CONTRIBUTIONS TO THE MODEL THEORY OF FIELDS AND COMPACT COMPLEX SPACES

BY

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THESIS

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To my parents

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Abstract

This thesis deals with the interaction between geometric model theory and the following two general themes: fields of positive characteristic and compact complex analytic spaces.

Two known results in the model theory of fields of characteristic zero are extended to all characteristics. Firstly, it is shown that the only difference fields admitting quantifier elimination in the language of difference fields are pure algebraically closed fields. Secondly, working in pure algebraically closed fields, the minimal field of definition for an algebraic set is obtained uniformly and definably from a given description of the algebraic set.

While the methods of geometric stability theory have traditionally been applied to algebraic rather than analytic geometry, it has been observed that these techniques are also meaningful when applied to compact complex analytic spaces. One obstacle is that these structures are not ω_1-saturated in their natural language of analytic sets. The notion of a Zariski closed set in an elementary extension of a compact complex space, as well as the associated complex dimension, are discussed. It is shown that a one-dimensional Zariski closed set in an elementary extension of a compact complex space is essentially algebraic. This provides a kind of nonstandard version of the Riemann Existence Theorem. The notion of a "saturated" compact complex analytic space, where it is not necessary to pass to elementary extensions, is also introduced. A characterisation of such spaces in terms of the Douady spaces of their cartesian powers, as well as an application to relative algebraic reductions, is given.

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Introduction

The interaction between geometric stability theory and algebraic geometry over the complex numbers is by now a familiar aspect of model theory. This thesis can be viewed as extending the applicability of stability-theoretic techniques to more general manifestations of algebraic geometry. The two particular contexts we have in mind are the model theory of fields in *positive* characteristic and especially the interaction of model theory with complex *analytic* geometry.

While much of the model theory of fields is independent of characteristic, there do occur certain situations where the positive characteristic case has to be dealt with differently from the characteristic zero case. This is usually due to the fact that an algebraically closed field of positive characteristic admits nontrivial definable automorphisms, namely, the integer powers of the Frobenius map. In Chapter 1 we consider two such situations. The first context is that of difference fields; fields equipped with a distinguished automorphism. The main result here is the following (Theorem 1.14):

Theorem A. *If (E, σ) is a difference field admitting quantifier elimination in the language of difference fields, then σ is either the identity or an integer power of the Frobenius automorphism.*

Since quantifier elimination for difference fields implies superstability, this can be seen as a partial extension to positive characteristic of the result that there are no nontrivial superstable difference fields in characteristic zero (Hrushovski [24]). Essentially, we reduce to the "uniformly algebraic" case, where we show that either a simple difference polynomial of a very particular form vanishes on all of E, or no difference polynomial identities hold on E (see Theorem 1.7).

The second situation we consider is that of pure algebraically closed fields. We are interested in how the minimal field of definition for an algebraic set varies with parameters. In characteristic zero, an immediate application of elimination of imaginaries implies that the minimal field of definition for an algebraic set can be obtained definably from a given description of the set. Modifying a construction of Weil's, we are able to prove this in all characteristics (Theorem 1.15):

Theorem B. *Let $\mathcal{L}$ be the language of fields, K an algebraically closed field, and $k \subset K$ the prime subfield. Let $\phi(x,y)$ be an $\mathcal{L}$-formula and $B \subset K^n$ a $\emptyset$-definable set, such that for all $b \in B$, $\phi(x,b)$ defines an algebraic set $V_b \subset K^m$. Then there exists a $\emptyset$-definable map $f \colon B \to K^N$ (some $N \geq 0$), and an $\mathcal{L}$-formula $\psi(x,z)$; such that for all $b \in B$, $V_b = \psi(x, f(b))^K$ and $k(f(b))$ is the minimal field of definition for V_b.*

This can be viewed an elementary account of the Hilbert scheme; one obtains a definable universal parameter space for the Zariski closed subsets of affine space over a field in any characteristic.

The greater part of this thesis is concerned with the model theory of compact complex analytic spaces. The methods of geometric stability theory have traditionally been more applicable to algebraic rather than analytic geometry. It has been observed, however, that these model-theoretic techniques are also meaningful when applied in the compact complex analytic category (see [25] and [45]). We view a compact complex analytic space model-theoretically by considering the structure induced on its cartesian powers by the closed analytic sets. In fact, it is convenient to consider all (reduced) irreducible compact complex analytic spaces at once, as the sorts of a many-sorted structure $\mathcal{A}$, with names for the closed analytic subsets of the (finite) cartesian products of the sorts. The model theory of $\mathcal{A}$ has in some cases successfully contributed to the study of compact complex spaces (see [34] and [35], for example).

In the opening sections of Chapter 2 we give an introduction to complex analytic spaces and their model theory. This exposition is written primarily to introduce model-theorists to the subject, and to translate between certain notions in complex analytic geometry and their model-theoretic counterparts. We also take this opportunity to provide the details of an observation of Pillay [34] that $\mathcal{A}$ admits the elimination of imaginaries. Elementary extensions of $\mathcal{A}$ are discussed, with a view to describing how facts about definable sets in these extensions correspond to facts about uniform families of definable sets in the standard model. In particular, we prove that all nonstandard curves are algebraic, thereby obtaining a nonstandard version of the Riemann Existence Theorem. This was asked for by Pillay in [34], and can be stated as follows (Theorem 2.38):

Theorem C. *Let $\mathcal{A}'$ be an elementary extension of $\mathcal{A}$. Then every absolutely irreducible Zariski closed set of dimension 1 in $\mathcal{A}'$ is Moishezon.*

The proof involves analysing the relationship between projective bundles and internality to projective space.

One obstacle to applying advanced model theoretic techniques directly and freely inside the structure $\mathcal{A}$, is that this structure is not sufficiently saturated in the natural language of analytic sets. One is thus often forced to pass to an elementary extension in order to see the behaviour of definable sets. In doing so, the underlying complex analytic structure is lost. The main goal of Chapter 2 is to introduce the notion of a saturated compact complex space – a space where it is possible to work model-theoretically without considering elementary extensions. We prove a number of properties of such spaces. The main result is a characterisation of saturated compact complex analytic spaces in terms of the universal parameter spaces for the analytic sets of their cartesian powers. If X is a compact complex analytic space, then the Douady space of X, $\mathcal{D}(X)$, parametrises all complex subspaces of X, and is itself equipped with a natural complex analytic structure (see the discussion beginning on page 45). Consider those irreducible components of $\mathcal{D}(X)_{red}$, D_α, such that for some $d \in D_\alpha$ the corresponding subspace of X is reduced and pure dimensional. By the *restricted Douady space of* X we will mean the subspace of $\mathcal{D}(X)$ obtained by taking the union of all such D_α. We are then able to show (Theorem 2.50):

Theorem D. *Let X be a compact complex analytic space. Then X is saturated if and only if for all $n > 0$, every irreducible component of the restricted Douady space of X^n is compact.*

The proof of this amounts to giving a definable interpretation of the Douady space. As the Douady space of a projective variety is its Hilbert Scheme, the result on minimal fields of definition from Theorem B, can be viewed as an algebraic example of obtaining Douady spaces in a definable fashion.

As a consequence of Theorem D, Douady spaces can be used to obtain a canonical language for the model-theoretic study of a saturated compact complex analytic space. For example, we show that saturated compact complex analytic spaces admit quantifier elimination and elimination of imaginaries with respect to the canonical language. We then apply these results to show (Theorem 2.77):

Theorem E. *If $f\colon X \to S$ is a fibre space of saturated compact complex analytic spaces, then a relative Moishezon reduction of f exists.*

Relative Moishezon reductions are a bimeromomorphic version of relative algebraic reductions. These latter reductions were shown to exist for Kähler-type complex spaces by Campana [5] and Fujiki [18] (independently). Since Kähler spaces are saturated, Theorem E includes this result.

What makes the category of compact complex analytic spaces particularly appealing from the model-theoretic perspective is that many of the more advanced notions of geometric stability theory (such as orthogonality and internality) seem to have meaning in complex analytic geometry. For example, the methods used by Fujiki [17] on the classification problem for Kähler-type spaces have a definite model-theoretic flavour. One is led to hope that a better model-theoretic understanding of saturated compact complex spaces may lead to new and useful techniques in the study of the structure of such spaces.

0 Some Model-Theoretic Preliminaries

Much of the motivation behind the results presented in this thesis is of a stability-theoretic nature. As such, frequent references are made to more involved ideas from model theory, even though they are often not necessary for the logical progression of the arguments. The purpose of this chapter is primarily to describe this general model-theoretic context. While we hope that the material presented here will improve the readability of the chapters that follow, Chapters 1 and 2 are nevertheless intended to be largely independent of these preliminaries. Some familiarity with first-order logic and the basic notions from model theory will be assumed. As a general reference on model theory, we suggest Hodges [23].

Let $\mathcal{L}$ be a fixed first order language, and T a fixed complete $\mathcal{L}$-theory. Given an $\mathcal{L}$-structure, M, we will generally not distinguish between the symbols in $\mathcal{L}$ and their interpretations in M. Moreover, the symbol M is used to denote both the $\mathcal{L}$-structure and its underlying universe. So a *definable set in* M is a subset of some finite cartesian power of M, that is definable with parameters. Notice that our notion of definability allows parameters. By a tuple from M we will always mean a finite tuple of elements of M, unless explicitly stated otherwise. Given tuples a, b from M, we will write ab for their concatenation. On the other hand, if $A \subset M$ and b is a tuple, then Ab will denote the union of A with the coordinates of b. If x is a tuple of variables, $\phi(x)$ is an $\mathcal{L}$-formula, and c is a tuple from M, then we will use the notation $M \models \phi(c)$, to mean that c realises $\phi(x)$ in M. By $\phi(x)^M$, we will mean the definable set of all tuples from M that realise $\phi(x)$.

Suppose $A \subset M$. The model-theoretic *definable closure of* A, denoted by $dcl(A)$, is the set of all elements c from M that are definable over A. That is, there is an $\mathcal{L}$-formula $\phi(x, y)$, and a tuple a from A, such that c is the unique realisation of $\phi(x, a)$. Similiarly, the model theoretic *algebraic closure* of A, denoted by $acl(A)$, is the set of all elements in M realising a formula over A that has only finitely many solutions. If c is a tuple from M, then we will write $c \in dcl(A)$ (respectively $c \in acl(A)$) to mean that each coordinate of c is in $dcl(A)$ (respectively $acl(A)$).

We use the symbol $\subset$, to denote the substructure relation on $\mathcal{L}$-structures. Suppose M, N are $\mathcal{L}$-structures, A, B are subsets of M and N respectively, and $f\colon A \to B$ is a map. Then f is an *elementary map between M and N* if for every tuple a from A and every $\mathcal{L}$-formula $\phi(x)$, $M \models \phi(a)$ if and only if $N \models \phi(f(a))$. An *elementary embedding* of M in N, is an elementary map $f\colon M \to N$. An *isomorphism* between M and N is a surjective elementary embedding of M in N. If $M \subset N$ and the inclusion map is an elementary embedding of M in N, then we write $M \prec N$, and say that M is an *elementary substructure* of N or N is an *elementary extension* of M. If M is an $\mathcal{L}$-structure and c is a tuple from M, the *complete type of c over A*, denoted by $tp(c/A)$, is the collection of all formulae with parameters from A, $\phi(x, a)$, such that $M \models \phi(c, a)$. By a *complete type over A* we mean the complete type of a tuple from some elementary extension of M, over A.

While our set-up in this preliminary chapter is essentially 1-sorted, we will eventually be working in the many-sorted context (Chapter 2). Let us recall that a many-sorted language consists of sorts which come with a supply of variables that pertain to that sort, relation symbols which relate specified sorts, and function symbols that map between specified sorts. All quantifiers are restricted so that they only quantify over specified sorts. A many-sorted structure for such a language is then a family of domains corresponding to the sorts of the language, equipped with functions and relations that interpret the symbols of the language (respecting the sorts specified by those symbols).

Saturation

Definition 0.1. Let M be an $\mathcal{L}$-structure and λ an infinite cardinal.

(a) M is *λ-saturated* if whenever x is a tuple of variables, $A \subset M$ is a subset of cardinality strictly less than λ, and $\Phi(x)$ is a collection of formulae over A all of whose finite subcollections have a realisation in M, then $\Phi(x)$ has a realisation in M. We say that M is *saturated* if it is $|M|$-saturated.

(b) M is *strongly λ-homogeneous* if for every pair of sets $A, B \subset M$ of cardinality strictly less than λ, and elementary bijection $f\colon A \to B$, f extends to an automorphism of M. We say that M is *strongly homogeneous* if it is strongly $|M|$-homogeneous.

Remark 0.2. In the definition of λ-saturation, we could actually allow x to be an infinite tuple of variables of length $\leq \lambda$.

Example 0.3. (i) Let K be an infinite field, and $\mathcal{L}$ the language of K-vector spaces. If M is a vector space over K of dimension $\geq |K|$, then M is a saturated $\mathcal{L}$-structure.

(*ii*) Let $\mathcal{L}$ be the language of fields and M an algebraically closed field of infinite transcendence degree over the prime subfield. Then M is a saturated $\mathcal{L}$-structure.

Every saturated $\mathcal{L}$-structure is strongly homogeneous. Saturated models can serve as universal domains where the general behaviour of definable sets is witnessed. This is partially illustrated by the following fact.

Fact 0.4. (See Chapter 10 of [23]). *Let M be a λ-saturated and strongly λ-homogeneous model of T, and $A \subset M$ with $|A| < \lambda$.*

(i) *Every type over A is realised in M.*

(ii) *If a and b are tuples from M, then $tp(a/A) = tp(b/A)$ if and only if there is an A-automorphism of M taking a to b.*

(iii) *Every model of T of cardinality less than or equal to λ is elementarily embeddable in M.*

(iv) *Suppose F is a definable set in M. Then F is definable over A if and only if every A-automorphism of M fixes F setwise.*

Without assuming the Generalised Continuum Hypothesis, it is not the case that every $\mathcal{L}$-theory has a saturated model. On the other hand, for all regular cardinals $\lambda > |\mathcal{L}|$, every $\mathcal{L}$-theory does have a λ-saturated and strongly λ-homogeneous model (Theorem 10.7.1, [23]). In any given context, one is generally only interested in models and sets of parameters of cardinality strictly less than some fixed λ. Usually any $\lambda > |\mathcal{L}|$ will do. Hence when we refer to a "sufficiently saturated" model of T, we will mean λ-saturated and strongly λ-homogeneous for some fixed $\lambda > |\mathcal{L}|$. A "small" set will then refer to a set of cardinality strictly less than λ.

There is a weaker notion that is also sometimes useful. An $\mathcal{L}$-structure M is said to be λ-*compact* if for any tuple of variables x, and any collection of formulae $\Phi(x)$ (with parameters), where $|\Phi(x)| < \lambda$, if every finite subcollection of $\Phi(x)$ is realised in M then $\Phi(x)$ is realised in M. Note that every λ-saturated $\mathcal{L}$-structure is λ-compact.

Quantifier Elimination

T is said to admit *quantifier elimination* if every $\mathcal{L}$-formula is equivalent modulo T to a quantifier-free $\mathcal{L}$-formula. If M is an $\mathcal{L}$-structure, then we say that M admits quantifier elimination if $Th(M)$ does. Given an $\mathcal{L}$-structure M, a tuple c from M, and a set $A \subset M$, by the *quantifier-free type of c over A*, denoted by $\mathrm{qftp}(c/A)$, we will mean the collection of all quantifier-free formulae over A, $\phi(x, a)$, such that $M \models \phi(c, a)$. Notice that quantifier-free types do not depend upon the structures in which they are computed. That is, if $M \subset N$, then the quantifier-free type of c over A with respect to M is the same as that with respect to N. The following fact yields a nice criterion for deducing quantifier elimination in a sufficiently saturated model of T.

Fact 0.5. (See Section 8.4 of [23]). *Let M be a sufficiently saturated model of T. The following are equivalent:*

(i) *T admits quantifier elimination.*

(ii) *If a, b are tuples from M, and A is a small subset of M, then $qftp(a/A) = qftp(b/A)$ if and only if $tp(a/A) = tp(b/A)$.*

Example 0.6. The theory of algebraically closed fields of a fixed characteristic admits quantifier elimination. For K a field, any complete theory of K-vector spaces admits quantifier elimination.

Elimination of Imaginaries

Given a model M of T, one often wishes to work with quotients of definable sets in M by definable equivalence relations. This is particularly easy to do when the quotients can themselves be represented by definable sets in M. The following condition on T ensures this:

Definition 0.7. T is said to admit *elimination of imaginaries* if given a model $M \models T$, a $\emptyset$-definable set F in M, and a $\emptyset$-definable equivalence relation E on F, there exists a finite $\emptyset$-definable cover of F, $F = F_1 \cup \ldots \cup F_m$, and $\emptyset$-definable functions $f_i \colon F_i \to M^{n_i}$ (some n_i), such that for all $u, v \in F_i$, $f_i(u) = f_i(v)$ if and only if uEv.

If M is an $\mathcal{L}$-structure, then we say that M *admits elimination of imaginaries* if $Th(M)$ does. It is not difficult to see that elimination of imaginaries is a property of the theory T and hence can be checked in any model of T.

Example 0.8. The theory of algebraically closed fields of a fixed characteristic admits elimination of imaginaries.

Remark 0.9. If T does not eliminate imaginaries, one can still deal with quotient objects by passing to the many-sorted $\mathcal{L}^{eq}$-structure M^{eq}. Essentially M^{eq} is obtained from M by adjoining additional sorts of the form M^n/E, E is a $\emptyset$-definable equivalence relation on M^n. One also equips M^{eq} with functions from M^n to M^n/E that pick out the E-class of an element in M^n. Elimination of imaginaries removes the need for passing to these enhanced structures.

Suppose M is an $\mathcal{L}$-structure, F is a definable set in M, and a is a tuple from M. We say that *a is a code for F* if there exists an $\mathcal{L}$-formula $\phi(x, y)$, such that $F = \phi(x, a)^M$, but $F \neq \phi(x, b)^M$ for any $b \neq a$.

Proposition 0.10. *Suppose M is a sufficiently saturated $\mathcal{L}$-structure, F is a definable set in M, and a is a tuple from M. Then the following are equivalent:*

(i) *a is a code for F*

(ii) *for all automorphisms α of M, $\alpha(F) = F$ if and only if $\alpha(a) = a$.*

Proof. It is clear that (i) implies (ii). For the other direction, assume (ii) holds. By saturation, F is definable over a, say $F = \psi(x, a)^M$. Moreover, for any tuple b, if $tp(a) = tp(b)$ and $F = \psi(x, b)^M$, then $a = b$. By saturation again, this is witnessed by some $\mathcal{L}$-formula. That is, there is a $\chi(y) \in tp(a)$, such that if b realises $\chi(y)$, and $F = \psi(x, b)^M$, then $a = b$. Let $\phi(x, y) = \chi(y) \wedge \psi(x, y)$. Then $\phi(x, y)$ witnesses that a is a code for F. □

Example 0.11. Let $\mathcal{L}$ be the language of fields, T the theory of algebraically closed fields of characteristic 0, M the field of complex numbers, $F \subset M^n$ an algebraic set, and a a tuple of complex numbers. Then a is a code for F if and only if $\mathbb{Q}(a)$ is the minimal field of definition for F.

Proposition 0.12. *Suppose M is a sufficiently saturated model of T. Then the following are equivalent:*

(i) *T admits elimination of imaginaries.*

(ii) *For every $\emptyset$-definable set F in M, $\emptyset$-definable equivalence relation E on F, and point $a \in F$, there exists a $\emptyset$-definable set $U \subset F$ containing a, and a $\emptyset$-definable function $f\colon U \to M^n$ (some n), such that for all $u, v \in U$, $f(u) = f(v)$ if and only if uEv.*

(iii) *Every definable set in M has a code.*

Proof. First of all, the equivalence of (i) and (ii) is a routine application of saturation. The point is that in an ω-saturated structure, if a definable set is covered by a collection of $\emptyset$-definable sets, then it is covered by only finitely many of them. We show the equivalence of (ii) and (iii).

Assume (ii) holds. We wish to prove that every definable set in M has a code. We first show that given a $\emptyset$-definable equivalence relation E on M^k (for some k), and a point $a \in M^k$, the E-class of a, denoted a/E, has a code. Let U be a $\emptyset$-definable subset of M^k containing a and $f\colon U \to M^n$ a $\emptyset$-definable function, as provided by (ii). We claim that $f(a)$ is a code for a/E. Clearly, any automorphism of M that fixes $f(a)$, must fix $f^{-1}(f(a)) = a/E \cap U$, and hence must fix a/E. On the other hand, if $\alpha(a/E) = a/E$, where α is an automorphism of M, then $\alpha(a) \in a/E \cap U = f^{-1}(f(a))$. It folows that $\alpha(f(a)) = f(\alpha(a)) = f(a)$. By Proposition 0.10, $f(a)$ is a code for a/E. Now suppose that X is any definable set in M (with parameters).

Let $X = \phi(x,a)^M$, where $a \in M^k$ (for some k). Let E be the $\emptyset$-definable equivalence relation on M^k, given by uEv if and only if $\phi(x,u)^M = \phi(x,v)^M$. It is not hard to see that a code for a/E will be a code for X. We have shown that every definable set in M has a code.

Finally, we show that (iii) implies (ii). Let F be a $\emptyset$-definable set, E a $\emptyset$-definable equivalence relation on F, and $a \in F$. Let $c \in M^n$ be a code for a/E. Any automorphism of M that fixes a fixes a/E, and hence must fix c. By saturation, this implies that c is definable over a, that is $c \in dcl(a)$. Let f be a $\emptyset$-definable function taking a to c, and let $\phi(x,c)$ witness that c is a code for a/E. So $a/E = \phi(x, f(a))^M$ and $a/E \neq \phi(x,b)^M$ for any $b \neq f(a)$. This is a $\emptyset$-definable property of a. Hence there is a $\emptyset$-definable subset of F, U, containing a, such that for all $u \in U$, $u/E = \phi(x, f(u))^M$ and $u/E \neq \phi(x,b)^M$ for any $b \neq f(u)$. It follows, that for all $u, v \in U$, $f(u) = f(v)$ if and only if uEv. $\square$

Remark 0.13. The equivalence of (i) and (iii) shows that we could have allowed parameters in the definition of elimination of imaginaries. Moreover, if we make the further assumption that $dcl(\emptyset)$ contains at least two distinct elements in M, then elimination of imaginaries would also be equivalent to: *For every definable set F and definable equivalence relation E on F, there exists a definable function $f\colon F \to M^n$, such that for all $u, v \in F$, $f(u) = f(v)$ if and only if uEv.* One can view the definable set $f(F) \subset M^n$ as the "quotient" of F by E.

Fact 0.14. (Corollary 10.5.9, [23]) *Suppose T admits elimination of imaginaries. Let M be a sufficiently saturated model of T, $A \subset M$ a small subset, and F a definable set in M. Then F is definable over $acl(A)$ if and only if it has finitely many conjugates under A-automorphisms of M.*

Remark 0.15. If T does not admit elimination of imaginaries, then we would have to replace $acl(A)$ by $acl^{eq}(A)$, where $acl^{eq}(A)$ refers to the algebraic closure of A in M^{eq} (see Remark 0.9).

Ranks and Stability

In this final section we discuss a few aspects of stability theory. Rather than work in full generality, we will use Morley rank and total transcendentality to introduce the essential concepts. A brief discussion of stability and U-rank will follow. Let us repeat, as a kind of preemptive apology, that we will not rely very heavily on this material in the chapters that follow. For a comprehensive treatment of the subject, we suggest Pillay [32].

Let M be a model of T, and F a definable set in M. The *Cantor-Bendixson rank of* F *with respect to* M, denoted by $RCB_M(F)$, is either an ordinal number or ∞, and is defined inductively as follows:

• $RCB_M(F) \geq 0$ if F is nonempty;

• $RCB_M(F) \geq \alpha + 1$ if for each $n \in \omega$, there are pairwise disjoint definable subsets $F_1, \ldots, F_n \subset F$, where $RCB_M(F_i) \geq \alpha$ for each i; and,

• $RCB_M(F) \geq \delta$, for a limit ordinal δ, if $RCB_M(F) \geq \alpha$ for all $\alpha < \delta$.

If $RCB_M(F) \geq \alpha$ for all α, then we say that $RCB_M(F) = \infty$.

Suppose $F = \phi(x,a)^M$. Let N be an ω-saturated elementary extension of M. Then $RCB_N(\phi(x,a)^N)$ is equal to the supremum of the values $RCB_{N'}(\phi(x,a)^{N'})$ as N' varies through all elementary extensions of M (Lemma 5.6.3, [23]). We call this the *Morley rank of* $\phi(x,a)$, and denote it by $RM(\phi(x,a))$. This is the rank that we are actually interested in. Note that the Morley rank does not depend on a particular model of T, it is just the Cantor-Bendixson rank computed in *any* ω-saturated elementary extension of M.

We say that T is *totally transcendental* if $RM(x = x) < \infty$, where x is a singleton variable. Similarly, T is *of finite Morley rank* if $RM(x = x) < \omega$. If T is totally transcendental (respectively of finite Morley rank), and M is a model of T, then *every* definable set in M (i.e. every definable subset of a finite cartesian power of M) has ordinal valued (respectively finite valued) Morley rank. The following proposition says that at least in the finite rank context, we can compute Morley rank inside any ω_1-compact model. Note that if $\mathcal{L}$ is uncountable, then an ω_1-compact $\mathcal{L}$-structure need not be ω-saturated. We will come across just such a situation in Chapter 2, and the following will be of use to us then.

Proposition 0.16. *Let M be an ω_1-compact model of T. Suppose all definable sets in M have finite Cantor-Bendixson rank with respect to M. Then T is of finite Morley rank and $RCB_M = RM$ on all definable sets in M.*

Proof. Let N be an ω-saturated elementary extension of M. We first prove something about RCB_N that has nothing to do with M. Suppose that $F = \phi(x,a)^N$ is a definable set in N with $RCB_N(F) \geq n \in \omega$. We claim that there is a countable collection of $\mathcal{L}$-formulae $\Phi(y)$, true of a, and such that if b realises $\Phi(y)$ then $RCB_N(\phi(x,b)^N) \geq n$. We do this by induction on $n \in \omega$. For $n = 0$, $\Phi(y)$ is just the formula $\exists x \phi(x,y)$. Suppose $RCB_N(F) \geq n + 1$. By definition, for each $m > 0$, there are m disjoint definable subsets of F of Cantor-Bendixson rank $\geq n$. Suppose that for a fixed m, these definable subsets are given by the formulae $\phi_1(x,a_1), \ldots, \phi_m(x,a_m)$. Let

$\Phi_i(y_i)$ be the countable collection of formulae associated to $\phi_i(x, a_i)$, given by the induction hypothesis for $RCB_N \geq n$. For any finite conjunction of formulae from $\bigcup_{i=1}^m \Phi_i(y_i)$, $\psi(y_1, \ldots, y_m)$, let $\chi_\psi(y)$ be the formula given by: $N \models \chi_\psi(b)$ if and only if there exists $c_1, \ldots, c_m$ realising ψ, such that $\phi_1(x, c_1)^N, \ldots, \phi_m(x, c_m)^N$ are disjoint definable subsets of $\phi(x, b)^N$. Let $\Phi^m(y)$ be the (countable) collection of all such $\chi_\psi(y)$, as ψ varies among the finite conjunctions from $\bigcup_{i=1}^m \Phi_i(y_i)$. By ω-saturation of N, if b realises $\Phi^m(y)$, then $\phi(x, b)^N$ has m disjoint definable subsets of Cantor-Bendixson rank $\geq n$. Let $\Phi(y)$ be the union of all the $\Phi^m(y)$'s as m varies. Then $\Phi(y)$ is true of a, and if b realises $\Phi(y)$, then $RCB_N(\phi(x, b)^N) \geq n+1$, as desired.

We next show that if F is a definable set in N over parameters in M, then $RCB_N(F) = RCB_M(F)$. For this it is sufficient to show that for all ordinals α, if $RCB_N(F) \geq \alpha$ then $RCB_M(F) \geq \alpha$. Since RCB_M is always finite by assumption, it is sufficient to check this for all $\alpha = n \in \omega$. For $n = 0$ this just states that F has a reallisation in M, which follows since F is definable over parameters in M and $M \prec N$. Suppose $RCB_N(F) \geq n+1$. Then for any $m > 0$, there are m disjoint definable sets $F_1, \ldots, F_m \subset F$, with $RCB_N(F_i) \geq n$. By the ω_1-compactness of M, and our previous claim, we may assume that the F_i's are definable with parameters in M. By induction, $RCB_M(F_i) \geq n$, for each i. Since m was arbitrary, this shows that $RCB_M(F) \geq n+1$, as desired.

In particular, $RCB_N(N) = RCB_M(M) < \omega$. Since N is an ω-saturated model of T, this shows that T is of finite Morley rank. Moreover, since the Morley rank of any definable set in M can be computed by considering its Cantor-Bendixson rank in N, we have also shown that $RM = RCB_M$ on all definable sets in M. This completes the proof of the proposition. $\square$

One of the main concerns in the early development of the theory was that of "counting types". For example, we have the following useful characterisation of countable totally transcendental theories:

Fact 0.17. *Suppose $\mathcal{L}$ is countable. Then the following are equivalent:*

(i) *T is totally transcendental.*

(ii) *T is ω-stable. That is, over any countable set of parameters there are only countably many distinct complete types.*

Let us also mention that in a totally transcendental theory one does not need the Generalised Continuum Hypothesis to obtain saturated models (i.e. a λ-saturated model of cardinality λ).

Fact 0.18. *Suppose T is totally transcendental. Then for all $\lambda > |\mathcal{L}|$, T has a saturated model of cardinality λ.*

For now, assume that T is totally transcendental and let M be a fixed saturated model of T, with $|M| > |\mathcal{L}|$. We view M as a universal domain for the model theory of T; all parameter sets will be assumed to be small subsets of M (unless explicitly stated otherwise), and all definable sets will be assumed to be in M.

Suppose F is a definable set over A. If $RM(F) = \alpha$, then the *Morley degree of* F, denoted by $dM(F)$, is the greatest positive integer m such that there are m disjoint definable subsets of F of Morley rank α.

Morley rank can be extended to (complete and partial) types as follows: if $\Phi(x)$ is a collection of formulae (over parameters), then $RM(\Phi(x))$ is the infimum of the values $RM(\phi(x)^M)$, as ϕ varies through all finite conjunctions from $\Phi(x)$. Given a tuple a and a set A, the *Morley rank of* a *over* A, denoted by $RM(a/A)$, is just the Morley rank of $tp(a/A)$.

We use Morley rank to define "nonforking independence" in M.

Definition 0.19. (T totally transcendental). Given a tuple a and sets $A \subset B$, we say that *a is independent from B over A* and that *$tp(a/B)$ is a nonforking extension of $tp(a/A)$*, if $RM(a/A) = RM(a/B)$. If b is another tuple, then we say that *a is independent of b over A* if a is independent from Ab over A.

