. On the other hand, if what we claim is true, then the number of edges resulting from the collection of all maximal D_{max} cliques should equate to

$$\frac{m(n-m)}{2}\binom{n}{m}.$$

By the preceding argument, each $(m-1)-$subset of $\mathcal{N}_n$ corresponds to a unique, maximal D_{max} clique of size $n-m+1$. This is because we have $n-(m-1)$ options for the final variable to include in each node belonging to the clique. Therefore, each of these cliques corresponds to $\binom{n-m+1}{2}$ edges. Lastly, we have $\binom{n}{m-1}$ distinct, maximal D_{max} cliques and hence in total, this collection results in

$$\binom{n-m+1}{2}\binom{n}{m-1} = \frac{(n-m+1)(n-m)}{2}\left[\frac{m}{n-m+1}\binom{n}{m}\right] = \frac{m(n-m)}{2}\binom{n}{m},$$

as claimed. □

TODO: Is the above prop in the best place?

The following corollary provides a closed-form expression for the number of triangles in a Johnson graph.

Corollary 3.10. *The number of triangles in $J_n(m, m-1)$ is given by*

$$\binom{n}{k}\binom{n-k}{3} + \binom{n}{k-1}\binom{n-k+1}{3} = \binom{m+1}{2}\frac{(n-2)}{3}\binom{n}{m+1}.$$

Proof. By Theorem 3.8, the number of triangles is given by

$$\binom{n}{k}\binom{n-k}{3} + \binom{n}{k-1}\binom{n-k+1}{3}.$$

This can be simplified as follows

$$\begin{aligned}
\binom{n}{k}\binom{n-k}{3} + \binom{n}{k-1}\binom{n-k+1}{3} &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{3!(n-k-3)!} \\
&+ \frac{n!}{(k-1)!(n-k+1)!}\frac{(n-k+1)!}{3!(n-k-2)!} \\
&= \frac{n!(n-k-2)+n!k}{3!k!(n-k-3)!(n-k-2)} \\
&= \frac{n!(n-k-2+k)}{3!k!(n-k-2)!} \\
&= \frac{n!(n-2)}{3!k!(n-k-2)!} \\
&= \frac{n!(n-2)}{3!k!(n-k-2)!} \\
&= \frac{(k+2)!}{2!k!}\frac{(n-2)}{3}\frac{n!}{(k+2)!(n-k-2)!} \\
&= \binom{k+2}{2}\frac{n-2}{3}\binom{n}{k+2} \\
&= \binom{m+1}{2}\frac{n-2}{3}\binom{n}{m+1},
\end{aligned}$$

as needed. □

We can use Theorem 3.8 to study the distribution of cliques in $J_n(m, m-1)$.

Corollary 3.11. *The total number of cliques of size* 3 *or larger in* $J_n(m, m-1)$ *is*

$$\binom{n}{m+1}\left(2^{m+1} - (m+2) - \binom{m+1}{2}\right) + \binom{n}{m-1}\left(2^{n-m+1} - (n-m+2) - \binom{n-m+1}{2}\right).$$

Proof. By Theorem 3.8, we know there are

$$\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}$$

cliques of size r. The result then follows by summing over all $r \geq 3$ and by applying the binomial theorem. □

4 Clique distribution of the Johnson graph

In this section, we concern ourselves with the distribution of cliques in $J_n(m, m-1)$. Our motivation for this stemmed from the idea of exploring data sets using navigation graphs, which can be viewed as subgraphs of Johnson graphs. Thus, we were interested in understanding what graph theory might say about the prevalence of one type of clique over the other or if there were perhaps other insights one might glean from the structure of the cliques that are present in a navigation graph.

We begin with a straightforward result that shows that for $m \in \mathbb{N}$ and $r \geq 3$ fixed, almost all $r-$cliques in $J_n(m, m-1)$ are of type D_{max}, as $n \to \infty$.

Proposition 4.1. *Fix $m \in \mathbb{N}$, and $r \geq 3$. Then the number of $r-$cliques in $J_n(m, m-1)$ is dominated by cliques of type D_{max}. That is,*

$$\frac{\binom{n}{m-1}\binom{n-m+1}{r}}{\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}} \to 1,$$

as $n \to \infty$.

Proof. First, we note that

$$1 \geq \frac{\binom{n}{m-1}\binom{n-m+1}{r}}{\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}}$$

and

$$\begin{aligned}\frac{\binom{n}{m-1}\binom{n-m+1}{r}}{\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}} &= 1 - \frac{\binom{n}{m+1}\binom{m+1}{r}}{\binom{n}{m+1}\binom{m+1}{r} + \binom{n}{m-1}\binom{n-m+1}{r}} \\ &\geq 1 - \frac{\binom{n}{m+1}\binom{m+1}{r}}{\binom{n}{m-1}\binom{n-m+1}{r}}.\end{aligned}$$

Therefore, it suffices to show that

$$\frac{\binom{n}{m+1}\binom{m+1}{r}}{\binom{n}{m-1}\binom{n-m+1}{r}} = o(1).$$

Recall that for $k \leq n$ fixed, we have

$$\binom{n}{k} = \Theta(n^k)$$

and so

$$\frac{\binom{n}{m+1}\binom{m+1}{r}}{\binom{n}{m-1}\binom{n-m+1}{r}} = \frac{\Theta(n^{m+1})\Theta(1)}{\Theta(n^{m-1})\Theta(n^r)} = \Theta(n^{2-r}) = O(n^{-1}) = o(1),$$

as claimed above. □

One might also wonder when we have balanced groups within the distribution. That is, for which values of n and m are the counts of D_{min} and D_{max} cliques equal? We begin by solving a related but easier problem which will motivate our technique for finding the solution to this problem.

Proposition 4.2. *The only solution to the system*

$$\binom{n}{m-1}\binom{n-m+1}{r} = \binom{n}{m+1}\binom{m+1}{r}$$

for $r \geq 0$ is given by $n = 2m$.

Proof. Let $F_{max}(q)$ denote the clique-type generating series for D_{max} and $F_{min}(q)$ denote the clique-type generating series for D_{min}. Then these generating series can be written compactly as follows

$$\begin{aligned} F_{max}(q) &= \binom{n}{m-1} \sum_{r=0}^{\infty} \binom{n-m+1}{r} q^r \\ &= \binom{n}{m-1} (1+q)^{n-m+1} \\ F_{min}(q) &= \binom{n}{m-1} \sum_{r=0}^{\infty} \binom{m+1}{r} q^r \\ &= \binom{n}{m-1} (1+q)^{m+1}. \end{aligned}$$

Thus, we have that the ratio is given by

$$\frac{F_{max}(q)}{F_{min}(q)} = \frac{\binom{n}{m-1}}{\binom{n}{m+1}} (1+q)^{n-2m}.$$

In order for this ratio to equal to 1, it must be that $[q^s]\frac{F_{max}(q)}{F_{min}(q)} = 0$ for all $s > 0$. If $n > 2m$, then we know that

$$(1+q)^{n-2m} = \sum_{r=0}^{n-2m} \binom{n-2m}{r} q^r,$$

and at least one positive power of q has a non zero coefficient since

$$[q^1](1+q)^{n-2m} = \frac{\binom{n}{m-1}}{\binom{n}{m+1}} \binom{n-2m}{1} > 0,$$

whenever $n > 2m$. On the other hand, if $n < 2m$, we set $\ell = 2m - n$ and then by the generalized binomial theorem, we have that

$$(1+q)^{n-2m} = \frac{1}{(1+q)^{\ell}} = \sum_{r \geq 0} \binom{r+\ell-1}{r} (-1)^r q^r.$$

Since ℓ is an integer and $\ell > 0$, we see that

$$\binom{r+\ell-1}{r} \neq 0,$$

for all $r \geq 0$. Therefore,

$$\frac{F_{max}(q)}{F_{min}(q)} = \sum_{r \geq 0} \binom{r+\ell-1}{r} (-1)^r q^r,$$

has a non-zero coefficient for all q^s with $s \geq 0$. If $\ell > 1$, then

$$[q^s]\frac{F_{max}(q)}{F_{min}(q)} = \frac{\binom{n}{m-1}}{\binom{n}{m+1}} \binom{s+\ell-1}{s} (-1)^s \neq 0,$$

and thus the ratio cannot be equal to 1. So, we can conclude that the only candidate for which we may see equality in the generating series is when $n = 2m$. We now show that $n = 2m$ ensures for equality of the two generating series.

If $n = 2m$, then since $2m - (m+1) = m - 1$, we have that $\binom{n}{m-1} = \binom{n}{m+1}$ and

$$\frac{F_{max}(q)}{F_{min}(q)} = \frac{\binom{n}{m-1}}{\binom{n}{m+1}}(1+q)^{2m-2m} = (1+q)^0 = 1,$$

as needed. □

Corollary 4.3. *Let $G = J_n(m, m-1)$. The distribution of cliques in G of type D_{max} is equal to the distribution of type D_{min} if and only if $n = 2m$.*

Proof. We remark that in order to solve the system

$$\binom{n}{m-1}\binom{n-m+1}{r} = \binom{n}{m+1}\binom{m+1}{r}$$

for all $r \geq 3$, it is sufficient to solve the system

$$\binom{n}{m-1}\binom{n-m+1}{r}r(r-1)(r-2) = \binom{n}{m+1}\binom{m+1}{r}r(r-1)(r-2), \qquad (1)$$

for all $r \geq 3$. The advantage of the latter system is that it has a generating function which has a nice factorization as a product of generating series, as we show below.

Let $F_{max}(q)$ and $F_{min}(q)$ be as in the proof of 4.2. Let $f_{max}(q)$ and $f_{min}(q)$ be the generating series defined by

$$\begin{aligned}
f_{max}(q) &= q^3 \frac{\partial^3}{\partial q^3} F_{max}(q) \\
&= q^3(n-m+1)(n-m)(n-m-1)\binom{n}{m-1}(1+q)^{n-m-2} \\
&= \binom{n}{m-1}\sum_{r=3}^{\infty}\binom{n-m+1}{r}r(r-1)(r-2)q^r, \\
f_{min}(q) &= q^3 \frac{\partial^3}{\partial q^3} F_{min}(q) \\
&= q^3(m+1)m(m-1)\binom{n}{m+1}(1+q)^{m-2} \\
&= \binom{n}{m+1}\sum_{r=3}^{\infty}\binom{m+1}{r}r(r-1)(r-2)q^r,
\end{aligned}$$

We note that n and m are solutions to system (1) if and only if $f_{max}(q) = f_{min}(q)$. So, we divide $f_{max}(q)$ by $f_{min}(q)$ and examine when this generating series equals 1.

$$\frac{f_{max}(q)}{f_{min}(q)} = \frac{q^3(n-m+1)(n-m)(n-m-1)\binom{n}{m-1}(1+q)^{n-m-2}}{q^3(m+1)m(m-1)\binom{n}{m+1}(1+q)^{m-2}}$$
$$= \frac{(n-m+1)(n-m)(n-m-1)\binom{n}{m-1}(1+q)^{n-2m}}{(m+1)m(m-1)\binom{n}{m+1}}$$

We may reuse the argument from Proposition 4.2 and note that since $[q^s]\frac{f_{max}(q)}{f_{min}(q)} = 0$, it must be that $(1+q)^{n-2m} = 1$ and $n = 2m$.

It is straightforward to verify that the other terms yield the proper cancellation when $n = 2m$. □

Additionally, when the distribution of D_{min} and D_{max} cliques are identical, we are immediately able to both identify the count and the size of the most common class of clique.

