A proof of ω-stability for m-DCF_0

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These notes give a proof of ω-stability for m-DCF_0. The theory of differential fields of characteristic zero in m commuting derivations, m-DF_0, has a model completion which we denote by m-DCF_0 the theory of differentially closed fields of characteristic zero in m commuting derivations. The usual proof, due to McGrail [2], involves establishing a bijection between complete 1-types over a model K and prime differential ideals in $K\{y\}$. In the lecture notes [5], Pillay gives an alternate argument for ω-stability of 1-DCF_0 using quantifier elimination and reducing to ACF_0. We act similarly, using quantifier elimination for m-DCF_0, ω-stability for ACF_0, and induction on m. Thus we avoid any serious differential algebra. A key ingredient that is absent in the case of $m = 1$ is the use of Kolchin's *differential-type*.

Let K be a Δ-field where $\Delta = \{\partial_1, \dots, \partial_m\}$ are m commuting derivations, we let $K\{\bar{y}\}$ denote the Δ-polynomial ring in differential indeterminates $\bar{y} = (y_0, \dots, y_l)$ and think of it as a Δ-ring in the natural way. Note that

$$K\{\bar{y}\} = K[\partial_m^{e_m} \cdots \partial_1^{e_1} y_i : \ i \leq l, \ e_j \in \omega]$$

thus we shall call the elements of the form $\partial_m^{e_m} \cdots \partial_1^{e_1} y_i$ the algebraic indeterminates. Let us fix an ordering of type ω on the algebraic indeterminates in $K\{\bar{y}\}$ satisfying:

1. $\left[\partial_m^{e_m} \cdots \partial_1^{e_1} y_i < \partial_m^{e'_m} y_i\right] \implies [e_m < e'_m]$
2. $[\sum_{i=1}^{m} e_i \leq e'_m] \implies \left[\partial_m^{e_m} \cdots \partial_1^{e_1} y_i \leq \partial_m^{e'_m} y_i\right]$

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For example, define

$$\partial_m^{e_m}\cdots\partial_1^{e_1}y_i < \partial_m^{e'_m}\cdots\partial_1^{e'_1}y_{i'} \Leftrightarrow \left(\sum_{i=1}^m e_i, i, e_m, ..., e_1\right) < \left(\sum_{i=1}^m e'_i, i', e'_m, ..., e'_1\right)$$

in the lexicographical order. Henceforth, enumerating the algebraic indeterminates according to this ordering, we denote them by $(Y_j)_{j\in\omega}$. If $Y = \partial_m^{e_m}\cdots\partial_1^{e_1}y_i$ and $\bar{a} \in K^l$ then we let $Y(\bar{a}) = \partial_m^{e_m}\cdots\partial_1^{e_1}a_i$. The field of constants of K, denoted by $\mathcal{C}_K$, is the intersection of the kernels of $\partial_1, \ldots, \partial_m$.

Lemma 1 Let $F \in K[x_0, ..., x_n]$ and consider $F(Y_0, \ldots, Y_n) \in K\{\bar{y}\}$. Then for any $\partial \in \Delta$,

$$\partial(F(Y_0, ..., Y_n)) = F^{\partial}(Y_0, ..., Y_n) + \sum_{j=0}^{n} \frac{\partial F}{\partial x_j}(Y_0, ..., Y_n)\,\partial Y_j$$

where $\frac{\partial F}{\partial x_j}$ denotes the formal partial derivative of F and F^{∂} denotes the polynomial obtained by applying the derivation to the coefficients of F.