Example 0.20. The theory T is called *strongly minimal* if $RM(x = x) = 1$ and $dM(x = x) = 1$, where x is a singleton variable. This is equivalent to every definable subset of M being either finite or cofinite. It is not hard to see that in a strongly minimal theory, nonforking independence comes from the algebraic closure relation. That is, a is dependent on B over A if and only if $a \in acl(B) \setminus acl(A)$. The Morley rank of a over A is then the least $n \geq 0$ such that $a \in acl(Ab)$ for some n-subtuple $b \subset a$. The theory of algebraically closed fields in a fixed characeristic is strongly minimal. Nonforking independence coincides exactly with (field-theoretic) algebraic independence in this case.

Fact 0.21. (T totally transcendental). *Fix tuples a, b, and sets $A \subset B \subset C$.*

(i) Existence. *If $p(x)$ is a complete type over A, then there exists a nonforking extension of $p(x)$ to B.*

(ii) Symmetry. *a is independent of b over A if and only if b is independent of a over A.*

(iii) Transitivity. *a is independent from C over A if and only if a is independent from C over B and from B over A.*

(iv) Finite Character. *a is independent from B over A if and only if a is independent of every (finite) tuple from B over A.*

(v) Finite Base. *There exists a finite subset $A_0 \subset A$, such that a is independent from A over A_0.*

The above fact describes some of the properties of nonforking independence that make it particularly useful. This is a manifestation in the totally transcendental context, of the more general phenomenon of stability. We drop our assumption that T is totally transcendental, but continue to work in a sufficiently saturated universal domain for T, M.

Let $p(x)$ be a complete type over A and $\phi(x,y)$ an $\mathcal{L}$-formula. A *ϕ-definition of p* is a formula $\psi(y)$ over A, such that $M \models \psi(a)$ if and only if $\phi(x,a) \in p(x)$, for all tuples a from A. We say that p *is definable* if for every $\mathcal{L}$-formula, $\phi(x,y)$, there is a ϕ-definition of p. The theory T is called *stable* if every complete type is definable. Totally transendental theories are stable. Let us point out a uniformity that one gets for free, and that we will make use of in Chapter 2 (see III.1.24, [2]).

Fact 0.22 (Uniform Definability of Types). *Suppose T is stable. Let $A \subset M$ be any set (not necessarilly small), x an n-tuple of variables, and $\phi(x,y)$ an $\mathcal{L}$-formula. Then there exists an $\mathcal{L}$-formula $\psi(y,z)$, such that for all n-tuples c from M, there is a tuple a from A, such that $\psi(y,a)$ is a ϕ-definition of $tp(c/A)$ over A.*

There is a notion of nonforking independence for stable theories, that coincides with Definition 0.19 in the totally transcendental case. Assume T is stable. Suppose A is a set and $\phi(x,b)$ is a formula over a tuple b. We say that $\phi(x,b)$ *forks over* A if there is a sequence $\{b_i | i \in \omega\}$ of realisations of $tp(b/A)$, and $k \in \omega$, such that any subset of $\{\phi(x,b_i) | i \in \omega\}$ of size k is inconsistent. If $p(x)$ is a complete type over some parameter set $B \supset A$, then we say that $p(x)$ *forks over* A, or that $p(x)$ is a *forking extension* of its restriction to A, if some formula in $p(x)$ forks over A. Given a tuple c, c is said to be *independent from B over A* if $tp(c/B)$ does not fork over A. The idea, though this may not be immediately clear from the definition, is that c is not any more related to B then it is to A. In any case, nonforking independence for stable theories does behave well. In particular, it satisfies properties (i) to (iv) of Fact 0.21.

A complete type $p(x)$ over A is called *stationary* if for every $B \supset A$, p has a unique nonforking extension to B. In this case we use the symbol $p|B$ to such an extension. Every complete type over a set $A = acl^{eq}(A)$ is stationary. Here are two further notions from stability that will be referred to from time to time in future chapters.

Definition 0.23. (T stable)

(i) Suppose p and q are complete stationary types over A. Then p is *orthogonal* to q if whenever $B \supset A$, and a and b are realisations of $p|B$ and $q|B$ respectively, a is independent of b over B.

(ii) Suppose p is a complete stationary type over A and F is a definable set over A. Then p is *internal* to F if for some $B \supset A$, some realisation a of $p|B$, and some tuple b from F, $a \in dcl(Bb)$.

Nonforking independence in stable theories gives rise to another rank function on complete types. The U-rank of a complete type, $p(x)$, is either an ordinal or ∞, and is defined inductively as follows: $U(p) \geq 0$ for all complete types p; $U(p) \geq \alpha + 1$ if there is a forking extension q of p with $U(q) \geq \alpha$; and $U(p) \geq \delta$ (limit ordinal) if $U(p) \geq \alpha$ for all $\alpha < \delta$. We write $U(p) = \infty$ to mean that $U(p) \geq \alpha$ for all ordinals α. For example, $U(p) = 0$ if and only if it is an algebraic type (has finitely many realisations), and $U(p) = 1$ if and only if every forking extension of p is algebraic.

We say that T is *superstable* if for all complete types $p(x)$, $U(p) < \infty$. Similarly, T is said to be of *finite U-rank* if every complete type has U-rank strictly less than ω. In a stable theory, U-rank is always bounded by Morley rank (for complete types). Hence totally transcendental theories are superstable, and theories of finite Morley rank are of finite U-rank.

Nonforking independence in superstable theories is witnessed by U-rank in the same way that Morley rank witnesses nonforking independence for totally transcendental theories. That is, if q is an extension of p, then q is a nonforking extension of p if and only if $U(q) = U(p)$. Moreover, property (v) of Fact 0.21 holds for superstable theories. In fact, among stable theories, superstable theories can be characterised by the property that a complete type does not fork over some finite subset of its domain.

We conclude these preliminaries with the following saturation result for theories of finite U-rank pointed out by Anand Pillay.

Proposition 0.24. *Suppose $\mathcal{L}$ is countable and T is of finite U-rank. Suppose $\kappa > |\mathcal{L}|$ and N is a model of T that satisfies the following strengthening of ω_1-saturation: every non-algebraic complete type over countably many parameters from N has κ-many realisations in N. Then N is κ-saturated.*

Proof. We may assume that N is an elementary substructure of our universal domain M of T. It suffices to prove that every complete type over less than κ-many parameters from N has a realisation in N. We do this by induction on the U-rank of such types. This is clear for the U-rank 0 case, since every algebraic type is realised in N. Let $B \subset N$ with $|B| < \kappa$, and $q(x)$ a complete type over B with U-rank $n+1$. Passing to M^{eq}, we have that $acl^{eq}(B) \subset N^{eq}$, and since $\mathcal{L}$ is countable, the cardinality of $acl^{eq}(B)$ is also less than κ. So we may as well assume that $B = acl^{eq}(B)$.

There is a finite subset of B over which q does not fork. Taking the algebraic closure of such a set in M^{eq}, we get a countable subset $A \subset B$, such that if $p(x)$ denotes the restriction of $q(x)$ to A, then p is stationary and q does not fork over A. In particular, q is the unique nonforking extension of p to B. So finding a realisation of q in N reduces to finding a realisation of p in N that is independent from B over A.

As $U(p) < \omega$, p is not orthogonal to some stationary type over A of U-rank 1 (Lemma 2.5.1, [32]), say $r(y)$. So for some countable set containing A, $C \subset M^{eq}$, there are realisations of $p|C$ and $r|C$, say a and b respectively, that are dependent over C. By the ω_1-saturation of N, we may assume that $C \subset N^{eq}$ and that a and b are in N. Increasing A and B by countably many points (if necessary), we may assume that $C = A$. So a is a realisation of p in N, b is a realisation of r in N, and a forks with b over A.

Let $s(x, y) = tp(ab/A)$. Let $s'(x, y)$ be the nonforking extension of s to B, and r' the nonforking extension of r to B. Suppose we found a realisation $a'b'$ of s' in N. Then a' realises p and is independent from B over A (as $a'b'$ is independent from B over A). So a' is a realisation of q in N, and we are done. It remains to show that s' has a realisation in N.

We first find a realisation of r' in N. As r' is the unique nonforking extension of r to B, all we need is a realisation of r in N that is independent from B over A. But since $U(r) = 1$, any realisation of r in N that is not algebraic over B, would be independent from B over A. Since $B = acl^{eq}(B)$, all we need is a realisation of r in N, that is not contained in B. But $|B| < \kappa$, and by assumption r has κ-many realisations in N. Hence there must be some b' from N that realises r'.

As a and b were dependent over A, $s(x, b)$ is a forking extension of p to Ab. Since b and b' realise the same complete type over A (namely $r(y)$), $s(x, b')$ is a forking extension of p to Ab'. Finally, $s'(x, b')$ extends $s(x, b')$ from Ab' to Bb'. By transitivity $s'(x, b')$ is a forking extension of p to Bb'. It follows that $U(s'(x, b')) < U(p) = U(q) = n + 1$. By induction, $s'(x, b')$ has a realisation in N, say a'. That is, $a'b'$ is a realisation of s' in N. $\square$

1 Fields of Positive Characteristic

In this chapter we extend, to positive characteristic, two known results in the model theory of fields of characteristic zero. The main result of Section 1.1 is that the only difference fields admitting quantifier elimination are those where the automorphism is an integer power of the Frobenius map (Theorem 1.14). In Section 1.2 we work in the context of pure algebraically closed fields and show that the minimal field of definition for a Zariski closed set can be obtained uniformly and definably from a given description of the Zariski closed set.

1.1 Superstable Difference Fields

A *difference field* is a field E together with a distinguished automorphism $\sigma: E \to E$. The model theory of difference fields can be described as the study of subsets of cartesian powers of E that are definable using the field operations together with a new unary function symbol representing the automorphism (also denoted by σ). For example, the most basic definable sets are the solution sets to *difference polynomial* equations. Given an n-tuple of indeterminates $X = (X_1, \ldots, X_n)$, a *difference polynomial* (or *σ-polynomial*) *in X over E*, is just an ordinary polynomial, f, with coefficients from E in the variables $X_1, \ldots, X_n$, $\sigma(X_1), \ldots, \sigma(X_n)$, $\sigma^2(X_1), \ldots$ One can evaluate f at an n-tuple from E in the natural way suggested by the symbols used to denote the indeterminates. Given a finite number of difference polynomials, the set of tuples from E at which they vanish is called a *difference closed* set (or *σ-closed* set). Just as the quantifier-free definable sets in pure fields are the boolean combinations of algebraic sets, a quantifier-free definable set in a difference field is a boolean combination of difference closed sets. Of course, an arbitrary definable set is obtained from difference closed sets by the usual boolean operations plus coordinate projection maps.

The basic model theory of difference fields is developed in [7] by the now familiar technique of studying the model companion for the theory of difference fields, which in this case is the theory of algebraically closed fields with a generic automorphism ($ACFA$). As in the cases of algebraically closed fields (the model companion for the theory of pure fields) and differentially closed fields (the model companion for the theory of differential fields), the models of $ACFA$ serve as "universal domains" in which the behaviour of the definable sets in general (i.e. as the difference fields vary) is exhibited. The crucial difference is that $ACFA$ is not a stable theory. This is witnessed, for example, by the fact that the fixed field of σ in a model of $ACFA$ (denoted by $Fix(\sigma)$) is a pseudo-finite field. However, one can still ask whether there are *particular* theories of difference fields that do have strong stability-theoretic properties.

The guiding conjecture in this regard is that there are essentially no superstable difference fields. In order to make this conjecture more precise, let us note that there are "trivial" ways of constructing superstable difference fields. Let E be a pure algebraically closed field, and let σ be an automorphism of E that is definable in the pure field language. Then $(E, +, \times, \sigma)$, as it has no additional structure, is still superstable. Now, as superstable fields are algebraically closed, the only automorphisms that can arise in this way are the identity in characteristic 0, and integer powers of the Frobenius in positive characteristic. (Recall that the Frobenius automorphism takes an element x of a field to the element x^p, where p is the characteristic of the field.) We call such difference fields *trivial*, as in this case the automorphism adds no new structure to the field. So the correct formulation of the conjecture would be that all superstable difference fields are trivial.

It is worth stating explicitly that we are working in the language of difference fields, i.e. the language of fields with a new unary function symbol. Hence, when we refer to a difference field E as being superstable we mean that $(E, +, \times, \sigma)$ is superstable. There is no real loss of generality here, since if E were superstable in some expanded language then it would already be superstable as a difference field.

Let us say a word about superstability in the presence of a group structure. In Chapter 0 we introduced the notion of a superstable theory and the associated U-rank on complete types, but we did not mention some particular facets of this theory for groups. Suppose G is a sufficiently saturated superstable group (with possibly additional structure). Then for a given set of parameters, A, there are 1-types over A of maximal U-rank in G. These types are usually called *generic for G over A*, but we will resist this terminology since "generic" will have a specific meaning for us in Chapter 2. In any case, by the U-rank of G we will mean this maximum U-rank of a type in G over $\emptyset$. A key property is that if $g \in G$ is an element whose type over

A is of maximum U-rank, then any A-definable subgroup of G containing g will be of finite index in G. Moreover, each A-definable subgroup of G of finite index gives rise to a type of maximal U-rank. In the following fact, we collect together various further properties of expansions of *fields* that are superstable.

Fact 1.1. *Let T be the theory of an infinite field with (possibly) additional structure. Suppose that T is superstable and E is a sufficiently saturated model of T. Then we have the following:*

(i) *E is algebraically closed.* (Cherlin, Shelah [8]).

(ii) *E has a unique 1-type of maximal U-rank, over any set of parameters. In particular, E has no proper definable additive or multiplicative subgroups of finite index.* (Poizat [37]).

(iii) *The U-rank of E is $\omega^\alpha m$, where α is some ordinal and m is some positive integer.* (Berline and Lascar [3]).

(iv) *For any tuples a and b from E and $A \subset E$, we have the inequality $U(a/bA) + U(b/A) \leq U(ab/A) \leq U(a/bA) \oplus U(b/A)$ (where $\oplus$ is the Cantor sum).* (Lascar [28]).

The strongest theorem about superstable *difference fields* is the following result of Hrushovski. The conclusion of the following proposition is the only real use of superstability that we will make. Since the proof of this result gives some insight into how superstability plays out in the setting of difference fields, we include a proof here.

Proposition 1.2. (Hrushovski [24]). *Let T be the theory of an infinite superstable difference field. If (E, σ) is a model of T, then either σ is the identity automorphism or the fixed field of σ is finite.*

Proof. The following argument is based on Hrushovski's idea as presented in [24], but rephrased along the lines of certain analogous arguments about superstable differential fields appearing in a paper by Pillay and Sokolovič [36]. Let (E, σ) be a saturated model of T. We will assume that $\mathrm{char} E \neq 2$ (a slight modification works for the characteristic 2 case and can be found in [24]).

$Fix(\sigma)$ is a $\emptyset$-subset of E. Equipped with the induced structure, $Fix(\sigma)$ is then a superstable field. If it is infinite, then by Fact 1.1 (i), it is an algebraically closed field. Assume this is the case. Moreover, if σ is not the identity automorphism on E, then $Fix(\sigma)$ is a proper algebraically closed subfield of E, and so E is an infinite dimensional vector space over $Fix(\sigma)$. Assume this is also the case, and seek a contradiction.

Let the U-rank of E be $\omega^\alpha m$, as in Fact 1.1 (iii). Since the vector space dimension of E over $Fix(\sigma)$ is infinite, for all $n > 0$, the nth cartesian power of $Fix(\sigma)$ is definably isomorphic to a definable subset of E. The U-rank inequality given by Fact 1.1 (iv) implies that the U-rank of E is greater than $U(Fix(\sigma))n$, for every $n > 0$. That is, $U(Fix(\sigma)) < \omega^\alpha$.

Let $\delta\colon E^\times \to E^\times$ be the definable endomorphism given by $\delta(x) = \frac{x}{\sigma(x)}$. Let $a \in E$ be an element whose type over $\emptyset$ is of maximal U-rank. Note that the kernel of δ is $Fix(\sigma)$, and hence $\delta^{-1}(\delta(a))$ is a multiplicative translate of $Fix(\sigma)$. This implies that $U(a/\delta(a)) < \omega^\alpha$. On the other hand, as $U(a) = \omega^\alpha m$ and $\delta(a)$ is definable over a, we have:

$$\omega^\alpha m = U(a\delta(a)) \leq U(a/\delta(a)) \oplus U(\delta(a))$$

This is only possible if $U(\delta(a)) = \omega^\alpha m$. That is, $\delta(a)$ is also of maximal U-rank in E. So $\delta(E^\times)$ is a multiplicative subgroup of $E^\times$ of finite index. By Fact 1.1 (ii), δ must be surjective.

Hence, there is $e \in E$ such that $\delta(e) = -1$. That is $\sigma(e) = -e$. Since $\mathrm{char} E \neq 2$, $e \in Fix(\sigma^2) \setminus Fix(\sigma)$. This means that $Fix(\sigma^2)$ is a proper extension of $Fix(\sigma)$. But every element of $Fix(\sigma^2)$ is algebraic over $Fix(\sigma)$ (of degree ≤ 2), and so $Fix(\sigma^2)$ is a proper algebraic extension of $Fix(\sigma)$. This contradicts the fact that $Fix(\sigma)$ is algebraically closed. Hence $Fix(\sigma)$ is finite or σ is the identity automorphism. $\square$

Corollary 1.3. *All characteristic* 0 *superstable difference fields are trivial.*

For positive characteristic superstable difference field, all that we can conclude from Proposition 1.2 is that the fixed field of σ must be finite. Our goal is to try to extend Corollary 1.3 by considering difference fields in arbitrary characteristic. We will obtain some partial results in this direction, the main one being that all difference fields with quantifier elimination (in the language of difference fields) are trivial (Theorem 1.14 below). The basic idea is that difference fields divide naturally into two groups, those that are "algebraic" and those that are "transcendental". We first study the former case and give a particularly simple description of them, implying their triviality under the assumption of superstablity. We then show that quantifier elimination implies "algebraic".

Transformal Algebraicity

Suppose (E, σ) is a difference field, $k \subset E$ a difference subfield and $a \in E$. The infinite tuple $(\ldots, \sigma^{-1}(a), a, \sigma(a), \ldots)$ is denoted by $(a)_\sigma$. The difference subfield of E generated by k and a is just the field generated by k and $(a)_\sigma$, and is denoted by $k(a)_\sigma$. An important invariant of the quantifier-free type of a over k is the transcendence degree of $k(a)_\sigma$ over k. This is called the *transformal transcendence degree of* a *over* k and is denoted by $deg_\sigma(a/k)$. If $deg_\sigma(a/k)$ is infinite then a is said to be *transformally transcendental over* k, if it is finite then we say that a is *transformally algebraic over* k. Note that $deg_\sigma(a/k) = 0$ if and only if a is field-theoretically algebraic over k. Finally, by the *order* of a difference polynomial f in the (single) indeterminate X, we mean the greatest n such that $\sigma^n(X)$ actually appears in f. So an order 0 difference polynomial is just an ordinary polynomial.

Lemma 1.4. *Let* (E, σ) *be a difference field,* k *a difference subfield of* E*,* a *an element of* E*, and* $n > 0$*. Then the following are equivalent.*

(i) $deg_\sigma(a/k) = n$.

(ii) $\{a, \ldots, \sigma^{n-1}(a)\}$ *is a transcendence basis for* $k(a)_\sigma$ *over* k*, of size* n.

(iii) *For some integer* i*,* $\{\sigma^i(a), \ldots, \sigma^{i+n-1}(a)\}$ *is a transcendence basis for* $k(a)_\sigma$ *over* k*, of size* n.

(iv) *For all integers* i*,* $\{\sigma^i(a), \ldots, \sigma^{i+n-1}(a)\}$ *is a transcendence basis for* $k(a)_\sigma$ *over* k*, of size* n.

(v) n *is the least positive integer for which there is a difference polynomial of order* n *that vanishes at* a.

Proof. We first prove the equivalence of (i) through (iv) and then use this to establish the equivalence of all those statements with (v). Now $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ are immediate. For $(iii) \Rightarrow (iv)$, note that all powers of σ are field automorphisms of $k(a)_\sigma$ that restrict to automorphisms of k. Moreover, any consecutive n-tuple from $(a)_\sigma$ can be obtained from any other consecutive n-tuple by applying an appropriate power of σ. Hence, *any* consecutive n members of $(a)_\sigma$ will be a transcendence basis for $k(a)_\sigma$ over k, as long as *some* consecutive n members are.

We show $(i) \Rightarrow (ii)$. Suppose $deg_\sigma(a/k) = n$. We can find a transcendence basis for $k(a)_\sigma$ over k inside the field $k(a, \sigma(a), \ldots)$; just apply an appropriate power of σ to any given transcendence basis. So $k(a)_\sigma$ is algebraic over $k(a, \sigma(a), \ldots)$. It is thus sufficient to show that $\{a, \ldots, \sigma^{n-1}(a)\}$ is a transcendence basis for $k(a, \sigma(a), \ldots)$ over k. Choose $m > 0$ minimally such that $\{a, \ldots, \sigma^m(a)\}$ is algebraically dependent over k. That is, $\{a, \ldots, \sigma^{m-1}(a)\}$ are algebraically independent over k while $\sigma^m(a)$ is algebraic over $k(a, \ldots, \sigma^{m-1}(a))$. Applying the field automorphism σ, we get that $\sigma^{m+1}(a)$ is algebraic over $k(\sigma(a), \ldots, \sigma^m(a))$, and hence also over $k(a, \ldots, \sigma^{m-1}(a))$ (by the transitivity of algebraic dependence). We can repeat this argument to conclude that $\sigma^i(a)$ is algebraic over $k(a, \ldots, \sigma^{m-1}(a))$ for all $i \geq m$. That is, $k(a, \sigma(a), \ldots)$ is algebraic over $k(a, \ldots, \sigma^{m-1}(a))$. Since the transcendence degree of $k(a, \sigma(a), \ldots)$ over k is n, m must equal n, and $\{a, \ldots, \sigma^{n-1}(a)\}$ is a transcendence basis for $k(a, \sigma(a), \ldots)$ over k as desired.

We now show the equivalence of all these statements with (v). First of all, notice that (v) is equivalent to saying that n is the least positive integer such that $\{a, \ldots, \sigma^n(a)\}$ is algebraically dependent over k. Hence $(ii) \Rightarrow (v)$. Moreover, (v) implies that a is tranformally algebraic over k. So for some m, $deg_\sigma(a) = m$. But then $\{a, \ldots, \sigma^{m-1}(a)\}$ is a transcendence basis for $k(a)_\sigma$ over k, and so m is the least positive integer such that $\{a, \ldots, \sigma^m(a)\}$ is algebraically dependent over k. Hence, m equals n, and (i) holds. We have shown that $(v) \Rightarrow (i)$, which is sufficient. $\square$

Even in one variable, the solution sets to difference polynomial equations can be rather complicated. Not only can such equations have infinitely many solutions (unlike the case for ordinary polynomial equations), but they may in fact hold for *all* elements of a difference field. For example, on any trivial difference field of characteristic p, some equation of the form $\sigma(X) - X^{p^n} = 0$ is satisfied by every element. On the other hand, the fixed field of σ^2 in a model of $ACFA$ gives an example of a nontrivial difference field on which a nonzero difference polynomial vanishes identically (in this case, $\sigma^2(X) - X$). We will eventually show that in some sense these are the only kinds of examples there are of one-variable difference polynomial identities (see Theorem 1.7 below). Let us first introduce some convenient terminology.

Definition 1.5. A difference field (E, σ) is said to be *uniformly algebraic* if there is some nonzero difference polynomial over E that vanishes on E.

Lemma 1.6. *Suppose (E, σ) is an ω_1-saturated difference field. Then the following are equivalent:*

(i) *(E, σ) is uniformly algebraic.*

(ii) *There is some countable difference subfield $k \subset E$ over which all elements of E are transformally algebraic.*

(iii) *There is some countable difference subfield $k \subset E$ and a nonnegative integer n such that $deg_\sigma(a/k) \leq n$ for all $a \in E$.*

Moreover, if (iii) *holds for some k and some n, then there is a difference polynomial over k of order $\leq n$ that vanishes on all of E.*

Proof. For $(i) \Rightarrow (ii)$, suppose f witnesses that (E, σ) is uniformly algebraic.

Let k be the countable difference field generated by the coefficients of f. Since f vanishes on E, every element of E is transformally algebraic over k.

Now assume that (ii) holds. Consider the following collection of difference polynomial inequations

$$\Phi(z) = \{f(z) \neq 0 | f \text{ a nonzero difference polynomial over } k\}$$

Since every element of E is the zero of some difference polynomial, $\Phi(z)$ is not realised in E. By ω_1-saturation, some finite subcollection of $\Phi(z)$ is not realised in E. Taking the product of these finitely many difference polynomials, we get a single difference polynomial over k that vanishes on all of E. Letting n be the order of this difference polynomial, we get (iii).

A similiar application of saturation yields $(iii) \Rightarrow (i)$. This time we consider the collection of inequations

$$\Phi(z) = \{f(z) \neq 0 | f \text{ a nonzero difference polynomial of order } \leq n \text{ over } k\}$$

The content of (iii) is that $\Phi(z)$ is not realised in E. By ω_1-saturation, some finite subcollection of $\Phi(z)$ is not realised in E. Taking the product of these finitely many difference polynomials, we get a single difference polynomial of order $\leq n$ over k that vanishes on all of E. This yields both (i) and the "moreover" clause of the Lemma. □

The following theorem yields a kind of dichotomy; either no difference polynomial identities hold in a given difference field, or a particularly simple one holds. Note that for characteristic 0 fields, the Frobenius automorphism is just the identity map.

Theorem 1.7. *If (E, σ) is a uniformly algebraic difference field, then for some $n > 0$ and some integer m, $\sigma^n = Fr^m$.*

Proof. We may assume that E is infinite and that (E, σ) is an ω_1-saturated difference field. The algebraic closure of E, denoted by $\overline{E}$, is of algebraic transcendence degree at least ω_1 over the prime field. Hence it is ω_1-saturated in the language of pure fields. We will view $\overline{E}$ as a universal domain for algebraic geometry.

Choose $n > 0$ minimally such that there is a difference polynomial of order n witnessing the uniform algebraicity of E. Our goal is to show that σ^n must be some integer power of the Frobenius automorphism on E. The first step is to show that E "lives" on some algebraic hypersurface in $\overline{E}^{n+1}$. We will then show that this hypersurface is of a very special form.

Let $H = \{(x, \sigma(x), \ldots, \sigma^n(x)) : x \in E\}$. Then H can be viewed as a copy of E in $\overline{E}^{n+1}$. Let V be the Zariski closure of H in $\overline{E}^{n+1}$. Then V is a (possibly reducible) affine subvariety of $\overline{E}^{n+1}$ defined over some finitely generated subfield of $\overline{E}$. The generators of this subfield are algebraic over some finitely generated subfield of E, say F. Let $k \subset E$ be the countable difference subfield generated by F. So V is a variety defined over the algebraic closure of k. By our minimal choice of n and Lemma 1.6, there is some $a \in E$ with $deg_\sigma(a/k) \geq n$. In particular (this is Lemma 1.4), $\{a, \ldots, \sigma^{n-1}(a)\}$ is algebraically independent over k. As an element of $\overline{E}^{n+1}$ then, $(a, \ldots, \sigma^n(a))$ has transcendence degree at least n over the algebraic closure of k as well. Since $(a, \ldots, \sigma^n(a)) \in H$, and $H \subset V$, the dimension of V is at least n. On the other hand, if we let $f(X, \sigma(X), \ldots, \sigma^n(X))$ be a difference polynomial of order n that vanishes on all of E, then H is contained in the zero set of the ordinary (nonzero) polynomial $F = f(X_0, \ldots, X_n)$. So V is a *proper* subvariety of $\overline{E}^{n+1}$. The dimension of V must be n.

The additive group structure on E is immediately inherited by H, since σ preserves addition. H is thus a subgroup of the algebraic group $\overline{E}^{n+1}$ under coordinatewise addition. Hence the Zariski closure of H, V, is an *algebraic* subgroup of $(\overline{E}^{n+1}, +)$. Moreover, H inherits the multiplicative structure of E as well, making it a subring of $\overline{E}^{n+1}$ under coordinatewise multiplication. We wish to show that V is also a subring of $\overline{E}^{n+1}$, closed under coordinatewise multiplication. Fix $w \in H$ and set $M_w = \{v \in V | vw \in V\}$. Then M_w is a Zariski closed subset of V containing H, hence $M_w = V$. As w in H was arbitrary, we have that for all $w \in H$ and $v \in V$, $vw \in V$. Now fix $v \in V$. Then $N_v = \{w \in V | vw \in V\}$ is also a Zariski closed subset of V. Moreover, we have just seen that N_v contains H. It follows that $N_v = V$. As v was arbitrary in V, we have that for all $v, w \in V$, $vw \in V$. So V is a subring of $\overline{E}^{n+1}$ under coordinatewise addition and multiplication.

Let $\pi_1 : \overline{E}^{n+1} \to \overline{E}^n$ be the projection map onto the first n coordinates and $\pi_2 : \overline{E}^{n+1} \to \overline{E}$ the projection map onto the last coordinate. We claim that restricted to V, both π_1 and π_2 are surjective. To begin with, as we have already seen, there is an $a \in E$ such that $\{a, \ldots, \sigma^{n-1}(a)\}$ is algebraically independent over the algebraic closure of k. Since $(a, \ldots, \sigma^n(a))$ is in V, $(a, \ldots, \sigma^{n-1}(a))$ is in $\pi_1(V)$. Hence $\pi_1(V)$ is an additive subgroup of $\overline{E}^n$, field-definable over the algebraic closure of k, and containing an element of transcendence degree n over the algebraic closure of k. Since the pure field type of a (field-theoretically) algebraically independent n-tuple from $\overline{E}$ is unique, $\pi_1(V) = \overline{E}^n$. Similarly, for any $a \in E$ transcendental over k, $\sigma^n(a) = \pi_2(a, \ldots, \sigma^n(a))$ is also transcendental over k. Just as before, $\pi_2(V)$

must be all of $\overline{E}$. So $\pi_1|_V: V \to \overline{E}^n$ and $\pi_2|_V: V \to \overline{E}$ are both surjective.

We wish to show that $V \subset \overline{E}^{n+1}$ is the graph of a surjective map from $\overline{E}^n$ to $\overline{E}$. For this it is sufficient to show that $\pi_1|_V: V \to \overline{E}^n$ is an isomorphism. Since the dimension of V is n, the kernel of $\pi_1|_V: V \to \overline{E}^n$ must be finite. Let $V_0 = \{y \in \overline{E} \mid (0, y) \in V\}$ be this kernel viewed as an additive subgroup of $\overline{E}$. Fix $x \in \overline{E}$ and $y \in V_0$. As $\pi_1|_V: V \to \overline{E}$ is surjective, there is a $v \in \overline{E}^n$ such that $(v, x) \in V$. Hence $(0, xy)$, which is equal to $(v, x) \times (0, y)$, is also in V. That is, for all $y \in V_0$ and $x \in \overline{E}$, the product xy is in V_0. So V_0 is an ideal of the field $\overline{E}$. As V_0 is finite, it must be the trivial ideal. We have shown that the kernel of $\pi_1|_V$ is the identity. That is, $\pi_1|_V: V \to \overline{E}^n$ is an isomorphism.