Proposition 4.4. *Let $G = J_{2m}(m, m-1)$. Then the largest count of $r-$cliques occurs when $r = \frac{m+1}{2}$, for m odd and for m even, the mode*

$$r = \begin{cases} \frac{m+1}{2} & \text{When } m \text{ is odd.} \\ \lceil \frac{m+1}{2} \rceil, \lfloor \frac{m+1}{2} \rfloor, & \text{Else} \end{cases}$$

Proof. Since $n = 2m$, the count of cliques of size r is given by

$$2\binom{2m}{m+1}\binom{m+1}{r}.$$

This is maximized when $\binom{m+1}{r}$ is maximized which occurs at $r = \frac{m+1}{2}$ for m odd and $\frac{m}{2}, \frac{m}{2}+1$, for m even. □

We shall consider two obvious mechanisms for picking cliques at random. First, we consider the distribution of picking a clique at random. Second, we consider the distribution of cliques given a fixed clique type (either D_{max} or D_{min}).

Under the first mechanism for sampling cliques, we sample uniformly at random from all possible cliques. Let R be the random variable which denotes the size of the clique selected. The total number of cliques in this setting is given by

$$S := 1 + \binom{n}{m} + \binom{n}{m}\frac{m(n-m)}{2} + \sum_{r=3}^{\max(n-m+1,m+1)} \binom{n}{m-1}\binom{n-m+1}{r} + \binom{n}{m+1}\binom{m+1}{r}.$$

Thus, R has a probability mass function given by

$$P(R = r) = \begin{cases} \binom{n}{m-1}\binom{n-m+1}{r} + \binom{n}{m+1}\binom{m+1}{r}, & 3 \le r \le \max(n-m+1, m+1) \\ \binom{n}{m}\frac{m(n-m)}{2}, & r = 2 \\ \binom{n}{m}, & r = 1 \\ 1, & r = 0. \end{cases}$$

Under the second mechanism, we begin by identifying the clique type of interest. For instance, if we are interested in sampling from cliques of type D_{min}, then we could proceed as follows.

1. Pick a maximal clique H of type D_{min}. This can be done in $\binom{n}{m+1}$ ways as it is sufficient to know the union of a clique of type D_{min} to identify its maximal clique.

2. Pick a subclique by sampling uniformly at random from the subcliques of H.

Let R_{min} be the random variable recording the size of the subclique selected. Under this scheme, we would obtain a clique of size r with probability

$$P(R_{min} = r) = \frac{\binom{n}{m+1}\binom{m+1}{r}}{\binom{n}{m+1}\sum_{r=0}^{m+1}\binom{m+1}{r}} = \binom{m+1}{r}\frac{1}{2^{m+1}} = \binom{m+1}{r}\frac{1}{2^r}\frac{1}{2^{m+1-r}}.$$

Thus, R_{min} is clearly Binomial$\left(m+1, \dfrac{1}{2}\right)$. Conceptually, we may explain this by noting that sampling cliques uniformly from a maximal clique of size $m+1$ is akin to allocating to node i a random variable $Y_i \sim$ Bernoulli $\left(\frac{1}{2}\right)$ and forming a clique using only nodes for which $Y_i = 1$.

In the case that we are interested in sampling from cliques of type D_{max}, our scheme is given below.

1. Pick a maximal clique H of type D_{max}. This can be done in $\binom{n}{m-1}$ ways as it is sufficient to know the intersection of a clique of type D_{max} to identify its maximal clique.

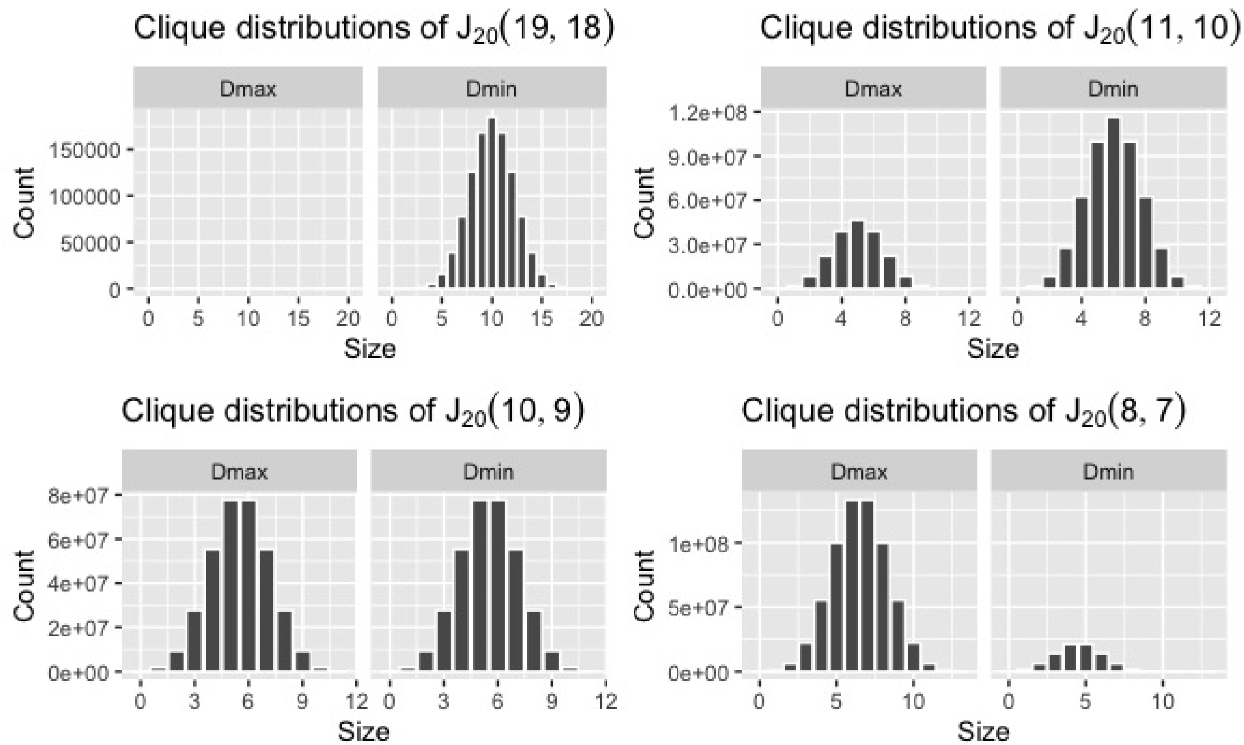
2. Pick a subclique by sampling uniformly at random from the subcliques of H.

Here, the random variable R_{max} which equals the size of the subclique selected is given by

$$P(R_{max} = r) = \frac{\binom{n}{m-1}\binom{n-m+1}{r}}{\binom{n}{m-1}\sum_{r=0}^{n-m+1}\binom{n-m+1}{r}} = \binom{n-m+1}{r}\frac{1}{2^{n-m+1}} = \binom{n-m+1}{r}\frac{1}{2^r}\frac{1}{2^{n-m+1-r}},$$

and hence R_{max} is a Binomial$\left(n-m+1, \frac{1}{2}\right)$.

This provides us with an additional proof for Corollary 4.2. The two clique type distributions are equal if and only if $R_{max} \overset{D}{=} R_{min}$. Hence, since two Binomial distributions are equal if and only if they agree in their parameters, we see that this is only true when $n = 2m$.



TODO: What do we learn from this? How is this used? Can we mention that we thought this was the way to go but realized that it only results in pretty math which we include here? TODO: Should size 0 cliques appear in the distribution? Feels like nonsense.

5 Cliques in generalized Johnson and generalized Kneser graphs

TODO: Need reintroduce the definitions
In this section, we focus on the generalized Johnson graph $J_n(m,k)$ and illustrate how one can enumerate all $(r+1)-$cliques that contain a particular $r-$clique as a subset. In the simple case of $r=3$, our theorem provides a closed form expression for the number of triangles in $J_n(m,k)$.

5.1 Triangle Counts in $J_n(m,k)$

In this section, we count the number of cliques of size r in $J_n(m,k)$. In order to motivate the intuition for the proof of the general theorem, we begin by considering $r=2,3$.

Proposition 5.1. *The number of edges in $J_n(m,k)$ is given by*

$$\frac{1}{2}\binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}.$$

Proof. Begin by fixing the variables to appear in both edges. Since this must be precisely k variables, this can be done in $\binom{n}{k}$ ways.

Next, we pick the variables that appear in one node but not in the intersection. This can be done in

$$\binom{n-k}{m-k}$$

ways as we only need $m-k$ additional variables and we have $n-k$ variables that we did not use yet. Similarly, the second node can be constructed in

$$\binom{n-k-(m-k)}{m-k} = \binom{n-m}{m-k}$$

different ways as we need only pick $m-k$ variables which we have not used yet.

Finally, since the order in which construct the two nodes does not matter, we divide by 2. $\square$

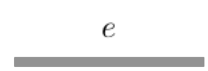
We are now in position provide a complete enumeration of triangles in $J_n(m,k)$.

Theorem 5.2. *The number of triangles in $J_n(m,k)$ is given by*

$$\frac{1}{3!}\binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}\sum_{s=0}^{k}\binom{k}{s}\binom{m-k}{k-s}\binom{m-k}{k-s}\binom{n-(2m-k)}{m-2k+s}.$$

Proof. We can build a triangle in $J_n(m,k)$ by constructing it from a single edge into two nodes and finally introducing the third node. Let $\{a,b,c\}$ denote the nodes in our triangle.

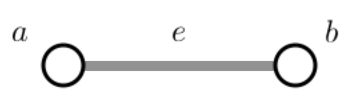
Choose the first edge: Fix the variables in an edge e which will connect a to b. This can be done in $\binom{n}{k}$ ways.



Choose the vertices incident to the first edge: Fix the other variables which appear in the two vertices. This can be done in

$$\binom{n-k}{m-k}\binom{n-k-(m-k)}{m-k}\frac{1}{2} = \binom{n-k}{m-k}\binom{n-m}{m-k}\frac{1}{2}$$

ways.



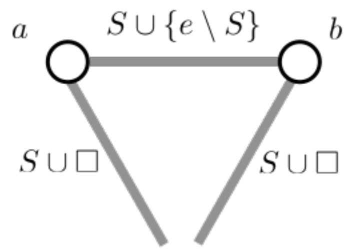
Let S denote the subset of the variables in our edge e to be the intersection of all three nodes in the triangle.

Pick the size for the intersection of the triangle: We know that s the size of S must satisfy $0 \leq s \leq k$ as it is a subset of the edge e. This means that we have $k+1$ nonoverlapping cases to consider for the size of the intersection of the triangle.

Finish constructing edges incident to c: Once we know s, we know that we must pick from the elements in a and b which are not in S to construct edges incident to a and b which are not e. This can be achieved in

$$\binom{m-k}{k-s}\binom{m-k}{k-s}$$

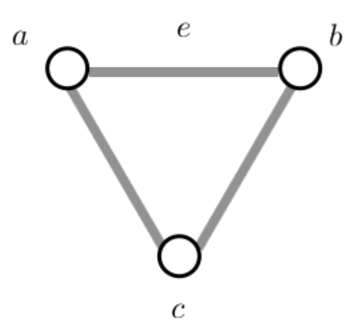
ways as the size of $\nu(a) \setminus (\nu(a) \cap \nu(b))$ is $m-k$ and we only need $k-s$ additional elements from each vertex.



Pick the remaining variables for c: By the time we complete the construction of the edges incident to c, we will have chosen exactly $s + (k-s) + (k-s) = 2k - s$ variables. Thus, we need to pick $m - (2k - s)$ variables from the variables that we did not use yet - variables that are not in $\nu(a) \cup \nu(b)$. This can be done in

$$\binom{n-(2m-k)}{m-2k+s}$$

ways since the size of $|\nu(a) \cup \nu(b)|$ is $2m - k$.