Proof. We shall prove this for polynomials of the form $F = cx_0^{e_0}...x_n^{e_n}$, where c is an element from K,

$$\begin{aligned}\partial F(Y_0, ..., Y_n) &= \partial(c)Y_0^{e_0}...Y_n^{e_n} + \sum_{j=0}^{n} cY_0^{e_1}\ldots e_jY_j^{e_j-1}\ldots Y_n^{e_n}\,\partial Y_j \\ &= F^{\partial}(Y_0, ..., Y_n) + \sum_{j=0}^{n}\frac{\partial F}{\partial x_j}(Y_0, ..., Y_n)\,\partial Y_j\end{aligned}$$

The linearity of ∂ ensures that the formula holds for a general polynomial. □

Let $K \subset L$ be a Δ-field extension. Recall that an element $a \in L$ is called *differentially algebraic* over K if there is a differential polynomial in $K\{y\}$ vanishing on a and differentially transcendental otherwise. We let $K < a >$ denote the Δ-field generated by $K \cup \{a\}$. For $j \leq m$, let $\Delta_j := \{\partial_1, \ldots, \partial_j\}$ and (K, Δ_j) denote K considered as a Δ_j-field. We let $K < a >_{\Delta_j}$ denote the Δ_j-field generated by $K \cup \{a\}$.

Recall that the *leader* of a differential polynomial $P \in K\{\bar{y}\}$ is the greatest algebraic indeterminant appearing nontrivially in P. The *separant* of P

is the formal partial derivative of P (considered as an ordinary polynomial) with respect to its leader.

Theorem 2 Let $K \preceq L \models m\text{-}DCF_0$ and let $a \in L$ be differentially algebraic over K. Suppose $P \in K\{y\}$ vanishes at a but $S_P(a) \neq 0$ (where S_P is the separant of P) and the leader of P is $\partial_m^s y$ for some $s \in \omega$, then:
(i)

$$K < a >= K < a, \partial_m a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$$

(ii) If $b \in L$ is such that

$$tp_{\Delta_{m-1}}(a, \partial_m a, \ldots, \partial_m^s a/K) = tp_{\Delta_{m-1}}(b, \partial_m b, \ldots, \partial_m^s b/K) \qquad (1)$$

then $tp(a/K) = tp(b/K)$.

Proof. (i) Note that $P(y) = F(Y_0, \ldots, Y_{n-1}, \partial_m^s y)$ for some $F \in K[x_0, \ldots, x_n]$. Using Lemma 1 we get,

$$\begin{aligned} 0 &= \partial_m(P(a)) \\ &= F^{\partial_m}(a) + \sum_{i=0}^{n-1} \frac{\partial F}{\partial x_i}(Y_0(a), \ldots, \partial_m^s a)\partial_m Y_i(a) + S_P(a)\partial_m^{s+1} a. \end{aligned}$$

Because $S_P(a) \neq 0$, we can rearrange to get:

$$\partial_m^{s+1} a = -(F^{\partial_m}(a) + \sum_{i=0}^{n-1} \frac{\partial F}{\partial x_i}(Y_0(a), \ldots, \partial_m^s a)\partial_m Y_i(a))(S_P(a))^{-1}$$

since the ∂_m-order of each $Y_0, ..., Y_{n-1}$ is $\leq s-1$ and the derivations commute, we get $\partial_m^{s+1} a = G_1(a, ..., \partial_m^s a)$ for some $G_1 \in K < x_0, ..., x_s >_{\Delta_{m-1}}$. Thus $\partial_m^{s+1} a \in K < a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$.

Next, for $r \geq 1$ inductively assume $\partial_m^{s+r} a \in K < a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$. This means

$$\partial_m^{s+r} a = G_r(a, \ldots, \partial_m^s a) = \frac{H_1(a, \ldots, \partial_m^s a)}{H_2(a, \ldots, \partial_m^s a)}$$

for some $G_r \in K < x_0, ..., x_s >_{\Delta_{m-1}}$. So

$$\partial_m^{s+r+1} a = \frac{\partial_m(H_1(a))H_2(a) - H_1(a)\partial_m(H_2(a))}{(H_2(a))^2}$$