What we have shown is that $V \subset \overline{E}^{n+1}$ is the graph of a surjective map $\phi: \overline{E}^n \to \overline{E}$; where $\phi(x_0, \ldots, x_{n-1}) = y$ if and only if $(x_0, \ldots, x_{n-1}, y) \in V$. Moreover, as V is a ring under coordinatewise addition and multiplication, it is not hard to see that ϕ is a ring homomorphism. The kernel of ϕ must be a maximal ideal of $\overline{E}^n$, since its image is a field. The only maximal ideals of $\overline{E}^n$ are products of $n-1$ of the factors with $\{0\}$ in one of the coordinates, say the ith coordinate. Hence ϕ induces an automorphism $\psi: \overline{E} \to \overline{E}$; where $\psi(x) = y$ if and only if there are $v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}$ in $\overline{E}$ such that $(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{n-1}, y) \in V$. Since V is algebraic, ψ is a field-definable automorphism of $\overline{E}$. But the only field-definable automorphisms of an algebraically closed field are the integer powers of the Frobenius automorphism. So there is an integer m such that $\psi = Fr^m$.

Let $a \in E$. Then $(a, \ldots, \sigma^n(a)) \in V$, and so by the above formula defining ψ, $\psi(\sigma^i(a)) = \sigma^n(a)$. Since $\psi = Fr^m$, we have that $Fr^m(\sigma^i(a)) = \sigma^n(a)$. Simplifying further we get that $Fr^m(a) = \sigma^{n-i}(a)$. Since $a \in E$ was arbitrary, the difference polynomial $Fr^m(X) - \sigma^{n-i}(X)$ vanishes on *all* of E. This would contradict the minimality of n, unless $i = 0$. Hence $Fr^m = \sigma^n$ on E, as desired. □

As we saw before, by considering the fixed field of σ^2 in a model of $ACFA$, uniformly algebraic difference fields need not be trivial. However, under the additional assumption of superstability this *is* the case. The following Corollary to Theorem 1.7 can be seen as a partial generalisation to arbitrary characteristic of the fact that all superstable difference fields in characteristic 0 are trivial (Corollary 1.3). Moreover, it will be used later to show that all difference fields with quantifier elimination are trivial.

Corollary 1.8. *Superstable uniformly algebraic difference fields are trivial.*

Proof. Fix a superstable uniformly algebraic difference field (E, σ). By Theorem 1.7, for some $n > 0$ and some integer m, $\sigma^n = Fr^m$ on E. We wish to show that σ is itself an integer power of the Frobenius automorphism.

By superstability E is an algebraically closed field. By Corollary 1.3 we may assume that E is of characteristic $p > 0$. Let $\mathbb{F}_p$ denote the prime field of E. Fix $i > 0$ and consider the restrictions $\sigma|_{\mathbb{F}_{p^i}}$ and $Fr|_{\mathbb{F}_{p^i}}$. Since the Galois group of $\mathbb{F}_{p^i}$ over $\mathbb{F}_p$ is cyclic of order i and generated by the Frobenius automorphism, $\sigma|^n_{\mathbb{F}_{p^i}} = Fr|^m_{\mathbb{F}_{p^i}}$ implies that n divides m modulo i. As i was arbitrary, we have that $m = nk$ for some integer k. Now, the absolute Galois group of $\mathbb{F}_p$ is torsion free, and so taking nth roots we obtain that $\sigma = Fr^k$ on the algebraic closure of $\mathbb{F}_p$. This in turn implies that the fixed field of $Fr^{-k}\sigma$ contains the algebraic closure of $\mathbb{F}_p$, and is thus infinite. Since $(E, Fr^{-k}\sigma)$ is still a superstable difference field, $Fr^{-k}\sigma$ must be the identity automorphism on all of E, by Proposition 1.2. That is, $\sigma = Fr^k$ on E. □

Given a difference field (E, σ) and a subset $A \subset E$, we let $acl_\sigma(A)$ denote the field-theoretic algebraic closure of the difference field generated by A (whereas $acl(A)$ is the model-theoretic algebraic closure of A). The following is also a kind of generalisation of Corollary 1.3 to arbitrary characteristic.

Corollary 1.9. *Let T be the theory of a superstable difference field. Suppose there is an ω_1-saturated model of T, (E, σ), where for all $a \in E$, $acl(a) = acl_\sigma(a)$. Then the models of T are trivial.*

Proof. Let (E, σ) be such a model of T. By Corollary 1.8 and Lemma 1.6, it is sufficient to prove that every element in E is transformally algebraic over the prime field $\mathbb{F}$. Suppose $a \in E$ is such that $deg_\sigma(a/\mathbb{F})$ is infinite, and seek a contradiction.

By Proposition 1.2, the fixed field of σ is finite. Hence the definable endomorphism $\delta\colon E^\times \to E^\times$ given by $\delta(x) = \frac{x}{\sigma(x)}$, has a finite kernel. It follows that a is model-theoretically algebraic over $b = \delta(a)$. By assumption, a is in $acl_\sigma(b)$, which is equal to the algebraic closure of the difference field $\mathbb{F}(b)_\sigma$. Hence for some n, $\{a, \sigma^{-n}(b), \ldots, b, \ldots, \sigma^n(b)\}$ is an algebraically dependent set over $\mathbb{F}$. That is, the transcendence degree of $\mathbb{F}(a, \sigma^{-n}(b), \ldots, b, \ldots, \sigma^n(b))$ is strictly less than $2n + 2$. But as $b = \frac{a}{\sigma(a)}$, we have that

$$\mathbb{F}(a, \sigma^{-n}(b), \ldots, b, \ldots, \sigma^n(b)) = \mathbb{F}(\sigma^{-n}(a), \ldots, a, \ldots, \sigma^{n+1}(a))$$

Since the transformal transcendence degree of a over $\mathbb{F}$ is infinite, the right hand side has transcendence degree $2n+2$ over $\mathbb{F}$. This contradiction proves the corollary. □

Quantifier Elimination

In order to examine the quantifier-free fragment of the theory of difference fields, we will consider complete quantifier-free types. Given a difference field (k, σ) and an n-tuple a from some difference field extension of (k, σ), recall that the quantifier-free type of a over k, denoted by $\mathrm{qftp}(a/k)$, is the collection of all quantifier-free formulae over k (in the language of difference fields) that are realised by a. Note that $\mathrm{qftp}(a/k)$ is determined by the collection of all difference polynomial equations and inequations, in n variables and with coefficients from k, that are realised by a.

Notice that $\mathrm{qftp}(a/k)$ is equivalent to the difference field isomorphism type of a over k. That is, if b is another n-tuple from some difference field extension of (k, σ), then $\mathrm{qftp}(a/k) = \mathrm{qftp}(b/k)$ if and only if there is an isomorphism of difference fields from $k(a)_\sigma$ to $k(b)_\sigma$ that fixes k pointwise and sends a to b. Now, the difference field isomorphism type of a over k is equivalent to the pure field isomorphism type of $(a)_\sigma$ over k; indeed, $\alpha: k(a)_\sigma \to k(b)_\sigma$ is a k-isomorphism of difference fields taking a to b if and only if it is a k-isomorphism of pure fields taking $\sigma^i(a)$ to $\sigma^i(b)$ for all integers i. Hence $\mathrm{qftp}(a/k)$ is determined by the quantifier-free type of $(a)_\sigma$ over k in the language of pure fields, denoted by $tp^-((a)_\sigma/k)$.

We will focus on complete quantifier-free 1-types. If a is a transformally transcendental element over k then $\mathrm{qftp}(a/k)$ is completely determined (independently of a); it is essentially the collection

$\{f(X) \neq 0 | f$ a nonzero difference polynomial over $k\}$.

We will therefore concentrate on the quantifier-free types of transformally algebraic elements. The following (rather easy) observations suggests that even though $(a)_\sigma$ is an infinite tuple, $tp^-((a)_\sigma/k)$ is somehow controlled by a finite amount of information.

Lemma 1.10. *Let (k, σ) be a difference field, (E, σ) a difference field extension, and $a \in E$ a transformally algebraic element over k. Then there exists a positive integer m such that for all $l \geq m$, $k(a, \ldots, \sigma^l(a))$ is a finite extension of $k(a, \ldots, \sigma^{l-1}(a))$ of constant degree (as $l \geq m$ varies). Moreover, if P_l denotes the minimal polynomial of $\sigma^l(a)$ over $k(a, \ldots, \sigma^{l-1}(a))$, then $P_{l+1} = \sigma(P_l)$ for all $l \geq m$.*

Proof. Let $n = deg_\sigma(a/k)$ and fix $l \geq n$. Let P_l denote the minimal polynomial of $\sigma^l(a)$ over $k(a, \ldots, \sigma^{l-1}(a))$. We wish to show that the degrees of these polynomials eventually become constant as l increases. Now $\sigma^{l+1}(a)$ is a zero of $\sigma(P_l)$ since σ is an automorphism of $k(a)_\sigma$. This implies that

P_{l+1} , the minimal polynomial of $\sigma^{l+1}(a)$ over $k(a,\ldots,\sigma^l(a))$, is of degree $\leq deg\sigma(P_l) = degP_l$. As $l \geq n$ was arbitrary, this decreasing chain of non-negative integers must eventually stabilise. That is there exists some $m > 0$, such that for all $l \geq m$, $degP_{l+1} = deg\sigma(P_l) = degP_l$. We have that $\sigma(P_l)$ is a monic polynomial over $k(a,\ldots,\sigma^l(a))$ that vanishes on $\sigma^{l+1}(a)$, and is of the same degree as the minimal polynomial of $\sigma^{l+1}(a)$ over $k(a,\ldots,\sigma^l(a))$. Hence, $\sigma(P_l) = P_{l+1}$, as desired. □

Lemma 1.11. *Let (k,σ) be a difference field, (E,σ) a difference field extension, and $a \in E$ a transformally algebraic element over k. Then qftp(a/k) is determined by the following data:*

(A) *The least positive integer for which the conclusion of Lemma 1.10 holds. That is, the least $m > 0$ such that for all $l \geq m$, $k(a,\ldots,\sigma^l(a))$ is a finite algebraic extension of $k(a,\ldots,\sigma^{l-1}(a))$, of constant degree.*

(B) *$tp^-((a,\ldots,\sigma^m(a))/k)$ (i.e. the quantifier-free type of $(a,\ldots,\sigma^m(a))$ over k in the language of pure fields), where m is given by* (A).

Proof. We first observe that qftp(a/k) is determined by $tp^-((a,\sigma(a),\ldots)/k)$. Suppose that b is another element of some difference field extension of $k(\sigma)$, and that there is a k-isomorphism of fields $\alpha\colon k(a,\sigma(a),\ldots) \to k(b,\sigma(b),\ldots)$ taking $\sigma^i(a)$ to $\sigma^i(b)$ for all $i \geq 0$. Now fix $n > 0$. Then $\beta_n\colon k(\sigma^{-n}(a),\ldots) \to k(\sigma^{-n}(b),\ldots)$ given by $\beta_n = \sigma^{-n}\alpha\sigma^n$, is a k-isomorphism of fields taking $\sigma^i(a)$ to $\sigma^i(b)$ for all $i \geq -n$. Moreover, for all $n_1 > n_2$, β_{n_1} extends β_{n_2}. Letting β be the union of the β_n's for all $n > 0$, we get that $\beta\colon k(a)_\sigma \to k(b)_\sigma$ is a k-isomorphism of fields taking $\sigma^i(a)$ to $\sigma^i(b)$ for all integers i. That is, $tp^-((a)_\sigma/k) = tp^-((b)_\sigma/k)$, and hence qftp$(a/k) =$ qftp(b/k).

We have reduced to showing that if a is transformally algebraic over k, then $tp^-((a,\sigma(a),\ldots)/k)$ is determined by (A) and (B). Let m be as in (A). Fix an element from a difference field extension of (k,σ), b, satisfying the same data from (A) and (B) as a. Let $\alpha\colon k(a,\ldots\sigma^m(a)) \to k(b,\ldots,\sigma^m(b))$ witness the fact that $(b,\ldots,\sigma^m(b))$ and $(a,\ldots,\sigma^m(a))$ have the same tp^- over k. For all $l \geq m$, let Q_l (respectively P_l) denote the minimal polynomial of $\sigma^l(b)$ over $k(b,\ldots,\sigma^{l-1}(b))$ (respectively $\sigma^l(a)$ over $k(a,\ldots,\sigma^{l-1}(a))$). We wish to show that $tp^-((a,\sigma(a),\ldots)/k) = tp^-((b,\sigma(b),\ldots)/k)$. The first step would be to extend α to an isomorphism from $k(a,\ldots,\sigma^{m+1}(a))$ to $k(b,\ldots,\sigma^{m+1}(b))$, that takes $\sigma^{m+1}(a)$ to $\sigma^{m+1}(b)$. For this it suffices to show that α takes the minimal polynomial of $\sigma^{m+1}(a)$ over $k(a,\ldots\sigma^m(a))$ to the minimal polynomial of $\sigma^{m+1}(b)$ over $k(b,\ldots,\sigma^m(b))$. That is, we need to show that $\alpha(P_{m+1}) = Q_{m+1}$. By assumption $\alpha(P_m) = Q_m$, $P_{m+1} = \sigma(P_m)$, and $Q_{m+1} = \sigma(Q_m)$. Now, as σ and α commute on $k(a, \ldots, \sigma^{m-1}(a))$,

$$\alpha(P_{m+1}) = \alpha\sigma(P_m) = \sigma\alpha(P_m) = \sigma(Q_m) = Q_{m+1}$$

as desired. Hence α extends to an isomorphism from $k(a, \ldots, \sigma^{m+1}(a))$ to $k(b, \ldots, \sigma^{m+1}(b))$, taking $\sigma^{m+1}(a)$ to $\sigma^{m+1}(b)$.

Continuing in this way and then taking the union of the partial isomorphisms constructed, we obtain a k-isomorphism of fields from $k(a, \sigma(a), \ldots)$ to $k(b, \sigma(b), \ldots)$, that takes $\sigma^i(a)$ to $\sigma^i(b)$ for all $i \geq 0$. In other words, $tp^-((a, \sigma(a), \ldots)/k) = tp^-((b, \sigma(b), \ldots)/k)$, as desired. □

The following Proposition was observed in [29] using a careful analysis of certain prime ideals in the difference polynomial ring over a difference field. The proof presented here avoids the difference algebra and follows a suggestion in [33].

Proposition 1.12. *Difference fields are quantifier-free ω-stable. That is, for any countable difference field (k, σ), there are only countably many quantifier-free types over k.*

Proof. It is sufficient to show this for quantifier-free 1-types. Indeed, if (E, σ) extends (k, σ), and $a_1, \ldots, a_n \in E$, then $\mathrm{qftp}((a_1, \ldots, a_n)/k)$ is determined by $\mathrm{qftp}(a_1/k)$, $\mathrm{qftp}(a_2/k(a_1)_\sigma)$, ..., $\mathrm{qftp}(a_n/k(a_1, \ldots, a_{n-1})_\sigma)$.

As we have already noted, there is a unique quantifier-free type for tranformally transcendental elements over k. Fix $m > 0$ and let Δ_m be the collection of quantifier free types of transformally algebraic elements over k for which m is the least positive integer satisfying the conclusion of Lemma 1.10. Then Lemma 1.11 reduces counting the size of Δ_m to counting the number of quantifier-free m-types over (the countable set) k, in the pure field language. But this is a countable set. Ranging over all $m > 0$ we get that there are only countably many quantifier-free 1-types over k. □

Corollary 1.13. *Suppose T is a theory of difference fields that admits quantifier elimination in the language of difference fields. Then T is totally transcendental (and hence superstable).*

Proof. Let (E, σ) be a sufficiently saturated model of T. Quantifier elimination implies that all complete types over small sets are determined by their quantifier-free restrictions. So if T admits quantifier elimination then by Proposition 1.12, for any countable $A \subset E$, there are only countably many complete types over A. Since we are in a countable language, this is equivalent to total transcendentality (Fact 0.17). □

Given the above corollary, the following theorem can be seen as resolving a special case of the conjecture that all superstable difference fields are trivial. It should be noted that Scanlon has given a nontrivial example of a difference *ring* that admits quantifier elimination, in [41].

Theorem 1.14. *All difference fields admitting quantifier elimination are trivial.*

Proof. It is sufficient to show that all difference fields admitting quantifier elimination are uniformly algebraic. Indeed, quantifier elimination implies superstability for difference fields and all superstable uniformly algebraic difference fields are trivial (Corollary 1.8). We will prove the contrapositive: suppose (E, σ) is a difference field that is not uniformly algebraic and let us show that the theory of (E, σ) does not admit quantifier elimination. We may assume that E is sufficiently saturated and algebraically closed (if it is not algebraically closed then it is already not superstable). Let $k \subset E$ be the algebraic closure of the prime field, it is thus a countable difference subfield of E. We will produce two distinct complete 2-types over k that have the same quantifier-free restrictions – implying that (E, σ) does not admit quantifier elimination.

Let q be a prime number different from char$E$. If one is willing to assume that char$E \neq 2$, then we can let $q = 2$ and some of the notation is simplified. Let ζ be a qth root of unity not equal to the identity. Since (E, σ) is not uniformly algebraic, by Lemma 1.6 there is some $a \in E$ transformally transcendental over k. Choose $b \in E$ such that $b^q = a$. Note that b is also transformally transcendental over k. Now, the multiplicative group of qth roots of unity is generated by ζ and is contained in k. Moreover, σ permutes the qth roots of unity and (as it fixes the identity) there is a positive integer n strictly less than q such that σ^n fixes ζ. As the action of σ on the qth roots of unity is determined by its action on ζ, σ^n fixes all the qth roots of unity. In the case that $q = 2$, ζ is -1 and n is 1.

Now let $p_1 = tp(\frac{b}{\sigma^n(b)}, a/k)$ and $p_2 = tp(\frac{b}{\sigma^n(\zeta^n b)}, a/k)$. Let $\phi(x, y)$ be the formula

$$\exists z(\frac{z}{\sigma^n(z)} = x \wedge z^q = y)$$

Clearly $E \models \phi(\frac{b}{\sigma^n(b)}, a)$ with $z = b$. On the other hand, if z is any qth root of a then $z = \zeta^i b$ for some $0 \leq i \leq q$. As σ^n fixes all qth roots of unity, we get that, $\frac{z}{\sigma^n(z)} = \frac{\zeta^i b}{\sigma^n(\zeta^i b)} = \frac{b}{\sigma^n(b)}$. Now, $\zeta^n \neq 1$, since n is strictly between 0 and q, and so $\frac{z}{\sigma^n(z)} \neq \frac{b}{\sigma^n(\zeta^n b)}$ for z any qth root of a. That is,$(\frac{b}{\sigma^n(\zeta^n b)}, a)$ does not satisfy the formula $\phi(x, y)$. So p_1 and p_2 are distinct 2-types over

k, witnessed by the fact that $p_1 \vdash \phi(x,y)$ and $p_2 \vdash \neg\phi(x,y)$.

Notice that the tuple $(\frac{b}{\sigma^n(b)}, a)$ generates the same difference field over k as $(\frac{b}{\sigma^n(\zeta^n b)}, a)$ (i.e. $k(\frac{b}{\sigma^n(b)}, a)_\sigma = k(\frac{b}{\sigma^n(\zeta^n b)}, a)_\sigma$). Let L denote this difference subfield of E. In order to show that p_1 and p_2 have the same quantifier-free restrictions, we will construct a k-automorphism of (L,σ) that takes $(\frac{b}{\sigma^n(b)}, a)$ to $(\frac{b}{\sigma^n(\zeta^n b)}, a)$. The automorphism will be constructed somewhat indirectly.

Since $a = b^q$, we have the following chain of difference field extensions: $k \subset k(a)_\sigma \subset L \subset k(b)_\sigma$. Now consider the infinite tuple from $k(b)_\sigma$ given by $(\sigma^i(\zeta b)|i$ an integer). It is clear that this tuple is algebraically independent over k and generates the field $k(b)_\sigma$ over k. Hence, the map τ, fixing k pointwise and taking $\sigma^i(b)$ to $\sigma^i(\zeta b)$ for all integers i, is an automorphism of the field $k(b)_\sigma$. We claim that τ fixes L pointwise. First of all, τ commutes with σ on $k(b)_\sigma$, since $\tau\sigma^i(b) = \sigma^i(\zeta b) = \sigma^i\tau(b)$ for all integers i. Hence it is sufficient to show that τ fixes both $\frac{b}{\sigma^n(b)}$ and a. But $\tau(\frac{b}{\sigma^n(b)}) = \frac{\zeta b}{\sigma^n(\zeta b)}$, which is equal to $\frac{b}{\sigma^n(b)}$ since σ^n fixes ζ. Also,

$$\tau(a) = \tau(b^q) = (\tau(b))^q = (\zeta b)^q = b^q = a$$

Hence τ restricted to L is the identity.

Now consider the composition $\alpha = \tau\sigma$. This is a k-automorphism of the field $k(b)_\sigma$. Moreover, for any fixed i, $\alpha^i(b) = \sigma^i(\zeta^i b)$. It is again clear that the infinite tuple $(\alpha^i(b)|i$ an integer) is algebraically independent over k and generates the field $k(b)_\sigma$ over k. Hence $k(b)_\sigma = k(b)_\alpha$ and α yields a new difference field structure on $k(b)_\sigma$, namely, $(k(b)_\alpha, \alpha)$. Since τ fixes L pointwise, α and σ agree on L. That is, (L,σ) is a common difference subfield of $(k(b)_\sigma, \sigma)$ and $(k(b)_\alpha, \alpha)$. Moreover, the map β, fixing k pointwise and taking $\sigma^i(b)$ to $\alpha^i(b)$ for all integers i, is an isomorphism of difference fields from $(k(b)_\sigma, \sigma)$ to $(k(b)_\alpha, \alpha)$ that takes b to b. Now,

$$\beta(a) = \beta(b^q) = (\beta(b))^q = b^q = a$$

and

$$\beta(\frac{b}{\sigma^n(b)}) = \frac{b}{\alpha^n(b)} = \frac{b}{\sigma^n(\zeta^n b)}$$

Hence $\beta|_L$ is a k-automorphism of (L,σ) taking $(\frac{b}{\sigma^n(b)}, a)$ to $(\frac{b}{\sigma^n(\zeta^n b)}, a)$. That is, $\mathrm{qftp}(\frac{b}{\sigma^n(b)}, a/k) = \mathrm{qftp}(\frac{b}{\sigma^n(\zeta^n b)}, a/k)$ as desired. $\square$

1.2 Minimal Fields of Definition

In this short section we are again concerned with how the model theory of fields in characteristic zero differs from that in positive characteristic – our context now being pure algebraically closed fields. Let $\mathcal{L}$ be the language of rings, K an algebraically closed field viewed as an $\mathcal{L}$-structure, and k the prime subfield of K. Recall that a field of definition for a Zariski closed set, $V \subset K^n$, is a subfield, $F \subset K$, such that the ideal of V is generated by polynomials with coefficients in F. In characteristic 0, this coincides with the model-theoretic notion of V being definable over F. However in positive characteristic, V is model-theoretically definable over F if and only if the perfect closure of F is a field of definition for V. Recall also that there is a smallest field of definition for any Zariski closed subset of a cartesian power of K, the *minimal field of definition*.

Our goal is to give an elementary account of the following phenomenon exhibited by the Hilbert scheme of projective space (see proposition 1.16 of [26], for example). Let $\mathbb{P}_n$ denote projective n-space, and $Hilb(\mathbb{P}_n)$ the Hilbert scheme of $\mathbb{P}_n$. Suppose $\mathcal{V} \subset \mathbb{P}_n(K) \times S$ is a flat family of subvarieties of $\mathbb{P}_n(K)$ parametrised by a variety S over k. Then for every member of $\mathcal{V}$, $Z \subset \mathbb{P}_n(K)$, if $[Z]$ denotes the Hilbert point of Z in $Hilb(\mathbb{P}_n)$, then the residue field of $Hilb(\mathbb{P}_n)$ at $[Z]$ is the minimal field of definition for Z. Intuitively, this says that as a family parametrised by $Hilb(\mathbb{P}_n(k))$, each fibre of $\mathcal{V}$ lies above a point whose "coordinates" generate its minimal field of definition. We will show:

Theorem 1.15. *Let $\phi(x, y)$ be an $\mathcal{L}$-formula, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$. Suppose $B \subset K^n$ is a $\emptyset$-definable set, such that for all $b \in B$, $\phi(x, b)$ defines a Zariski closed subset of K^m, denoted by V_b. Then there exists a $\emptyset$-definable map $f\colon B \to K^N$ (for some $N \geq 0$), and an $\mathcal{L}$-formula $\psi(x, z)$ (where $z = (z_1, \dots, z_N)$); such that for all $b \in B$, $V_b = \psi(x, f(b))^K$ and $k(f(b))$ is the minimal field of definition for V_b.*

The above theorem essentially yields a definable version of the Hilbert scheme. In Chapter 2 we will be considering compact complex analytic spaces, and a key step will be to give a definable interpretation of the Douady space. The Douady space is the analogue of the Hilbert scheme in the complex analytic category. This section can thus be viewed as an introduction to the theme of obtaining universal parameter spaces in a definable fashion.

We give a quick proof of Theorem 1.15 in the characteristic zero case. First of all, we may assume without loss of generality that K is ω_1-saturated.

Since definability in the model-theoretic sense corresponds to definability in the field-theoretic sense, the minimal field of definition for a Zariski closed set $V \subset K^m$ can be characterised as the unique subfield F of K satisfying the following property: an automorphism of K fixes F pointwise if and only if it fixes V setwise. Let E be the definable equivalence relation on B given by: aEb if and only if $V_a = V_b$. By elimination of imaginaries in algebraically closed fields, there exists a $\emptyset$-definable map $f\colon B \to K^N$ (for some $N \geq 0$), such that for all $a, b \in B$, $f(a) = f(b)$ if and only if aEb. It is clear that for all $b \in B$, an automorphism of K fixes V_b setwise if and only if it fixes $f(b)$. Hence $k(f(b))$ is the minimal field of definition for V_b, for all $b \in B$. Now fix $a \in B$. There is a formula $\psi_a(x, z)$ such that $\psi_a(x, f(a)))$ witnesses the fact that V_a is defined over $f(a)$. Let U_a be the set of all $b \in B$ such that $V_b = \psi_a(x, f(b))^K$. This is a $\emptyset$-definable subset of B containing a. By saturation, finitely many such U cover B. Hence by case distinction we can obtain a formula $\psi(x, z)$ such that for all $b \in B$, $V_b = \psi(x, f(b))^K$. This completes the proof of Theorem 1.15 in the case that $\mathrm{char}K = 0$.

In order to prove the general case we will make use of certain bounds in polynomial ring theory obtained by van den Dries and Schmidt [44]. These bounds will allow us to make Weil's original construction, of the minimal field of definition for an ideal, uniform in parameters. This will yield a proof that is independent of the characteristic.

We now establish some notation. We will generally use lower case greek letters γ, λ and ν for monomials in the variables x, upper case roman letters F, G, and H to denote polynomials in x over K, and lower case roman letters c, d, and k for tuples from K. For a Zariski closed set $V \subset K^m$, let $I(V)$ denote the ideal of V in the polynomial ring $K[x]$. For each $a \in B$, let $K[V_a]$ denote the coordinate ring of V_a, i.e. the quotient ring $K[x]/I(V_a)$.

Notice that since the Zariski closed sets V_a for $a \in B$ are given to us uniformly (by the formula $\phi(x, y)$), there is a bound D' such that for each $a \in B$, V_b can be expressed as the set of common zeros to finitely many polynomials of total degree $\leq D'$. As we have also fixed the number of indeterminates (i.e. m), the results in [44] mentioned above, yield a bound D, such that for each $a \in B$, the ideal of V_a is generated by polynomials of total degree $\leq D$.

Let $\Gamma = \{$monomials in $K[x]$ of (total) degree $\leq D\}$. For each $a \in B$, let $K[\Gamma_a]$ denote the finite dimensional K-vector subspace generated by the image of Γ in $K[V_a]$. By a *monomial basis* for $K[\Gamma_a]$ we mean a subset $\Lambda \subseteq \Gamma$ such that $(\lambda + I(V_a))_{\lambda \in \Lambda}$ is a basis for $K[\Gamma_a] \subseteq K[V_a]$.

As before, we may assume that K is ω_1-saturated. Using saturation, it will be sufficient to prove the theorem locally. That is, for each $a \in B$, we must find a $\emptyset$-definable set U contained in B and containing a, a $\emptyset$-definable map $f\colon U \to K^N$ (for some $N \geq 0$), and a formula $\psi(x, z)$, such that for all $b \in U$, $V_b = \psi(x, f(b))^K$ and $k(f(b))$ is the minimal field of definition for V_b. In fact we will also obtain $\psi(x, z)$ as a system of polynomial equations such that for each $b \in U$, the polynomials occurring in $\psi(x, f(b))$ will generate $I(V_b)$.

Fix $a \in B$. Let $\Lambda \subseteq \Gamma$ be a monomial basis for $K[\Gamma_a] \subseteq K[V_a]$. Consider the following subset of B:

$$U \quad = \quad \{b : \Lambda \text{ is a monomial basis for } K[\Gamma_b]\}$$

Clearly $a \in U$. We would like to show that U is a $\emptyset$-definable set. Now, U is the set of all b such that $(\lambda + I(V_b))_{\lambda \in \Lambda}$ is K-linearly independent in $K[V_b]$ and spans $K[\Gamma_b]$. That is,

$$U = \left\{ b \in B : \begin{array}{c} \text{given } \{k_\lambda \in K\}_{\lambda \in \Lambda}, \text{ if } \sum_{\lambda \in \Lambda} k_\lambda \lambda \\ \text{vanishes on } V_b, \text{ then } k_\lambda = 0 \text{ for all } \lambda \in \Lambda; \\ \text{and for each } g \in \Gamma, \text{there exist } \{k_\lambda \in \ K\}_{\lambda \in \Lambda}, \\ \text{such that } \sum_{\lambda \in \Lambda} k_\lambda \lambda - g \text{ vanishes on } V_b \end{array} \right\}$$

Since Γ is finite, U is definable. An inspection of the above description of U shows that it is $\emptyset$-definable.

We would like to construct the minimal fields of definition for V_b, $b \in U$, in such a way that our procedure is uniform as b varies in U. After some modification, Weil's original construction of the minimal field of definition for an ideal (as described by Lang in [27]) can be made uniform. We now describe this construction.