Lastly, we must divide by 3 as we have three edges in our triangle and any one of them could have been chosen to be the first edge constructed which yields the same graph.

To summarize the steps above, we see that the number of triangles in $J_n(m,k)$ is

$$\frac{1}{2}\binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}\sum_{s=0}^{k}\frac{1}{3}\binom{k}{s}\binom{m-k}{k-s}\binom{m-k}{k-s}\binom{n-(2m-k)}{m-2k+s}.$$

□

As a consequence of Theorem 5.2, we may bound the size of the intersection of all triangles in $J_n(m,k)$.

Corollary 5.3. *Let $H = \{a,b,c\}$ be any triangle in $J_n(m,k)$ and let s be the size of the intersection*

$$s = \left|\nu(a)\bigcap\nu(b)\bigcap\nu(c)\right|.$$

Then s must satisfy

$$2k - m \leq s \leq \min(k, n-3(m-k)).$$

Proof. As evident in Theorem 5.2, the set of possible values that s can take is $\{0,1,\ldots,k\}$. Moreover, we know that there exists a triangle which satisfies an intersection of size s if and only if the binomial coefficients in the summand are nonzero.

Thus, we need that s satisfies

$$\binom{k}{s}\binom{m-k}{k-s}\binom{m-k}{k-s}\binom{n-(2m-k)}{m-2k+s} \geq 0.$$

This implies that we need s to satisfy

$$\begin{aligned} k &\geq s \\ m-k &\geq k-s \\ n-(2m-k) &\geq m-2k+s. \end{aligned}$$

Combining these together gives us the inequality

$$2k - m \leq s \leq \min(k, n-3(m-k)).$$

□

We can use Theorem 5.2 together with Corollary 5.3 to derive an alternative proof to Theorem 3.10.
TODO: Add remark about how this agrees with our classification of cliques in Johnson because intersection $m-2$ is for D_{min}

Corollary 5.4. *For the Johnson graph $G = J_n(m, k = m-1)$, the intersection of any triangle can only be of size $k-1$ or $k-2$. Moreover, the number of triangles in G is given by*

$$\binom{m+1}{2}\frac{(n-2)}{3}\binom{n}{m+1}.$$

Proof. We know that by Corollary 5.3, s must satisfy $2k-m \le s \le \min(k, n-3(m-k))$. Since $k=m-1$, this simplifies to

$$2(m-1)-m = m-2 \le s \le \min(m-1, n-3) \le m-1,$$

as claimed.

By specializing Theorem 5.2 to the case where $k=m-1$, we have that the number of triangles t is given by

$$\begin{aligned}
t &= \binom{n}{m-1}\binom{n-m+1}{1}\binom{n-m}{1}\frac{1}{3!}\sum_{s=0}^{k-1}\binom{m-1}{s}\binom{1}{m-1-s}^2\binom{n-(2m-m+1)}{m-2(m-1)+s} \\
&= \frac{1}{3!}\binom{n}{m-1}(n-m+1)(n-m)\sum_{s=0}^{m-1}\binom{m-1}{s}\mathbb{1}_{\{m-1-s\le 1\}}\binom{n-m-1}{s-m+2} \\
&= \frac{1}{3!}\binom{n}{m-1}(n-m+1)(n-m)\left[\binom{n-m-1}{1}+\binom{m-1}{m-2}\right] \\
&= \frac{1}{3!}\binom{n}{m-1}(n-m+1)(n-m)\left[n-m-1+m-1\right] \\
&= \frac{1}{3!}\binom{n}{m-1}(n-m+1)(n-m)(n-2).
\end{aligned}$$

Furthermore, we can simplify t as

$$t = \frac{1}{3!}\frac{n!}{(m-1)!(n-m+1)!}(n-m+1)(n-m)(n-2)$$

On the other hand we may simplify the expression

$$\binom{m+1}{2}\frac{(n-2)}{3}\binom{n}{m+1} = \frac{(m+1)m}{2}\frac{(n-2)}{3}\frac{n!}{(m+1)!(n-m-1)!}$$

After elimination of terms that appear in both expressions, we find that the theorem holds if and only if

$$\frac{(m+1)m}{(m+1)!(n-m-1)!} = \frac{(n-m+1)(n-m)}{(m-1)!(n-m+1)!},$$

which is true as

$$\begin{aligned}
\frac{(m+1)m}{(m+1)!(n-m-1)!} &= \frac{1}{(m-1)!(n-m-1)!} \\
&= \frac{(n-m+1)(n-m)}{(m-1)!(n-m+1)!},
\end{aligned}$$

as needed to be shown.

□

5.2 r-cliques counts in $J_n(m,k)$

In order to motivate the upcoming theorem, we start by generalizing our construction in Theorem 5.2. Here, we demonstrate how to construct a K_4 in an arbitrary $J_n(m,k)$ by first building a triangle.

In our construction of r-cliques in $J_n(m,k)$, we make use of compositions of integers which we define as follows.

Definition 5.5. Let $r \in \mathbb{N}$ be the size of a set of variables. We say that $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}, \ldots, \boldsymbol{\gamma}_\ell)$ is a **composition** of m if $\boldsymbol{\gamma}_i \geq 0$ for all $i \in [\ell]$ and $\sum_{i=1}^{\ell} \boldsymbol{\gamma}_i = m$.

Here, we are interested in integer compositions which are indexed by subsets of the set $\mathcal{N}_r$. That is, we will view $\boldsymbol{\gamma}$ as a function $\boldsymbol{\gamma} : \mathcal{P}(\mathcal{N}_r) \to \mathbb{N}_0$ where we write $\boldsymbol{\gamma}_A := \boldsymbol{\gamma}(A)$ for $A \subseteq \mathcal{N}_r$. In other words, we view a composition as a function from a set of size 2^r into the nonnegative integers whose sum is equal to some predetermined positive integer m.

Now, we are in position to state the result for the number of 4-cliques within $J_n(m,k)$:

Theorem 5.6. *The number of 4-cliques in $J_n(m,k)$ that contain a particular triangle $\{a,b,c\}$ with intersection size s is given by*

$$\frac{1}{4}\sum_{\boldsymbol{\gamma}\in\mathcal{A}} \binom{s}{\boldsymbol{\gamma}_{abc}}\binom{k-s}{\boldsymbol{\gamma}_{ab}}\binom{k-s}{\boldsymbol{\gamma}_{ac}}\binom{k-s}{\boldsymbol{\gamma}_{bc}}\binom{m-2k+s}{\boldsymbol{\gamma}_{a}}\binom{m-2k+s}{\boldsymbol{\gamma}_{b}}\binom{m-2k+s}{\boldsymbol{\gamma}_{c}}\binom{n-(3m-3k+s)}{\boldsymbol{\gamma}_{\varnothing}},$$

where $\mathcal{A}$ is the set of all compositions $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_{abc}, \boldsymbol{\gamma}_{ab}, \boldsymbol{\gamma}_{ac}, \boldsymbol{\gamma}_{bc}, \boldsymbol{\gamma}_{a}, \boldsymbol{\gamma}_{b}, \boldsymbol{\gamma}_{c}, \boldsymbol{\gamma}_{\varnothing}) \vDash m$ satisfying the constraints

$$\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_{ac} + \boldsymbol{\gamma}_{a} = k \tag{2}$$
$$\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_{bc} + \boldsymbol{\gamma}_{b} = k \tag{3}$$
$$\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ac} + \boldsymbol{\gamma}_{bc} + \boldsymbol{\gamma}_{c} = k \tag{4}$$

Moreover, the number of 4-cliques in $J_n(m,k)$ is

$$\frac{1}{4!}\binom{n}{k}\binom{n-k}{m-k}\binom{n-m}{m-k}\sum_{s=0}^{k}\binom{k}{s}\binom{m-k}{k-s}\binom{m-k}{k-s}\binom{n-(2m-k)}{m-2k+s}\sum_{\boldsymbol{\gamma}\in\mathcal{A}}\binom{s}{\boldsymbol{\gamma}_{abc}}\binom{k-s}{\boldsymbol{\gamma}_{ab}}$$
$$\binom{k-s}{\boldsymbol{\gamma}_{ac}}\binom{k-s}{\boldsymbol{\gamma}_{bc}}\binom{m-2k+s}{\boldsymbol{\gamma}_{a}}\binom{m-2k+s}{\boldsymbol{\gamma}_{b}}\binom{m-2k+s}{\boldsymbol{\gamma}_{c}}\binom{n-(3m-3k+s)}{\boldsymbol{\gamma}_{\varnothing}}.$$

Proof. In order to construct a 4-clique which contains the triangle $\{a,b,c\}$, we need to consider how we could uniquely construct a vertex d such that the pairwise intersection of $\{a,b,c\}$ is k. To that end, we propose the following approach.

Pick from the intersection of the triangle: First, we pick $\boldsymbol{\gamma}_{abc} \geq 0$ from the intersection S of $\{a,b,c\}$. This can be done in

$$\binom{s}{\boldsymbol{\gamma}_{abc}}$$

different ways.

Pick from the pairwise intersections and outside of the triangle intersection: We may also pick elements from edges that do not appear in our triangle intersection S. For instance, we may pick $\boldsymbol{\gamma}_{ab}$ variables from the edge $e_{ab}\backslash S := (\nu(a)\cap\nu(b))\backslash(\nu(a)\cap\nu(b)\cap\nu(c))$. Since the size of this edge is k and we are excluding the intersection which has size s, we can pick from this edge in $\binom{k-s}{\boldsymbol{\gamma}_{ab}}$ different ways.

Similarly, we can pick elements from edges e_{ac}, e_{bc} just as done above. This means that accounting for all pairwise intersections of variables in our triangle, we can choose elements in

$$\binom{k-s}{\boldsymbol{\gamma}_{ab}}\binom{k-s}{\boldsymbol{\gamma}_{ac}}\binom{k-s}{\boldsymbol{\gamma}_{bc}}$$

different ways, where $\boldsymbol{\gamma}_{ab}, \boldsymbol{\gamma}_{ac}, \boldsymbol{\gamma}_{bc} \geq 0$.

Pick elements unique to a node: Next, we may also pick variables that are unique to a node. For instance, we may pick $\boldsymbol{\gamma}_a \geq 0$ elements from $a \setminus (b \cup c)$. Since this set has $m - 2k + s$ elements, such a selection can be done in

$$\binom{m-2k+s}{\boldsymbol{\gamma}_a}$$

different ways. Since we can do the same with the other two nodes, in total the unique node elements may be selected in

$$\binom{m-2k+s}{\boldsymbol{\gamma}_a}\binom{m-2k+s}{\boldsymbol{\gamma}_b}\binom{m-2k+s}{\boldsymbol{\gamma}_c}$$

different ways, where $\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b, \boldsymbol{\gamma}_c \geq 0$.

Pick elements outside of the triangle: We may have new variables which did not appear in the triangle yet. Since the size of $|\nu(a) \cup \nu(b) \cup \nu(c)|$ is given by $3m - 3k + s$ (by principle of inclusion/exclusion) and there are n variables available to us, we may pick new elements in

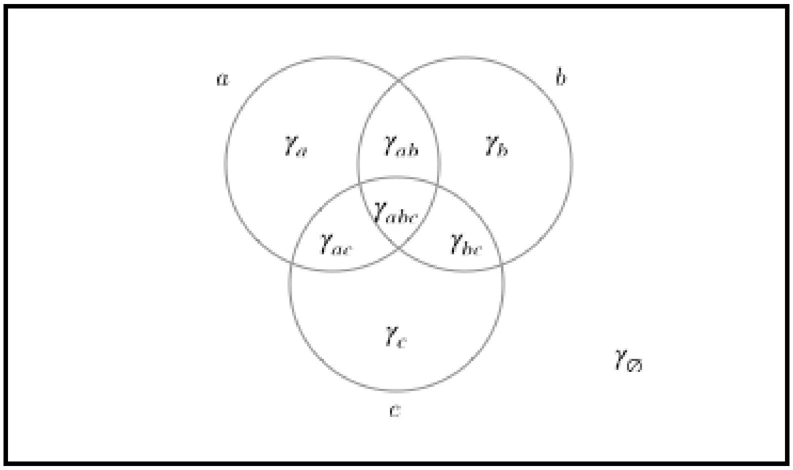
$$\binom{n-(3m-3k+s)}{\boldsymbol{\gamma}_\varnothing}$$

different ways, where $\boldsymbol{\gamma}_\varnothing \geq 0$.