but $\partial_m(H_i(a))$ is in $K < a, \ldots, \partial_m^{s+1} a >_{\Delta_{m-1}} = K < a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$. Thus $\partial_m^{s+r+1} a \in K < a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$ and there is a Δ_{m-1}-polynomial $G_{r+1}(x_0, \ldots, x_s)$ witnessing this. So

$$\partial_m^{s+r} a = G_r(a, \ldots, \partial_m^s a) \in K < a, \ldots, \partial_m^s a >_{\Delta_{m-1}},$$

for all $r = 1, 2, \ldots,$ and so

$$K < a >= K < a, \partial_m a, \ldots, \partial_m^s a >_{\Delta_{m-1}}$$

(ii) Since P and S_P both have ∂_m-order s, our assumption on b implies that $P(b) = 0$ and $S_P(b) \neq 0$. In part (i), the constuction of the G_i's used only the fact that $P(a) = 0$ and $S_P(a) \neq 0$, and so the same G_i's will work for b. That it, $\partial_m^{s+r}(b) = G_r(b, \ldots, \partial_m^s b)$ for $r = 1, 2, \ldots$.

By quantifier elimination for m-DCF_0 we only need to show that for $F \in K\{y\}$, $F(a) = 0$ if and only if $F(b) = 0$. Note that there is $F^* \in K\{x_0, ..., x_l\}_{\Delta_{m-1}}$ such that $F(y) = F^*(y, \partial_m y, ..., \partial_m^l y)$. If $l \leq s$ the result follows immediately by (1). Thus assume $l > s$

$$\begin{aligned}
F(a) = 0 \quad &\Leftrightarrow \quad F^*(a, ..., \partial_m^s a, \partial_m^{s+1} a, ..., \partial_m^l a) = 0 \\
&\Leftrightarrow \quad F^*(a, ..., \partial_m^s a, G_1(a, ..., \partial_m^s a), ..., G_l(a, ..., \partial_m^s a)) = 0 \\
\text{by (1)} \quad &\Leftrightarrow \quad F^*(b, ..., \partial_m^s b, G_1(b, ..., \partial_m^s b), ..., G_l(b, ..., \partial_m^s b)) = 0 \\
&\Leftrightarrow \quad F^*(b, ..., \partial_m^s b, \partial_m^{s+1} b, ..., \partial_m^l b) = 0 \\
&\Leftrightarrow \quad F(b) = 0
\end{aligned}$$

Thus $tp(a/K) = tp(b/K)$. □

The following theorem is actually a step in Kolchin's theorem on "differential-type", see [1,§2.11].

Theorem 3 Let $a \in L$ be differentially algebraic over K. Suppose $P \in K\{y\}$ vanishes at a but $S_P(a) \neq 0$. Then we can find derivations $\Delta' = \{\partial'_1, \ldots, \partial'_m\}$ such that $\Delta = C\Delta'$ for some $C = (c_{ij}) \in GL_m(\mathcal{C}_K)$, and if $P^{\Delta'}$ denotes P viewed as a Δ'-polynomial then the leader of $P^{\Delta'}$ is of the form $\partial_m'^s y$ for some $s \in \omega$ and $S_{P^{\Delta'}}(a) \neq 0$.

Proof. Fix a matrix $C = (c_{ij}) \in GL_m(\mathcal{C}_K)$ and let $\Delta' = \{\partial_1, \ldots, \partial'_m\}$ be the derivations on K given by $\Delta' = C^{-1}\Delta$. So for each i, $\partial_i = \sum c_{ij}\partial'_j$. We need to show that C can be chosen so that the conclusion of Theorem 3 holds.