Fix an arbitrary $b \in U$. As Λ is a monomial basis for $K[\Gamma_b]$, for each $\gamma \in \Gamma$ there are unique $(c^b_{\gamma,\lambda} \in K : \lambda \in \Lambda)$ such that,

$$\gamma \quad = \quad \left(\sum_{\lambda \in \Lambda} c^b_{\gamma,\lambda} \lambda \right) \operatorname{mod} I(V_b) \tag{1.1}$$

Claim 1.16. $I(V_b)$ is generated by $\left\{ \gamma - \sum_{\lambda \in \Lambda} c^b_{\gamma,\lambda} \lambda \right\}_{\gamma \in \Gamma}$

Proof. By equation (1.1), $\gamma - \sum_{\lambda \in \Lambda} c^b_{\gamma,\lambda} \lambda$ is in $I(V_b)$ for all $\gamma \in \Gamma$. By our choice of D, we can choose a generating set of polynomials for $I(V_b)$, $F_1, \ldots, F_r$, where for each i, $degF_i \leq D$. Fix $i \leq r$. Then, since each monomial occurring in F_i comes from Γ, there are constants $\{a_\gamma\}_{\gamma\in\Gamma}$, such that,

$$\begin{aligned} F_i &= \sum_{\gamma\in\Gamma} a_\gamma \gamma \\ &= \sum_{\gamma\in\Gamma} a_\gamma \left(\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda\right) + \sum_{\gamma\in\Gamma} a_\gamma \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda \\ &= \sum_{\gamma\in\Gamma} a_\gamma \left(\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda\right) + \sum_{\lambda\in\Lambda} d_\lambda \lambda \end{aligned}$$

where $d_\lambda \in K$. Now, since F_i and $\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda$ are in $I(V_b)$, we get that $\sum_{\lambda\in\Lambda} d_\lambda\lambda$ is in $I(V_b)$. But $(\lambda + I(V_b))_{\lambda\in\Lambda}$ is K-linearly independent in $K[V_b]$, and hence $d_\lambda = 0$ for all $\lambda \in \Lambda$. Thus,

$$F_i = \sum_{\gamma\in\Gamma} a_\gamma \left(\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda\right)$$

Since this is true for each $i = 1, \ldots, r$, and $F_1, \ldots, F_r$ generate $I(V_b)$, $\left\{\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda\right\}_{\gamma\in\Gamma}$ generates $I(V_b)$. $\square$

Claim 1.17. $\{c^b_{\gamma,\lambda} : \lambda \in \Lambda, \gamma \in \Gamma\}$ generates the minimal field of definition for V_b.

Proof. By the previous claim it is clear that these constants generate a field of definition for V_b. Hence it is sufficient to show that if $L \subset K$ is a field of definition for V_b, then $\{c^b_{\gamma,\lambda} : \lambda \in \Lambda, \gamma \in \Gamma\} \subset L$. Assume $I(V_b)$ is generated by $G_1, \ldots, G_t$, where each $G_i \in L[x]$.

Fix $\gamma \in \Gamma$. As $\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda$ is in $I(V_b)$, there are polynomials $H_1, \ldots, H_t$ over K, such that,

$$\gamma - \sum_{\lambda\in\Lambda} c^b_{\gamma,\lambda}\lambda = H_1G_1 + \cdots + H_tG_t \tag{1.2}$$

Let H_i^* be obtained from H_i by replacing the coefficients with new indeterminates, say x^*. Then we can rewrite $H_1^*G_1 + \cdots + H_t^*G_t$ as:

$$H_1^*G_1 + \cdots + H_t^*G_t = \sum_{\nu} l_\nu(x^*)\nu \tag{1.3}$$

where ν are monomials in x (not necessarily in Γ), and $l_\nu(x^*)$ are linear forms in x^* with coefficients in L. Now, the system of linear equations:

$$\left\{\begin{array}{ll} l_\gamma(x^*) = 1 & \\ l_\nu(x^*) = 0 & \nu \notin \Lambda \end{array}\right\}$$

has a solution in K by (1.2). So there must be a solution in L, that is, there is a tuple d from L such that $l_\gamma(d) = 1$ and $l_\nu(d) = 0$ for all $\nu \notin \Lambda$. Letting H_i' be obtained from H_i^* by substituting d back in for x^*, we get from (1.3),

$$H_1'G_1 + \cdots + H_t'G_t \quad = \quad \gamma + \sum_{\lambda\in\Lambda} l_\lambda(d)\lambda$$

This implies that $\gamma + \sum_{\lambda\in\Lambda} l_\lambda(d)\lambda$ is in $I(V_b)$, and hence,

$$\gamma \quad = \quad \left(\sum_{\lambda\in\Lambda} -l_\lambda(d)\lambda\right) \ \mathrm{mod} I(V_b)$$

Now, Λ is a monomial basis for $K[\Gamma_b]$, and so comparing the above with equation (1.1), we have that $c^b_{\gamma,\lambda} = -l_\lambda(d)$ for all $\lambda \in \Lambda$. Since the l_λ have coefficients in L and d is from L, it follows that $c^b_{\gamma,\lambda} \in L$ for all $\lambda \in \Lambda$. As $\gamma \in \Gamma$ was arbitrary, $\{c^b_{\gamma,\lambda} : \lambda \in \Lambda, \gamma \in \Gamma\} \subset L$, as desired. $\square$

Let $N = |\Gamma||\Lambda|$ and define $f: U \to K^N$ by $f(b) = (c^b_{\gamma,\lambda} : \lambda \in \Lambda\ ,\ \gamma \in \Gamma)$. Then, by Claim 1.17, for all $b \in U$, the co-ordinates of $f(b)$ generate the minimal field of definition for V_b. Let

$$\psi(x, z) = \bigwedge_{\gamma\in\Gamma} (\gamma - \sum_{\lambda\in\Lambda} z_{\gamma,\lambda}\lambda = 0)$$

Claim 1.16 yields that for all $b \in U$, $V_b = \psi(x, f(b))^K$. In fact, the polynomials occurring in $\psi(x, f(b))$ are generators for $I(V_b)$. All that remains to be demonstrated is that f is a $\emptyset$-definable function. This follows from the following characterisation of f: for $b \in U$, $f(b) = c$ where $c = (c_{\gamma,\lambda} : \gamma \in \Gamma, \lambda \in \Lambda)$, if and only if for each $\gamma \in \Gamma$, $\gamma - \sum_{\lambda\in\Lambda} c_{\gamma,\lambda}\lambda$ vanishes on V_b. This completes the proof of Theorem 1.15 $\square$

Remark 1.18. Quantifier elimination will ensure that we can choose U so that f is the composition of a rational function over k with iterations of the inverse of Frobenius. The question arises as to when f can be chosen to be rational. This will not be the case in general. However, if the original formula $\phi(x, y)$ was such that $\phi(x, a) = \bigwedge_i (F_i = 0)$, where the F_i's generate $I(V_a)$, then this can be accomplished. Indeed, in that case, $k(a)$ is a field of definition for V_a and so $k(f(a)) \subset k(a)$. Hence $f(a) = g(a)$ where g is a rational function over k. We can shrink U to the $\emptyset$-definable subset U', given by $U' = \{b \in U : f(b) = g(b)\}$, on which f *is* rational. In general, we would first have to prepare the given formula $\phi(x, y)$ to satisfy the above condition. This is possible, also using the bounds on the generators of radical ideals mentioned earlier [44]. However, in doing so, we may have to introduce the inverse of the Frobenius map.

2 Compact Complex Analytic Spaces

One of the obstacles in the model theory of compact complex analytic spaces is that these spaces are not sufficiently saturated in their natural language of analytic sets. In this chapter we introduce the notion of a saturated compact complex analytic space – a space where it is possible to work model-theoretically without considering elementary extensions. Among the main results here are a characterisation of saturated compact complex spaces in terms of Douady spaces, an application of saturation to relative Moishezon reductions, and a nonstandard version of the Riemann Existence Theorem.

Section 2.1 is a brief introduction to the theory of complex analytic spaces. We take this opportunity to clarify certain notions that we found to be rather obscure in the literature. In Section 2.2 we discuss how compact complex analytic spaces can be studied from the model-theoretic point of view. An exposition of the first model-theoretic results in this area is given. We also provide the details of an argument sketched by Pillay in [34], showing that compact complex spaces admit elimination of imaginaries. We then consider elementary extensions in Section 2.3, and describe the correspondence between model-theoretic statements about such extensions and geometric notions in the standard model. We are then able to prove a version of the Riemann Existence Theorem for such extensions (Theorem 2.38), asked for by Pillay in [34]. The notion of a saturated compact complex analytic space, is introduced in Section 2.4. The main result is a characterisation of such spaces as those compact complex analytic spaces all of whose cartesian powers have Douady spaces with compact irreducible components (Theorem 2.50). We also show that a saturated compact complex analytic space is endowed with a canonical language (Corollary 2.59), and that they eliminate imaginaries with respect to that language (Proposition 2.60). Finally, in Section 2.5 we apply these techniques to prove that fibre spaces of saturated compact complex spaces admit relative Moishezon reductions (Theorem 2.77). In doing so we recover a bimeromorphic version of a result of Fujiki, stating that fibre spaces in the class $\mathcal{C}$ (examples of saturated spaces) admit relative algebraic reductions.

2.1 Preliminaries on Complex Spaces

The purpose of this section is to provide an introduction to those notions in complex analytic geometry that will be relevant to us, as well as to fix our notation. While some detailed explanations are provided, many of the fundamental notions will only be briefly introduced, and proofs will be omitted. The reader will be referred to various sources along the way, and as a general reference for the basic theory of complex spaces we suggest [21].

In the previous chapter, we took as our fundamental objects of study the zero sets of polynomials in the case of pure fields, and the zero sets of difference polynomial equations in the case of difference fields. In a similar, though more indirect manner, the model theory of compact complex spaces considers objects that are controlled by the zero sets of holomorphic functions. Let D be a domain in $\mathbb{C}^n$, and denote by $\mathcal{O}_D$ the sheaf of germs of holomorphic functions on D. Given finitely many holomorphic functions on D, $f_1, \ldots, f_n$, the set of common zeros of these functions, $V = V(f_1, \ldots, f_n)$, defines an *analytic set in* D. The sheaf structure on V, denoted by $\mathcal{O}_V$, is obtained as the quotient of $\mathcal{O}_D$ by the ideal sheaf of all holomorphic functions vanishing on V. Every section of the structure sheaf gives rise to a unique continuous $\mathbb{C}$-valued function on V – a *holomorphic function* on V. If W is another analytic set in some domain, then a *holomorphic map* from V to W is a morphism between the ringed spaces $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$. Similiarly, an isomorphism of these ringed spaces yields a *bihlomorphic map* (or an *isomorphism*) between V and W.

A 2nd countable Hausdorff topological space X is said to be a *complex analytic space* if there exists an open cover $\{X_\alpha\}$ of X, such that for each α there is a homeomorphism $\phi_\alpha : X_\alpha \to V_\alpha$, where V_α is an analytic set in a domain of $\mathbb{C}^{n_\alpha}$; and such that for all α and β for which $X_\alpha \cap X_\beta \neq \emptyset$, the induced transition function

$$\phi_\beta \phi_\alpha^{-1} : V_\alpha \cap \phi_\alpha(X_\beta) \to V_\beta \cap \phi_\beta(X_\alpha)$$

is a biholomorphism. In other words, a complex analytic space is locally modeled by the zero sets of finitely many holomorphic functions in domains of $\mathbb{C}^n$ (various n). We obtain the structure sheaf of holomorphic functions on X by pulling back the $\mathcal{O}_{V_\alpha}$'s and using the transition functions to glue them together. The structure sheaf is denoted by $\mathcal{O}_X$, and its section are the *holomorphic functions* on X. Again, a *holomorphic map* (respectively *biholomorphic map* or *isomorphism*) between analytic spaces $f : X \to Y$, is a morphism (respectively isomorphism) of the ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Let X be a complex analytic space. A subset A of X is an *analytic set in X* if for all $x \in X$ there is a neighbourhood U of x in X, and finitely many holomorphic functions on U, $f_1, \ldots, f_n$, such that $A \cap U = V(f_1, \ldots, f_n)$. Note that analytic sets are closed and that an analytic set in X is a complex analytic space in its own right. The Baire category theorem implies essential uncountability; an irreducible complex analytic space cannot be written as a countable union of proper analytic subsets.

Given a complex analytic space X and a point $x \in X$, the (complex) *dimension of X at x*, denoted $dim_x X$, is the least $d \geq 0$ such that there exists a finite to one holomorphic map $f: U \to D$, where U is a neighbourhood around x in X, and D is a domain in $\mathbb{C}^d$. The *tangential dimension of X at x*, denoted $dimt_x X$ is the least $d \geq 0$ for which there is an injective holomorphic map $f: U \to \mathbb{C}^d$, where U is a neighbourhood of x in X. We say that X is *smooth at x* if $dim_x X = dimt_x X$. In this case, there is a neighbourhood about x that is biholomorphic with a domain in $\mathbb{C}^d$, where $d = dimt_x X = dim_x X$. A complex space is said to be *smooth* if X is smooth at x for all $x \in X$. A complex analytic space X is *pure dimensional* if $dim_x X$ is constant for all $x \in X$; and we let the *dimension of X*, $dim X$, be this quantity. Every ireducible complex analytic space is pure dimensional.

It should be noted that our treatment of complex analytic spaces differs somewhat from the standard one in modern complex analytic geometry. The structure sheaf on the analytic sets $V_\alpha = V(f_1, \ldots, f_n)$ that are part of the local data defining a complex analytic space, is usually obtained by only quotienting out the ideal sheaf generated by $f_1, \ldots, f_n$ (as opposed to all holomorphic functions vanishing on V_α). This would give rise to possibly nonreduced analytic spaces; that is, spaces whose structure sheaf may have a nonzero nilradical ideal sheaf. If $(X, \mathcal{O}_X)$ is possibly nonreduced, then we denote by X_{red} the reduced complex analytic space $(X, \mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}$ is the nilradical ideal sheaf of $\mathcal{O}_X$. Notice that the underlying sets and the analytic subsets of X and X_{red} are the same – it is only the structure sheaf that differs. We refer the reader to [40] for a more thorough treatment of the possibly nonreduced context. Since our point of view will be essentially set-theoretic (as opposed to sheaf-theoretic), we do not generally need to distinguish between X and X_{red}. However, even in our context, nonreduced spaces will come up naturally when we consider families of analytic sets and Douady spaces. We will be careful to point out explicitly when such situations do arise, and restrict our attention to reduced complex analytic spaces in all other cases.

Complex Varieties

We now restrict our attention almost exclusively to the compact case, even though a lot of what we say holds in a more general setting. To this end we follow Ueno in [43], and introduce some convenient, though possibly misleading, terminology: a *complex variety* will mean a reduced irreducible compact complex analytic space. An irreducible analytic set in a complex variety is a complex variety in its own right and will thus be referred to as a *complex subvariety.* A smooth complex variety is a *complex manifold.* Hence for us, complex manifolds are always compact (unless explicitly stated otherwise). Note that we recover the description of a complex manifold as a compact Hausdorff topological space that is locally modeled after domains in $\mathbb{C}^d$, where d is dimension of the complex manifold.

Let X be a complex variety. The analytic sets equip X with a new coarser topology referred to as the *Zariski topology* on X. We will use the terms 'analytic set' and 'Zariski closed set' interchangeably. If P is a property of points in X, then we say that *P holds for general $x \in X$* if it holds in some nonempty Zariski open subset. More generally, following Fujiki in [17], we write that *P holds for 'general' $x \in X$* if it holds in some countable intersection of nonempty Zariski open subsets. It would not be too inaccurate to describe our model-theoretic point of view as being that of ignoring the underlying complex topology on X and considering only the structure induced by the Zariski topology.

What makes the compact case particularly accessible to model-theoretic analysis is that the Zariski topology is rather well behaved. For example we have the following theorem of Remmert's on proper maps between complex analytic spaces (note that in the compact case, all holomorphic maps are proper):

Theorem 2.1 (Proper Mapping Theorem). *Let $f: X \to Y$ be a proper holomorphic map between complex analytic spaces. Then $f(X)$ is an analytic set in Y.*

A proof of this theorem can be found in [21] (p. 162). It tells us that if we work inside the category of complex varieties then all holomorphic maps are closed with respect to the Zariski toplogy. That is, the image of a Zariski closed set is Zariski closed.

Compactness also gives us a nice dimension theory on the Zariski closed sets. If X is a complex variety then every Zariski closed set in X is a finite union of complex subvarieties, its *irreducible components.* We can define the *dimension* of a Zariski closed set in X to be the maximum of the dimensions of its irreducible components. Moreover, if $A \subset X$ is a complex subvariety and $B \subset A$ is a Zariski closed set, then $dimB < dimA$ if and only if $B \neq A$. This yields the descending chain condition on Zariski closed sets, showing that the Zariski topology on X is noetherian.

Suppose X is a compact complex analytic space, Y is a complex variety, and $f: X \to Y$ is a surjective holomorphic map. For all $y \in Y$, the set-theoretic fibre of f above y will be denoted by X_y. That is, X_y is the analytic set $f^{-1}(y) \subset X$. The following theorem of Remmert relates the dimension of the fibres with the dimension of the domain and range. A proof can be found in chapter 3 of [11].

Theorem 2.2 (Dimension Formula). *Suppose X and Y are complex varieties and $f: X \to Y$ is a surjective holomorphic map. Then there exists a nonempty Zariski open set $U \subset Y$, such that for all $y \in U$ and $x \in X_y$, $dim_x X_y = dimX - dimY$. In particular, each irreducible component of a general fibre of f is of dimension $dimX - dimY$. Moreover, this is the least dimension of any fibre of f.*

Remark 2.3. Recall that a holomorphism of complex analytic spaces is said to be *open* if the image of an open set is open. Remmert has shown that for $f: X \to Y$, a holomorphic map between irreducible complex analytic spaces, openness is equivalent to the above dimension formula being satisfied everywhere; for all $x \in X$, $dim_x X_{f(x)} = dimX - dimY$. Hence Theorem 2.2 says that a holomorphic map between complex varieties is open when restricted to some Zariski open subset of the base.

Let X and Y be complex varieties. Then $f: X \to Y$ is a holomorphic map if and only if the graph of f, $\Gamma(f) \subset Y \times X$, is a complex subvariety and the second coordinate projection map $\Gamma(f) \to X$ is biholomorphic. We will often identify a holomorphic map with its graph. If X_y is irreducible for general $y \in Y$, then we call $f: X \to Y$ a *fibre space.* A surjective holomorphic map of complex varieties $f: X \to Y$, is a *modification* if there are proper Zariski closed subsets $A \subset X$ and $B \subset Y$, such that f induces a biholomorphism between $X \setminus A$ and $Y \setminus B$. A *meromorphic map* $f: X \to Y$, is a complex subvariety of $Y \times X$, denoted $\Gamma(f)$, such that the second coordinate projection map $\Gamma(f) \to X$ is a modification. In other words, there is a nonempty Zariski open subset in X on which $\Gamma(f)$ defines a holomorphism. We will alternate between thinking of a meromorphic map as the multi-valued map defined by $\Gamma(f)$ and as the partial map given by $\Gamma(f)$ restricted to an appropriate Zariski open subset of X. A meromorphic map, $f: X \to Y$, is *surjective* if the coordinate projection $\Gamma(f) \to Y$ is surjective.

This is equivalent to saying that the image of a Zariski open set in X on which f is well defined is Zariski dense in Y. Finally, a *bimeromorphism* is a meromorphic map $f: X \to Y$ such that the first coordinate projection $\Gamma(f) \to Y$ is also a modification. Again, this means that f induces a biholomorphism between nonempty Zariski open subsets of X and Y. In these terms, a modification is a holomorphic bimeromorphism.

Families of Analytic Sets

Suppose that X and S are complex varieties and $G \subset S \times X$ is an analytic set that projects onto S. Then the fibres of $G \to S$ can be identified (by the second projection map) with analytic sets in X. Hence we often think of G as defining a family of analytic sets in X parametrised by S, namely $\{G_s | s \in S\}$. The prototypical example of this is when G is the graph of a holomorphic surjection $f: X \to S$. In this case, the family that G defines is just $\{X_s | s \in S\}$, and we refer to it as the family of analytic sets in X given by f.

More generally, suppose $f: X \to S$ and $g: T \to S$ are holomorphic maps of complex varieties, where g is surjective. Then $(T \times_S X)_{red}$, the set-theoretic fibred product of X and T over S, will be denoted by X_T:

$$X_T = (T \times_S X)_{red} = \{(t, x) | f(x) = g(t)\} \subset T \times X$$

The first coordinate projection map restricted to X_T will be denoted by $f_T: X_T \to T$. As before, $X_T \subset T \times X$ yields a family of analytic sets in X parametrised by T. Under this identification, $(X_T)_t = X_{g(t)}$ for all $t \in T$. So $f_T: X_T \to T$ is just the lifting by g of the original family given by $f: X \to S$ to a new set of parameters. The following commutative diagram is illustrative:

$$\begin{tikzcd} X \arrow[d, "f"] & X_T \subset T \times X \arrow[l] \arrow[d, "f_T"] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

This process is referred to as *base change*. Note that if $T = S$ and g is the identity map, then X_S is just the graph of f – and we recover the previous example. Also, if $T = \{s\}$ is a point in S and $g: \{s\} \hookrightarrow S$ is the identity embedding of this point, then X_s is just the fibre of f above s and we have agreement with our earlier notation.

One complication is that in general X_T need no longer be irreducible. However, if $f: X \to S$ is a fibre space, then the general fibre of $f_T: X_T \to T$ is irreducible, and there is a unique irreducible component of X_T containing the general fibre. That is, there is a unique irreducible component of X_T

that contains $f_T^{-1}g^{-1}(U)$ for some Zariski open $U \subset S$. We call this complex variety the *strict pull back* of X in X_T and denote it by $X_{(T)}$. Note that $X_{(T)} \subset T \times X$ is a fibre space over T that defines a family of analytic sets in X parametrised by T. It is no longer the case, however, that this family is just the lifting of f to T by g. Only for the *general* fibres is this still true. That is, for general $t \in T$, $(X_{(T)})_t$ is isomorphic to $X_{g(t)}$ under the second coordinate projection map.

Let $f\colon X \to S$ be a holomorphic map between complex analytic spaces, and suppose $\mathcal{F}$ is a coherent analytic sheaf on X. (We suggest section 6 of [40] as a reference to the theory of coherent analytic sheaves). Then $\mathcal{F}$ is said to be *f-flat* if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module. We say that f itself is *flat* if the structure sheaf $\mathcal{O}_X$ is f-flat. If $S = X$ and f is the identity map, then a coherent anlaytic sheaf on X is f-flat if and only if it is locally-free. For more details on the concept of flatness in complex analytic geometry see Section 2 of [31]. Essentially, flatness of a holomorphic surjection between complex varieties means that the family of analytic sets it defines varies in a smooth manner. For example, a flat map is open and hence the fibres are pure dimensional and of constant dimension (see Remark 2.3). The following theorem of Hironaka [22] often allows us to reduce to the flat case.

Theorem 2.4 (Hironaka's Flattening Theorem). *Let $f\colon X \to S$ be a holomorphic map of complex analytic spaces and $\mathcal{F}$ a coherent analytic sheaf on X. Then there exists a modification $g\colon T \to S$, such that if g' and f' are the natural projections:*

$$\begin{tikzcd} X & X_T \arrow[l, "g'"'] \arrow[d, "f'"] \\ S \arrow[from=u, "f"'] & T \arrow[l, "g"] \end{tikzcd}$$

then $(g')^*\mathcal{F}$ modulo its f'-torsion is f'-flat.

In the above theorem, $(g')^*\mathcal{F}$ refers to the pull-back of $\mathcal{F}$ to X_T.

Let us try to understand this theorem by restricting our attention to the context in which we will make most use of it. Suppose X and S are complex varieties, and $G \subset S \times X$ is a complex subvariety such that $G \to S$ is a fibre space. So G is a (not necessarily flat) family of analytic sets in X. Hironaka's Flattening Theorem says that after changing $G \to S$ bimeromorphically, we can obtain a *flat* family of analytic sets in X. Indeed, applying Thereom 2.4 to $G \to S$ we obtain a modification $g\colon T \to S$ such that the strict pull back $G_{(T)} \to T$ is now flat. Again a diagram is illustrative:

$$\begin{tikzcd} S \times X \supset G \arrow[d] & G_{(T)} \subset T \times_S (S \times X) = T \times X \arrow[l, "g \times id"'] \arrow[d] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

By the definition of the strict pull back, the general fibres of $G \to S$ and $G_{(T)} \to T$ agree. Moreover, as g is a modification, there are Zariski open sets $U \subset S$ and $V \subset T$ such that $g: V \to U$ is biholomorphic and lifts to a biholomorphic map $g \times id: G_{(T)_V} \to G_U$. That is, $G_{(T)} \to T$ is a bimeromorphic copy of $G \to S$, that now defines a flat family of analytic sets in X parametrised by T.

Douady Spaces and Universal Families

A notion from complex analytic geometry that will be of particular interest to us is that of the universal flat family of analytic subsets of a complex analytic space, constructed by Douady in [10]. The parameter spaces of such families need not be compact. In fact, the question of when these spaces are compact will be an essential aspect of our study of complex varieties.

Theorem 2.5 (Existence of Douady Spaces). *Let X be a complex analytic space. Then there exists a possibly nonreduced complex analytic space $\mathcal{D} = \mathcal{D}(X)$ and an analytic subspace $\mathcal{Z} = \mathcal{Z}(X) \subset \mathcal{D} \times X$ such that:*

(a) *The projection $\mathcal{Z} \to \mathcal{D}$ is a flat and proper surjection.*

(b) *If S is a complex analytic space and G is an analytic set in $S \times X$ that is flat and proper over S, then there exists a unique holomorphic map $g: S \to \mathcal{D}$ such that $G \simeq S \times_{\mathcal{D}} \mathcal{Z}$ canonically.*

$\mathcal{D}(X)$ is called the Douady Space of *X, $\mathcal{Z}(X)$ is called the* universal family of *X, and $g: S \to \mathcal{D}(X)$ as in* (b) *is called the* Douady map associated to *$G \subset S \times X$.*

Once again let us try to understand this theorem by restricting to the case we are most concerned with. Suppose X and S are complex varieties and $G \subset S \times X$ is a complex subvariety such that $G \to S$ is a flat fibre space. So G is a flat family of analytic sets in X parametrised by S. Then Theorem 2.5 says that there exists a unique holomorphic map, the Douady map, $g: S \to \mathcal{D}(X)$, such that $G \simeq S \times_{\mathcal{D}(X)} \mathcal{Z}(X)$ in a canonical fashion. Set-theoretically speaking, this means that g extends by identity to a holomorphic map from G to $\mathcal{Z}(X)$,

$$\begin{tikzcd}
S \times X \supset G \arrow[r, "g \times id"] \arrow[d] & \mathcal{Z}(X) \subset D(X) \times X \arrow[d] \\
S \arrow[r, "g"] & \mathcal{D}(X)
\end{tikzcd}$$

such that $G_s = \mathcal{Z}(X)_{g(s)}$ for all $s \in S$. Essentially, every flat family of analytic sets in X lives in $\mathcal{Z}(X) \to \mathcal{D}(X)$.

The assumption of flatness is not very restrictive because of Hironaka's Flattening Theorem. Suppose $G \subset S \times X$ is as above except that $G \to S$ is not necessarily flat. First of all, we already know that there is a Zariski open set U in S such that $G_U \to U$ is flat. Hence we can apply Theorem 2.5 to this restriction. This means that the general fibres of $G \to S$ do live uniformly in $\mathcal{Z}(X) \to \mathcal{D}(X)$ even though the whole family is not flat. However, Hironaka's Flattening Theorem tells us even more. There is a bimeromorphic copy of $G \to S$ that is flat, namely $G_{(T)} \to T$ for some modification $h: T \to S$, and where $G_{(T)}$ is the associated strict pull back of G. Applying Theorem 2.5 to $G_{(T)} \to T$, we then obtain a Douady map $T \to \mathcal{D}(X)$. If we compose the meromorphic map $h^{-1}: S \to T$ with this Douady map $T \to \mathcal{D}(X)$, we obtain a meromorphic $g: S \to \mathcal{D}(X)$. By the uniqueness part of Theorem 2.5, if U is chosen small enough, g restricts to the Douady map associated to $G_U \to U$. In this sense, g is the "meromorphic Douady map" associated to $G \to S$. Hence, bimeromorphically speaking, *all* families of analytic sets in X can be viewed as living in $\mathcal{Z}(X) \to \mathcal{D}(X)$ (not just the flat ones).

Let us point out a particular case. Suppose $A \subset X$ is any given analytic set in X. Take S to be a fixed point $\{s\}$, and G to be the trivial product $\{s\} \times A \subset \{s\} \times X$. G is a one-member family of analytic sets in X, and is trivially flat over $\{s\}$. Hence there is a Douady map $g: \{s\} \to \mathcal{D}(X)$ such that the sheaf-theoretic fibre of $\mathcal{Z}(X)$ over $g(s)$, $\{g(s)\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$, is A. Since A is reduced, so is $\{g(s)\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$. In other words, $\mathcal{Z}(X)_{g(s)} = A$. As A was arbitrary, we have that *every* analytic set in X occurs (uniquely) as a reduced fibre of $\mathcal{Z}(X) \to \mathcal{D}(X)$.

It is already becoming clear that when dealing with Douady spaces, nonreduced spaces naturally enter the picture. We have pointed out that the reduced fibres of the Douady space are in bijective correspondence to the analytic subsets of X. The nonreduced fibres, correspond to subspaces of X equipped with possibly nonreduced structure sheaves. The underlying set of such subspaces will coincide with analytic sets that have already appeared as reduced fibres elsewhere in $\mathcal{Z}(X) \to \mathcal{D}(X)$. Hence, if we consider all the set-theoretic fibres of $\mathcal{Z}(X) \to \mathcal{D}(X)$, then their will be redundancies. We get around this problem, following Fujiki in [12], by considering a certain subspace of the Douady space:

Definition 2.6. Suppose X is a complex variety. Let $D(X)$ be the subspace of $\mathcal{D}(X)$ obtained as the union of all the irreducible components, D_α, of $\mathcal{D}(X)_{red}$ such that for some $d \in D_\alpha$, the (sheaf-theoretic) fibre $\{d\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$ is reduced and pure dimensional. Let $Z(X)$ be the subspace of $\mathcal{Z}(X)$ obtained by restricting to $D(X)$. We call $D(X)$ the *restricted Douady space of* X, and $Z(X)$ the *restricted universal family of* X.

By passing from $\mathcal{Z}(X) \to \mathcal{D}(X)$ to $Z(X) \to D(X)$, we are focusing on only certain components of the Douady space. Our choice of which components (those that have at least one reduced and pure dimensional fibre) is justified by the following fact (see Lemma 1.4 of [12] and Lemma 3 of [15]):

Fact 2.7. *Let D_α be an irreducible component of $\mathcal{D}(X)_{red}$ and Z_α the restriction of $\mathcal{Z}(X)$ to D_α. Then the folowing are equivalent:*

(i) *D_α is a component of the restricted Douady space, $D(X)$,*

(ii) *Z_α is reduced and pure dimensional,*

(iii) *there is a dense Zariski open subset $U \subset D_\alpha$ such that for all $u \in U$, the (sheaf-theoretic) fibre of Z_α over u is reduced and pure dimensional.*

A consequence is that the collection of pure dimensional Zariski closed subsets of X are in bijective correspondence with a dense Zariski open subset of $D(X)$. Moreover, if S is a complex variety and $G \subset S \times X$ is a pure dimensional Zariski closed subset that is flat and surjective over S, then the universal Douady map associated to $G \to S$, maps S to $D(X)$ (this is because a general (sheaf-theoretic) fibre of $G \to S$ will be reduced and pure dimensional). The restricted Douady space and the corresponding restricted universal family is sufficiently rich for our purposes.

Observe that $D(X)$ is not necessarily a complex variety. Of course it would be too much to expect $D(X)$ to be irreducible. However, one can ask whether the irreducible components of $D(X)$ are compact (and hence complex varieties). This is not in general the case. Those complex varieties whose restricted Douady spaces have compact components (or rather all of whose cartesian powers have restricted Douady spaces with compact components) are of particular interest from the model-theoretic point of view, and will be considered in some detail later. A crucial property of Douady spaces that we will make much use of is stated in the following theorem.

