ZOÉ CHATZIDAKIS AND THE CANONICAL BASE PROPERTY

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A special feature of model theory is the role it often plays of recognising and facilitating analogies between different areas of mathematics. The Canonical Base Property (CBP), a topic that Chatzidakis made fundamental contributions to, is a striking illustration of this phenomenon. The property (which I will explain later) originates in theorems of Campana [1] and Fujiki [4] (independently) from complex analytic geometry in the nineteen eighties, was then abstracted to a model-theoretic setting by Pillay [5] in the early naughts, and then soon after shown to hold in both differential-algebraic and difference-algebraic geometry by Pillay and Ziegler [6]. That is, model theory was the conduit whereby theorems about compact complex analytic spaces begot theorems about rational vector fields and dynamical systems. Chatzidakis, working in the abstract model-theoretic setting, proved some remarkable consequences of the CBP which then specialise to theorems in differential- and difference-algebraic geometry. I will try to explain these applications first, and then later discuss the CBP itself.

1. Three geometric contexts

First let me say a few words about the three relevant disciplines we will be working in, and what the main geometric objects of study are in each case:

(1) Complex analytic geometry. Here we are interested in (reduced) compact complex analytic spaces that are of Kähler-type. Among these are the projective algebraic varieties, and more generally those that are bimeromorphically equivalent to such, called Moishezon varieties. We view the Moishezon varieties as the locus of algebraic geometry within complex analytic geometry.

(2) Differential-algebraic geometry. We work over an algebraically closed field k of characteristic zero equipped with a derivation δ, and consider *rational D-varieties* over (k,σ). These are pairs (V,ϕ) where V is an algebraic variety and $\phi : V \to \tau V$ is a rational section to the δ-twisted tangent bundle. If V is in affine n-space, for example, then τV is a subvariety of affine $2n$-space defined by polynomials of the form $f^\delta(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) y_i$ where $f(x)$ vanishes on V. Here f^δ refers to the polynomial obtained from f by appying δ to the coefficients – this is the "δ-twist". In particular, if δ is the zero derivation on k, then $\tau V = TV$ is just the tangent bundle and ϕ is a rational vector field on V. This category also expands pure algebraic geometry which is represented by the trivial vector fields (where ϕ is the zero section). Actually, for reasons that are natural (and important) from the model-theoretic point of view, we allow finite covers and base extension in our analogue of Moishezon: let us simply say that a D-variety is *algebraic* if it is a finite cover of a D-variety which over some differential field extension of (k,δ) is birationally equivalent to a trivial vector field.

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(3) Difference-algebraic geometry. Now work over an algebraically closed field k of characteristic zero[1] equipped with an endomorphism σ, and consider *rational σ-varieties*: also pairs (V, ϕ) with V an algebraic variety but now with $\phi : V \to V^\sigma$ a dominant rational map to the transform of V by σ. In particular, if σ is the identity on k then $V^\sigma = V$ and (V, ϕ) is a rational dynamical system. Trivial dynamics is when $\phi = \text{id}$, and we say that a σ-variety is *algebraic* – the analogue here of Moishezon – if it is a finite cover of a σ-variety which over some difference field extension of (k, σ) is birationally equivalent to trivial dynamics.

2. Two theorems

Chatzidakis' work on the CBP lead to a pair of theorems in an abstract model-theoretic setting whose specialisations to differential- and difference-algebraic geometry I would like to articulate. These theorems appear in [3].

2.1. **Existence of algebraic coreductions.** In all three geometric contexts, a natural approach to study the structure of an arbitrary object is to understand how far it is from being Moishezon/algebraic; that is how far it is from coming from algebraic geometry proper. A first such measure might be to consider the *algebraic reduction* of an object: its maximal Moishezon/algebraic image. Dually, one can look for an *algebraic coreduction*; a map whose fibres are maximally algebraic. While the existence of algebraic reductions is relatively straightforward, the same is not true of algebraic coreductions. Campana [2] proves that every compact complex analytic space of Kähler-type admits an algebraic coreduction. Chatzidakis, through her work on the CBP, is able to transport the theorem to differential- and difference-algebraic geometry. I state it here somewhat informally, and in a way that deals simultaneously with both categories.