Let s be the order of P, if $\partial_1^{e_1}\cdots\partial_m^{e_m}$ is a derivative of order s then

$$\begin{aligned}\partial_1^{e_1}\cdots\partial_m^{e_m}y &= (\sum_{i=1}^m c_{1i}\partial_i')^{e_1}\cdots(\sum_{i=1}^m c_{mi}\partial_i')^{e_m}y \\ &= c_{1m}^{e_1}\cdots c_{mm}^{e_m}\partial_m'^s y + Q(y)\end{aligned}$$

where Q is a Δ'-polynomial with ∂_m'-order $< s$. Using the chain rule and letting $\{\gamma_k\}_{k\in I}$ be a basis of L over $\mathcal{C}_K$ we get

$$\begin{aligned}\frac{\partial P^{\Delta'}}{\partial(\partial_m'^s y)}(a) &= \sum_{\sum e_j=s}\frac{\partial P}{\partial(\partial_1^{e_1}\cdots\partial_m^{e_m}y)}(a)c_{1m}^{e_1}\cdots c_{mm}^{e_m} \\ &= \sum_{\sum e_j=s}(\sum_i \beta_{e_1,\ldots,e_m}^{(i)}\gamma_i)c_{1m}^{e_1}\cdots c_{mm}^{e_m} \\ &= \sum_i(\sum_{\sum e_j=s}\beta_{e_1,\ldots,e_m}^{(i)}c_{1m}^{e_1}\cdots c_{mm}^{e_m})\gamma_i \\ &= \sum_i g_i(c_{1m},\ldots,c_{mm})\gamma_i\end{aligned}$$

where the g_i's are homogeneous (algebraic) polynomials over $\mathcal{C}_K$. Note that the leader of P is a derivative of order s, so one of the formal partial derivatives in the first equality is the separant of P, and by assumption it doesn't vanish at a. So for some i and $e_1,...,e_m$, $\beta_{e_1,...e_m}^{(i)}\neq 0$, and hence g_i is a non-zero polynomial. Let g be such a g_i. Since we are in characteristic zero, $\mathcal{C}_K$ is an infinite field and so we can find an invertible matrix $C=(c_{ij})$ such that $g(c_{1m},\ldots,c_{mm})\neq 0$.

In fact, this is the matrix we are looking for. Recall P has order s and hence so does $P^{\Delta'}$. On the other hand,

$$\frac{\partial P^{\Delta'}}{\partial(\partial_m'^s y)}(a) = \sum_i g_i(c_{1m},\ldots,c_{mm})\gamma_i \neq 0$$

since $g(c_{1m},\ldots,c_{mm})\neq 0$. So $\partial_m'^s(y)$ must appear in $P^{\Delta'}$. It follows, by the nature of the ordering, that $\partial_m'^s$ must be the leader of $P^{\Delta'}$. Also, it is clear that $P^{\Delta'}(a)=0$ and the last inequation tells us that the separant of $P^{\Delta'}$ does not vanish at a. □

Lemma 4 Let $(K,\Delta)\models m\text{-}DCF_0$ and let $\Delta'=\{\partial_1',\ldots,\partial_m'\}$ be derivations given by $\Delta=C\Delta'$ for some $C\in GL_m(\mathcal{C}_K)$. Then $(K,\Delta')\models m\text{-}DCF_0$.

Proof. Let $\delta_1, \ldots, \delta_m$ be the derivative symbols in L the language of fields with m derivations. Let $\phi(\bar{x})$ be a quantifier free L_K-formula. Suppose $(K, \Delta') \subset (M, \Delta')$ where $(M, \Delta') \models m\text{-}DF_0$. Let $\bar{a}$ be a tuple in M such that $(M, \Delta') \models \phi(\bar{a})$.