Theorem 2.8. (Fujiki [14]) *The Douady space of a complex analytic space has only countably many irreducible components.*

There is also a relative version of the Douady space for complex analytic spaces over a given complex analytic space (due to Pourcin [39]). Essentially it provides uniform Douady spaces for a family of complex analytic spaces. We state it here for future reference, without discussion.

Theorem 2.9 (Relative Douady Spaces). *Let X and S be complex analytic spaces and $f: X \to S$ a holomorphic map. Then there exists a (possible nonreduced) complex analytic space over S, $\mathcal{D} = \mathcal{D}(X/S) \to S$, and an analytic sunspace $\mathcal{Z} = \mathcal{Z}(X/S) \subset \mathcal{D}(X/S) \times_S X$ such that:*

(a) *$\mathcal{Z}$ is flat and proper over D.*

(b) *If T is a complex analytic space over S and $G \subset T \times_S X$ is an analytic set that is flat and proper over T, then there exists a unique holomorphic S-map $g: T \to \mathcal{D}$ such that $G \simeq T \times_{\mathcal{D}} \mathcal{Z}$ canonically.*

$\mathcal{D}(X/S)$ is called the relative Douady space of X over S.

Projective Varieties and Morphisms

We use $\mathbb{P}_n(\mathbb{C})$, or just $\mathbb{P}_n$, to denote projective n-space viewed as a complex variety. Chow's Theorem says that every analytic set in projective space is algebraic. Hence in this case the analytic Zariski topology we defined for complex varieties in general coincides with the usual algebraic Zariski topology. A *projective variety* is a complex variety which embeds in $\mathbb{P}_n$, for some $n \geq 0$. The analytic structure on any projective variety is just the algebraic one. The Douady space of a projective variety has compact irreducible components. Indeed, if X is a projective variety then $\mathcal{D}(X)$ is exactly the Hilbert scheme of X and hence the components of $\mathcal{D}(X)$ are themselves projective varieties.

More generally, a *Moishezon variety* is a complex variety that is bimeromorphic to a projective variety. In classifying complex varieties one is often only interested in the bimeromorphism class of a complex variety. From this point of view then, Moishezon varieties are part of the "algebraic universe". It is a nontrivial fact that a Moishezon variety can also be characterised as a complex variety that is the holomorphic image of a projective variety. Moishezon varieties also possess the structure of a complex *algebraic space* in the sense of M. Artin.

Fix a complex variety S and consider the category of complex varieties over S. The analogue of projective space in this relative category is the notion of a projective linear space over S. Let $\mathcal{F}$ be a coherent analytic sheaf on S. We sketch a (local) construction of the *projective linear space over S associated to $\mathcal{F}$*, denoted by $\pi: \mathbb{P}(\mathcal{F}) \to S$. Let $U \subset S$ be a small open subset for which there exists a resolution of $\mathcal{F}_U$ as follows:

$$\mathcal{O}_U^p \xrightarrow{\alpha} \mathcal{O}_U^q \longrightarrow \mathcal{F}_U \longrightarrow 0$$

As an $\mathcal{O}_U$-linear homomorphism, α can be represented by a $q \times p$ matrix $M = (m_{ij})$, where each $m_{ij} \in \mathcal{O}_U$. Letting X be coordinates for U and $(Y_1 : \ldots : Y_q)$ homogeneous coordinates for $\mathbb{P}_{q-1}$, $\mathbb{P}(\mathcal{F})_U$ is the analytic subset of $U \times \mathbb{P}_{q-1}$ defined by the equations:

$$m_{1i}(X)Y_1 + \ldots + m_{qi}(X)Y_q = 0$$

for $i = 1, \ldots, p$. One checks that $\mathbb{P}(\mathcal{F})_U$ depends only on the coherent sheaf $\mathcal{F}_U$, and not on the particular resolution chosen above. We then patch the $\mathbb{P}(\mathcal{F})_U$ to obtain $\mathbb{P}(\mathcal{F})$. The structure of a fibre space over S is induced on $\mathbb{P}(\mathcal{F})$ by the coordinate projection maps $U \times \mathbb{P}_{q-1} \to U$.

For each $s \in S$, the fibre $\mathbb{P}(\mathcal{F})_s$ is isomorphic to $\mathbb{P}_r$, where $r + 1$ is the rank of $\mathcal{F}$ at s. In the special case when $\mathcal{F}$ is locally free, this rank is constant and $\pi: \mathbb{P}(\mathcal{F}) \to S$ is called a *projective bundle over S*. Projective bundles are locally trivial in the following strong sense. There is an open cover $\{U_i\}$ of S and local trivialisations

$$\begin{tikzcd}
\mathbb{P}(\mathcal{F})_{U_i} \arrow[rr, "h_i", "\simeq"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{P}_r \arrow[dl] \\
& U_i &
\end{tikzcd}$$

such that the induced transition functions

$$\begin{tikzcd}
U_i \cap U_j \times \mathbb{P}_r \arrow[rr, "h_i h_j^{-1}", "\simeq"'] \arrow[dr] & & U_i \cap U_j \times \mathbb{P}_r \arrow[dl] \\
& U_i \cap U_j &
\end{tikzcd}$$

are of the form $h_i h_j^{-1}(x,p) = (x, g_{ij}(x)(p))$, where $g_{ij}: U_i \cap U_j \to PGL(r, \mathbb{C})$ is holomorphic. In other words, a projective bundle over S of rank r is locally a product of the base with projective r-space, where the transition functions fix the base points while permuting the fibres as elements of the projective general linear group. For more details on projective linear spaces and projective bundles, see Section 3 of [31].

A *projective morphism*, $f: X \to S$, of complex varieties is then a holomorphic map that factors through an embedding into a projective linear space over S. That is, there is a coherent analytic sheaf $\mathcal{F}$ on S and an embedding of X into $\mathbb{P}(\mathcal{F})$ over S. We say that $f: X \to S$ is a *Moishezon morphism* if it is bimeromorphic over S to a projective morphism. That is, there is a projective morphism $g: Y \to S$ and a bimeromorphism $\sigma: X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

That is, for general $s \in S$, $\sigma_s: X_s \to Y_s$ is a bimeromorphism. For example, any holomorphic map between projective varieties (respectively Moishezon varieties) is a projective (respectively Moishezon) morphism.

Given a projective variety V, the category of coherent analytic sheaves on V is equivalent to the category of coherent algebraic sheaves on V (this is by GAGA [42]). Hence a projective linear space over V is itself a projective variety. This in turn implies that the preimage of a projective (respectively Moishezon) variety under a projective (respectively Moishezon) morphism is again projective (respectively, Moishezon).

These definitions seem somewhat weak. Naively, one would expect a projective morphism to factor through an embedding into a product of the base with a projective space. Ambiguous notation in [20] actually suggests (incorrectly) that this is the case. In fact, it was partially in order to clarify these issues that we have given a few details on projective linear spaces and projective bundles. The point is that while a projective morphism need not embed into a product of the base with projective space, it does embed bimeromorphically into an object that is at least locally of that form – namely, a projective bundle over the base. Indeed, suppose we have a fibre space of complex varieties $f: X \to S$ that is a projective morphism, and that this is witnessed by a coherent analytic sheaf $\mathcal{F}$ on S such that X is a subspace of $\mathbb{P}(\mathcal{F})$ over S. By Hironaka's Flattening Theorem, there is a modification $\sigma: T \to S$ such that the quotient of $\sigma^*(\mathcal{F})$ by its torsion elements is a flat, and hence locally-free, sheaf on T. Let $\mathcal{E}$ denote this locally-free analytic sheaf on T. Then the strict pull back $X_{(T)}$ is a subspace of the projective bundle $\mathbb{P}(\mathcal{E})$ over T. Since $T \to S$ was a modification, $X_{(T)} \to T$ is a bimeromorphic copy of $X \to S$. We have shown that upto bimeromorphic equivalence, every projective morphism does factor through an embedding into a projective bundle over the base. As Moishezon morphisms are bimeromorphic (over their bases) to projective morphisms, the same holds for Moishezon morphisms as well.

Kähler-type Varieties and Morphisms

There is a class of complex varieties, more general than the projective varieties, to which many of the methods of algebraic geometry are still applicable. This class plays a particularly important role from the model-theoretic point of view. A *Kähler manifold* is a (compact) complex manifold that admits a Hermitian metric whose associated differential 2-form of type $(1,1)$ is closed. A complex variety is said to be of *Kähler-type* if it is the holomorphic image of a Kähler manifold. Moishezon varieties and complex tori are all of Kähler-type. The class of Kähler-type varieties is often referred to as the class $\mathcal{C}$, and has been extensively studied by Fujiki among others (see [17]).

Our use of Kähler-type varieties will be rather axiomatic, in the sense that we will use their properties as a class without appealing to the definition. For example, the class $\mathcal{C}$ is closed under bimeromorphism, images under meromorphic maps, and taking products. A very important fact is that the components of the Douady space of a Kähler-type variety are compact, and even of Kähler-type themselves ([12] and [15]).

As in the projective case there is a relativisation of this notion to complex varieties over a given complex variety. This gives rise to the notion of a *Kähler morphism* that includes the Moishezon morphisms. The class $\mathcal{C}$ is also closed under taking preimages by Kähler morphisms. These morphisms are defined in [15] where it is proven that if $f: X \to S$ is a Kähler morphism of complex varieties, then the irreducible components of the relative Douady space of X over S are compact.

2.2 The Basic Model Theory of Complex Varieties

We will consider a complex variety as a structure, in the sense of model theory, by taking the Zariski closed subsets of all its cartesian powers as the basic relations. So if X is a complex variety, then a quantifier-free definable set from X is a constructible subset of X^n for some $n \geq 0$; that is, it is a finite boolean combination of Zariski closed sets. In order to work with more than one complex variety at a time, it is convenient to consider the many-sorted structure $\mathcal{A}$ where there is a sort for each complex variety, and where the relations are the Zariski closed subsets of the (finite) cartesian products of the sorts. That is, we consider all complex varieties at once equipped with the structure induced by the Zariski topology. We let

$\mathcal{L}$ denote the corresponding many-sorted language consisting of predicates for these relations. $\mathcal{A}$ is then naturally an $\mathcal{L}$-structure. A quantifier-free definable set in $\mathcal{A}$ is a constructible set in some cartesian product of its sorts. An arbitrary definable set would be obtained from the Zariski closed sets by the usual boolean operations plus the coordinate projection maps. Notice that all definable sets in $\mathcal{A}$ are definable over the empty set; all points in $\mathcal{A}$, being themselves Zariski closed sets, are named in $\mathcal{L}$.

One of the special sorts of $\mathcal{A}$ is $\mathbb{P}(\mathbb{C})$, the projective line. We will use the (possibly unusual) notation $\mathbb{P}(\mathbb{C})^n$, or just $\mathbb{P}^n$, to denote the nth cartesian power of the projective line (as opposed to $\mathbb{P}_n$ which denotes projective n-space). The complex field, $(\mathbb{C}, +, \times)$, is definable in $\mathbb{P}(\mathbb{C})$; where an identification of $\mathbb{C}$ with an affine open subset of $\mathbb{P}(\mathbb{C})$ is fixed once and for all. Moreover, every projective variety is naturally interpretable in $(\mathbb{C}, +, \times)$; i.e. an affine atlas and the associated transition functions are definable in the structure $(\mathbb{C}, +, \times)$. Hence all of algebraic geometry lives naturally in $\mathcal{A}$, on the sort $\mathbb{P}(\mathbb{C})$.

In this section we discuss some of the basic model-theoretic proerties of $\mathcal{A}$. For the sake of completeness we will provide proofs of these facts even when discussions are already present in the literature.

Proposition 2.10. (Łojasiewicz [38], Zilber [45]) *$\mathcal{A}$ admits quantifier elimination. That is, every definable set is a boolean combination of Zariski closed sets.*

Proof. This is a consequence of Remmert's Proper Mapping Theorem and the dimension formula. Let X and Y be complex varieties and $Q \subset X \times Y$ a quantifier-free definable subset. Then Q is of the form $\bigcup_{i=1}^{k} S_i \setminus T_i$ for some k, where each S_i is an irreducible Zariski closed set and each T_i is a proper Zariski closed subset of S_i. We need to show that $\pi(Q)$ is also constructible, where $\pi: X \times Y \to Y$ is the coordinate projection. We use induction on the dimension of the Zariski closure of Q, namely $\bigcup_{i=1}^{k} S_i$.

Without loss of generality, we may assume that $k = 1$ and so $Q = S \setminus T$. By Remmert's Proper Mapping Theorem, $F = \pi(S)$ and $G = \pi(T)$ are Zariski closed subsets of Y. Suppose $G \neq F$. Then, $S' = \pi^{-1}(G) \cap S$ is a proper Zariski closed subset of S, and hence of lower dimension (as S is irreducible). Moreover $\pi(Q) = (F \setminus G) \cup \pi(S' \setminus T)$. But by induction, $\pi(S' \setminus T)$ is constructible. Hence $\pi(Q)$ is constructible.

Now assume that $F = G$. Let d be the dimension of the general fibre of S. Since T is a proper Zariski closed subset of S, each irreducible component of T is of lower dimension than S. By the dimension formula, the general fibre of each irreducible component of T that projects onto F must be of dimension strictly less than d. Hence the dimension of the general fibres of T are of dimension $< d$. Let $F' \subset F$ be a proper Zariski closed subset such that for all $a \in F \setminus F'$, $dimT_a < d$ and $dimS_a = d$. Then clearly, T_a is a proper subset of S_a, and so $a \in \pi(Q)$. Moreover, it is clear that

$$\pi(Q) = (F \setminus F') \cup \pi((\pi^{-1}(F') \cap S) \setminus (\pi^{-1}(F') \cap T))$$

But $\pi^{-1}(F') \cap S$ is of lower dimension than S and so by induction again, $\pi((\pi^{-1}(F') \cap S) \setminus (\pi^{-1}(F') \cap T))$ is constructible. Hence $\pi(Q)$ is constructible, as desired. $\square$

Remark 2.11. Consider the sort $\mathbb{P}(\mathbb{C})$. Chow's Theorem implies that every analytic set in $\mathbb{P}^n$, for $n \geq 0$, is algebraic. Hence, by quantifier elimination, every definable subset of $\mathbb{P}^n$ is a finite boolean combination of algebraic sets. Moreover, as we noted earlier, $(\mathbb{C}, +, \times)$ is definable in $\mathbb{P}(\mathbb{C})$, and every algebraic set in $\mathbb{P}^n$ is naturally interpretable in $(\mathbb{C}, +, \times)$. So the structure induced by $\mathcal{A}$ on the sort $\mathbb{P}(\mathbb{C})$ is bi-interpretable with $(\mathbb{C}, +, \times)$. A consequence of this is that $\mathbb{P}(\mathbb{C})$ is strongly minimal and admits elimination of imaginaries. We will eventually show that the second property and a weakening of the first property extend to *all* of $\mathcal{A}$.

Suppose F is a definable subset of some complex variety X. By quantifier elimination, F has a unique expression of the following form:

$$F = (S_1 \setminus T_1) \cup (S_2 \setminus T_2) \cup \ldots \cup (S_k \setminus T_k)$$

where the S_i are irreducible Zariski closed subsets of X and T_i is a proper Zariski closed subset of S_i. The Zariski closure of F, denoted by $\overline{F}$, is then $S_1 \cup \ldots \cup S_k$. We say that F is *irreducible* if $k = 1$, which is equivalent to $\overline{F}$ being irreducible. For each i, $S_i \setminus T_i$ in the above expression is called an *irreducible component of* F. By the *dimension of* F we mean the dimension of $\overline{F}$, and denote it by $dimF$. Note that the dimension of a definable set is equal to the maximum of the dimensions of its irreducible components.

Corollary 2.12. *Every definable map in $\mathcal{A}$ is piecewise meromorphic.*

Proof. Let $f: A \to B$ be a definable map between definable sets in $\mathcal{A}$. We may assume that $X = \overline{A}$ and $Y = \overline{B}$ are complex varieties, since by quantifier elimination every definable set is a finite union of such. By induction on the dimension of X, it is sufficient to show that there exists a meromorphic map, $g: X \to Y$, that agrees with f on a nonempty Zariski open subset of X. Let $\Gamma \subset Y \times X$ be the graph of f. By quantifier elimination, $\Gamma = S_1 \setminus T_1 \cup \ldots \cup S_n \setminus T_n$, where S_i is a complex subvariety of $Y \times X$ and T_i is a proper Zariski closed subset of S_i. Since f is defined on A, and A is a nonempty Zariski dense constructible subset of X, Γ_x is a singleton for general $x \in X$. It follows that exactly one of the S_i's projects onto X, say S_1. Moreover, the general fibres of $S_1 \to X$ are singletons. This implies that $S_1 \to X$ is a modification, and hence the graph of a meromorphic map, $g\colon X \to Y$. Since the general fibres of $S_1 \to X$ and $\Gamma \to X$ agree, g agrees with f on a nonempty Zariski open subset of X, as desired. □

Proposition 2.13. (Zilber [45]) *$\mathcal{A}$ is ω_1-compact. That is, the intersection of any countable collection of definable sets is nonempty as long as all of its finite subcollections have nonempty intersection.*

Proof. This is equivalent to every countable descending chain of nonempty definable sets having a common point. Suppose that $A_0 \supset A_1 \supset \ldots$ is a descending chain of nonempty definable subsets of some complex variety X. By quantifier elimination, each A_i is of the form $\bigcup_{j=1}^{k_i} S_{ij} \setminus T_{ij}$ where the S_{ij}'s are irreducible Zariski closed sets and the the T_{ij}'s are proper Zariski closed subsets of these. By the noetherianity of the Zariski topology, the $\overline{A_i}$'s will stabilise. So without loss of generality, we may assume that $A_i = (S_1 \setminus T_{i1}) \cup \ldots \cup (S_n \setminus T_{in})$, where $S_1, \ldots, S_n$ are fixed irreducible Zariski closed sets and the T_{ij}'s are proper Zariski closed subsets of S_j, for $j = 1, \ldots, n$. If $\bigcap_{i \geq 0} A_i = \emptyset$, then $S_j = \bigcup_{i \geq 0} T_{ij}$, for each $j = 1, \ldots, n$. But as we have pointed out in 2.1, an irreducible complex analytic space cannot be written as a countable union of proper analytic subsets (essential uncountability). Hence the A_i's must have a common point, as desired. □

We wish to show that $Th(\mathcal{A})$ is of finite Morley rank (see the discussion on page 11, Chapter 0). In principle, Morley rank must be computed in a model that is at least ω-saturated. However, by Proposition 0.16, if the Cantor-Bendixson rank of all definable sets in an ω_1-compact model are finite, then the theory is of finite Morley rank and Morley rank is equal to Cantor-Bendixson rank in that model. Since $\mathcal{A}$ is ω_1-compact, we will not distinguish between Cantor-Bendixson rank and Morley rank in the following proposition. This will only be justified once the proposition is proved.

Proposition 2.14. (Zilber [45]) *Every definable set in $\mathcal{A}$ is of finite Morley rank. In fact, the Morley rank is bounded by the dimension.*

Proof. Suppose F is a definable set. We prove by induction on n that if $RM(F) \geq n$, then $dim F \geq n$. We may assume that F is irreducible.

For the case $n = 0$ this is clear. Suppose $RM(F) \geq n+1$. Then there exist infinitely many disjoint definable sets $F_i \subset F$ with $RM(F_i) \geq n$. By induction $dim F_i \geq n$. Now $\overline{F_i} \subset \overline{F}$. If $\overline{F_i} = \overline{F}$ then by quantifier elimination, F_i contains a Zariski open subset of $\overline{F}$. As $\overline{F}$ is irreducible there are no disjoint Zariski open sets and hence for some i, $\overline{F_i} \neq \overline{F}$. So $dim F = dim \overline{F} \geq dim \overline{F_i} + 1 \geq n+1$, as desired. □

Remark 2.15. It is worth pointing out that $\mathcal{A}$ satisfies the axioms, developed by Hrushovski and Zilber, of a Zariski-type structure (see [45]). Quantifier elimination, finite Morley rank, and ω_1-compactness can all be seen as consequences of this observation.

Quantifier elimination, finite Morley rank, and ω_1-compactness are essentially the reasons why there is a good model theory of complex varieties. We have a concrete description of the definable sets and the techniques of geometric stability theory will apply. One complication however, is that $\mathcal{A}$ is not ω-saturated; that is, it is not the case that every collection of definable sets over finitely many parameters has a nonempty intersection as long as all of its finite subcollections have nonempty intersections. For example, given an infinite sort X of $\mathcal{A}$, the collection of formulas,

$$\Phi(x) = \{x \in X\} \ \cup \ \{x \neq c | c \in X\}$$

uses no parameters since every singleton from X is named in $\mathcal{L}$. Every finite subcollection of $\Phi(x)$ can be realised in $\mathcal{A}$ since X is infinite. However $\Phi(x)$ itself has no realisation in $\mathcal{A}$. In order to recognise the behaviour of definable sets it is therefore often necessary to pass to an elementary extension of $\mathcal{A}$. The problem is that the sorts of such an extension are no longer complex varieties; the underlying complex structure is lost and only the Zariski structure remains. The main results of this chapter will be attempts to avoid this obstacle. For now, we give a brief description of what tuples from an elementary extension look like and how they relate to the standard model $\mathcal{A}$.

By total transcendentality, $\mathcal{A}$ has a saturated elementary extension of cardinality κ, for all $\kappa > |\mathcal{L}|$. Fix such a saturated elementary extension $\mathcal{A} \prec \mathcal{A}'$. The category of $\emptyset$-definable sets remains stable as we pass from $\mathcal{A}$ to $\mathcal{A}'$. That is, every $\emptyset$-definable set in $\mathcal{A}'$ is the interpretation in $\mathcal{A}'$ of a definable set in $\mathcal{A}$. We will denote by $F^{\mathcal{A}'}$ the interpretation in $\mathcal{A}'$ of the definable set F in $\mathcal{A}$. Similarly, if F is a $\emptyset$-definable set in $\mathcal{A}'$, then the corresponding definable set in $\mathcal{A}$ will be denoted by $F^{\mathcal{A}}$ (it is just the intersection of F with the standard model). The Zariski topology on $\mathcal{A}$ endows the sorts of $\mathcal{A}'$ with a "$\emptyset$-definable Zariski topology":

Definition 2.16. Given a sort $X^{\mathcal{A}'}$ of $\mathcal{A}'$, $F \subset (X^{\mathcal{A}'})^n$ is an *$\emptyset$-definable Zariski closed set* if $F^{\mathcal{A}} = G$ (or equivalently if $F = G^{\mathcal{A}'}$) for some Zariski closed set G in X^n.

Suppose c is an n-tuple of elements from a sort $X^{\mathcal{A}'}$. By quantifier elimination, $tp(c)$ is determined by which $\emptyset$-definable Zariski closed subsets of $(X^{\mathcal{A}'})^n$ c is in and which ones it is not in. The noetherianity of the Zariski topology implies that there is a unique complex subvariety $F \subset X^n$ such that $c \in F^{\mathcal{A}'}$ and for all proper Zariski closed subsets $G \subset F$, $c \notin G^{\mathcal{A}'}$. Conversely, by saturation of $\mathcal{A}'$, for every complex subvariety, $F \subset X^n$, there is a $c \in (X^{\mathcal{A}'})^n$, such that $c \in F^{\mathcal{A}'}$ but $c \notin G^{\mathcal{A}'}$ for all proper Zariski closed $G \subset F$.

Definition 2.17. Suppose X is a complex variety, $F \subset X^n$ a complex subvariety, and $c \in (X^{\mathcal{A}'})^n$. We say that c is a *generic point of F over $\emptyset$*, and F is the *$\emptyset$-locus of* c, if $c \in F^{\mathcal{A}'}$ but $c \notin G^{\mathcal{A}'}$ for all proper Zariski closed $G \subset F$. By the *dimension of c over $\emptyset$* we mean the dimension of its $\emptyset$-locus. If $F \subset X^n$ is an irreducible definable set, then by a *generic point of F* we mean a generic point of $\overline{F}$. If F is any definable set, then a *generic point of* F refers to a generic point of a maximal dimensional component of F.

This yields a correspondence between tuples from $\mathcal{A}'$ and irreducible Zariski closed sets in the standard model. Moreover, since any two generic points have the same type (namely; "I am in this irreducible Zariski closed set but in no proper Zariski closed subset of it"), we will often refer to *the* generic point. Note the relationship between this notion and that of a general or 'general' point of an irreducible Zariski closed set (introduced on page 41): if P is a definable property of points in F, then P holds for the generic point of F if and only if P holds for general $x \in F$. Similarly, if P is a property that is definable by countably many formulas, then P holds for the generic point of F if and only if P holds for 'general' $x \in F$.

The generic point of a complex variety is a bimeromorphic invariant in the following strong sense: if X, Y are complex varieties, then X and Y are bimeromorphic if and only if for some generic points a and b, of X and Y respectively, a and b are interdefinable over $\emptyset$. This will follow from the following lemma by symmetry:

Lemma 2.18. *Suppose X and Y are complex varieties, a is generic in X over $\emptyset$, and b is generic in Y over $\emptyset$. Then $b \in dcl(a)$ if and only if there is a meromorphic surjection $h: X \to Y$ such that $h^{\mathcal{A}'}(a) = b$.*

Proof. The right-to-left direction is immediate. For the other direction, assume $b \in dcl(a)$. There is a partial $\emptyset$-definable map $f: X^{\mathcal{A}'} \to Y^{\mathcal{A}'}$ that takes a to b. Since a is generic in X over $\emptyset$, $f^{\mathcal{A}}$ is defined on a nonempty Zariski open subset of X. Similarly, since b is generic in Y over $\emptyset$, the image of $f^{\mathcal{A}}$ contains a nonempty Zariski open subset of Y. By Corollary 2.12, there is a meromorphic surjection $h: X \to Y$ that agrees with $f^{\mathcal{A}}$ on a nonempty Zariski open subset of X. Clearly $h^{\mathcal{A}'}(a) = b$. $\square$

Genericity can also be used to examine more complicated sets; sets definable with parameters also coming from $\mathcal{A}'$. Before continuing along this line, however, we discuss another basic model-theoretic property of complex varieties in some detail.

Elimination of Imaginaries

In [34], Pillay gives a broad sketch of how Grauert's theorem on meromorphic equivalence relations can in principle be used to show that $\mathcal{A}$ admits elimination of imaginaries. We provide the details of this argument here. Recall that $\mathcal{A}$ admits elimination of imaginaries if for every definable set F, and definable equivalence relation E on F, there is a finite cover of F by definable sets $F = F_1 \cup \ldots \cup F_n$, and definable functions f_i on F_i, such that for all $a, b \in F_i$, $f_i(a) = f_i(b)$ if and only if aEb. By quantifier elimination, it is sufficient to show this when $F = X$ is a complex variety. By induction on the dimension of X, we reduce to the following statement: if X is a complex variety and E is a definable equivalence relation on X, then there exists a nonempty Zariski open subset $U \subset X$, a complex variety Q, and a definable map $f: U \to Q$, such that for all $a, b \in U$, $f(a) = f(b)$ if and only if aEb. Since every complex variety is bimeromorphic to a complex manifold (resolution of singularities [1]), it suffices to consider definable equivalence relations on complex manifolds.

Definition 2.19. Let X be a complex manifold. Supose $R \subset X \times X$ is a Zariski closed set that contains the diagonal of $X \times X$ and is symmetric. R is a *meromorphic equivalence relation* on X if there is a proper Zariski closed subset $P \subset X$ such that,

(i) no irreducible component of R is contained in $X \times P$;

(ii) $R^\circ \equiv R \cap (X \setminus P)^2$ is an equivalence relation on $X \setminus P$;

(iii) the dimensions of the R°-classes in $X \setminus P$ are constant; and,

(iv) the coordinate projections, $R^\circ \to X \setminus P$, are open maps.

We call P a *degeneracy set* for R.

The theorem of Grauert referred to above and given below, essentially states that the "quotient" of a complex manifold by a meromorphic equivalence relation exists.

Theorem 2.20. (Grauert [19]) *Suppose X is a complex manifold and R is a meromorphic equivalence relation on X with degeneracy set P. Then there exist complex varieties X' and Q, a modification $\sigma: X' \to X$, and a holomorphic map $q: X' \to Q$, such that*

(i) *for each $s \in Q$, $\sigma(q^{-1}(s)) \cap (X \setminus P)$ is either empty or an R°-class,*

(ii) *the association $s \longmapsto \sigma(q^{-1}(s)) \cap (X \setminus P)$ yields a bijection between $\{s \in Q | \sigma(q^{-1}(s)) \cap (X \setminus P) \neq \emptyset\}$ and the set of R°-classes.*

Grauert works in a rather general setting and makes the further assumption of *regularity* on R. In our compact context this comes for free (see proposition 10.1 in [9]). We will use the above theorem to obtain elimination of imaginaries. In order to do this, we first have to show that definable equivalence relations are essentially meromorphic.

Proposition 2.21. *Suppose X is a complex manifold and E is a definable equivalence relation on X. Then there exists a meromorphic equivalence relation on X that agrees with E outside of its degeneracy set.*

Proof. By quantifier elimination, we have an irredundant expression of E as $E = \bigcup_{i=0}^n S_i \setminus Q_i$, where for each $i \leq n$, Q_i is a proper Zariski closed subset of the complex subvariety $S_i \subset X \times X$. Since E contains the diagonal, some S_i must contain the diagonal and hence must project onto X in both coordinates. After re-indexing the $\{S_0, \ldots, S_n\}$ we may assume that there is an m, $0 \leq m \leq n$, such that S_i projects onto X in both coordinates if and only if $i \leq m$.

Let $R = \bigcup_{i=0}^m S_i$ and $S = \bigcup_{i=0}^n S_i$. It is clear that R is an analytic set in $X \times X$ that contains the diagonal. Since reflection across the diagonal is an analytic isomorphism of $X \times X$, the symmetry of E implies that S is symmetric. Moreover, since reflection preserves the property of projecting onto X in both coordinates, it will permute $\{S_0, \ldots, S_m\}$. So R is symmetric.

We will eventually construct a proper closed analytic subset of X, P, such that $(i) - (iv)$ of Definition 2.19 hold and R agrees with E on $X \setminus P$. First of all, notice that $R \subset S$ and the only components of S that are not contained in R are those that project (in one or both directions) onto a proper Zariski closed subset of X. Letting P' be the union of these projections, we get that P' is a proper analytic set in X and that R and S agree on $(X \setminus P')^2$.

Let $\pi: X \times X \to X$ be the second coordinate projection map, and fix $i \leq m$. Let $P_i \subset X$ be a proper Zariski closed subset of X such that $\pi: S_i \cap (X \times (X \setminus P_i)) \to X \setminus P_i$ and $\pi: Q_i \cap (X \times (X \setminus P_i)) \to X \setminus P_i$ are open (see Remark 2.3). Note that while $Q_i \cap (X \times (X \setminus P_i))$ may be empty (if Q_i does not project onto X), $S_i \cap (X \times (X \setminus P_i))$ is not empty (as S_i does project onto X).