Theorem 2.1 (Chatzidakis)**.** *Suppose (V, ϕ) is either a D-variety over (k, δ) or a σ-variety over (k, σ). Then (V, ϕ) admits an algebraic coreduction. That is, there is a dominant rational map $f : (V, \phi) \to (W, \psi)$ whose general fibre is algebraic, and such that f factors through any other such map on (V, ϕ).*

2.2. **Descent.** In all three geometric contexts, one is often concerned with uniform families of objects, and it is important to be able to detect when the family is obtained by base extension from another family over a smaller parameterising space. As an application of the CBP, Chatzidakis proves a general descent theorem of this kind. Actually, the specialisation to complex analytic geometry of this theorem is also new[2], but I will only state it here for the differential- and difference-algebraic settings. Moreover, I restrict attention to the special case of descent to a trivial family. My formulation may not be immediately recognisable as an articulation of Chatzidakis' theorems; I leave a verification of that to the knowledgeable reader.

Theorem 2.2 (Chatzidakis)**.** *Suppose we are given the following commuting diagram of dominant rational maps of either D-varieties or σ-varieties, with π the*

[1] Here the charactersitic zero assumption is made only to simplify definitions; Chatzidakis, notably, works in arbitrary characteristics.

[2] Though, in private communication in 2012, Frederic Campana, upon my showing him such a specialisation, did provide me with a complex geometric proof; one that turned out to be very much in the spirit of Chatzidakis' model-theoretic proof.

natural co-ordinate projection:

$$(*)\qquad \begin{array}{ccc} (X_1,\theta_1)\times(W,\psi) & \longrightarrow & (V,\phi) \\ & \searrow^{\pi} & \downarrow f \\ & & (W,\psi) \end{array}$$

Then there exists a commuting diagram of dominant rational maps of the form

$$\begin{array}{ccc} (V,\phi) & \xrightarrow{g} & (X_2,\theta_2)\times(W,\psi) \\ f\downarrow & \swarrow_{\pi} & \\ (W,\psi) & & \end{array}$$

where the general fibres of g are algebraic.

In particular, if the algebraic coreduction of (V,ϕ) is itself then the existence of $(*)$ implies that (V,ϕ) is birationally equivalent to a product over (W,ψ).

3. The CBP itself

Let me give an informal account of the Canonical Base Property in model theory.

Each of the geometric contexts I have discussed above corresponds to a certain tame first order theory, and in each of these we have seen that there is a part that plays the role of a pure "algebraic geometric reduct". What we have been calling Moishezon/algebraic are those objects that are closely tied to this reduct. All of this can be abstracted to any first order theory T satisfying certain tameness conditions (namely, *supersimplicity*) where the algebraic geometry reduct is replaced by the family of *non-locally-modular minimal types*, and the role of Moishezon/algebraic is played by those definable sets (or types) that are *almost internal* to this family. Indeed, the Zilber conjecture would have that in general the non-locally-modular minimal types really are algebraic geometry. (The conjecture turned out to be false, but very influential.) For convenience let us continue to use the term *algebraic* for almost-internality to non-locally-modular minimal types, and to continue to think of them as an abstract avatar of algebraic geometry in T.

The CBP is a condition that the theory T may or may not satisfy. It asserts that the minimal parameter space of a definable family of sets passing through a fixed point is algebraic. A little more precisely, if $\mathcal{Y}=(Y_b)_{b\in B}$ is a definable family of subsets of X passing through $a\in X$, each of some fixed finite dimension d (in a sense that can be made precise in supersimple theories), and such that if $b\neq b'$ then $Y_b\cap Y_{b'}$ is of dimension strictly less than d; then B must be algebraic. To see, very roughly, why one should expect this to be true of complex geometry, consider the case when X and B are compact complex manifolds, and $\mathcal{Y}$ is an analytic family of irreducible closed analytic subspaces. If all the higher order tangent spaces at a of Y_b and $Y_{b'}$ agree, as subspaces of the ambient tangent spaces of X at a, then $Y_b=Y_{b'}$. By compactness one obtains a bound on what order of tangency at a one has to check, and this gives an embedding of B into a grassmanian of a fixed finite dimensional complex vector space, witnessing the algebraicity of B.

Chatzidakis discovered that, under the CBP, the algebraicity of B arises with a certain strong uniformity. Pillay and I had noticed that the theorems of Campana and Fujiki gave the stronger property (which we called Uniform CBP) in the case of complex geometry, and we had begun to investigate its abstract consequences. But Chatzidakis showed directly, in a technical tour de force, that the CBP always implies the UCBP. Her theorems on algebraic coreductions and descent are applications of this specialised to the relevant geometric contexts.

References

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