Lastly, we must ensure that our $\boldsymbol{\gamma}$ satisfies that the sum of the variables we picked is m (since that is the number of variables in our node d) and that the pairwise intersection property holds. This can be ensured by picking compositions $\boldsymbol{\gamma}$ for which $\boldsymbol{\gamma} \vDash m$ and

$$\begin{aligned}
\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_{ac} + \boldsymbol{\gamma}_a &= k\\
\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_{bc} + \boldsymbol{\gamma}_b &= k\\
\boldsymbol{\gamma}_{abc} + \boldsymbol{\gamma}_{ac} + \boldsymbol{\gamma}_{bc} + \boldsymbol{\gamma}_c &= k
\end{aligned}$$

Since there are $4 = \binom{4}{3}$ different triangles that would have yield the same 4-clique, we divide by 4 to account for overcounting.



□

Before continuing to the general case, we prove that for any collection of subsets of a finite set $\mathcal{N}_n$, there is a natural partition of $\mathcal{N}_n$ induced by the collection.

Lemma 5.7. *For any $r \geq 1$, given any collection $(A_i)_{i=1}^r$ of subsets of $\mathcal{N}_n$, the collection given by*

$$\mathcal{A} := \left\{ \bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \notin J} A_i \right) : J \subseteq \mathcal{N}_r \right\}$$

is a partition of $\mathcal{N}_n$. Moreover, for any $j \in \mathcal{N}_r$, we have that

$$A_i = \bigcup_{\substack{J \subseteq \mathcal{N}_r \\ J \ni i}} \left[\bigcap_{i \in J} A_i \setminus \bigcup_{i \notin J} A_i \right].$$

Proof. First, we show that

$$\mathcal{N}_n \subseteq \bigcup_{J \subseteq \mathcal{N}_r} \left[\bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \notin J} A_i \right) \right].$$

Fix $x \in \mathcal{N}_n$ and let $J_x = \{i : x \in A_i\} \subseteq \mathcal{N}_r$. Since $x \in A_i$ for all $i \in J_x$, we have that $x \in \cap_{i \in J_x} A_i$. If $i \notin J_x$, then we find that $x \notin J_x$ and hence

$$x \notin \bigcup_{i \notin J_x} A_i$$

and we can conclude

$$x \in \left[\bigcap_{i \in J_x} A_i \setminus \left(\bigcup_{i \notin J_x} A_i \right) \right].$$

All that we have left to show is that the intersection of any two distinct members of $\mathcal{A}$ is empty. To that end, fix $J, H \subseteq \mathcal{N}_r$ distinct and suppose that

$$x \in \left[\bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \notin J} A_i \right) \right] \bigcap \left[\bigcap_{i \in H} A_i \setminus \left(\bigcup_{i \notin H} A_i \right) \right].$$

Since J and H are distinct, without loss of generality, we may assume that there is some $i \in J \setminus H$. Thus, we see that $x \in A_i$ as $i \in J$ and $x \notin A_i$ as $i \notin H$, which then implies $x \in \varnothing$ - which gives us the contradiction we seek. $\square$

Since we will be using this partition of $\mathcal{N}_n$ repeatedly in our proof, we introduce the following shorthand notation. For a given collection of subsets $(A_i)_{i=1}^r$ of $\mathcal{N}_n$, we let

$$\Gamma(J) := \left[\bigcap_{i\in J} A_i \setminus \left(\bigcup_{i\notin J} A_i\right)\right] = \left[\bigcap_{i\in J} A_i \cap \left(\bigcap_{i\notin J} A_i^c\right)\right],$$

for any $J \subseteq \mathcal{N}_r$.

Remark 5.8. For instance, if a collection of subsets is given by (A_1, A_2),

$$\begin{aligned}
\Gamma(\{1,2\}) &= A_1 \bigcap A_2 \\
\Gamma(\{1\}) &= A_1 \bigcap A_2^c \\
\Gamma(\{2\}) &= A_2 \bigcap A_1^c \\
\Gamma(\varnothing) &= \mathcal{N}_n \bigcap A_1^c \bigcap A_2^c.
\end{aligned}$$

In the generalization to our clique counting propositions, we will be interested in nonnegative integer solutions to a particular system of equations which we will define as follows.

For an $r-$clique H with nodes $\{A_1, \ldots, A_r\}$, let $\mathcal{C}_{n,m,k}(H)$ denote the set of all compositions $\boldsymbol{\gamma}$ indexed by subsets of $\mathcal{N}_r$ satisfying

$$\sum_{J\subseteq \mathcal{N}_r} \boldsymbol{\gamma}_J = m \qquad \text{(Subset size condition)}$$

$$\sum_{\substack{J\subseteq \mathcal{N}_r \\ i\in J}} \boldsymbol{\gamma}_J = k, \qquad \text{(Intersection condition } i\text{)}$$

where intersection condition holds for all $i \in \mathcal{N}_r$.

Now, we can prove our main result regarding the number of r-cliques in general.

Theorem 5.9. *The number of $(r+1)$-cliques in $J_n(m,k)$ which contain a particular $r-$clique H with nodes $\{A_1, \ldots, A_r\}$ is given by*

$$\frac{1}{r+1} \sum_{\boldsymbol{\gamma}\in \mathcal{C}_{n,m,k}(H)} \prod_{J\subseteq \mathcal{N}_r} \binom{|\Gamma(J)|}{\boldsymbol{\gamma}_J},$$

where $\mathcal{C}_{n,m,k}(H)$ is the set of all compositions $\boldsymbol{\gamma} : \mathcal{P}(\mathcal{N}_r) \to \mathbb{N}_0$ of m satisfying that for all $l \in \{1, 2, \ldots, r\}$,

$$\sum_{J\subset\{1,\ldots,r\}: j\in J} \boldsymbol{\gamma}_J = k. \tag{5}$$

Proof. Fix $A_{r+1} \in V_m$ adjacent to all of H and let $\boldsymbol{\gamma}_J := |\nu(A_{r+1}) \cap \Gamma(J)|$. By 5.7, $(\Gamma(J))_{J \subseteq \mathcal{N}_r}$ forms a partition of $\mathcal{N}_n$ and thus

$$\sum_{J \subseteq \mathcal{N}_r} \boldsymbol{\gamma}_J = \left| \bigcup_{J \subseteq \mathcal{N}_r} A_{r+1} \cap \Gamma(J) \right| = |\nu(A_{r+1})| = m.$$

Similarly, since A_{r+1} is adjacent to $A_i \in H$, we have that $|\nu(A_i) \cap \nu(A_{r+1})| = k$ for all i and hence

$$\sum_{i \in J \subseteq \mathcal{N}_r} \boldsymbol{\gamma}_J = \left| \bigcup_{i \in J \subseteq \mathcal{N}_r} A_{r+1} \cap \Gamma(J) \right| = |\nu(A_{r+1}) \cap \nu(A_i)| = k.$$

Thus, $\boldsymbol{\gamma}$ meets the subset size and intersection size condition. Furthermore, such a subset A_{r+1} can be constructed in

$$\prod_{J \subseteq H} \binom{|\Gamma(J)|}{\boldsymbol{\gamma}_J}$$

ways.

Conversely, given an integer composition $\boldsymbol{\gamma}$ in $\mathcal{C}_{n,m,k}(H)$, we can choose a corresponding node adjacent to H in

$$\prod_{J \subseteq H} \binom{|\Gamma(J)|}{\boldsymbol{\gamma}_J}$$

ways. Since $(\Gamma(J))_{J \subseteq \mathcal{N}_r}$ partitions $\mathcal{N}_n$ and $\boldsymbol{\gamma}$ satisfies both subset size condition and intersection condition for A_i, the corresponding node A_{r+1} will be adjacent to all of H and $|\nu(A_{r+1})| = m$.

Finally, since there are $r + 1 = \binom{r+1}{r}$ different r-cliques that we could have chosen to be H and would have yielded the same $(r+1)$-clique, we divide by $r+1$ to account for overcounting. □

Let's examine an equivalent formulation for our counts for the number of $2-$cliques and $3-$cliques which can be constructed given a particular $1-$clique and $2-$clique, respectively.

Remark 5.10. Consider Lemma 5.7 when $r = 1$ and suppose that $A_1 \subset \mathcal{N}_n$ with $|A_1| = m$. Then our partition of $\mathcal{N}_n$ is given by $\mathcal{N}_n = A_1 \sqcup A_1^c$, where $\boldsymbol{\gamma}(\{1\}) = A_1$ and $\boldsymbol{\gamma}(\varnothing) = A_1^c$.

Now, for a given $1-$clique with node A_1, the number of $2-$cliques which contain A_1 is given by

$$\sum_{\substack{\boldsymbol{\gamma}: \boldsymbol{\gamma}_{A_1} + \boldsymbol{\gamma}_{\varnothing} = m \\ \boldsymbol{\gamma}_{A_1} = k}} \prod_{J \subseteq \{1\}} \binom{|\boldsymbol{\gamma}(J)|}{\boldsymbol{\gamma}_J} = \sum_{\substack{\boldsymbol{\gamma}: \boldsymbol{\gamma}_{A_1} + \boldsymbol{\gamma}_{\varnothing} = m \\ \boldsymbol{\gamma}_{A_1} = k}} \binom{|A_1|}{\boldsymbol{\gamma}_{A_1}} \binom{|A^c|)}{\boldsymbol{\gamma}_{\varnothing}} = \binom{m}{k} \binom{n-m}{m-k},$$

which gives us an additional characterization of the number of edges in Proposition 5.1.

Remark 5.11. Given a 2-clique $H = \{A_1, A_2\}$, suppose we are interested in counting the number of 3-cliques which contain H. First, note that Lemma 5.7 states that $\Omega := \{A_1 \cap A_2, A_1 \setminus A_2, A_2 \setminus A_1, \mathcal{N}_n \setminus (A_1 \cup A_2)\}$ forms a partition of $\mathcal{N}_n$. To find all possible triangles which contain H, we can proceed by considering all viable subsets of the partition Ω that meet our intersection requirement and set size requirement:

$$\frac{1}{3} \sum_{\mathcal{C}_{n,m,k}(H)} \prod_{J \subseteq \{1,2\}} \binom{|\boldsymbol{\gamma}(J)|}{\boldsymbol{\gamma}_J} = \frac{1}{3} \sum \binom{|A_1 \cap A_2|}{\boldsymbol{\gamma}_{A_1 A_2}} \binom{|A_1 \setminus A_2|}{\boldsymbol{\gamma}_{A_1}} \binom{|A_2 \setminus A_1|}{\boldsymbol{\gamma}_{A_2}} \binom{|\mathcal{N}_n \setminus (A_1 \cup A_2)|)}{\boldsymbol{\gamma}_{\varnothing}},$$

where the sum on the right hand-side is over all $\boldsymbol{\gamma}$ for which

$$\begin{aligned}
\boldsymbol{\gamma}_{A_1 A_2} + \boldsymbol{\gamma}_{A_1} \quad + \boldsymbol{\gamma}_{A_2} + \boldsymbol{\gamma}_{\varnothing} &= m \\
\boldsymbol{\gamma}_{A_1 A_2} + \boldsymbol{\gamma}_{A_1} &= k \\
\boldsymbol{\gamma}_{A_1 A_2} \quad + \boldsymbol{\gamma}_{A_2} &= k \\
|\boldsymbol{\gamma}(J)| \geq \boldsymbol{\gamma}_J &\geq 0
\end{aligned}$$

Now, we show that this provides us with an equivalent formulation to the one in Theorem 5.2.