Let $P = P' \cup P_0 \cup \ldots \cup P_m$, $U = X \setminus P$, and $R^\circ = R \cap (U \times U)$. As P is a proper analytic set in X, no component of R is contained in $X \times P$. Since $\pi: S_i \cap (X \times (X \setminus P_i)) \to X \setminus P_i$ is open for each $i \leq m$, it follows that π restricted to each irreducible component of R° is an open map onto U. Hence $R^\circ \to U$ is open and its fibres are of constant dimension. By symmetry, this is also true of the other (first) coordinate projection. All that remains to be shown then, is that R and E agree on $U \times U$; that is, $R^\circ = E \cap (U \times U)$.

Claim 2.22. For all $a \in U$, $E_a \cap U$ is Zariski dense in R°_a.

Proof. Since E is Zariski dense in S, and R and S agree on $U \times U$, $E \cap (U \times U)$ is Zariski dense in $R \cap (U \times U)$. But this does not imply directly that the same is true fibrewise (which is the content of the claim). Fix $i \leq m$. Since S_i is irreducible and Q_i is a proper closed subset set of S_i, for general $a \in X$, Q_{i_a} does not contain any irreducible component of S_{i_a}. But both $\pi: S_i \cap (X \times (X \setminus P_i)) \to X \setminus P_i$ and $\pi: Q_i \cap (X \times (X \setminus P_i)) \to X \setminus P_i$ are open by choice of P_i, and hence their fibres are pure dimensional and of constant dimension. This implies that for *all* $a \in X \setminus P_i$, Q_{i_a} cannot contain any irreducible component of S_{i_a}. In particular, for $a \in U$, $E_a = S_a \setminus Q_a$ is Zariski dense in S_a. Intersecting with U we get that $E_a \cap U$ is Zariski dense in $S_a \cap U$, for all $a \in U$. But since R and S agree on $U \times U$, this yields that $E_a \cap U$ is Zariski dense in $R_a \cap U = R^\circ_a$. $\square$

We now show that $R^\circ = E \cap (U \times U)$. Fix $(b, a) \in R^\circ$. So $a \in U$ and $b \in R^\circ_a = R_a \cap U$. We need to show that $b \in E_a$. Let c be a generic point of $E_a \cap U$ in a sufficiently saturated elementary extension $\mathcal{A} \prec \mathcal{A}'$. Since E is an equivalence relation, $E_c^{\mathcal{A}'} = E_a^{\mathcal{A}'}$. But then by Claim 2.22, transferred to $\mathcal{A}'$, we have that $R_c^{\mathcal{A}'} \cap U^{\mathcal{A}'} = R_a^{\mathcal{A}'} \cap U^{\mathcal{A}'}$. So $b \in R_c^{\mathcal{A}'} \cap U^{\mathcal{A}'}$ and by symmetry $c \in R_b^{\mathcal{A}'} \cap U^{\mathcal{A}'}$, this last being the interpretation of R°_b in $\mathcal{A}'$. We claim that c is generic in R°_b. In other words, the $\emptyset$-locus of c is a maximal dimensional component of $\overline{R^\circ_b}$. For this it is sufficient to show that the dimension of c over $\emptyset$ is equal to $dim R^\circ_b$. Now $dim R^\circ_b = dim R^\circ_a$. On the other hand, since

$E_a \cap U$ is Zariski dense in R_a°, and c is generic in $E_a \cap U$, the dimension of c over $\emptyset$ is also $dim R_a^\circ$. Hence c is a generic point of R_b°. By Claim 2.22 again, $c \in E_b^{\mathcal{A}'} \cap U^{\mathcal{A}'}$. We have shown that $c \in E_a^{\mathcal{A}'} \cap E_b^{\mathcal{A}'}$. As E is an equivalence relation, and transferring back to the standard model, we have that $E_a = E_b$. In particular, $b \in E_a$ as desired. This completes the proof of the proposition. □

Corollary 2.23. *$\mathcal{A}$ has elimination of imaginaries.*

Proof. Let X be a complex manifold and E a definable equivalence relation on X. By Proposition 2.21 we have a meromorphic equivalence relation on X, R, with degeneracy set P, such that R agrees with E on the Zariski open subset $U = X \setminus P$. Let $\sigma\colon X' \to X$ and $q\colon X' \to Q$ be as in Theorem 2.20. We define a function $f\colon U \to Q$ by $f(a) = s$ if there exists $x \in q^{-1}(s)$ such that $\sigma(x) = a$. Then f is a definable function, and for all $a, b \in U$, $f(a) = f(b)$ if and only if aEb. As noted earlier, this is sufficient to prove that $\mathcal{A}$ eliminates imaginaries. □

2.3 Nonstandard Models and the Riemann Existence Theorem

Passing to elementary extensions of the standard universe is characteristic of the model-theoretic approach. In the case of complex varieties it is in some sense the "new" technique that model theory brings to complex analytic geometry. In classical algebraic geometry the notion of a universal domain in which all the objects live, and where "generic" points can be found, is (or at least was) familiar. There does not seem to be an analogy to this in complex analytic geometry. In the previous section we have already seen how a sufficiently saturated elementary extension of $\mathcal{A}$ provides a universal domain for the study of the Zariski structure on complex varieties, and that it is in such extensions that the general (and 'general') behaviour of definable sets is witnessed. In this section we continue our study of definable sets in an elementary extension, working now over parameters, and considering the geometry induced by the generalised Zariski topology on this nonstandard universe. In particular, we will prove a version of the Riemann Existence Theorem for elementary extensions of $\mathcal{A}$ (asked for by Pillay in [34]) that describes all nonstandard curves as "algebraic".

As before, fix a saturated elementary extension $\mathcal{A} \prec \mathcal{A}'$ (of cardinality $\kappa > |\mathcal{L}|$). All parameter sets will be assumed to be relatively small (i.e. of cardinality less than κ). All definable sets in $\mathcal{A}'$ are obtained as fibres of $\emptyset$-definable sets. That is, they are sets of the form $G_s^{\mathcal{A}'}$, where $G \subset Y^m \times X^n$ is definable, X and Y are complex varieties, and s is an m-tuple from $Y^{\mathcal{A}'}$. In particular, $G_s^{\mathcal{A}'}$ is by definition *Zariski closed* if G is a Zariski closed set in $Y^m \times X^n$. By quantifier elimination, every definable set (with parameters) is a boolean combination of Zariski closed sets.

There is a more canonical description of a Zariski closed set in $\mathcal{A}'$. Suppose $F = G_s^{\mathcal{A}'}$ is a Zariski closed set, where G and s are as above. By taking the $\emptyset$-locus of s, $S \subset Y^m$, and restricting G to S, we obtain F as the *generic* fibre of $G \to S$. Such a description is somewhat more stable in the following sense: if F is also the generic fibre of some other analytic family $H \to S$, then G and H have the same general fibres, and hence they share those irreducible components that project onto S. Hence such a description of F (for a fixed S at least) is unique up to irreducible components whose projections are proper subsets of S.

The above description also gives us a canonical way of going from one set of parameters to another. Suppse that t is an $m+k$ tuple from $Y^{\mathcal{A}'}$ that extends s, and that $T \subset Y^{m+k}$ is the $\emptyset$-locus of t. Then F can be viewed as being definable over t as well, and as such is obtained as the generic fibre of some Zariski closed subset of $T \times X^n$. How do these two descriptions of F compare? Let π be the coordinate projection map $(Y^{\mathcal{A}'})^{m+k} \to (Y^{\mathcal{A}'})^m$ that takes t to s. Then $\pi(T)$ is a $\emptyset$-definable Zariski closed set containing the generic point of S, and hence containing S. Similiarly, $\pi^{-1}(S) = T$. That is, π induces a surjection $T^{\mathcal{A}'} \to S^{\mathcal{A}'}$. Transferring back to the standard model, we get a surjective holomorphic map $T \to S$. After base change (see the discussion on page 43) we have,

$$\begin{array}{ccc} S \times X^n \supset G & & G_T = (T \times_S G)_{red} \subset T \times X^n \\ \downarrow & & \downarrow \\ S & \longleftarrow & T \end{array}$$

Now the fibres of $G_T \to T$ and $G \to S$ are the same. Hence F is also obtained as the generic fibre of $G_T \to T$. In other words, working with additional parameters in $\mathcal{A}'$ corresponds to base change in the standard model.

Using this description of Zariski closed sets, it is not hard to deduce that they also form a noetherian topology on the sorts of $\mathcal{A}'$. First of all, suppose X is a complex variety and $F_1, F_2 \subset X^{\mathcal{A}'}$ are Zariski closed sets obtained as the generic fibres of $G_1 \to S_1$ and $G_2 \to S_2$ respectively. Let $S = S_1 \times S_2$. Then $F_1 \cap F_2$ (respectively $F_1 \cup F_2$) is the generic fibre of ${G_1}_S \cap {G_2}_S \to S$ (respectively ${G_1}_S \cup {G_2}_S \to S$). Hence the Zariski closed sets in $\mathcal{A}'$ are closed under finite unions and intersections.

Moreover, the descending chain condition holds. Indeed, suppose this were not the case and that for some complex variety X there were infinitely many Zariski closed subsets $F_i \subset X^{\mathcal{A}'}$, $i \in \omega$, where each F_{i+1} is a proper subset of F_i. For each i, let F_i be obtained as the generic fibre of $G_i \to S_i$, where S_i is a complex variety and $G_i \subset S_i \times X$ is a Zariski closed subset. By base change, we may as well assume that there are surjective holomorphic maps $S_{i+1} \to S_i$. Since $F_{i+1} \subset F_i$, the general fibres of $G_{i+1} \to S_{i+1}$ are proper subsets of the general fibres of $G_i \to S_i$. This then yields an infinite properly descending chain of Zariski closed subsets in the 'general' fibre of $G_0 \to S_0$ (i.e. in the fibres above a countable intersection of nonempty Zariski open subsets of S_0) – contradicting the descending chain condition of the Zariski topology on X.

Remark 2.24. That the Zariski topology on $\mathcal{A}'$ is noetherian could have been deduced from the general fact that an elementary extension of a Zariski-type structure is again a Zariski-type structure (see Remark 2.15).

A consequence of the descending chain condition is that every Zariski closed set in $\mathcal{A}'$ has a unique expression as the union of finitely many irreducible Zariski closed sets. We will often use the term *absolutely irreducible* (instead of just irreducible) Zariski closed set in order to distinguish it from the following relative notion:

Definition 2.25. Suppose X is a complex variety, A is a set of parameters, and $F \subset X^{\mathcal{A}'}$ is a Zariski closed set definable over A. Then F is said to be *irreducible over* A if it cannot be written as the union of two proper A-definable Zariski closed sets.

Remark 2.26. F is irreducible over A if and only if for any tuple s from A over which F is defined, there is an irreducible Zariski closed set $G \subset S \times X$ where S is the $\emptyset$-locus of s and $F = G_s^{\mathcal{A}'}$. On the other hand, F is absolutely irreducible if and only if $G \to S$ can be chosen to be a fibre space. The independence of absolute irreducibility from the parameter set is reflected in the fact that the strict transform of a fibre space is again a fibre space (see the discussion on page 43).

Suppose F is an A-definable Zariski closed set in $\mathcal{A}'$. Then its absolutely irreducible components are defined over $acl(A)$. Indeed, since there are only finitely many such components, and automorphisms of $\mathcal{A}'$ that fix A pointwise must permute them, each absolutely irreducible component of F has only finitely many A-conjugates. By elimination of imaginaries and saturation, they are definable over $acl(A)$. Hence, over algebraically closed sets of parameters, irreducibility and absolute irreducibility agree.

Definition 2.27. Suppose F is a Zariski closed set in $\mathcal{A}'$ obtained as the generic fibre of a holomorphic surjection $G \to S$. The *dimension* of F is the dimension of the general fibres of $G \to S$.

Remark 2.28. While it is not obvious from 2.27 that the dimension of a Zariski closed set in $\mathcal{A}'$ is well-defined, this follows from the definability of dimension in $\mathcal{A}$. Indeed, by the dimension formula (2.2) the general fibres of each irreducible component of G that projects onto S are of constant dimension, and hence the general fibres of $G \to S$ are also of constant dimension. Moreover, if F is obtained as the generic fibre of another analytic family $H \to T$, then by base change to $S \times T$, we see that the general fibres of $G \to S$ and $H \to T$ have the same dimension. Hence the dimension of F is well-defined.

Definition 2.29. Suppose A is a set of parameters, c is a tuple of elements from $\mathcal{A}'$, and F is an A-definable Zariski closed in $\mathcal{A}'$. We say that F is the *A-locus* of c, and that *c is a generic point in F over A*, if F is the smallest Zariski closed set over A that contains c. The *dimension of c over A* (or of $tp(c/a)$) is the dimension of the A-locus of c.

Remark 2.30. By quantifier elimination, c is generic in F over A if and only if $c \in F$, and $c \notin G$ for any proper A-definable Zariski closed subset of F. By saturation, every A-definable A-irreducible Zariski closed set has a generic point. Moreover, the type of this generic element over A is unique.

We recover nonforking independence from these notions. That is, if c and d are tuples in $\mathcal{A}'$, and A is a set of parameters, then *c is independent from d over A* if $dim(c/A) = dim(c/Ad)$. In this case the $acl(A)$-locus of (c, d) is just the product of the $acl(A)$-locus of c with the $acl(A)$-locus of d.

We often find ourselves in the following situation: a, b, and s are tuples from $\mathcal{A}'$ and $b \in dcl(s, a)$. We wish to describe how this translates into a statement about the standard model.

Lemma 2.31. *Suppose X and Y are complex varieties, $a \in X^{\mathcal{A}'}$, $b \in X^{\mathcal{A}'}$, and s is a tuple from $\mathcal{A}'$. Let S be the $\emptyset$-locus of s, $F \subset S \times X$ the $\emptyset$-locus of (s, a), and $G \subset S \times Y$ the $\emptyset$-locus of (s, b). Then the following are equivalent:*

(i) $b \in dcl(s, a)$

(ii) *there is a surjective meromorphic map $h: F \to G$,*

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr] & & G \arrow[dl] \\ & S & \end{tikzcd}$$

such that for general $s' \in S$, $h_{s'}$ is a meromorphic surjection from $F_{s'}$ to $G_{s'}$. Moreover $h_s^{\mathcal{A}'}(a) = b$.

Proof. This is just a uniform version of Lemma 2.18. The direction (ii) implies (i) is immediate. For the other direction, let $H \subset S \times Y \times X$ be the $\emptyset$-locus of (s, b, a). Then $H_{(s,a)}^{\mathcal{A}'} \subset G_s^{\mathcal{A}'}$ is the (s, a)-locus of b, and hence is a singleton by assumption. Since a is generic in $F_s^{\mathcal{A}'}$ over s and b is generic in $G_s^{\mathcal{A}'}$ over s, it follows that $H_s^{\mathcal{A}'} \subset G_s^{\mathcal{A}'} \times F_s^{\mathcal{A}'}$ is one-to-one over an s-definable Zariski dense subset of $F_s^{\mathcal{A}'}$ and onto $G_s^{\mathcal{A}'}$. As s is generic in S over $\emptyset$, for general $s' \in S$, $H_{s'} \subset G_{s'} \times F_{s'}$ is one-to-one over a nonempty Zariski open subset of $F_{s'}$ and onto $G_{s'}$. This implies that $H_{s'}$ is the graph of a surjective meromorphic map $F_{s'} \to G_{s'}$. That is, H induces a meromorphic map $h: F \to G$, such that for general $s' \in S$, $h_{s'}: F_{s'} \to G_{s'}$ is a meromorphic surjection. By construction $h_s^{\mathcal{A}'}(a) = b$. □

Remark 2.32. Note that if a and b were interdefinable over s, then we would obtain a bimeromorphism $h: F \to G$ over S.

The following proposition is a good example of how computations involving generic points and independence in $\mathcal{A}'$ can be used to make conclusions about the standard model. The proof given here is based on a more model-theoretic argument suggested by Anand Pillay.

Proposition 2.33. *Suppose X is a complex variety. Then exactly one of the following two conditions must hold:*

(i) *For all n and m, every complex subvariety of $X^n \times \mathbb{P}(\mathbb{C})^m$ is the product of a complex subvariety of X^n with a complex subvariety of $\mathbb{P}(\mathbb{C})^m$.*

(ii) *For some n, there exists a complex subvariety $Y \subset X^n$ and a surjective meromorphic map $f: Y \to \mathbb{P}(\mathbb{C})$.*

Proof. Suppose that for every tuple a from $X^{\mathcal{A}'}$ and b from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$, a is independent from b over $\emptyset$. In stability-theoretic terms, this says that X and $\mathbb{P}(\mathbb{C})$ are fully orthogonal. We claim that this imples (i). Suppose $Z \subset X^n \times \mathbb{P}^m$ is a complex subvariety, and let A (respectively B) be the image of its projection to X^n (respectively $\mathbb{P}^m$). Then Z is a Zariski closed subset of $A \times B$, and we wish to show it is all of $A \times B$. Let (a, b) be a generic point of Z over $\emptyset$. Then a is generic in A over $\emptyset$ and b is generic in B over $\emptyset$. By assumption, a and b are independent, and hence (a, b) is a generic point of $A \times B$ over $\emptyset$. So $Z = A \times B$, as desired.

Now suppose that X and $\mathbb{P}(\mathbb{C})$ are not fully orthogonal. That is, for some n-tuple y from $X^{\mathcal{A}'}$, y is dependent on a tuple from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ over $\emptyset$. We claim that (ii) holds. By strong minimality of $\mathbb{P}(\mathbb{C})$, there are generic independent $b_1, \ldots, b_m$ in $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$, such that $b_1, \ldots, b_{m-1}, y$ are independent, and $b_m \in acl(b_1, \ldots, b_{m-1}, y)$. Let $\{c_1, \ldots, c_k\}$ be the $(b_1, \ldots, b_{m-1}, y)$-conjugates of b_m. By elimination of imaginaries in $\mathbb{P}(\mathbb{C})$, there is a tuple c from $\mathbb{P}(\mathbb{C})$ that is interdefinable with the finite set $\{c_1, \ldots, c_k\}$. Since b_m was generic in $\mathbb{P}(\mathbb{C})$, $dim(c) \geq 1$. So one of the coordinates of c must be a generic point of $\mathbb{P}(\mathbb{C})$, say $p \in \mathbb{P}(\mathbb{C})^{\mathcal{A}'}$. Now, c is independent of $\{b_1, \ldots, b_{m-1}\}$, and so the same holds for p. The $\emptyset$-locus of $(b_1, \ldots, b_{m-1}, p)$ is therefore $\mathbb{P}^m$. Let $\mathbb{P}^m \to \mathbb{P}^{m-1}$ be the projection onto the first $m-1$ coordinates. Since $b_1, \ldots, b_{m-1}, y$ are independent, the $\emptyset$-locus of $(b_1, \ldots, b_{m-1}, y)$ is $\mathbb{P}^{m-1} \times Y$. Finally, by choice of c, $c \in dcl(b_1, \ldots, b_{m-1}, y)$, and hence $p \in dcl(b_1, \ldots, b_{m-1}, y)$ as well. By Lemma 2.31, there is a meromorphic surjection $f\colon \mathbb{P}^{m-1} \times Y \to \mathbb{P}^m$ such that

$$\begin{tikzcd} \mathbb{P}^{m-1} \times Y \arrow[rr, "f"] \arrow[dr] & & \mathbb{P}^m \arrow[dl] \\ & \mathbb{P}^{m-1} & \end{tikzcd}$$

commutes. Restricting f to any general fibre, we obtain the desired meromorphic surjection from Y to $\mathbb{P}(\mathbb{C})$. $\square$

One approach to the classification problem for complex varieties up to bimeromorphism is to understand them in so far as they differ from projective varieties. From this perspective the simplest complex varieties are the Moishezon varieties (page 48). We now define what the correct analogue of Moishezon should be in $\mathcal{A}'$.

Definition 2.34. Suppose F is an absolutely irreducible Zariski closed set over A. Then we say that F is *Moishezon* if there exists parameters $B \supset A$ and a generic point c of F over B, such that c is interdefinable with a tuple from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ over B. Given tuples c and s from $\mathcal{A}'$, we say that $tp(c/s)$ is *Moishezon* if the $acl(s)$-locus of c is Moishezon.

Remark 2.35. Given a stationary type $p(x)$ over A, recall that $p(x)$ is internal to $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ if there exists $B \supset A$, a realisation c of the nonforking extension of $p(x)$ to B, and a tuple b from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$, such that $c \in dcl(Bb)$. By elimination of imaginaries in $\mathbb{P}(\mathbb{C})$, this is equivalent to c being interdefinable with a tuple from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ over B. So if c and s are tuples from $\mathcal{A}'$, then $tp(c/s)$ is Moishezon if and only if $tp(c/acl(s))$ is internal to $\mathbb{P}(\mathbb{C})$.

The following lemma interprets the meaning of Moishezon in terms of fibre spaces in $\mathcal{A}$.

Lemma 2.36. *Suppose that F is the generic fibre of a fibre space $G \to S$. Then F is Moishezon if and only if for some complex variety T, and holomorphic surjection $T \to S$, the strict transform $G_{(T)}$ is bimeromorphic over T to a subspace of $T \times \mathbb{P}_n(\mathbb{C})$ for some n. That is, there is a meromorphic embedding g that makes the following diagram commute:*

$$\begin{tikzcd}
G \arrow[d] & G_{(T)} \arrow[r, hook, "g"] \arrow[d] & T \times \mathbb{P}_n(\mathbb{C}) \arrow[dl] \\
S & T \arrow[l] &
\end{tikzcd}$$

Proof. Assume such a T and g exist. Now F is also the generic fibre of $G_{(T)}$. Let t be generic in T over $\emptyset$ such that $F = G^{\mathcal{A}'}_{(T)_t}$. Then $g^{\mathcal{A}'}$ witnesses that a generic point of $G^{\mathcal{A}'}_{(T)_t}$ over t is interdefinable with some tuple in $\mathbb{P}^{\mathcal{A}'}_n$ over t. Since $F = G^{\mathcal{A}'}_{(T)_t}$ and $\mathbb{P}_n$ lives definable in $\mathbb{P}(\mathbb{C})$, this implies that a generic point of F over t is interdefinable with a tuple from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ over t. That is, F is Moishezon.

Conversely, suppose F is Moishezon, and let s be generic in S over $\emptyset$ such that $F = G^{\mathcal{A}'}_s$. Let t be a tuple from $\mathcal{A}'$ extending s, c a generic point of F over t, and d an element of $\mathbb{P}^m(\mathbb{C})^{\mathcal{A}'}$ that is interdefinable with c over t. By identifying $\mathbb{P}^m$ with a subvariety of some $\mathbb{P}_n$, we may assume that $d \in \mathbb{P}^{\mathcal{A}'}_n$. Letting T be the $\emptyset$-locus of t and $T \to S$ the induced holomorphic surjection, we have that the $\emptyset$-locus of (t, c) is $G_{(T)}$ and the $\emptyset$-locus of (t, d) is a subvariety of $T \times \mathbb{P}_n$. By Lemma 2.31 (and the remark following it), there is a meromorphic T-embedding $g: G_{(T)} \to T \times \mathbb{P}_n$, as desired. $\square$

The above lemma suggests that for a type to be Moishezon is a rather strong algebraicity condition. Recall that a Moishezon fibre space (see page 49) need not be bimeromorphic to a subspace of the product of the base with projective space; this only happens locally (ie. it is bimeromorphic to a projective bundle over the base). However, because we have allowed for base change (ie. additional parameters from $\mathcal{A}'$), a Moishezon type does in fact correspond to a Moishezon fibre space in the standard model:

Proposition 2.37. *Suppose c and s are tuples from $\mathcal{A}'$. Then $tp(c/s)$ is Moishezon if and only if the $acl(s)$-locus of c is the generic fibre of a Moishezon fibre space.*

Proof. If $tp(c/s)$ is Moishezon, then by Lemma 2.36, the $acl(s)$-locus of c is the generic fibre of a fibre space $G_{(T)} \to T$ that is bimeromorphic to a subspace of $T \times \mathbb{P}_n \to T$. Since the latter is (trivially) a projective morphism, $G_{(T)} \to T$ is Moishezon.

Conversely, suppose the $acl(s)$-locus of c is the generic fibre of a fibre space $G \to S$ which is Moishezon. Then by definition there is a coherent analytic sheaf $\mathcal{F}$ on S such that $G \to S$ is bimeromorphic to a subspace of the projective linear space $\mathbb{P}(\mathcal{F}) \to S$. Recall that $\mathcal{F}$ can be taken to be locally-free (upto bimeromorphic equivalence, page 50), and hence $\mathbb{P}(\mathcal{F}) \to S$ can be taken to be a projective bundle. By Lemma 2.36, it suffices to show that every projective bundle is after base change bimeromorphic to a product of the base with projective space. That is, for some holomorphic surjection of complex varieties, $T_0 \to S$, $T_0 \times_S \mathbb{P}(\mathcal{F})$ is bimeromorphic to $T_0 \times \mathbb{P}_n$ over T_0, for some n. We could not find a reference for this last fact in the literature, though it seems implicit in [17] (proof of Proposition 5.1). In any case, we use Fujiki's work on holomorphic fibre bundles ([16]) to give a proof.

Let rank$(\mathcal{F})= n+1$. For fixed $s \in S$, $\mathbb{P}(\mathcal{F})_s$ is isomorphic to $\mathbb{P}_n$. Any such isomorphism can be identified with its graph, which can in turn be identified with a point in the Douady space of $\mathbb{P}(\mathcal{F})_s \times \mathbb{P}_n$ (see page 45). The set of all such isomorphisms (for fixed s) form a subset U_s of $D(\mathbb{P}(\mathcal{F})_s \times \mathbb{P}_n)$. Varying s in S we get a subset $Isom_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)$ of the relative Douady space of $\mathbb{P}(\mathcal{F}) \times_S (S \times \mathbb{P}_n)$ over S (see page 48). The induced map $Isom_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n) \to S$ is surjective, and for all $s \in S$, $Isom_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)_s = U_s$. We let $U = Isom_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)$. Fujiki has shown that U is a Zariski open subset of $D(\mathbb{P}(\mathcal{F}) \times_S (S \times \mathbb{P}_n)/S)$, which accordingly inherits the structure of a (possibly noncompact) complex analytic space.

Moreover, since $\mathbb{P}(\mathcal{F}) \times_S (\mathbb{P}_n \times S) \to S$ is Kähler (in fact projective), the components of its relative Douady space are compact. This implies that U has a natural compactification, $T = Isom^*_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)$, all of whose irreducible components project onto S. Let T_0 be an irreducible component of T, and let U_0 be the Zariski open subset of T_0 obtained by intersecting with U. We argue that this T_0 works, namely that $T_0 \times_S \mathbb{P}(\mathcal{F})$ is bimeromorphic to $T_0 \times \mathbb{P}_n$ over T_0.

We first point out that $U_0 \times_S \mathbb{P}(\mathcal{F})$ is isomorphic to $U_0 \times \mathbb{P}_n$ over U_0. It is not hard to see, by the definition of U, that what is required is a holomorphic section to

$$Isom_{U_0}(U_0 \times_S \mathbb{P}(\mathcal{F}), U_0 \times \mathbb{P}_n) \longrightarrow U_0$$

But by the functoriality of $Isom$ (see 1.1(b) of [16])

$$Isom_{U_0}(U_0 \times_S \mathbb{P}(\mathcal{F}), U_0 \times \mathbb{P}_n) \approx U_0 \times_S Isom_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)$$

over U_0. The right hand side is just $U_0 \times_S U$. Since U_0 is an irreducible component of U (intersection with a component of T), the diagonal map

$d: U_0 \to U_0 \times_S U$ yields the desired section.

We want the above isomorphism to extend to a bimeromorphism between $T_0 \times_S \mathbb{P}(\mathcal{F})$ and $T_0 \times \mathbb{P}_n$ over T_0. For this it is sufficient (and necessary) that d extend to a meromorphic section to

$$Isom^*_{T_0}(T_0 \times_S \mathbb{P}(\mathcal{F}), T_0 \times \mathbb{P}_n) \longrightarrow T_0$$

(see Remark 2 to Definition 1 of [16]). But by the functoriality of $Isom^*$ this time (Remark 4 to Definition 1 of the same paper),

$$Isom^*_{T_0}(T_0 \times_S \mathbb{P}(\mathcal{F}), T_0 \times \mathbb{P}_n) \approx T_0 \times_S Isom^*_S(\mathbb{P}(\mathcal{F}), S \times \mathbb{P}_n)$$

over T_0. The right hand side is just $T_0 \times_S T$. As T_0 is an irreducible component of T, we have the diagonal map $T_0 \to T_0 \times_S T$, yielding the desired extension to d. □

Recall that the ordinary Riemann Existence Theorem states that any compact complex curve is projective. The following theorem is an extension of this to $\mathcal{A}'$. It implies that with respect to 1 dimensional definable sets, $\mathcal{A}'$ is no more mysterious than $\mathcal{A}$.

Theorem 2.38 (Nonstandard Riemann Existence Theorem). *Every absolutely irreducible Zariski closed set of dimension* 1 *in* $\mathcal{A}'$ *is Moishezon.*

Proof. Suppose that F is a nonstandard curve in $\mathcal{A}'$, that is, an absolutely irreducible 1-dimensional Zariski closed set. Then F is the generic fibre of a fibre space $f: X \to S$ of complex varieties whose general fibres are complex curves. By Proposition 2.37, it is suficient to show that for some holomorphic surjection $T \to S$ of complex varieties, $X_{(T)} \to T$ is a Moishezon morphism.

To begin with, consider the strict transform of f with itself, $X_{(X)} \to X$. The diagonal map yields a holomorphic section to this fibre space. Hence we may assume that $f: X \to S$ already admits a holomorphic section. The theorem then follows from the following fact:

Fact 2.39. (Lemma 3.10 in [6]) *Suppose* $f: X \to S$ *is a fibre space of complex varieties whose general fibres are complex curves and which admits a holomorphic section. Then* f *is Moishezon.*

A proof of this fact can be found in Campana [4] (Théorème 2′). We briefly discuss the idea of the proof here. Assume that f is smooth. Fixing general $s \in S$, we know by the ordinary Riemann Existence Theorem that X_s is embeddable in projective space. Such an embedding is given by a very ample line bundle on X_s. Now if $\sigma: S \to X$ is a holomorphic section to f, then $\sigma(S)$ is a Cartier divisor on X and hence gives rise to a line bundle $\mathcal{E}$ on X. If g is the genus of the general fibres of f, let $\mathcal{F} = \mathcal{E}^{\otimes 2g+1}$. For general $s \in S$, $\mathcal{F}_s$ is a very ample line bundle on X_s. The main point is that the embeddings of the X_s's into projective space associated to the $\mathcal{F}_s$'s are given uniformly by a meromorphic map

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathbb{P}(f_*\mathcal{F}) \arrow[dl] \\ & S & \end{tikzcd}$$

where $\mathbb{P}(f_*\mathcal{F}) \to S$ is the projective linear space associated to the coherent analytic sheaf $f_*\mathcal{F}$ on S (of rank $g+2$). See page 19 of Ueno [43] for details on the construction of h. Since h_s is an embedding for general $s \in S$, h is a bimeromorphism onto its image. Hence $f\colon X \to S$ is Moishezon. □

Remark 2.40. (Need for additional parameters) The nonstandard Riemann Existence Theorem would fail if we did not allow additional parameters in the definition of Moishezon. Given tuples c and s from $\mathcal{A}'$, where c is the generic point over s of an absolutely irreducible 1-dimensional Zariski closed set, it is not necessarily the case that c is interdefinable with a tuple from $\mathbb{P}(\mathbb{C})$ over s. Indeed, let G be a non-Moishezon surface that admits a meromorphic surjection $G \to S$, where S is a projective curve and where the general fibres are projective curves. Any surface of algebraic dimension 1 is of this form (see Section 2.5 below for the definition of algebraic dimension). Examples of such surfaces exist (even among complex tori), and can be found in [43]. Let s be generic in S over $\emptyset$, and c generic in $G_s^{\mathcal{A}'}$ over s. Since the general fibres of $G \to S$ are curves, $dim(c/s) = 1$. However, if c were interdefinable with a tuple from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$ over s, then there would be a meromorphic S-embedding of G into $S \times \mathbb{P}^m$, for some m. Since S is projective, so is $S \times \mathbb{P}^m$, and this would contradict the fact that G is non-Moishezon.