Note $(M, \Delta) \models m\text{-}DF_0$. Write $C^{-1} = (c_{ij})$. Let $\psi(\bar{x})$ be the quantifier free L_K-formula obtained by replacing each occurrence of δ_i in $\phi(\bar{x})$ by the L_K-term $t_i = \sum_{j=1}^{m} c_{ij}\delta_j$. In (K, Δ') and (M, Δ'), each δ_i is interpreted as ∂_i' and so t_i is interpreted as ∂_i. Hence, for any $\bar{c}$ from M

$$(M, \Delta') \models \phi(\bar{c}) \iff (M, \Delta) \models \psi(\bar{c})$$

and similarly for K. Thus $(M, \Delta) \models \psi(\bar{a})$, but since (K, Δ) is existentially closed there is a tuple $\bar{b}$ in K such that $(K, \Delta) \models \psi(\bar{b})$. This implies $(K, \Delta') \models \phi(\bar{b})$ and hence $(K, \Delta') \models m\text{-}DCF_0$. □

Recall that an L-theory T is ω-stable if for any $\mathcal{M} \models$ T, and any countable A $\subset$ M, we have that $S_n^{\mathcal{M}}(A)$ is also countable. In fact, it suffices to consider the case $n = 1$.

Lemma 5 Let L be a countable language, and T an L-theory. If for all countable $\mathcal{N} \models T$ we have $|\mathcal{N}| = |S_1^{\mathcal{N}}(N)|$, then T is ω-stable.

Proof. Let $\mathcal{M} \models T$, and let A be a countable subset of M. By the Downward Lowenhiem-Skolem Theorem, there exists a countable $\mathcal{N} \preceq \mathcal{M}$ containing A. Thus $\omega \geq |S_1^{\mathcal{N}}(N)| \geq |S_1^{\mathcal{N}}(A)| = |S_1^{\mathcal{M}}(A)|$, and hence T is ω-stable. □

Theorem 6 The theory $m\text{-}DCF_0$ is ω-stable.

Proof. We will prove ω-stability by induction on the number of derivations, the base case is when $m = 0$ and so we get ACF_0 which we know is ω-stable. Assume $(m-1)\text{-}DCF_0$ is ω-stable. Let $K \models m\text{-}DCF_0$ be countable and $K \preceq L$ sufficiently saturated, so counting $S_1^K(K)$ amounts to counting $\{tp(a/K) : a \in L\}$.

Fix $a \in L$.

Case 1 Suppose a is differentially transcendental. Then the only atomic formulas realized by a are equivalent to $(0 = 0)$, so by QE, $tp(a/K)$ is completely determined.

Case 2 Otherwise a is differentially algebraic over K. If we establish an embedding of the types of differentially algebraic elements into

$$\bigcup_{\Delta'_{m-1}} \bigcup_{n} S_n^{(K, \Delta'_{m-1})}(K) \tag{2}$$

where we range over all Δ' such that $\Delta = C\Delta'$ for some $C \in GL_m(\mathcal{C}_K)$, then we will have shown m-DCF_0 is ω-stable. Indeed, by Lemma 4

$$(K, \Delta'_{m-1}) \models (m-1)\text{-}DCF_0$$

and so by induction is ω-stable. Hence (2) is a countable set. Since a is differentially algebraic over K we can always pick a differential polynomial vanishing at a that is minimal with respect to the degree of its leader, so its separant will not vanish at a. Thus, by Theorem 3, we can find Δ' and $P \in K\{y\}_{\Delta'}$ such that the leader of P is of the form $\partial_m'^s y$ and the separant S_P does not vanish on a. Send

$$tp(a/K) \longmapsto tp_{\Delta'_{m-1}}(a, \partial'_m a, \dots, \partial_m'^s a/K)$$

we will show this map is injective. By Theorem 2(ii) applied to (K, Δ'), if

$$tp_{\Delta'_{m-1}}(a, \partial'_m a, \dots, \partial_m'^s a/K) = tp_{\Delta'_{m-1}}(b, \partial'_m b, \dots, \partial_m'^s b/K)$$

then $tp_{\Delta'}(a/K) = tp_{\Delta'}(b/K)$. But, clearly

$$tp_{\Delta'}(a/K) = tp_{\Delta'}(b/K) \Leftrightarrow tp(a/K) = tp(b/K)$$

and so the map is indeed injective. □

References

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