First, we begin with the most restrictive component of the third node of A_3: the subset of A_3 within $A_1 \cap A_2$. This corresponds to picking a value $0 \leq \boldsymbol{\gamma}_{A_1 A_2} = s \leq |A_1 \cap A_2| = k$. Once this is determined, the two intersection constraints are solved uniquely by $\boldsymbol{\gamma}_{A_1} = \boldsymbol{\gamma}_{A_2} = k - s$. Lastly, we have that the set size constraint gives us that

$$\boldsymbol{\gamma}_{\varnothing} = m - 2(k - s) - s.$$

Next, after we know the intersection sizes we can proceed with selecting subsets of our partition to build A_3:

$$\sum_{s=0}^{k} \binom{2m-k}{s} \binom{m-k}{k-s} \binom{m-k}{k-s} \binom{n-2m+k}{m-2(k-s)-s}$$

Now, since there are $\binom{3}{2} = 3$ different 2-cliques contained within any 3-clique and we could have started with any one of them to obtain the same 3-clique, we must divide by 3.

5.3 Types

In Theorem 3.7 and Proposition 3.6, we implicitly used the intersection of a clique and union of a clique as a decision rule for classifying the type of a clique. In this section, we generalize this idea and provide an intuitive explanation for what clique type means and when two $r-$cliques are semantically different despite being graph isomorphic.

5.3.1 Equivalence classes of cliques

In the previous sections, we saw that nonnegative integer solutions to a system of equations dictate the feasibility of constructing an $(r+1)-$clique from an $r-$clique. In this section, we explore the different types of cliques that exist in $J_n(m, k)$, regardless of the variable labelling of the nodes.

Definition 5.12. We say that two $r-$cliques $H_1 = \{A_1, \dots, A_r\}$ and $H_2 = \{B_1 \dots, B_r\}$ have the same **type** if there exists a bijection $f : \mathcal{N}_n \to \mathcal{N}_n$ for which $f(A_i) \in H_2$ for all $i \in \mathcal{N}_r$. If such an f exists, we say that H_1 and H_2 are **equivalent** as $r-$cliques or that they have the same **type**. In such case, we call the map f a **type isomorphism**.

For an $r-$clique H, we let the set $[H]$ denote the set of all $r-$cliques which share the same type as H.

Example 5.13. It is clear here that any two $1-$cliques (nodes) are equivalent as there are $m!$ different bijections between one $m-$set and another $m-$set. For the remaining $n - m$ variables in $\mathcal{N}_n$, we may permute them in $(n - m)!$ ways. Therefore, there are

$$m!(n - m)!$$

bijections $f : \mathcal{N}_n \to \mathcal{N}_n$ which certify the type equivalence of nodes.

Additionally, we can show that any two $2-$cliques (edges) are equivalent as follows. Fix $H_A = \{A_1, A_2\}$ and $H_B = \{B_1, B_2\}$ two edges in $J_n(m, k)$. We claim that there are

$$2 \times k!(m - k)!(m - k)!(n - 2m + k)!$$

type isomorphisms between H_A and H_B. We now demonstrate how one may construct one of these isomorphisms. First, let $S_A := \nu(A_1) \cap \nu(A_2)$ and $S_B := \nu(B_1) \cap \nu(B_2)$. Let $f_S : S_A \to S_B$ be any bijection. This can be chosen in $k!$ ways as $|S_A| = k = |S_B|$. Now, extend f_S into a mapping $f_1 : A_1 \to B_1$ (without loss of generality) where $f_1|_{A_1 \setminus S_A}$ is a bijection onto $B_1 \setminus S_B$. This may be done in $(m - k)!$ ways as $|A_1| = (m - k)! = |B_2|!$. Next, we need to extend f_1 into $f_{12} : A_1 \cup A_2 \to B_1 \cup B_2$ such that f_{12} is a bijection. We can do this in $(m - k)!$ ways as it suffices to only decide which bijection to choose to be $f_{12}|_{A_2 \setminus S_A} : A_2 \setminus S_A \, toB_2 \setminus S_B$. Finally, we need to extend f_{12} to the whole space $\mathcal{N}_n$. This can be done in $(n - 2m + k)!$ ways as any bijection from $\mathcal{N}_n \setminus (A_1 \cup A_2)$ onto $\mathcal{N}_n \setminus (B_1 \cup B_2)$ would do. Lastly, we note that when we constructed f_1 above, we chose without loss of generality that A_1 must map to B_1. However, A_1 could have also mapped to B_1 to obtain a valid type isomorphism. Thus, there are

$$2 \times k!(m - k)!(m - k)!(n - 2m + k)!$$

type isomorphisms between any two edges in $J_n(m, k)$.

Proposition 5.14. *The type relation induces an equivalence relation on the set of all cliques in* $J_n(m, k)$.

Proof. Let $H_1 \sim H_2$ denote that H_1 and H_2 have the same type. We are required to show that $\sim$ is reflexive, symmetric and transitive.

Reflexivity Fix H_1 a clique in $J_n(m, k)$. Consider the identity mapping $f : \mathcal{N}_n \to \mathcal{N}_n$. This is a bijection on $\mathcal{N}_n$ and moreover, $f(A_i) = A_i$ for all $A_i \in H_1$.

Symmetry Fix $H_1 = \{A_i : i \in \mathcal{N}_r\} \sim H_2 = \{B_i : i \in \mathcal{N}_r\}$ and let $f : \mathcal{N}_n \to \mathcal{N}_n$ be a corresponding bijection. Then the map $g := f^{-1}$ is a bijection on $\mathcal{N}_n$ and we see that $g(B_i) = A_i$ for all $i \in \mathcal{N}_r$.

Transitivity Suppose that $H_1 \sim H_2$ and $H_2 \sim H_3$ and let f, g be bijections for which $f(H_1) = H_2$, $g(H_2) = H_3$. Consider the mapping $g \circ f$. As f and g are bijections, so is $g \circ f$. Moreover, it is clear that

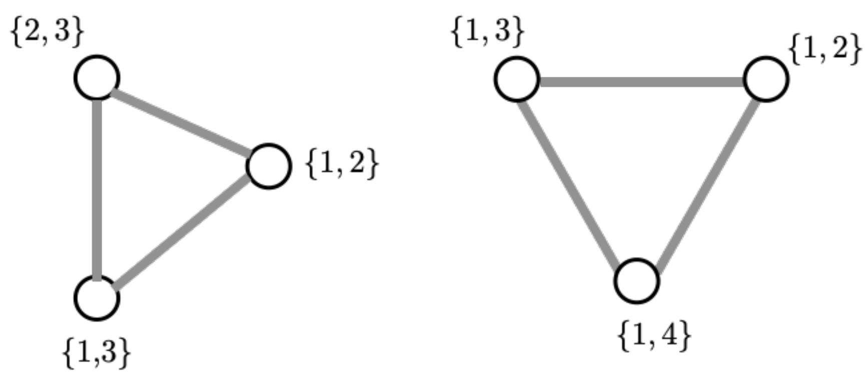
$$g(f(H_1)) = H_3,$$

and hence $H_1 \sim H_3$.

□

Example 5.15 (Isomorphic and non isomorphic triangles)**.** Consider the two triangles $T_1 := \{\{1,2\},\{1,3\},\{1,4\}\}, T_2 := \{\{1,2\},\{2,3\},\{1,3\}$ in $J_4(2,1)$. These are both cliques of size 3 but they have different types. In fact, T_1 is of type D_{max} and T_2 is of type D_{min}. To show explicitly that there is no graph isomorphism that preserves the node structure, it suffices to consider where an isomorphism $f : V(T_1) \to V(T_2)$ would have to map the element 1 to. It cannot be anything in $\{1,2,3,4\}$ because $1 \in v$, for all $v \in V(T_2)$ but no element in $\mathcal{N}_4$ appears in all of $V(T_2)$.

Now, if we have a triangle $T_3 := \{\{2,1\},\{2,3\},\{2,4\}\}$ in $J_4(2,1)$, we have $3! = 6$ different choices for isomorphisms of type. Indeed, we see that any such isomorphism f must map 1 to 2 but then any permutation on $\{2,3,4\}$ achieves the desired isomorphism.



The following theorem demonstrates how generating series can be used to count the number of different types of triangles in $J_n(m,k)$.

Theorem 5.16. *Let $\Phi_J(q)$ be the generating series defined by*

$$\Phi_J(q) = 1 + q + q^2 + \cdots + q^{|\Gamma(J)|}.$$

Then the type generating series for 3−cliques in $J_n(m,k)$ is given by

$$[q^{2(m-k)}]\Phi_a(q)\Phi_b(x)\Phi_\varnothing(q^2) = [q^{2(m-k)}]\frac{(1-q^{(m-k)+1})^2}{(1-q)^2}\frac{1-q^{2(n-2m+k+1)}}{1-q^2}.$$

Proof. Each type of triangle corresponds to a unique solution to the system

$$\begin{aligned}
\gamma_{A_1A_2} + \gamma_{A_1} \quad & + \gamma_{A_2} + \gamma_\varnothing &= m \\
\gamma_{A_1A_2} + \gamma_{A_1} \quad & &= k \\
\gamma_{A_1A_2} \quad & + \gamma_{A_2} &= k
\end{aligned}$$

with integer solutions $|\boldsymbol{\gamma}(J)| \geq \boldsymbol{\gamma}_J \geq 0$.

We solve the system by multiplying the first constraint by $r = 2$ and then substituting a copy of each of the other constraints into the first. Thus, the original system of equations will be equivalent to

$$\begin{gathered}\boldsymbol{\gamma}_{A_1} + \boldsymbol{\gamma}_{A_2} + 2\boldsymbol{\gamma}_{\varnothing} = 2(m-k) \\ |\boldsymbol{\gamma}(J)| \geq \boldsymbol{\gamma}_J \geq 0\end{gathered}$$

after simplification. We note that the second constraint gives us that the the generating function $\Phi_J(q)$ is given by

$$\Phi_J(q) = 1 + q + \cdots + q^{|\Gamma(J)|} = \frac{1 - q^{|\Gamma(J)|+1}}{1-q}$$

since $\mathcal{N}_{|\Gamma(J)|}$ is the set of values that $\boldsymbol{\gamma}_J$ can take. Since addition of components of the $\boldsymbol{\gamma}'_J s$ corresponds to a product of the generating functions of their parts (e.g. [4], [6]), we have that the number of solutions to the system above is given by

$$\begin{aligned}[q^{2(m-k)}]\Phi_a(q)\Phi_b(x)\Phi_{\varnothing}(q^2) &= [q^{2(m-k)}]\frac{1-q^{|\Gamma(a)|+1}}{1-q}\frac{1-q^{|\Gamma(b)|+1}}{1-q}\frac{1-q^{2(|\Gamma(\varnothing)|+1)}}{1-q^2} \\ &= [q^{2(m-k)}]\frac{1-q^{(m-k)+1}}{1-q}\frac{1-q^{(m-k)+1}}{1-q}\frac{1-q^{2(n-2m+k+1)}}{1-q^2} \\ &= [q^{2(m-k)}]\frac{(1-q^{(m-k)+1})^2}{(1-q)^2}\frac{1-q^{2(n-2m+k+1)}}{1-q^2}.\end{aligned}$$

□

Example 5.17. Consider the graph $J_5(m=2, k=1)$. The generating series for types of triangles is then given by

$$\begin{aligned}\frac{(1-q^{(m-k)+1})^2}{(1-q)^2}\frac{1-q^{2(n-2m+k+1)}}{1-q^2} &= \frac{(1-q^2)^2}{(1-q)^2}\frac{1-q^6}{1-q^2} \\ &= \frac{(1-q^2)(1-q^6)}{(1-q)^2}.\end{aligned}$$

Upon expanding this series, we find that the coefficient of $q^{2(m-k)}$ is given by 2:

$$\frac{(1-q^2)(1-q^6)}{(1-q)^2} = 1 + 2q + 2q^2 + 2q^3 + 2q^4 + 2q^5 + O\left(q^6\right),$$

and hence the number of triangles in $J_5(2,1)$ is 2.