In the standard model, definable fields are very well understood. Suppose F is an infinite definable field in $\mathcal{A}$. That is, F is an infinite definable set with a field structure $(F, +, \times)$, where $+$ and $\times$ are definable functions from $F \times F$ to F. For example, the field $(\mathbb{C}, +, \times)$ lives definably in the sort $\mathbb{P}(\mathbb{C})$. Pillay uses the Riemann Existence Theorem to show that $(F, +, \times)$ is definably isomorphic to $(\mathbb{C}, +, \times)$ (see the discussion after Remark 3.10 in [34]). By Chow's Theorem, it follows that F is a pure algebraically closed field; every $\mathcal{L}$-definable subset of a cartesian power of F is definable in the structure $(F, +, \times)$. One is lead to ask whether every definable field in $\mathcal{A}'$ is definably isomorphic to $(\mathbb{C}^{\mathcal{A}'}, +, \times)$. The only known argument for this, obtained in conjunction with Pillay, is the following corollary of the nonstandard Riemann Existence Theorem.

Corollary 2.41. *Any definable field in $\mathcal{A}'$ is definably isomorphic to the field $(\mathbb{C}^{\mathcal{A}'}, +, \times)$.*

Proof. Let $(F, +, \times)$ be a definable field in $\mathcal{A}'$. Suppose we already know that the set F is definably isomorphic to some $G \subset \mathbb{P}^n(\mathbb{C})^{\mathcal{A}'}$. We can pull the field structure on F over to G, so that $(G, +, \times)$ is a definable copy of $(F, +, \times)$. By definability of types, $(G, +, \times)$ is definable with parameters also in the sort $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$. Since the full structure on the sort $\mathbb{P}(\mathbb{C})$ comes from the field $(\mathbb{C}, +, \times)$, and since there is a unique definable field in any pure algebraically closed field up to definable isomorphism, $(G, +, \times)$ must be definably isomorphic to $(\mathbb{C}^{\mathcal{A}'}, +, \times)$. Hence $(F, +, \times)$ is definably isomorphic to $(\mathbb{C}^{\mathcal{A}'}, +, \times)$. It suffices, therefore, to show that F is definably ismorphic to a definable set from the sort $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$. Clearly, it is then sufficient to show that F minus a finite set of points is definably isomorphic to a set from the sort $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$.

Suppose $F \subset X^{\mathcal{A}'}$, for some complex variety X. Let $\overline{F}$ be the (absolute) Zariski closure of F in $X^{\mathcal{A}'}$. Let t be a tuple from $\mathcal{A}'$ over which F, its field structure, and all the absolutely irreducible components of $\overline{F}$ are definable. Let S be the $\emptyset$-locus of t. Back in the standard model, there is a definable subset $G \subset S \times X$, such that $G_t^{\mathcal{A}'} = F$. If $\overline{G}$ denotes the Zariski closure of G in $S \times X$, then $\overline{G}_t^{\mathcal{A}'} = \overline{F}$. So for general $s \in S$, G_s is a definable field that is Zariski dense in $\overline{G}_s$. Now each such G_s is definably isomorphic to $\mathbb{C}$. It follows that $dim\overline{G}_s = 1$, for general $s \in S$. Hence $dim\overline{F} = 1$.

We claim that $\overline{F}$ has a unique maximal dimensional component. That is, $\overline{F}$ consists of a 1-dimensional absolutely irreducible Zariski closed set plus a finite set of points. Indeed, since $dim\overline{F} = 1$, the Morley rank of F is also 1. As F is a definable field, it follows that F has a unique complete non-algebraic type over t (see Fact 1.1, for example). Now if $\overline{F}$ has two distinct 1-dimensional components, then intersecting with F would produce two disjoint infinite definable subsets over t. This in turn would give rise to distinct complete non-algebraic types over t in F.

So $\overline{F}$ must be equal to C plus a finite set of points, where C is a 1-dimensional absolutely irreducible Zariski closed set. By Theorem 2.38, C is Moishezon. By Lemma 2.36, and the fact that $dimC = 1$, C minus a finite set of points can be definably embedded into a cartesian power $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$. Hence F minus a finite set of points is definably isomorphic to a definable set from $\mathbb{P}(\mathbb{C})^{\mathcal{A}'}$. This proves the corollary. $\square$

2.4 Saturated Complex Varieties

The main obstacle to applying model-theoretic techniques directly to $\mathcal{A}$ is that $\mathcal{A}$ is not ω_1-saturated. Since ω_1-saturation and ω_1-compactness are equivalent for countable languages, the reduct of $\mathcal{A}$ to any countable sublanguage of $\mathcal{L}$ is ω_1-saturated. Of course, in reducing the language we lose some of the structure on $\mathcal{A}$ that we are interested in. It may be the case, nevertheless, that for certain sorts of $\mathcal{A}$ the full structure (allowing parameters now) is induced by some countable sublanguage of $\mathcal{L}$.

The sort $\mathbb{P}(\mathbb{C})$ is a typical example of this phenomenon. As noted earlier, the full structure induced by $\mathcal{A}$ on $\mathbb{P}(\mathbb{C})$ comes from the complex field $(\mathbb{C}, +, \times)$. In particular, there is a finite language, $\mathcal{L}_0 \subset \mathcal{L}$, such that every $\mathcal{L}$-definable set in a cartesian power of $\mathbb{P}(\mathbb{C})$ is $\mathcal{L}_0$-definable. Thus $\mathbb{P}(\mathbb{C})$ both retains all of its structure and becomes ω_1-saturated, when it is considered as an $\mathcal{L}_0$-structure. With this example in mind we define:

Definition 2.42. Let X be a complex variety. A *full countable language for* X is a countable sublanguage $\mathcal{L}_0$ of $\mathcal{L}$, such that every $\mathcal{L}$-definable subset of a cartesian power of X is $\mathcal{L}_0$-definable with parameters from $\mathcal{A}$. More generally, if $\mathcal{X} = (X_i)_{i \in I}$ is a countable collection of sorts from $\mathcal{A}$, then a *full countable language for* $\mathcal{X}$ is a countable sublanguage $\mathcal{L}_0$ of $\mathcal{L}$, such that every $\mathcal{L}$-definable set from $\mathcal{X}$ is $\mathcal{L}_0$-definable with parameters from $\mathcal{A}$.

Remark 2.43. We are considering $\mathcal{X} = (X_i)_{i \in I}$ with the structure induced by $\mathcal{A}$. Note that $\mathcal{X}$ has a full countable language if and only if every finite cartesian product of the X_i's has a full countable language if and only if every irreducible Zariski closed set from $\mathcal{X}$ has a full countable language.

In the case of $\mathbb{P}(\mathbb{C})$, the structure induced by $\mathcal{A}$ comes from the (finite) field language, using parameters also from $\mathbb{P}(\mathbb{C})$. In Definition 2.42, on the other hand, we only required that the full structure is $\mathcal{L}_0$-definable using parameters from anywhere in the universe. However, uniform definability of types for stable theories will imply that no parameters from outside the complex variety in question are needed.

Proposition 2.44. *Let X be a complex variety. The following conditions are are equivalent:*

(i) *X has a full countable language.*

(ii) *There exists a countable language $\mathcal{L}(X)$ and an $\mathcal{L}(X)$-structure on X such that for all $F \subset X^n$, F is $\mathcal{L}$-definable if and only if F is $\mathcal{L}(X)$-definable with parameters from X.*

Proof. It is clear that $(ii) \to (i)$. Indeed, since every $\mathcal{L}(X)$-definable set is $\mathcal{L}$-definable, and since $\mathcal{L}(X)$ is countable, there is a countable $\mathcal{L}_0 \subset \mathcal{L}$ from which one can define all sets that are $\mathcal{L}(X)$-definable over $\emptyset$. Thus $\mathcal{L}_0$ will be a full countable language for X.

For the converse, assume that X has a full countable language, witnessed by some $\mathcal{L}_0 \subset \mathcal{L}$. $\mathcal{L}_0$ may involve other sorts from $\mathcal{A}$. Let $\mathcal{M}$ be the $\mathcal{L}_0$-structure consisting of the sort X together with all the other sorts appearing in $\mathcal{L}_0$. Since $\mathcal{A}$ is stable (in fact of finite Morley rank), so is $\mathcal{M}$.

Let $\phi(x, z)$ be any $\mathcal{L}_0$-formula where x is a tuple of variables belonging to the sort X and z is a tuple of variables belonging to some other sort Z in $\mathcal{M}$. By uniform definablility of types (Fact 0.22), there exists an $\mathcal{L}_0$-formula $\psi(x, y)$, where y is now a tuple of variables also belonging to the sort X, with the following property: for all b from Z there exists c from X such that $\psi(x, c)$ is the ϕ-definition of $tp_{\mathcal{L}_0}(b/X)$ in $\mathcal{M}$. That is, for all a from X, $\mathcal{M} \models \psi(a, c)$ iff $\mathcal{M} \models \phi(a, b)$. So $\psi(x, c)$ is equivalent to $\phi(x, b)$. We have pulled the parameter b into the sort X.

Let $\mathcal{L}(X)$ be the language made up of a predicate for each $\psi(x, y)$ obtained in the above manner. Endow X with the natural $\mathcal{L}(X)$-structure. Since $\mathcal{L}_0$ is countable, so is $\mathcal{L}(X)$. Clearly, every $\mathcal{L}(X)$-definable set is $\mathcal{L}$-definable. On the other hand, as X is a saturated complex variety, every $\mathcal{L}$-definable subset of X^n is $\mathcal{L}_0$-definable with parameters; and hence defined by some relation in $\mathcal{L}(X)$ with parameters in X. We have shown that (i) implies (ii), as desired. □

Remark 2.45. Let $\mathcal{X} = (X_i)_{i \in I}$ be a countable collection of sorts from $\mathcal{A}$, that has a full countable language. Applying the above proposition to the cartesian products of sorts from $\mathcal{X}$, we obtain a corresponding many-sorted countable language $\mathcal{L}(\mathcal{X})$ and an $\mathcal{L}(\mathcal{X})$-strucuture on $\mathcal{X}$ satisfying condition (ii) of the proposition.

Proposition 2.46. *Suppose $\mathcal{X} = (X_i)_{i \in I}$ has a full countable language, and let $\mathcal{L}(\mathcal{X})$ be as in part (ii) of Proposition 2.44. Then,*

(i) *Every $\mathcal{L}(\mathcal{X})$-definable set in $\mathcal{X}$ is a finite boolean combination of Zariski closed sets.*

(ii) *$\mathcal{X}$ is of finite Morley rank as an $\mathcal{L}(\mathcal{X})$-structure.*

(iii) *$\mathcal{X}$ is a saturated $\mathcal{L}(\mathcal{X})$-structure.*

Proof. Part (i) and (ii) are just inherited from $\mathcal{A}$, since $\mathcal{L}$ and $\mathcal{L}(\mathcal{X})$ produce the same definable sets (with parameters) in $\mathcal{X}$.

For part (iii) notice that since $\mathcal{A}$ is ω_1-compact and $\mathcal{L}(\mathcal{X})$ is countable, $\mathcal{X}$ is at least ω_1-saturated as an $\mathcal{L}(\mathcal{X})$-structure. We wish to show that $\mathcal{X}$ is actually $|\mathcal{X}|$-saturated. Since $|\mathcal{X}| = 2^{\omega}$, this could be obtained by assuming the Continuum Hypothesis. Fortunately, using the fact that $\mathcal{X}$ is of finite U-rank, we can avoid such an assumpion. By Proposition 0.24, showing that $\mathcal{X}$ is 2^{ω}-saturated is reduced to showing that every complete non-algebraic $\mathcal{L}(\mathcal{X})$-type over countably many parameters in $\mathcal{X}$ has 2^{ω}-many realisations in $\mathcal{X}$. But part (i) of this proposition (plus the noetherianity of the Zariski topology and the countability of $\mathcal{L}(\mathcal{X})$), implies that any such type will be realised by a countable intersection of nonempty Zariski open subsets of a complex variety. A slight modification of the proof of the Baire Category Theorem in this context, yields that any such intersection is of size the continuum. Hence, $\mathcal{X}$ is a saturated $\mathcal{L}(\mathcal{X})$-structure. $\square$

Given Proposition 2.46, we will allow ourselves the following convenient terminology:

Definition 2.47. A complex variety, X, will be called *saturated* if it has a full countable language. Moreover, if $\mathcal{X} = (X_i)_{i \in I}$ is a countable collection of sorts from $\mathcal{A}$, then we will say that $\mathcal{X}$ is a *saturated reduct of* $\mathcal{A}$ to mean that $\mathcal{X}$ has a full countable language.

Remark 2.48. The terminology above is somewhat misleading. Having a full countable language is not in principle equivalent to saturation with respect to some "full" language. There may exist a complex variety X without a full *countable* language, but with a full *uncountable* langugae $\mathcal{L}_0$, such that X is saturated as an $\mathcal{L}_0$-structure. We are not aware of any such examples. In any case, no confusion should arise in our context.

Remark 2.49. Note that part (i) of the above proposition is something like quantifier elimination. The point is that we do not as yet have an explicit choice for $\mathcal{L}(\mathcal{X})$, with respect to which we could check for quantifier elimination. We will eventually obtain a canonical language for a saturated complex variety, and will prove quantifier elimination in that language.

We give a geometric characterisation of saturated complex varieties in terms of the restricted Douady spaces of their cartesian powers (see page 47 for a discussion on restricted Douady spaces).

Theorem 2.50. *A complex variety X is saturated if and only if for all $n > 0$, every irreducible component of the restricted Douady space of X^n is compact.*

Proof. Suppose that for all $n > 0$ each component, C, of $D(X^n)$ is compact. Then C is a sort in $\mathcal{A}$. Let Z_C denote the restriction of $Z(X^n)$ to C. Then

Z_C is a Zariski closed subset of $C \times X^n$, and hence a basic relation in $\mathcal{A}$. Let $\mathcal{L}_0$ be the sublanguage of $\mathcal{L}$ consisting of all such relations Z_C, as C and n vary. Since $D(X^n)$ has only countably many components for each n (Theorem 2.8), $\mathcal{L}_0$ is countable. We have seen that every pure dimensional Zariski closed subset of X^n occurs as a fibre of $Z(X^n) \to D(X^n)$ (page 47). In particular, for all n, every complex subvariety of X^n, is $\mathcal{L}_0$-definable with parameters (where the parameters come from the components of $D(X^n)$). By quantifier elimination, $\mathcal{L}_0$ witnesses that X is saturated.

We wish to prove the converse, but we begin with a discussion of families of Zariski closed subsets in saturated complex varieties. Let $\mathcal{X}$ be any countable collection of sorts from $\mathcal{A}$ and X a cartesian product of sorts from $\mathcal{X}$. A *definable family of Zariski closed subsets of X^n in $\mathcal{X}$* is a definable subset $F \subset B \times X^n$ where B is definable in $\mathcal{X}$, and such that the natural projection $F \to B$ is surjective with Zariski closed fibres. Such a family is *analytic*, if B is locally Zariski closed and irreducible (i.e., B is a Zariski open subset of an irreducible Zariski closed subset of some cartesian product of sorts from $\mathcal{X}$), and F is Zariski closed in $B \times X^n$. An analytic family is *flat* if the natural projection $F \to B$ is flat.

Lemma 2.51. *Suppose $\mathcal{X}$ is a saturated reduct of $\mathcal{A}$ and X is a cartesian product of sorts from $\mathcal{X}$. For any $n > 0$, there are countably many flat analytic families of pure dimensional Zariski closed subsets of X^n in $\mathcal{X}$, $(F_i \to B_i)_i$ such that:*

(i) *every pure dimensional Zariski closed subset of X^n occurs as a member of (at least) one of these families, and*

(ii) *for each i, the Douady map $g_i\colon B_i \to D(X^n)$, given by the universality of $D(X^n)$, is definable. That is, if $\overline{B_i}$ denotes the Zariski closure of B_i and $\overline{g(B_i)}$ denotes the Zariski closure of $g(B_i)$ in $D(X^n)$, then $\overline{g(B_i)}$ is compact and g_i extends to a meromorphic map $\overline{B_i} \to \overline{g_i(B_i)}$.*

Proof. Fix $n > 0$ and a countable language $\mathcal{L}(\mathcal{X})$ witnessing the saturation of $\mathcal{X}$. It is more or less clear from the countability of $\mathcal{L}(\mathcal{X})$ and the fact that every closed analytic subset of X^n is $\mathcal{L}(\mathcal{X})$-definable, that there are countably many definable families of pure dimensional Zariski closed subsets of X^n in $\mathcal{X}$, such that every pure dimensional Zariski closed subset of X^n is a member of one of them. Hence it is sufficient to show that any such $F \to B$ can be decomposed into finitely many flat analytic families satisfying condition (*ii*) of the lemma. First of all, we may assume that B is locally Zariski closed and irreducible, since by quantifier elimination any definable set is a finite union of such. Also by quantifier elimination, $F \to B$ is analytic over some Zariski open subset of B, and hence by induction on the dimension of B, $F \to B$ can be decomposed into finitely many analytic families. So it remains to show that an analytic family of Zariski closed subsets of X^n, $F \to B$, in $\mathcal{X}$ can be decomposed into finitely many flat families for which condition (ii) holds. Consider $\overline{F} \to \overline{B}$. On page 46, we observed that Hironaka's Flattening Theorem gives us a "meromorphic" Douady map on $\overline{B}$, that agrees with the actual Douady map on a Zariski open subset of $\overline{B}$. We can choose this Zariski open set to be in B. In particular, the image of $\overline{B}$ under this meromorphic map is compact. By induction on $dim B$ again, we can decompose $F \to B$ into finitely many flat analytic families for which condition (ii) holds. $\square$

Now for the remaining direction of Theorem 2.50. Suppose X is a saturated complex variety. Fix $n > 0$. We need to show that every irreducible component of $D(X^n)$ is compact. Let $(F_i \to B_i)_i$ be the countably many flat analytic families of pure dimensional Zariski closed subsets of X^n given by Lemma 2.51, and $g_i\colon B_i \to D(X^n)$ the Douady maps associated to them. Let $D_0(X^n)$ be the dense Zariski open subset of $D(X^n)$ above which the fibres of $Z(X^n)$ are reduced (see 2.7). Then $D_0(X^n)$ is in bijective correspondence with the collection of all pure dimensional Zariski closed subsets of X^n. Since every pure dimensional Zariski closed subset of X^n is a member of one of the $F_i \to B_i$, $D_0(X^n)$ is covered by the $g_i(B_i)$'s. Fix an irreducible component, C, of $D(X^n)$, and let C_0 be its intersection with $D_0(X^n)$. Then C_0 is also covered by countably many $g_i(B_i)$'s. Since no irreducible complex analytic space can be written as the countable union of proper analytic subsets, one of the $g_i(B_i)$'s has a dense intersection with C. But as the $\overline{g_i(B_i)}$'s are compact, C is compact. This completes the proof of Theorem 2.50. $\square$

Example 2.52. (The class $\mathcal{C}$) Recall that the irreducible components of the Douady space of a Kähler-type variety are compact (see the discusion on page 51). Moreover the class of Kähler-type varieties, $\mathcal{C}$, is closed under taking cartesian products. So by Theorem 2.50 they are saturated. In particular, all complex tori and Moishezon varieties are saturated.

Example 2.53. (Hopf manifolds) We consider the original Hopf surface, H, defined as follows: Let W be the analytic space $\mathbb{C}^2 - \{(0,0)\}$ and $g\colon W \to W$ the analytic automorphism of W given by $g(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{2}z_2)$. Then the Hopf manifold $H = W/ < g >$ is a (compact) complex surface. In [13] Fujiki points out that $D(H \times H)$ has a noncompact component (by analysing the group of analytic automorphisms of H). By Theorem 2.50, this implies that H is not a saturated complex variety. In particular, H is not of Kähler-type.

One of the open questions about saturated complex varieties is whether they are stable under cartesian products. This gives rise to the issue of when several saturated complex varieties can occur as sorts in a single saturated reduct of $\mathcal{A}$. The following lemma yields a criterion for being able to extend a saturated reduct of $\mathcal{A}$ to include a given complex variety.

Lemma 2.54. *Suppose $\mathcal{X}$ is a saturated reduct of $\mathcal{A}$, and X is a complex variety. If there is a set B, definable in $\mathcal{X}$, and an $\mathcal{L}$-definable surjective map $g: B \to X$, then $\mathcal{X} \cup \{X\}$ is again saturated. In particular X is saturated.*

Proof. Let $\mathcal{L}(\mathcal{X})$ be a countable language witnessing the saturation of $\mathcal{X}$, and let $\mathcal{X}'$ be the countable collection of sorts $\mathcal{X} \cup \{X\}$. We wish to show that $\mathcal{X}'$ is a saturated reduct of $\mathcal{A}$. Let $\mathcal{L}(\mathcal{X}')$ be obtained from $\mathcal{L}(\mathcal{X})$ by adjoining a predicate for the graph of g, $\Gamma(g) \subset B \times X$. Clearly $\mathcal{L}(\mathcal{X}')$ is countable and every $\mathcal{L}(\mathcal{X}')$-definable subset of $\mathcal{X}'$ is $\mathcal{L}$-definable. For the converse, let F be an $\mathcal{L}$-definable subset of $\mathcal{X}'$. We may assume that for some cartesian product of sorts from $\mathcal{X}$, Y, and some $m > 0$, F is a subset of $Y \times X^m$. Let $h: Y \times B^m \to Y \times X^m$ be the map $id \times g^{\times m}$. Then h is $\mathcal{L}(\mathcal{X}')$-definable. Now, $h^{-1}(F) \subset Y \times B^m$, is an $\mathcal{L}$-definable subset of $\mathcal{X}$. Hence, it is $\mathcal{L}(\mathcal{X})$-definable in $\mathcal{X}$ (possibly with parameters). But then, $F = h(h^{-1}(F))$ is $\mathcal{L}(\mathcal{X}')$-definable in $\mathcal{X}'$. So every $\mathcal{L}$-definable subset of $\mathcal{X}'$ is $\mathcal{L}(\mathcal{X}')$-definable (possibly with parameters). We have shown that $\mathcal{X}'$ is a saturated reduct of $\mathcal{A}$, as desired. $\square$

The following proposition is useful because it allows us to assume that the sort $\mathbb{P}(\mathbb{C})$ is contained in any given saturated reduct of $\mathcal{A}$.

Proposition 2.55. *If $\mathcal{X}$ is a saturated reduct of $\mathcal{A}$, then $\mathcal{X} \cup \{\mathbb{P}(\mathbb{C})\}$ is again saturated.*

Proof. It is sufficient to show that if X is a saturated complex variety then $\{X, \mathbb{P}(\mathbb{C})\}$ is a saturated reduct. By Proposition 2.33, there are two cases to consider. The first case, when X is fully orthogonal to $\mathbb{P}(\mathbb{C})$, is that every complex subvariety of $X^n \times \mathbb{P}(\mathbb{C})^m$ is a product of complex subvarieties in X^n and $\mathbb{P}(\mathbb{C})^m$. In this case, the saturation of $\{X, \mathbb{P}(\mathbb{C})\}$ is witnessed by the union of the languages witnessing the saturation of X and of $\mathbb{P}(\mathbb{C})$.

If X is not fully orthogonal to $\mathbb{P}(\mathbb{C})$, then by Proposition 2.33, there is a complex subvariety $Y \subset X^n$ (for some n), and a meromorphic surjection $f: Y \to \mathbb{P}(\mathbb{C})$. Hence, $\mathbb{P}(\mathbb{C})$ minus a finite set of points is the definable image of a definable set from the sort X. We can extend this trivially to obtain a definable sujective map $g: F \to \mathbb{P}(\mathbb{C})$, where F is a definable set from the sort X. If $\mathcal{L}(X)$ witnesses the saturation of X, we let $\mathcal{L}(X)'$ be the countable language obtained by adding a symbol for g to $\mathcal{L}(X)$. By Lemma 2.54, we obtain that $\mathcal{L}(X)'$ witnesses the saturation of $\{X, \mathbb{P}(\mathbb{C})\}$. $\square$

Corollary 2.56. *Supose $\mathcal{X}$ is a saturated reduct of $\mathcal{A}$ and W is a Moishezon variety. Then $\mathcal{X} \cup \{W\}$ is again saturated.*

Proof. Recall that a Moishezon variety is the image under a holomorphic map of a projective variety. Let $f: V \to W$ be such a holomorphic map between the projective variety V and W. Recall that V is definably isomorphic to a definable set from the sort $\mathbb{P}(\mathbb{C})$. Let B denote this definable set, and $g: B \to W$ the corresponding definable surjection onto W. Now $\mathcal{X}' = \mathcal{X} \cup \{\mathbb{P}(\mathbb{C})\}$ is saturated by Proposition 2.55, and B is definable in $\mathcal{X}'$. Hence by Lemma 2.54, $\mathcal{X} \cup \{\mathbb{P}(\mathbb{C}), W\}$ is saturated. In particular, $\mathcal{X} \cup \{W\}$ is a saturated reduct of $\mathcal{A}$. $\square$

The Douady Language

We now deal with the issue of obtaining a canonical language to witness the saturation of a saturated complex variety. We first define a particular class of saturated reducts of $\mathcal{A}$ that do admit such a canonical language.

Definition 2.57. Suppose $\mathcal{X}$ is a countable collection of sorts from $\mathcal{A}$. We say that $\mathcal{X}$ is a *Douady-stable* reduct of $\mathcal{A}$ if for each cartesian product of sorts from $\mathcal{X}$, X, each component of the restricted Douady space, $D(X)$, is again a sort in $\mathcal{X}$. In this case, the *Douady language* of $\mathcal{X}$, denoted $\mathcal{L}_D(\mathcal{X})$, is defined as follows: for each cartesian product of sorts from $\mathcal{X}$, X; and for each component C of $D(X)$, we have a predicate for $(Z(X) \cap (C \times X))$, the restricted universal family of X over C.

Since every sort in $\mathcal{A}$ is compact, a Douady-stable reduct of $\mathcal{A}$ is saturated by Theorem 2.50. The Douady language of a Douady-stable reduct is a countable sublanguage of $\mathcal{L}$ that witnesses its saturation. Recall that if X is a complex variety and $F \subset X$ is a complex subvariety, then for some $a \in D(X)$, $F = Z(X)_a$. By quantifier elimination for $\mathcal{A}$, if $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$, then every $\mathcal{L}$-definable set in $\mathcal{X}$ is $\mathcal{L}_D(\mathcal{X})$-definable with parameters *in* $\mathcal{X}$. Finally, since the Douady space of a complex variety is unique, the Douady language of a Douady-stable reduct is a canonical witness to its saturation.

Of course, it is not clear how meaningful this definition is, since it was clearly designed to make a canonical language possible. It turns out, however, that we can always embed a saturated complex variety into a Douady-stable reduct of $\mathcal{A}$. The first step is to show that saturation is stable under taking components of Douady spaces.

Proposition 2.58. *Suppose $\mathcal{X}$ is a saturated reduct of $\mathcal{A}$, X is a cartesian product of sorts from $\mathcal{X}$, and C is a component of $D(X)$. Then $\mathcal{X} \cup \{C\}$ is again a saturated reduct of $\mathcal{A}$.*

Proof. Consider $Z_C \subset C \times X$, the restricted universal family of X over C. The set-theoretic fibres of $Z_C \to C$ are an irredundant collection of pure dimensional Zariski closed subsets of X (though some small set of fibres may appear in other components of $Z(X) \to D(X)$). Since $\mathcal{X}$ has a full countable language, the fibres of $Z_C \to C$ fit into countably many uniformly definable families of pure dimensional Zariski closed subsets of X, $F_i \to B_i$, in $\mathcal{X}$. For each i, there is an $\mathcal{L}$-definable map $g_i \colon B_i \to C$ given by $g_i(b) = c$ if $F_{i_b} = Z_{C_c}$. It follows that C is covered by the $g_i(B_i)$'s. By ω_1-compactness of $\mathcal{A}$, finitely many of them cover C. That is $C = \bigcup_{i \leq n} g_i(B_i)$, where B_i is a definable set in $\mathcal{X}$ and g_i is an $\mathcal{L}$-definable map.

Now, each B_i is contained in some finite cartesian product of sorts Y_i from $\mathcal{X}$. By taking appropriate definably isomorphic copies of the B_i's in $Y = Y_1 \times \ldots \times Y_n$, we may assume that they are disjoint subsets of Y. Then, letting $B = \bigcup_{i \leq n} B_i$ and $g = \bigcup_{i \leq n} g_i$, we obtain a surjective definable map $g \colon B \to C$. Hence by Lemma 2.54, $\mathcal{X} \cup \{C\}$ is saturated. $\square$

Corollary 2.59. *Every saturated reduct of $\mathcal{A}$ is contained in a Douady-stable reduct of $\mathcal{A}$.*

Proof. Just iterate the procedure of adjoining the irreducible components of the restricted Douady spaces of the cartesian products of sorts from $\mathcal{X}$. By Proposition 2.58 we have a saturated reduct of $\mathcal{A}$ at each stage and hence can continue the construction. In the limit we obtain a Douady-stable reduct. $\square$

In Section 2.2 we showed that $\mathcal{A}$ admits elimination of imaginaries (Corollary 2.23). It does *not* follow (at least not immediately) that a Douady-stable reduct of $\mathcal{A}$ will eliminate imaginaries in the Douady language. We will show directly that Douady-stable reducts admit elimination of imaginaries. The argument in this context is considerably easier than it was for $\mathcal{A}$. For example, one does not require the theory of meromorphic equivalence relations.

Suppose that $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$. Since $\mathcal{X}$ is a saturated $\mathcal{L}_D(\mathcal{X})$-structure, elimination of imaginaries is equivalent to every definable set in $\mathcal{X}$ having a code in $\mathcal{X}$ (Proposition 0.12). That is, for every $\mathcal{L}_D(\mathcal{X})$-definable set F, there is a tuple c from $\mathcal{X}$, such that if α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$, then $\alpha(F) = F$ if and only if $\alpha(c) = c$. The main point is that the Douady language is canonical; the parameters involved in the natural description of a definable set are already its code.

Proposition 2.60. *Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$. Then as an $\mathcal{L}_D(\mathcal{X})$-structure, $\mathcal{X}$ admits elimination of imaginaries.*

Proof. Suppose Y is a cartesian product of sorts from $\mathcal{X}$, and $F \subset Y$ is a pure dimensional Zariski closed subset. Then for a unique $a \in D(Y)$, the (sheaf-theoretic) fibre of $Z(Y)$ is reduced and $F = Z(Y)_a$. Since $\mathcal{X}$ is Douady-stable, the components of $Z(Y) \to D(Y)$ are in $\mathcal{X}$. Moreover, by choice of the Douady language, these components are $\mathcal{L}_D(\mathcal{X})$-definable over $\emptyset$. Let $Z \to D$ be the component of $Z(Y) \to D(Y)$ in which a lies. Suppose α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$. Clearly, if $\alpha(a) = a$ then

$$\alpha(F) = \alpha(Z)_{\alpha(a)} = Z_a = F$$

On the other hand, suppose that $\alpha(F) = F$. Then

$$Z_{\alpha(a)} = \alpha(Z)_{\alpha(a)} = \alpha(F) = F$$

Since $\alpha(a) \in D$, the irredundency of the set-theoretic fibres of $Z \to D$, implies that α must fix a. Hence a is a code for F. We have shown that every pure dimensional Zariski closed set from $\mathcal{X}$ has a code in $\mathcal{X}$.

Now consider and arbitrary Zariski closed subset of Y, F. Then F is the finite union of complex subvarieties $F_1, \ldots, F_n$. Let a_i be a code for F_i. If α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$, then α fixes F setwise if and only if it permutes the F_i if and only if it fixes the finite set of distinct points $\{a_1, \ldots, a_n\}$. Since $\{a_1, \ldots, a_n\}$ is a pure dimensional Zariski closed set in $\mathcal{X}$, it has its own code, say c. It follows that c is a code for F. So every Zariski closed set in $\mathcal{X}$ has a code.

Next, suppose that $F = S \setminus T$, where S is a complex subvariety of Y and T is a proper Zariski closed subset of S. In particular, S is the Zariski closure of F. Let a be a code for S, and b a code for T. Suppose α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$. Clearly, if $\alpha(ab) = ab$ then $\alpha(S) = S$ and $\alpha(T) = T$, and so $\alpha(F) = F$. On the other hand, suppose $\alpha(F) = F$, and let $S' = \alpha(S)$. Then $S \cap S'$ is a Zariski closed set in Y that contains F – hence the Zariski closure of F, S, is contained in $S \cap S'$. This implies that

$S' = S$, and so α fixes S setwise. But $T = S \setminus F$, and so $\alpha(T) = T$ as well. Hence $\alpha(ab) = ab$. So ab is a code for F.

Finally, let F be an arbitrary definable set in Y. By quantifier elimination, F can be expressed irredundantly as a finite union of the form $(S_1 \setminus T_1) \cup \ldots \cup (S_n \setminus T_n)$, where each S_i is a complex subvariety of Y and each T_i is a proper Zariski closed subset of S_i. Let a_i be a code for $S_i \setminus T_i$. Just as before, a code for the finite set $\{a_1, \ldots, a_n\}$ will be a code for F. We have shown that every definable set in $\mathcal{X}$ has a code in $\mathcal{X}$. This proves the proposition. $\square$

Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$, and A is a countable set from $\mathcal{X}$. The algebraic closure of A (respectively definable closure of A) in the sense of $\mathcal{L}_D(\mathcal{X})$ is denoted by $acl_{\mathcal{X}}(A)$ (respectively $dcl_{\mathcal{X}}(A)$). One consequence of elimination of imaginaries (and saturation) is that for any definable set F in $\mathcal{X}$, F is $\mathcal{L}_D(\mathcal{X})$-definable over $acl_{\mathcal{X}}(A)$ if and only if F has finitely many A-conjugates by $\mathcal{L}_D(\mathcal{X})$-automorphisms.

We already know that every definable set in $\mathcal{X}$ is a boolean combination of Zariski closed sets. This is very much like quantifier elimination, since Zariski closed sets are quantifier-free $\mathcal{L}_D(\mathcal{X})$-definable. However, as usual, one has to pay attention to parameters.

Proposition 2.61. *Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$. Then $\mathcal{X}$ admits quantifier elimination in the language obtained from $\mathcal{L}_D(\mathcal{X})$ by adding names for all the elements of $acl_{\mathcal{X}}(\emptyset)$.*

Proof. Let Y be a finite cartesian product of sorts from $\mathcal{X}$, and $F \subset Y$ an $\mathcal{L}_D(\mathcal{X})$-definable set over $\emptyset$. By quantifie elimination in $\mathcal{A}$, we can express F irredundantly as $(S_1 \setminus (T_1^1 \cup \ldots \cup T_{n_1}^1)) \cup \ldots \cup (S_m \setminus (T_1^m \cup \ldots \cup T_{n_m}^m))$, where each S_i is a complex subvariety of Y and each T_j^i is a proper complex subvariety of S_i. Writing each S_i as $Z(Y)_{a_i}$, and each T_j^i as $Z(Y)_{b_j^i}$, for appropriate codes a_i's and b_j^i's from $D(Y)$, we obtain a quantifier-free $\mathcal{L}_D(\mathcal{X})$-expression for F over the parameters $P = \{a_1, \ldots, a_m, b_1^1, \ldots, b_{n_1}^1, \ldots, b_1^m, \ldots, b_{n_m}^m\}$. Now if α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$, then α will fix F, and so α will permute P. By saturation and elimination of imaginaries, this implies that $P \subset acl_{\mathcal{X}}(\emptyset)$. If the elements of $acl_{\mathcal{X}}(\emptyset)$ were named, then we would find a quantifier-free expression for F without parameters. This proves the corollary. $\square$

Remark 2.62. From now on, we will just assume that the elements of $acl_{\mathcal{X}}(\emptyset)$ have already been added as constants to the language. Since $\mathcal{L}_D(\mathcal{X})$ is countable, this is only a countable collection of new constants, and we lose nothing by making this assumption.

Fix a Douady-stable reduct $\mathcal{X}$ of $\mathcal{A}$. Since we often shift between considering $\mathcal{X}$ as an $\mathcal{L}$-structure and as an $\mathcal{L}_D(\mathcal{X})$-structure, we will use notation that specifies the language, in order to avoid confusion. For example, $tp_{\mathcal{X}}(c/A)$ will denote the complete type of c over A with respect to $\mathcal{X}$.

The advantage of saturation is that $\mathcal{X}$ acts as a universal domain in the sense that the generic behaviour of definable sets is witnessed in $\mathcal{X}$ without having to pass to an elementary extension. The price we have to pay is that parameters now come into play. This leads us naturally to relative notions of irreducibility and genericity, just as it did in the context of an elementary extension of $\mathcal{A}$ (compare with Definitions 2.25 and 2.29).

Definition 2.63. Suppose A is a countable set of parameters from $\mathcal{X}$, and F is a Zariski closed set in $\mathcal{X}$ that is $\mathcal{L}_D(\mathcal{X})$-definable over A. Then F is said to be *$\mathcal{L}_D(\mathcal{X})$-irreducible over A* if it cannot be written as the union of two proper Zariski closed subsets that are $\mathcal{L}_D(\mathcal{X})$-definable over A.

Remark 2.64. If the context is not clear, we will use the term *absolutely irreducible* to describe irreducible Zariski closed sets (in the ordinary sense). By elimination of imaginaries and saturation, the absolutely irreducible components of a Zariski closed set that is $\mathcal{L}_D(\mathcal{X})$-definable over A, are themselves $\mathcal{L}_D(\mathcal{X})$-definable over $acl_{\mathcal{X}}(A)$

Definition 2.65. Suppose A is a countable set of parameters from $\mathcal{X}$, c is a tuple of elements from $\mathcal{X}$, and F is a Zariski closed set that is $\mathcal{L}_D(\mathcal{X})$-definable over A. We say that F is the *A-locus of c with respect to $\mathcal{L}_D(\mathcal{X})$* and that *$c$ is $\mathcal{L}_D(\mathcal{X})$-generic in F over A*, if F is the smallest Zariski closed set containing c that is $\mathcal{L}_D(\mathcal{X})$-definable over A. The *$\mathcal{L}_D(\mathcal{X})$-dimension of c over A* (or of $tp_{\mathcal{X}}(c/a)$) is the dimension of the A-locus of c with respect to $\mathcal{L}_D(\mathcal{X})$.

Remark 2.66. By saturation, every Zariski closed set that is $\mathcal{L}_D(\mathcal{X})$-definable and $\mathcal{L}_D(\mathcal{X})$-irreducible over A has an $\mathcal{L}_D(\mathcal{X})$-generic point in $\mathcal{X}$. By quantifier elimination, the $\mathcal{L}_D(\mathcal{X})$-type of a generic element over A is unique.

The notion of genericity in a Douady-stable reduct of $\mathcal{A}$ is a concrete manifestation of what we have been calling *general* and *'general'* behaviour (see page 41). Indeed, suppose F is an absolutely irreducible Zariski closed set in $\mathcal{X}$ that is $\mathcal{L}_D(\mathcal{X})$-definable over A, c is an $\mathcal{L}_D(\mathcal{X})$-generic point of F over A, and P is a property of points in F. If P is $\mathcal{L}_D(\mathcal{X})$-definable over A, then P holds for c if and only if it holds for general $x \in F$. If P is $\mathcal{L}_D(\mathcal{X})$-type-definable over A, then P holds for c if and only if it holds for 'general' $x \in F$.

We have often mentioned that the reason for introducing saturated complex varieties was to avoid having to pass to elementary extensions. A more precise way of putting this is that elementary extensions of a saturated complex variety are completely controlled by the standard model. As an illustration of how this works, we give a quick proof of the nonstandard Riemann Existence Theorem for saturated complex varieties (compare to Theorem 2.38). Let $\mathcal{A}'$ be a sufficiently saturated elementary extension of $\mathcal{A}$. Recall that $\mathbb{P}_n(\mathbb{C})^{\mathcal{A}'}$ is just projective n-space over the saturated algebraically closed field $(\mathbb{C}^{\mathcal{A}'}, +, \times)$. By Chow's Theorem every irreducible Zariski closed subset of $\mathbb{P}_n(\mathbb{C})^{\mathcal{A}'}$ is a projective variety over $\mathbb{C}^{\mathcal{A}'}$.

Proposition 2.67. *Suppose $f: X \to Y$ is a fibre space of complex varieties in a saturated reduct of $\mathcal{A}$, whose general fibres are complex curves. Then the generic fibre of f in $\mathcal{A}'$ (in the sense of $\mathcal{L}$) is definably isomorphic to a projective curve in $\mathbb{P}_n(\mathbb{C})^{\mathcal{A}'}$, for some $n > 0$.*

Proof. Let $\mathcal{X}$ be a Douady-stable reduct of $\mathcal{A}$ containing $f: X \to Y$ and the sort $\mathbb{P}(\mathbb{C})$ (this is possible by Proposition 2.55 and Corollary 2.59). There is a nonempty Zariski open set $U \subset Y$, such that for all $u \in U$, X_u is a complex curve. Let A be a countable set of paramters from $\mathcal{X}$ over which $f: X \to Y$ and U are $\mathcal{L}_D(\mathcal{X})$-definable.

Now let s be a generic point of Y in $\mathcal{A}'$ over $\emptyset$ in the sense of $\mathcal{L}$ (see Definition 2.17). Since $\mathcal{L}_D(\mathcal{X}) \subset \mathcal{L}$, it makes sense to evaluate $\mathcal{L}_D(\mathcal{X})$-formulas in $\mathcal{A}'$. In particular, we may consider $tp_{\mathcal{X}}(s/A)$. By saturation of $\mathcal{X}$, there is a point in $\mathcal{X}$ that realises this type, say t. Now t is in U, and so X_t is a complex curve. By the Riemann Existence Theorem there is a holomorphic embedding $g: X_t \to \mathbb{P}_n$, for some n. Now g is $\mathcal{L}_D(\mathcal{X})$-definable, say by a formula $\phi(z_1, z_2, b)$. Of course, the parameter b used to define g may be outside of A. In any case, $tp_{\mathcal{X}}(t/A)$ does say that there *exists* a tuple b such that $\phi(z_1, z_2, b)$ defines an isomorphism of X_t with a projective curve in $\mathbb{P}_n$. Since t realises $tp_{\mathcal{X}}(s/A)$, we obtain a definable isomorphism of $X_s^{\mathcal{A}'}$ with a projective curve in $\mathbb{P}_n(\mathbb{C})^{\mathcal{A}'}$. As s was chosen to be a generic point of Y, this proves the proposition. $\square$

Remark 2.68. Notice that while Proposition 2.67 above only applies in the saturated context, it yeilds a conclusion that is stronger than that of the more general Theorem 2.38: the "nonstandard curve", $X_s^{\mathcal{A}'}$, is shown to be definably isomorphic to a projective curve. In Theorem 2.38 this was only demonstrated up to finitely many points; nonstandard curves were shown to be "generically definably isomorphic" to projective curves (i.e. they were "nonstandard" Moishezon).

Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$, and Y is a cartesian product of sorts from $\mathcal{X}$. Then the components of $D(Y)$ are again sorts in $\mathcal{X}$. Now suppose $X \subset Y$ is a complex subvariety. We are interested in where and how the restricted Douady space of X lies. Given an irreducible component C of $D(X)$, let Z_C denote the restriction of $Z(X)$ to C. As X is a complex subvariety of Y, $Z_C \to C$ is a flat family of pure dimensional Zariski closed subsets in Y. Hence there is a Douady map $C \to D(Y)$, that is in fact an embedding. Moreover, as C is compact, its image in $D(Y)$ is a complex subvariety of some component of $D(Y)$. Identifying C with this subspace of $D(Y)$, we have that Z_C is just the restriction of $Z(Y)$ to C. In particular, the components of the restricted Douady space of all complex varieties from $\mathcal{X}$ also live definably in $\mathcal{X}$. The following proposition tells us that no new parameters are needed to define them.

Proposition 2.69. *Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$, Y is a finite cartesian product of sorts from $\mathcal{X}$, and $X \subset Y$ is a complex subvariety that is $\mathcal{L}_D(\mathcal{X})$-definable over a countable set of parameters A. Then*

(i) *the restricted universal family, $Z(X) \to D(X)$, is invariant under the action of $\mathcal{L}_D(\mathcal{X})$-automorphisms that fix A pointwise; and,*

(ii) *if C is an irreducible component of $D(X)$, and Z_C is the restriction of $Z(X)$ to C, then $Z_C \to C$ is $\mathcal{L}_D(\mathcal{X})$-definable over $acl_{\mathcal{X}}(A)$.*

Proof. We have already observed that $Z(X) \to D(X)$ can be witnessed as a subspace of $Z(Y) \to D(Y)$. For any irreducible component D of $D(Y)$, let $F_D = \{a \in D \mid Z(Y)_a \subset X\}$. Then $Z(X) \to D(X)$ is the union of the restrictions of $Z(Y) \to D(Y)$ to F_D, as D varies through all irreducible components of $D(Y)$. Moreover, it is clear from the above expression that F_D is $\mathcal{L}_D(\mathcal{X})$-definable over A. Any automorphism of $\mathcal{X}$ that fixes A pointwise will fix F_D, and hence the restriction of $Z(Y) \to D(Y)$ to F_D. That is, $Z(X) \to D(X)$ is A-invariant under the action of $\mathcal{L}_D(\mathcal{X})$-automorphisms.

Now let C be an irreducible component of $D(X)$, and Z_C the restriction of $Z(X)$ to C. So $C \subset F_D$ for some irreducible component D of $D(Y)$. If α is an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$, then $\alpha(C)$ is also an irreducible component of $D(X)$, and also contained in F_D. Now F_D is a definable subset of D that is covered by the irreducible components of $D(X)$. Since $D(X)$ has countably many components, F_D must be covered by only finitely many of them, by ω_1-compactness. Hence the orbit of C under $\mathcal{L}_D(\mathcal{X})$-automorphisms of $\mathcal{X}$, that fix A pointwise, is finite. So $Z_C \to C$ has only finitely many A-conjugates. This implies that $Z_C \to C$ is $\mathcal{L}_D(\mathcal{X})$-definable over $acl_{\mathcal{X}}(A)$. □

Relative Saturation

We end this section with a brief aside on a relative notion of saturation. We return to the notation of Section 2.3: $\mathcal{A}'$ is a saturated elementary extension of $\mathcal{A}$ of cardinality $\kappa > |\mathcal{L}|$. Moreover, all parameter sets are assumed to be of cardinality less than κ.

Definition 2.70. Suppose F is an absolutely irreducible Zariski closed set in $\mathcal{A}'$ over a parameter set A. Then we say that F is *generically saturated* if there exists parameters $B \supset A$, a generic point c of F over B, and a saturated complex variety Z, such that c is interdefinable with an element in $Z^{\mathcal{A}'}$ over B. Given a tuple c from $\mathcal{A}'$, we say that $tp(c/A)$ is *saturated* (or *c is saturated over A*) if the $acl(A)$-locus of c is generically saturated. If F is the generic fibre of a fibre space $G \to S$, then we say that $G \to S$ is a *saturated morphism* (or that *G is relatively saturated over S*), if F is generically saturated.

Remark 2.71. Comparing to Definition 2.34, and recalling that projective space is saturated, we have that if F is Moishezon then F is generically saturated. By Proposition 2.37, every Moishezon fibre space is a saturated morphism. Moreover, the nonstandard Riemann Existence Theorem (2.38) tells us that all 1-dimensional types are saturated.

Remark 2.72. Notice that $tp(c/A)$ is saturated if and only if $tp(c/acl(A))$ is internal to a saturated complex variety. This follows exactly as in Remark 2.35 using elimination of imaginaries in Douady-stable reducts of $\mathcal{A}$.

Lemma 2.73. *Suppose $G \to S$ is a fibre space of complex varities. Then G is relatively saturated over S if and only if for some complex variety T, and holomorphic surjection $T \to S$, the strict transform $G_{(T)}$ is bimeromorphic over T to a subspace of $T \times Z$, for some saturated complex variety Z. That is, there is a meromorphic embedding, g, that makes the following diagram commute:*

$$\begin{tikzcd}
G \arrow[d] & G_{(T)} \arrow[r, hook, "g"] \arrow[d] & T \times Z \arrow[dl] \\
S & T \arrow[l] &
\end{tikzcd}$$

Proof. This is just a translation of generic saturation by now familiar methods to the standard model. For example, the argument goes exactly as in the proof of Lemma 2.36. □

As a consequence, a complex variety is generically saturated if and only if it is bimeromorphic to a saturated complex variety. Indeed, if X is a generically saturated complex variety, then we have a complex variety T, a saturated complex variety Z, and a meromorphic embedding of $T \times X$ into $T \times Z$ over T. Hence the general fibre of $T \times X$, which is just X, is bimeromorphic to a complex subvariety of the general fubre of $T \times Z$, which is just Z. Since Z is saturated so is this subvariety.

We do not know whether saturation for complex varieties is stable under taking bimeromoprhisms. If this were the case, then Definition 2.70 would generalise the notion of a saturated complex variety to the relative category of complex varieties over a fixed base. Notice that if X and Y are bimeromorphic complex varieties, then there exists a complex variety Z, and modifications $Z \to X$ and $Z \to Y$. Indeed, let Z be the graph of a bimeromorphism between X and Y. Since the holomorphic image of a saturated complex variety is saturated (Lemma 2.54), to show that saturation is closed under bimeromorphism it is sufficient to show that the preimage of a complex variety under a modification is saturated. Furthermore, by a consequence of Hironaka's Flattening Theorem (Chow's Lemma), every modification is dominated by a finite sequence of blow-ups. That is, if $f: X \to Y$ is a modification then there is a complex variety Z and a commutative diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & Z \arrow[l] \arrow[dl] \\ Y & \end{tikzcd}$$

where $Z \to Y$ is a finite sequence of blow-ups and $Z \to X$ is a holomorphic surjection. For details on blow-ups, we refer the reader to [30]. Hence the question of whether saturation is stable under bimeromorphism reduces to:

Question 2.74. Suppose Y is a saturated complex variety and $X \to Y$ is a blow-up. Is it the case that X is saturated?

2.5 Moishezon Reductions: an Application

Suppose X is a complex variety. The meromorphic function field of X, denoted by $\mathbb{C}(X)$, is a finitely generated extension of $\mathbb{C}$ of transcendence degree $\leq dimX$ over $\mathbb{C}$. This transcendence degree is called the *algebraic dimension* of X, and is denoted by $a(X)$. Since any bimeromorphism induces an isomorphism of meromorphic function fields, $\mathbb{C}(X)$ and $a(X)$ are bimeromorphic invariants of X. However, $\mathbb{C}(X)$ does not capture the bimeromorphism type of X; in the sense that if $f: X \to Y$ is a meromorphic surjection with $f^*: \mathbb{C}(Y) \to \mathbb{C}(X)$ an isomorphism, it is not necessarily the case that f is a bimeromorphism. It is only the "algebraic part" of X that is encoded by $\mathbb{C}(X)$. This can be explained using the notion of "algebraic reductions".

Since $\mathbb{C}(X)$ is finitely generated over $\mathbb{C}$, there is a projective manifold of dimension $a(X)$, V, whose rational function field is $\mathbb{C}(X)$. Let $\mathbb{C}[P_0, \ldots P_n]$ be a homogeneous coordinate ring for V such that $f_1 = \frac{P_1}{P_0}, \ldots, f_n = \frac{P_n}{P_0}$ generate $\mathbb{C}(X)$. We can view $f_1, \ldots, f_n$ as meromorphic functions on X. The induced meromorphic map from X to $\mathbb{P}_n$ given by

$x \mapsto (1 : f_1(x) : \ldots : f_n(x))$,

defines a meromorphic surjection $f\colon X \to V$, such that $f^*\colon \mathbb{C}(V) \to \mathbb{C}(X)$ is an isomorphism. Such a map is called an *algebraic reduction* of X.

In order to stabilise this notion under bimeromorphic equivalence, we say that $f\colon X \to W$ is a *Moishezon reduction* if W is a Moishezon variety of dimension $a(X)$, and f is a surjective meromorphism such that $f^*\colon \mathbb{C}(W) \to \mathbb{C}(X)$ is an isomorphism. Moishezon reductions satisfy the following universal property: if f' is a meromorphic surjection of X onto another Moishezon variety W', then there is a meromorphic map $\alpha\colon W' \to W$, such that f' factors through f and α. In particular, W is uniquely determined up to bimeromorphism; and we will denote it by $A(X)$. Note that X is a Moishezon variety if and only if $a(X) = dim X$.

The Moishezon reduction of X is exactly what the meromorphic function field encodes. That is, if $f\colon X \to Y$ is a meromorphic surjection with $f^*\colon \mathbb{C}(Y) \to \mathbb{C}(X)$ an isomorphism, then there exists a bimeromorphism $f'\colon A(X) \to A(Y)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ A(X) \arrow[r, "f'"] & A(Y) \end{tikzcd}$$

It is in this sense that the "algebraic part" of a complex variety is encoded by the meromorphic function field, and hence the algebraic dimension, of a complex variety. We refer the reader to [6] for a more complete discussion of these notions.

The following lemma essentially says that in the saturated context there is a canonical Moishezon reduction.

Lemma 2.75. *Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$, A is a countable set of parameters in $\mathcal{X}$, and X is an irreducible Zariski closed set in $\mathcal{X}$ that is $\mathcal{L}_D(\mathcal{X})$-definable over A. Then there is a Moishezon reduction of X in $\mathcal{X}$ that is also $\mathcal{L}_D(\mathcal{X})$-definable over A.*

Proof. Let $g\colon X \to W$ be an algebraic reduction of X, and let $\Gamma(g) \subset W \times X$ be the graph of g. Using Hironaka's Flattening Theorem we can find a bimeromorphic copy $\Gamma(g)' \subset W' \times X$ that is flat over W'. Now $\Gamma(g)'$ is the graph of a Moishezon reduction $g' : X \to W'$. We may as well assume that $g = g'$ and that $\Gamma(g) \to W$ is flat.

Let V be the image of W in the restricted Douady Space of X, $D(X)$. Let G be the restriction of $Z(X)$ to V. Since $\Gamma(g)$ restricted to a Zariski open set of W is the graph of a map, the Douady map, $W \to D(X)$, is a bimeromorphism with its image, namely V. Hence, $G \to V$ is a bimeromorphic copy of $\Gamma(g) \to W$. So $G \subset V \times X$ is also the graph of a Moishezon reduction of X, say $h: X \to V$. Moreover, as V is in $\mathcal{X}$, $h: X \to V$ is $\mathcal{L}_D(\mathcal{X})$-definable. Our goal is to show that h is $\mathcal{L}_D(\mathcal{X})$-definable over A.

Let $G' \to V'$ be a conjugate of $G \to V$, by an $\mathcal{L}_D(\mathcal{X})$-automorphism of $\mathcal{X}$ that fixes A pointwise. Clearly, $G' \subset V' \times X$ is also the graph of a Moishezon reduction of X. Moreover, since $Z(X) \to D(X)$ is A-invariant under $\mathcal{L}_D(\mathcal{X})$-automorphisms (Proposition 2.69), $V' \subset D(X)$ and G' is the restriction of $Z(X)$ to V'. Now, there are Zariski open subsets U and U' of V and V' respectively, such that G and G' restricted to U and U' induce the same fibrations on X (this is by the universal property for Moishezon reductions). By the uniqueness of the Douady maps, $U = U'$, and so V and V' intersect densely in $D(X)$. It follows, by counting dimensions, that $V = V'$. Hence $G \to V$ and $G' \to V'$ are identical.

We have shown that $G \to V$ is fixed by all $\mathcal{L}_D(\mathcal{X})$-automorphisms of $\mathcal{X}$ that fix A pointwise. By ω_1-saturation, $G \to V$ is $\mathcal{L}_D(\mathcal{X})$-definable over A. This implies that $h: X \to V$ is a Moishezon reduction of X that is $\mathcal{L}_D(\mathcal{X})$-definable over A, as desired. $\square$

Given a fixed complex variety S, there is a natural way to relativise the notion of an algebraic reduction to the category of complex varieties over S (see, Fujiki [18]). Interested only in bimeromorphic data, we give the Moishezon version of this definition.

Definition 2.76. Let $f: X \to S$ be a fibre space of complex varieties. A *relative Moishezon reduction* of f is a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

where Y is a complex variety, h is a meromorphic fibre space, g is a fibre space, and for 'general' $s \in S$, $h_s: X_s \to Y_s$ is a Moishezon reduction of X_s.

It is not necessarily the case that every fibre space over S has a relative Moishezon reduction. Campana [5] and Fujiki [18] have shown (independently) that relative Moishezon reductions (in fact *relative algeraic* *reductions*) for fibre spaces in the class $\mathcal{C}$ exist. In the saturated context, the existence of relative Moishezon reductions follows easily from model-theoretic arguments, as the following theorem demonstrates. Notice that since Kähler-type varieties are saturated, one recovers from this theorem the existence of Moishezon reductions for fibre spaces in $\mathcal{C}$.

Theorem 2.77. *Suppose $f: X \to S$ is a fibre space of complex varieties in a saturated reduct of $\mathcal{A}$. Then a relative Moishezon reduction of f exists (also in a saturated reduct of $\mathcal{A}$).*

Proof. Suppose $\mathcal{X}$ is a Douady-stable reduct of $\mathcal{A}$ in which $f: X \to S$ is $\mathcal{L}_D(\mathcal{X})$-definable over a countable set of parameters, say $A = acl_{\mathcal{X}}(A)$. Let $s \in S$ be an $\mathcal{L}_D(\mathcal{X})$-generic over A (see the discussion on page 81). By Lemma 2.75, there is a Moishezon reduction of X_s in $\mathcal{X}$ that is $\mathcal{L}_D(\mathcal{X})$-definable over As. Denote it by $g_s: X_s \to Y_s$. Let $a \in X_s$ be $\mathcal{L}_D(\mathcal{X})$-generic over As and $b = g_s(a)$. Then b is $\mathcal{L}_D(\mathcal{X})$-generic in Y_s over As. Let Y be the A-locus of (b, s), and $Y \to S$ the corresponding coordinate projection. Since s is $\mathcal{L}_D(\mathcal{X})$-generic in S over A, letting s vary in S, we obtain an extension of g_s to a partial map $g: X \to Y$, that is $\mathcal{L}_D(\mathcal{X})$-definable over A, and such that for general $s' \in S$, $g_{s'}$ is a definable map between the $X_{s'}$ and $Y_{s'}$. Moreover, for 'general' $t \in S$ (namely realisations of the $\mathcal{L}_D(\mathcal{X})$-type of s over A), $g_t: X_t \to Y_t$ is a Moishezon reduction. Indeed, g_s is a Moishezon reduction, and this is preserved under $\mathcal{L}_D(\mathcal{X})$-automorphisms. Since g is piecewise meromorphic, there is a meromorphic surjection $h: X \to Y$ that makes the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

and such that h agrees with g on a nonempty Zariski open subset of X. In particular, for 'general' $t \in S$, $h_t: X_t \to Y_t$ is a Moishezon reduction of X_t. We have thus produced a relative Moishezon reduction of f in $\mathcal{X}$. $\square$

Possibilities for Further Study

Moishezon reductions are relevant to the classification problem for saturated complex varieties. One is interested in understanding a saturated complex variety X, in so far as it differs from a Moishezon variety. The first step would clearly be to take a Moishezon reduction of X, say $f: X \to V$. Now the study of X has in some sense been reduced to the study of the 'general' fibres of f (modulo an understanding of projective varieties). The next step would then be to take a relative Moishezon reduction of $f: X \to V$, thereby isolating the algebraic parts of these fibres. Continuing in this manner, one obtains a succession of relative Moishezon reductions that eventually stabilises as follows:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "f_1"] \arrow[drrr, "f_n"] & & & \\ Y & Y_1 \arrow[l, "g_1"] & \cdots \arrow[l, "g_2"] & Y_n \arrow[l, "g_n"] \end{tikzcd}$$

where the 'general' fibres of Y_{i+1} over Y_i are Moishezon, and the 'general' fibres of $f_n: X \to Y_n$ are of algebraic dimension 0.

This approach to classification is very natural from the model-theoretic perspective; it essentially corresponds to a $\mathbb{P}(\mathbb{C})$-*analysis* of the generic type of X. One reduces the study of saturated complex varieties to (A), those that are $\mathbb{P}(\mathbb{C})$-analysable (the 'general' fibres of $Y_n \to Y$ above); and (B), those that are orthogonal to $\mathbb{P}(\mathbb{C})$ (the 'general' fibres of $X \to Y_n$ above).

Fujiki begins to carry out such an analysis for Kähler-type varieties in [17]. He is able to describe case (A) (more or less completely) in terms of Albanese reductions and homogeneous actions of linear algebraic groups (see Theorem 1 of [17]). This seems to be a combination of the definable "Galois" groups that appear in stability theory (see Theorem 7.4.8 in [32]) as well as a version of Chevalley's Theorem for definable groups (see [35]). Indeed, for the case $n = 1$ in the above diagram (i.e. $\mathbb{P}(\mathbb{C})$-analysable in 1-step), the above methods can in fact be used to give a model-theoretic understanding of Fujiki's theorem. It would be interesting, and seems natural, to give a thorough model-theoretic treatment of these methods in the saturated context. In the same paper, Fujiki also begins to analyse case (B) in terms of relative Kummer and Albanese reductions. A model-theoretic treatment of such relative reductions should be possible using methods analogous to the ones used to obtain relative Moishezon reductions in this thesis. It seems that both stability theory and complex analytic geometry could benefit from an exploration of the interactions between them.

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Vita

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