This example brings us to the following corollary. This agrees with our results from section 2 where we found that there are at most two types of cliques in $J_n(m, m-1)$: D_{max} cliques categorized by having a total intersection of $m-1$ and D_{min} cliques categorized by having a total intersection of 0.

Corollary 5.18. *The number of types of triangles in $J_n(m, k = m-1)$ is given by*

$$\begin{cases} 0, & n \leq m \\ 1, & n = m+1 \\ 2, & n \geq m+2 \end{cases}$$

Proof. The case when $n < m$ is clear as the graph does not exist then. As $k = m-1$, by Theorem 5.16, the number of types of triangles is given by

$$\begin{aligned}
[q^{2(m-k)}]\frac{(1-q^{(m-k)+1})^2}{(1-q)^2}\frac{1-q^{2(n-2m+k+1)}}{1-q^2} &= [q^2]\frac{(1-q^{(1)+1})^2}{(1-q)^2}\frac{1-q^{2(n-m)}}{1-q^2} \\
&= [q^2]\frac{(1-q^2)(1-q^{2(n-m)})}{(1-q)^2} \\
&= [q^2]\frac{(1-q^2-q^{2(n-m)}+q^{2(n-m+1)})}{(1-q)^2}.
\end{aligned}$$

Since

$$\frac{1}{(1-q)^2} = \sum_{n\geq 0}\binom{n+1}{n}q^n,$$

we have that

$$[q^2]\frac{(1-q^2-q^{2(n-m)}+q^{2(n-m+1)})}{(1-q)^2} = [q^2](1-q^2-q^{2(n-m)}+q^{2(n-m+1)})\sum_{n\geq 0}\binom{n+1}{n}q^n \tag{6}$$

$$= [q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}-q^{n+2(n-m)}+q^{n+2(n-m+1)}). \tag{7}$$

If $n = m$, then Equation (7) becomes

$$[q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}-q^{n+2(n-m)}+q^{n+2(n-m+1)}) = [q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}-q^n+q^{n+2}) = 0.$$

If $n = m+1$, then Equation (7) becomes

$$[q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}-q^{n+2}+q^{n+2(2)}) = [q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-2q^{n+2}) = \binom{3}{2}-2\binom{1}{0} = 1.$$

If $n = m+2$, then Equation (7) becomes

$$\begin{aligned}
[q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}-q^{n+2\cdot 2}+q^{n+2(2+1)}) &= [q^2]\sum_{n\geq 0}\binom{n+1}{n}(q^n-q^{n+2}) \\
&= \binom{3}{2}-\binom{1}{0} = 2.
\end{aligned}$$

□

We could do more with generating series as a tool for approaching these problems but first we require some heavier machinery.

6 Crude, refined generating series and MacMahon calculus

In [5], the author introduced Partition Analysis as a computational framework for solving problems concerning linear homogeneous diophantine inequalities and demonstrated how they relate to the theory of integer partitions. In [2], the authors revitalized the idea by implementing a package for the evaluation of Omega calculus computations and demonstrating how the theory can be used to solve challenging counting problems in number theory and computational geometry.

Definition 6.1. The operator $\underset{\geqslant}{\Omega}$ is defined on functions with absolutely convergent multisum expansions

$$\sum_{s_1\in\mathbb{Z}}\cdots\sum_{s_r\in\mathbb{Z}} A_{s_1,\ldots,s_r}\lambda_1^{s_1}\cdots\lambda_r^{s_r},$$

in an open neighborhood of the complex circles $|\lambda_i| = 1$. The action of $\underset{\geqslant}{\Omega}$ is given by

$$\underset{\geqslant}{\Omega}\sum_{s_1\in\mathbb{Z}}\cdots\sum_{s_r\in\mathbb{Z}} A_{s_1,\ldots,s_r}\lambda_1^{s_1}\cdots\lambda_r^{s_r} = \sum_{s_1\in\mathbb{Z}}\cdots\sum_{s_r\in\mathbb{Z}} A_{s_1,\ldots,s_r}$$

The Omega operator has been used extensively in [2], [7] and [1]. The usefulness of the operator is rooted in the realization that many rational functions have a closed form expression for which the simplification under $\underset{\geqslant}{\Omega}$ is readily available. We illustrate an instance of its power below.

Example 6.2. Consider the problem of finding nonnegative integer solutions (a_1, a_2, a_3) to the system

$$\begin{aligned} a_1 + a_2 + a_3 &= k \\ a_1 + a_2 - a_3 &\geq 0, \end{aligned}$$

for some $k \geq 0$ fixed. One way to approach this problem is to identify the generating series for tuples (a_1, a_2, a_3) which records both the sum $a_1 + a_2 + a_3$ and the constraint $a_1 + a_2 - a_3$.

If we let $F(x, y, z, \lambda)$ denote the generating series for all integer three tuples (a_1, a_2, a_3) which marks the difference $a_1 + a_2 - a_3 \geq 0$, then we know that it has the form

$$F(x, y, z, \lambda) = \sum_{a_1,a_2,a_3\geq 0} x^{a_1} y^{a_2} z^{a_3} \lambda^{a_1+a_2-a_3}.$$

Moreover, we can simplify the generating series as follows

$$\begin{aligned} F(x, y, z, \lambda) &= \sum_{a_1,a_2,a_3\geq 0} x^{a_1} y^{a_2} z^{a_3} \lambda^{a_1+a_2-a_3} \\ &= \sum_{a_1\geq 0}(\lambda x)^{a_1}\sum_{a_2\geq 0}(\lambda y)^{a_2}\sum_{a_3\geq 0}(\lambda^{-1}z)^{a_3} \\ &= \frac{1}{1-x\lambda}\frac{1}{1-y\lambda}\frac{1}{1-\frac{z}{\lambda}}. \end{aligned}$$

If we let $f(x, y, z)$ denote the generating series

$$f(x, y, z) \sum_{\substack{a_1, a_2, a_3 \geq 0 \\ a_1 + a_2 - a_3 \geq 0}} x^{a_1} y^{a_2} z^{a_3},$$

then by definition we know that

$$f(x, y, z) = \underset{\geqslant}{\Omega}\, F(x, y, z, \lambda) = \underset{\geqslant}{\Omega} \frac{1}{1 - x\lambda} \frac{1}{1 - y\lambda} \frac{1}{1 - \frac{z}{\lambda}}.$$

In order to evaluate the above expression, one may use theorem 2.1 from [2] in the special case where $n = 2, m = 1, a = 0$ to obtain that

$$f(x, y, z) = \underset{\geqslant}{\Omega} \frac{1}{1 - x\lambda} \frac{1}{1 - y\lambda} \frac{1}{1 - \frac{z}{\lambda}} = \frac{1 - xyz}{(1 - x)(1 - y)(1 - xz)(1 - yz)}.$$

Finally, we find that the number of integer solutions to $a_1 + a_2 + a_3 = k$ for a fixed $k \geq 0$ where $a_1 + a_2 \geq a_3$ is given by

$$[q^n] f(q, q, q) = [q^k] \frac{1 - q^3}{(1 - q)^2 (1 - q^2)^2} = [q^k] \frac{1 - q^3}{(1 - q - q^2 + q^3)^2}.$$

The above example illustrates that one may obtain counts of complicated systems of linear diophantine equations by lifting the system to a more general space and then specializing using the Omega operator.

The function F in the example above which also contains nonnegative integers which we wish to discard is sometimes called a *crude generating series*. This is because its refinement under the $\underset{\geqslant}{\Omega}$ operator produces the generating series we are interested in.

Definition 6.3. The operator $\underset{=}{\Omega}$ is defined on functions with absolutely convergent multi-sum expansions

$$\sum_{s_1 \in \mathbb{Z}} \cdots \sum_{s_r \in \mathbb{Z}} A_{s_1 \cdots, s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r},$$

in an open neighborhood of the complex circles $|\lambda_i| = 1$. The action of $\underset{=}{\Omega}$ is given by

$$\underset{=}{\Omega} \sum_{s_1 \in \mathbb{Z}} \cdots \sum_{s_r \in \mathbb{Z}} A_{s_1, \ldots, s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} = A_{0, \ldots, 0}.$$

That is, $\underset{=}{\Omega}$ extracts the constant term of the series after setting the λ's to 1.

Example 6.4. Consider the problem of finding solutions to the system

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= k \\ a_1 + a_2 - a_3 - a_4 &= 0, \end{aligned}$$

for some $k \geq 0$ fixed. We may begin by writing the crude generating series $F(w, x, y, z, \lambda)$ as

$$F(w, x, y, z, \lambda) = \sum_{a_1, a_2, a_3, a_4 \geq 0} w^{a_1} x^{a_2} y^{a_3} z^{a_4} \lambda^{a_1 + a_2 - a_3 - a_4}.$$

Now, we express F as

$$F(w,x,y,z,\lambda) = \left(\sum_{a_1\geq 0} w^{a_1}\lambda^{a_1}\right)\left(\sum_{a_2\geq 0} x^{a_2}\lambda^{a_2}\right)\left(\sum_{a_3\geq 0} y^{a_3}\lambda^{-a_3}\right)\left(\sum_{a_4\geq 0} w^{a_4}\lambda^{-a_4}\right)$$
$$= \frac{1}{1-w\lambda}\frac{1}{1-x\lambda}\frac{1}{1-\frac{y}{\lambda}}\frac{1}{1-\frac{z}{\lambda}}$$

Now, one may use the observation that for all $F(\lambda)$

$$\underset{=}{\Omega} F(\lambda) = \underset{\geqslant}{\Omega} F(\lambda) + \underset{\geqslant}{\Omega} F(\lambda^{-1}) - F(1),$$

and Theorem 2.1 from [2], we find that the refined generating series is given by

$$\begin{aligned} f(w,x,y,z) &= \underset{=}{\Omega} F(w,x,y,z) \\ &= \underset{=}{\Omega} \frac{1}{1-w\lambda}\frac{1}{1-x\lambda}\frac{1}{1-\frac{y}{\lambda}}\frac{1}{1-\frac{z}{\lambda}} \\ &= \frac{1-wxyz}{(1-wy)(1-xy)(1-wz)(1-xz)}. \end{aligned}$$

Thus, the solutions to the linear dipohantine system is given by

$$[q^k]f(q,q,q,q) = [q^k]\frac{1-q^4}{(1-q^2)^4}.$$

6.1 Edge type generating series

We recall that the type of a 2-clique in $J_n(m,k)$ corresponds to a solution of the system

$$\begin{cases} n = \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_a + \boldsymbol{\gamma}_b + \boldsymbol{\gamma}_\varnothing \\ m = \boldsymbol{\gamma}_{ab} + \boldsymbol{\gamma}_a \\ m = \boldsymbol{\gamma}_{ab} \qquad\quad + \boldsymbol{\gamma}_b \\ k = \boldsymbol{\gamma}_{ab} \end{cases},$$

where $\boldsymbol{\gamma}_{ab} \leq k$ and $\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b \leq m-k$. The crude generating series $F(\boldsymbol{x}, y, z, \boldsymbol{\lambda})$ for the above system, with flexibility in choice for k and m, is given by

$$F(\boldsymbol{x}, y, z, \boldsymbol{\lambda}, \varepsilon) = \sum_{m\geq 0} z^m \sum_{k\geq 0} y^k \sum_{\substack{\boldsymbol{\gamma}_{ab},\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b,\boldsymbol{\gamma}_\varnothing\geq 0 \\ \boldsymbol{\gamma}_{ab}\leq k \\ \boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b\leq m-k \\ 0\leq k\leq m}} \varepsilon_{ab}^{k-\boldsymbol{\gamma}_{ab}} \lambda_a^{m-\boldsymbol{\gamma}_{ab}-\boldsymbol{\gamma}_a} \lambda_b^{m-\boldsymbol{\gamma}_{ab}-\boldsymbol{\gamma}_b} x_{ab}^{\boldsymbol{\gamma}_{ab}} x_a^{\boldsymbol{\gamma}_a} x_b^{\boldsymbol{\gamma}_b} x_\varnothing^{\boldsymbol{\gamma}_\varnothing},$$

where $\boldsymbol{x}$ records the tuples $\boldsymbol{\gamma}$, y records the intersection size constraint and z records the size of the node in the graph and $\boldsymbol{\lambda}$ records the feasibility of the tuple. After simplifying,

$$\begin{aligned}
F(\boldsymbol{x}, y, z, \boldsymbol{\lambda}, \varepsilon) &= \sum_{m\geq 0} z^m \sum_{k\geq 0} y^k(\lambda_a\lambda_b)^{-k} \sum_{\gamma_{ab}\geq 0}(\lambda_a\lambda_b x_{ab})^{\gamma_{ab}} \sum_{\gamma_a=0}^{m-k}(\lambda_a x_a)^{\gamma_a} \sum_{\gamma_b=0}^{m-k}(\lambda_b x_b)^{\gamma_b} \sum_{\gamma_\varnothing\geq 0} x_\varnothing^{\gamma_\varnothing} \\
&= \sum_{m\geq 0} z^m \frac{1}{1-\frac{y}{\lambda_a\lambda_b}} \frac{1}{1-\lambda_a\lambda_b x_{ab}} \left[\frac{1-(\lambda_a x_a)^{m-k+1}}{1-\lambda_a x_a}\right] \left[\frac{1-(\lambda_b x_b)^{m-k+1}}{1-\lambda_b x_b}\right] \left[\frac{1}{1-x_\varnothing}\right] \\
&= \left(\frac{1}{1-z} - \frac{1}{(\lambda_b x_b)^{k-1}} \frac{1}{1-z\lambda_b x_b} - \frac{1}{(\lambda_a x_a)^{k-1}} \frac{1}{1-z\lambda_a x_a} + \right. \\
&\qquad \left. \frac{1}{(x_a x_b \lambda_a\lambda_b)^{k-1}} \frac{1}{1-z\lambda_a\lambda_b x_a x_b}\right) \frac{1}{1-\frac{y}{\lambda_a\lambda_b}} \frac{1}{1-\lambda_a\lambda_b x_{ab}} \left[\frac{1}{1-x_\varnothing}\right].
\end{aligned}$$

Now, we see that the type generating function for the 2-cliques is given by

$$f(\boldsymbol{x}, y) = \underset{=}{\Omega}\, F(\boldsymbol{x}, y, z, \boldsymbol{\lambda}).$$

Moreover, since we are only interested in the count of types, we may simplify $F(\boldsymbol{x}, y, z, \boldsymbol{\lambda})$ by substituting q in all occurrences of $\boldsymbol{x}$ and find that

$$\begin{aligned}
F(\boldsymbol{x}, y, z, \boldsymbol{\lambda})|_{\boldsymbol{x}=q} = &\left(\frac{1}{1-z} - \frac{1}{(\lambda_b q)^{k-1}} \frac{1}{1-z\lambda_b q} - \frac{1}{(\lambda_a q)^{k-1}} \frac{1}{1-z\lambda_a q} + \frac{1}{(q^2\lambda_a\lambda_b)^{k-1}} \frac{1}{1-z\lambda_a\lambda_b q^2}\right) \\
&\times \frac{1}{1-\frac{y}{\lambda_a\lambda_b}} \frac{1}{1-\lambda_a\lambda_b q} \left[\frac{1}{1-q}\right],
\end{aligned}$$

and the number of types of a 2-clique in $J_n(m,k)$ is given by

$$[q^m y^k z^m]\, \underset{=}{\Omega}\, F(\boldsymbol{x}, y, z, \boldsymbol{\lambda})|_{\boldsymbol{x}=q}.$$

Remark 6.5. Suppose we have a series $F(\boldsymbol{x}, \lambda_1, \lambda_2)$. In general, the operators of $\underset{\geqslant}{\Omega}$ and $\underset{=}{\Omega}$ do not commute and

$$\underset{\substack{\geqslant\\ \lambda_2}}{\Omega}\, \underset{\substack{=\\ \lambda_1}}{\Omega} \neq \underset{\substack{=\\ \lambda_1}}{\Omega}\, \underset{\substack{\geqslant\\ \lambda_2}}{\Omega}.$$

For our purposes, we will always apply $\underset{\geqslant}{\Omega}$ before we apply $\underset{=}{\Omega}$ so as to avoid removing valid summands from the computation. That is, we will always apply the less restrictive map prior to applying the more restrictive map.

6.2 Triangle type series in generality

We are interested in nonnegative integer solutions to the system

$$\begin{cases}
n = \gamma_{abc} + \gamma_{ab} + \gamma_{ac} + \gamma_{bc} + \gamma_a + \gamma_b + \gamma_c + \gamma_\varnothing \\
m = \gamma_{abc} + \gamma_{ab} + \gamma_{ac} + \gamma_a \\
m = \gamma_{abc} + \gamma_{ab} + \gamma_{bc} + \gamma_b \\
m = \gamma_{abc} + \gamma_{ac} + \gamma_{bc} + \gamma_c \\
k = \gamma_{abc} + \gamma_{ab} \\
k = \gamma_{abc} + \gamma_{ac} \\
k = \gamma_{abc} + \gamma_{bc}
\end{cases},$$

Consider the crude generating series

$$\Psi(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon}) = \sum_{\gamma\geq 0} \lambda_a^{m-\gamma_{abc}-\gamma_{ab}-\gamma_{ac}-\gamma_a}\cdots\lambda_c^{m-\gamma_{abc}-\gamma_{bc}-\gamma_{ac}-\gamma_c}\varepsilon_{ab}^{k-\gamma_{abc}-\gamma_{ab}}\cdots\varepsilon_{bc}^{k-\gamma_{abc}-\gamma_{bc}}x_{abc}^{\gamma_{abc}}x_{ab}^{\gamma_{ab}}\cdots x_c^{\gamma_c}$$

$$= \left[\frac{1}{1-\frac{x_{abc}}{\lambda_a\lambda_b\lambda_c\varepsilon_{ab}\varepsilon_{ac}\varepsilon_{bc}}}\right]\left[\frac{1}{1-\frac{x_{ab}}{\lambda_a\lambda_b\varepsilon_{ab}}}\right]\left[\frac{1}{1-\frac{x_{ac}}{\lambda_a\lambda_c\varepsilon_{ac}}}\right]\left[\frac{1}{1-\frac{x_{bc}}{\lambda_b\lambda_c\varepsilon_{bc}}}\right]\times$$

$$\left[\frac{1}{1-\frac{x_a}{\lambda_a}}\right]\left[\frac{1}{1-\frac{x_b}{\lambda_b}}\right]\left[\frac{1}{1-\frac{x_c}{\lambda_c}}\right]\left[\frac{1}{1-x_\varnothing}\right](\lambda_a\lambda_b\lambda_c)^m(\varepsilon_{ab}\varepsilon_{ac}\varepsilon_{bc})^k,$$

where $\boldsymbol{\lambda}$ measures the feasibility of the node size being m, $\boldsymbol{\varepsilon}$ measures the feasibility of the intersection size being k and $\boldsymbol{x}$ records the values of the tuples of $\boldsymbol{\lambda}$.

Now, in order to proceed in obtaining the triangle type generating series, we proceed as follows

1. Apply $\underset{=\lambda}{\Omega}$ to $\Psi(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})$.
2. (a) Derive $\psi_K(\boldsymbol{x})$ by applying $\underset{\geqslant\varepsilon}{\Omega}$ to $\underset{=\lambda}{\Omega}\,\Psi(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})$ if we are interested in $KG_{n,m}$.
   (b) Derive $\psi_J(\boldsymbol{x})$ by applying $\underset{=\varepsilon}{\Omega}$ to $\underset{=\lambda}{\Omega}\,\Psi(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\varepsilon})$ if we are interested in $J_n(m,k)$.
3. Extract the coefficient of q^n in $\psi(\boldsymbol{x})|_{x=q}$ for the generating series of interest.

6.3 $r-$clique type generating series for $r \geq 3$

We know that for $r \in \mathbb{N}$, the type of an $r-$clique is a solution to the system

$$\begin{aligned}
\sum_{J\subseteq[r]} \boldsymbol{\gamma}_J &= n \\
\sum_{i\in J\subseteq[r]} \boldsymbol{\gamma}_J &= m, && \forall i \\
\sum_{i,j\in J\subseteq[r]} \boldsymbol{\gamma}_J &= k, && \forall i \neq j
\end{aligned}$$

If we let $\boldsymbol{\lambda}$ record the node constraint and $\boldsymbol{\varepsilon}$ record the intersection constraint, then we obtain the crude generating series

$$\boldsymbol{\lambda}^m\boldsymbol{\varepsilon}^k \prod_{J\subseteq[r]} \frac{1}{1-\frac{x_J}{(\prod_{i\in J}\lambda_i)(\prod_{i,j\in J, i\neq j}\varepsilon_{i,j})}}.$$

If one were to apply the operator $\underset{=\boldsymbol{\varepsilon}}{\Omega}\,\underset{=\boldsymbol{\lambda}}{\Omega}$ and then apply the substitution $\boldsymbol{x}|_{x=q}$, one would obtain the refined generating series for the number of types of $r-$cliques in $J_n(m,k)$ for any $r \geq 3$.

6.4 Type generating series in full generality

The most general crude generating series for types we have derived contains *all* possible configurations of types for all n, m, k and r. We shall denote it by Φ and it is given by

$$\begin{aligned}
\Phi(w,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\varepsilon}) &= \sum_{n,m,k\geq 0}\sum_{r\geq 0}\sum_{\substack{\gamma_J\geq 0\\ J\subseteq\mathcal{N}_r}}\left(\prod_{i=1}^{r}\lambda_i^{m-\sum_{i\in J}\gamma_J}\right)\prod_{i\neq j}\varepsilon_{i,j}^{k-\sum_{\substack{i,j\in J\\ i\neq j}}\gamma_J}\prod_{J\in\mathcal{N}_r}x_J^{\gamma_J}y_1^m y_2^k w^r\\
&= \sum_{k\geq 0}y_2^k\sum_{s\geq 0}y_1^{k+s}\sum_{r\geq 0}\sum_{\substack{\gamma_J\geq 0\\ J\subseteq\mathcal{N}_r}}\prod_{i=1}^{r}\lambda_i^{k+s-\sum_{i\in J}\gamma_J}\prod_{i\neq j}\varepsilon_{i,j}^{-\gamma_J}x_J^{\gamma_J}\left(\prod_{J\subseteq\mathcal{N}_r}x_J^{\gamma_J}\right)w^r\\
&= \sum_{r\geq 0}w^r\sum_{k\geq 0}y_2^k y_1^k\prod_{i=1}^{r}\lambda_i^k\prod_{i\neq j}\varepsilon_{i,j}^k\sum_{s\geq 0}y_1^s\prod_{i=1}^{r}\lambda_i^s\sum_{J\subseteq\mathcal{N}_r}\sum_{\gamma_J\geq 0}\prod_{i\in J}\lambda_i^{-\gamma_J}\prod_{i,j\in J,i\neq j}\varepsilon_{i,j}^{-\gamma_J}x_J^{\gamma_J}\\
&= \sum_{r\geq 0}w^r\left(\frac{1}{1-y_1y_2\prod_{i=1}^{r}\lambda_i\prod_{i,j\in\mathcal{N}_r i\neq j}\varepsilon_{i,j}}\right)\left(\frac{1}{1-y_1\prod_{i=1}^{r}\lambda_i}\right)\\
&\qquad\prod_{J\subseteq\mathcal{N}_r}\left[\frac{1}{1-\dfrac{x_J}{\prod_{i\in J}\lambda_i\prod_{i,j\in J;i\neq j}\varepsilon_{i,j}}}\right],
\end{aligned}$$

where y_1 records the node size feasibility constraint, y_2 records the intersection size feasibility constraint and w records the clique size of interest.

We would like to apply $\underset{=\boldsymbol{\lambda}}{\Omega}$ and $\underset{=\boldsymbol{\varepsilon}}{\Omega}$ to $\Phi(w,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\varepsilon})$. This is one of the main challenges we have left.

Why do we want to be able to apply it? Suppose we had a closed form expression for

$$\phi_J(w,\boldsymbol{x},\boldsymbol{y}) := \underset{\substack{=\\ \boldsymbol{\varepsilon}}}{\Omega}\,\underset{\substack{=\\ \boldsymbol{\lambda}}}{\Omega}\,\Phi(w,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\varepsilon}),$$

then $[q^n y_1^m y_2^k]\phi_J(w,\boldsymbol{x}|_{x=q},\boldsymbol{y})$ will be a finite degree polynomial in w. In particular, the degree of the polynomial will correspond to the clique number (the size of largest clique) of $J_n(m,k)$, by construction.

6.5 Johnson graph coclique problem

Moreover, similar technique can be used to obtain the clique number of $KG_n(m,k)$. That is, if we instead were able to obtain a closed form for

$$\phi_K(w,\boldsymbol{x},\boldsymbol{y}) := \underset{\substack{\geqslant\\ \boldsymbol{\varepsilon}}}{\Omega}\,\underset{\substack{=\\ \boldsymbol{\lambda}}}{\Omega}\,\Phi(w,\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{\varepsilon}),$$

then $[q^n y_1^m y_2^k]\phi_K(w,\boldsymbol{x}|_{x=q},\boldsymbol{y})$ will be a finite degree polynomial in w whose degree is the clique number of $KG_{n,m}$.

We begin by recalling the definition of a coclique.

Definition 6.6. Let G be a graph and H be a subset of the vertices of G. We say that H is a coclique in G if for any distinct $x, y \in H$, x and y are not adjacent in G. Equivalently, the subgraph induced by H in G is empty.

In [3], the authors pose the following open problem on page 312:

Problem 6.7. *What is the size of the largest coclique in the Johnson graph $J_n(k, k-1)$ for all n and k?*

TODO: Not sure if we can call our work a theorem

We now explain how the above theorem addresses this open problem. Since H is a coclique if and only if H is a clique in the complement in the complement of $J_n(m, m-1)$. Furthermore, we note that x, y in $J_n(m, m-1)$ are adjacent if and only if $|\nu(x) \bigcap \nu(y)| = m-1$. Therefore, x and y are not adjacent in $J_n(m, m-1)$ if and only if $|\nu(x) \bigcap \nu(y)| \leq m-2$. Thus, the complement of $J_n(m, m-1)$ is $KG_n(m, m-2)$ and H must be a clique in $KG_n(m, m-2)$ to be a coclique in $J_n(m, m-1)$.

7 Connection to Random Graphs

TODO: Define $G(n, p)$, mention motivation

Theorem 7.1. *If G is a subgraph of the complete graph K_n, then $L(G)$ is a subgraph of the Johnson graph $J_n(2, 1)$. Thus, since $G(n, p)$ always produces a subgraph of the complete graph, the line graph of $G(n, p)$ will always be a subgraph of $J_n(2, 1)$.*

Proof. First, we show that the line graph of a complete graph is $J_n(2, 1)$. Suppose that the vertex set of K_n is labelled $\{1, 2, \ldots, n\}$. Therefore, the vertices line graph of K_n are identified as a subset of the 2−subsets of $\{1, 2, \ldots, n\}$. Now, since every $i, j \in \{1, \ldots, n\}$ are adjacent in K_n, $\{i, j\}$ is an edge in K_n and hence each of the 2−subsets of $\{1, 2, \ldots, n\}$ give to a vertex in $L(K_n)$.

Now, fix two vertices $\{i, j\}$, $\{k, l\}$ in $L(K_n)$. In order for them to be adjacent in $L(K_n)$, it would mean that the corresponding edges in must share exactly one vertex in K_n (as multiedges are not allowed). Thus, $\{i, j\}$ and $\{k, l\}$ are adjacent in $L(K_n)$ if and only if

$$|\{i, j\} \cap \{k, l\}| = 1,$$

which agrees with the classic definition of $J_n(2, 1)$.

Finally, since the line graph of a subgraph H of G is a subgraph of $L(G)$, the claim follows. □

Therefore, ER graphs form a model for how we may construct our navigation graphs, at least in the case for when we only visit two variables at a time.

Definition 7.2. Let $\mathcal{R}$ denote the projection operator from the set of all multisets of $\mathcal{N}_n$ onto the set of all subsets of $\mathcal{N}_n$ which acts on a multiset by replacing multiplicities greater than 1 with 1. That is,

$$\mathcal{R}(\{i_1^{(\ell_1)}, i_2^{(\ell_2)}, \dots, i_r^{(\ell_r)}\}) = \{i_1, i_2, \dots, i_r\}.$$

This operator acts on line graphs of $J_n(m,k)$ by contracting vertices. That is, suppose one has a collection of vertices of the form

$$\{\ \{i_1^{(\ell_1)}, \dots, i_r^{(\ell_r)}\} :\ \ell_j \geq 1, \forall j = 1, \dots, r\}\},$$

then all such sets would contract into a single vertex corresponding to the $m-$set $\{i_1, i_2, \dots, i_r\}$.

Theorem 7.3. *For all $n > m > k$ positive integers,*

$$\mathcal{R}(L(J_n(m,k)) \cong J_n(2m-k, m) + J_n(2m-k, m+1) + \cdots + J_n(2m-k, 2m-k-1),$$

where the addition operation is the usual graph addition (edge union of vertices).

Proof. Let H denote the graph $L(J_n(m,k))$. To prove the claim, we must show that $\mathcal{R}(H)$ consists of all of the $(2m-k)-$subsets of $\mathcal{N}_n$ and that two nodes are adjacent if and only if the two corresponding sets have an intersection of size m.

First, we show that $\mathcal{R}(H)$ consists of nodes of the form $A \uplus e \uplus B$, where e is a subset of $\mathcal{N}_n$ of size k and $A, B \subset \mathcal{N}_n \setminus e$ are disjoint sets of size $m-k$. To this end, fix $e \subseteq \mathcal{N}_n$ of size k let A, B be two disjoint sets $A, B \subset \mathcal{N}_n \setminus e$ of size $m-k$. Then $\nu(v_1) = A \uplus e$ and $\nu(v_2) = B \uplus e$ are two adjacent nodes in $J_n(m,k)$. Applying the line graph operation to this edge produces a node in H which has the form

$$\nu(v) = \{x^{(1)} : x \in A\} \uplus \{y^{(1)} : y \in B\} \uplus \{i^{(2)} : i \in e\}.$$

Applying the reduction operator $\mathcal{R}$, we find

$$\mathcal{R}(\nu(v)) = A \uplus e \uplus B.$$

Since every node in $\mathcal{R}(H)$ is constructed through identifying it with an edge in $J_n(m,k)$, the first claim follows:

$$|\nu(v)| = |A \uplus e \uplus B| = |A| + |B| + |e| = 2(m-k) + k = 2m-k,$$

for all $v \in V(\mathcal{R}(H))$.

Next, we must show that v_1 and v_2 are adjacent in $\mathcal{R}(H)$ if and only if they intersect in at least m elements.

Suppose that $|\nu(v_1) \cap \nu(v_2)| \geq m$. Fix $e \subset \nu(v_1) \cap \nu(v_2)$ of size m and suppose that $v_1 = A_1 \uplus e$, $v_2 = A_2 \uplus e$, where A_i is the complement of e in $\nu(v_i), i = 1, 2$ and hence has cardinality $m-k$. Fix $f \subset e$ of size $(m-k)$. Since e is of size m, it corresponds to a unique node in $J_n(m,k)$. Now, consider the set $u_1 = (e \setminus f) \uplus A_1$. This is a set of size m and hence also a node in $J_n(m,k)$. Since $|e \setminus f| = k$ and A_1 is disjoint from e, we know that the two nodes corresponding to e and u_1 are adjacent in $J_n(m,k)$. Similarly, the two nodes corresponding to e and $u_2 = (e \setminus f) \uplus A_2$ are adjacent in $J_n(m,k)$. Therefore, the two edges that connect e with u_1 and e with u_2 must be adjacent in H. However, these edges are precisely v_1 and v_2 after applying the projection operator $\mathcal{R}$.

Conversely, suppose that v_1, v_2 are adjacent in $\mathcal{R}(H)$. Then there exists x, y, z some $m-$subsets of $\mathcal{N}_n$ for which $x \sim y, x \sim z$ in $J_n(m,k)$ and $\nu(v_1) = x \cup y, \nu(v_2) = x \cup z$. Now, we claim that $|\nu(v_1) \cap \nu(v_2)| \geq m$. Clearly, $x \subseteq (x \cup y) \bigcap (x \cup z)$ and $|x| = m$ and hence we are done. □

Corollary 7.4. *For all positive integers $n > m$,*

$$\mathcal{R}(L(J_n(m, m-1))) = J_n(m+1, m).$$

Proof. Follows immediately from Theorem 7.3 as when $k = m - 1$, the right handside becomes

$$J_n(2m - (m-1), m) = J_n(m+1, m).$$

□

TODO: Find a home for this

References

[1] G. E. Andrews and P. Paule. Macmahon's dream. In *Partitions, q-series, and Modular Forms*, pages 1–12. Springer, 2012.

[2] G. E. Andrews, P. Paule, and A. Riese. Macmahon's partition analysis: the omega package. *European Journal of Combinatorics*, 22(7):887–904, 2001.

[3] C. Godsil and K. Meagher. *Erdos-Ko-Rado theorems: algebraic approaches*. Number 149 in Cambridge studies in advanced mathematics. Cambridge University Press, 2016.

[4] I. P. Goulden and D. M. Jackson. *Combinatorial enumeration*. Courier Corporation, 2004.

[5] P. A. MacMahon. *Combinatory Analysis, Volumes I and II*, volume 137. American Mathematical Soc., 2001.

[6] H. S. Wilf. *generatingfunctionology*. AK Peters/CRC Press, 2005.

[7] G. Xin. A fast algorithm for macmahon's partition analysis. *the electronic journal of combinatorics*, 11(1):58, 2004.