

## A note on subvarieties of powers of OT-manifolds

by  
RAHIM MOOSA<sup>(1)\*</sup> AND MATEI TOMA<sup>(2)</sup>

### Abstract

It is shown that the space of finite-to-finite holomorphic correspondences on an OT-manifold is discrete. When the OT-manifold has no proper infinite complex-analytic subsets, it then follows by known model-theoretic results that its cartesian powers have no interesting complex-analytic families of subvarieties. The methods of proof, which are similar to [Moosa, Moraru, and Toma “An essentially saturated surface not of Kähler-type”, *Bull. of the LMS*, 40(5):845–854, 2008], require studying finite unramified covers of OT-manifolds.

**Key Words:** Complex subvariety, correspondence, essential saturation, OT-manifold, strongly minimal compact complex manifold.

**2010 Mathematics Subject Classification:** Primary 32J18, 03C98.

### 1 Introduction

This note is concerned with complex-analytic families of subvarieties in cartesian powers of the compact complex manifolds introduced by Oeljeklaus and the second author in [7], here referred to as OT-manifolds. These manifolds are higher dimensional analogues of Inoue surfaces of type  $S_M$ . In [4], we, along with Ruxandra Moraru, showed that if  $X$  is an Inoue surface of type  $S_M$  then  $X^n$  contains no infinite complex-analytic families of subvarieties, except for the obvious ones such as  $(\{a\} \times V : a \in X^m)$  where  $V$  is a fixed subvariety of  $X^{n-m}$ . Using model-theoretic techniques we were able to reduce the problem to considering only the case of  $n = 2$ . That case amounted to showing that the set of finite-to-finite holomorphic correspondences on  $X$ , viewed as subvarieties of  $X^2$ , is discrete. Here we extend this result to OT-manifolds in general. Actually, it is useful to consider the following higher arity version of correspondences: for any compact complex manifold  $X$ , let  $\text{Corr}_n(X)$  denote the set of irreducible complex-analytic  $S \subset X^n$  such that the co-ordinate projections  $\text{pr}_i : S \rightarrow X$  are surjective and finite for all  $i = 1, \dots, n$ . So  $\text{Corr}_2(X)$  is the set of finite-to-finite holomorphic correspondences.<sup>1</sup>

---

\*R. Moosa was partially supported by an NSERC Discovery Grant. M. Toma was partially supported by the ANR project MNGNK, decision NANR-10-BLAN-0118.

<sup>1</sup>It may be worth pointing out that the elements of  $\text{Corr}_n(X)$  are simply components of intersections of pull-backs of finite-to-finite holomorphic correspondences. That is, for  $n > 1$ , if  $S \in \text{Corr}_n(X)$  and  $\pi_i : X^n \rightarrow X^2$  is

**Theorem 1.** *If  $X$  is an OT-manifold then  $\text{Corr}_n(X)$  is discrete for all  $n > 0$ .*

The proof, which we will give in Section 3, follows to some extent what was done for Inoue surfaces of type  $S_M$  in [4]. But this approach leads naturally to the consideration of finite unramified coverings of OT-manifolds, and the latter are not formally instances of the original construction in [7]. However, we show in Section 2 that a mild generalisation of that construction leads to a class of manifolds which is closed under finite unramified coverings. We call these manifolds also OT-manifolds and the theorem is valid for this larger class.

The theorem is particularly significant when  $X$  has no proper positive dimensional subvarieties, because of the following fact coming from model theory.

**Fact 2.** *Suppose  $X$  is a compact complex manifold that is not an algebraic curve, is not a complex torus, and has no proper infinite complex-analytic subsets. Then every irreducible complex-analytic subset of a cartesian power of  $X$  is a cartesian product of points and elements of  $\text{Corr}_n(X)$  for various  $n > 0$ .*

**Proof:** This is Proposition 5.1 of [9] together with Lemma 3.3(b) of [5]. □

That OT-manifolds without proper positive dimensional subvarieties are ubiquitous in all dimensions follows from work of Ornea and Verbitsky [8] showing that we get examples whenever  $X$  is the OT-manifold corresponding to a number field that has precisely two complex embeddings which are not real.

Putting together the Theorem and the Fact, we conclude:

**Corollary 3.** *Suppose  $X$  is an OT-manifold that has no proper infinite complex-analytic subsets. Then, for all  $n > 0$ ,  $X^n$  has no infinite complex-analytic families of subvarieties that project onto each co-ordinate.*

**Remark 4.** The model theorist should note that for  $X$  to have no proper infinite complex-analytic subsets is exactly *strong minimality* of  $X$  as a first-order structure in the language of complex-analytic sets. Strongly minimal OT-manifolds are of *trivial acl-geometry* by the manifestation of the Zilber trichotomy in this context. By [5, Proposition 3.5], the discreteness of  $\text{Corr}_2(X)$  implies that strongly minimal OT-manifolds are *essentially saturated* in the sense of [3]. In particular, we obtain in every dimension examples of essentially saturated manifolds that are not of Kähler-type. This was the original motivation for both [4] and the current note.

## 2 Finite covers of OT-manifolds

We will quickly review the original construction of OT-manifolds from [7] and then describe how to generalise it.

the co-ordinate projection  $(x_1, \dots, x_n) \mapsto (x_1, x_i)$ , for  $i = 2, \dots, n$ , then each  $\pi_i(S) \subset X^2$  is a correspondence and  $S$  is an irreducible component of  $\bigcap_{i=2}^n \pi_i^{-1}(\pi_i(S))$ . This is an easy dimension calculation, see [5, Lemma 3.2].

Fix a number field  $K$  admitting  $n = s + 2t$  distinct embeddings into  $\mathbb{C}$ , which we will denote by  $\sigma_1, \dots, \sigma_n$  where  $\sigma_1, \dots, \sigma_s$  are real and each  $\sigma_{s+i}$  is complex conjugate to  $\sigma_{s+i+t}$ . Assume that  $s$  and  $t$  are positive. By Dirichlet's Theorem the multiplicative group of units  $\mathcal{O}_K^*$  of the ring of integers  $\mathcal{O}_K$  of  $K$  has rank  $s + t - 1$ . The subgroup

$$\mathcal{O}_K^{*,+} := \{a \in \mathcal{O}_K^* : \sigma_i(a) > 0 \text{ for all } 1 \leq i \leq s\}$$

of "positive" units is free abelian of finite index in  $\mathcal{O}_K^*$ . Let  $U$  be a rank  $s$  subgroup of  $\mathcal{O}_K^{*,+}$  that is admissible for  $K$  in the sense of [7]. With respect to the natural action of  $U$  on the additive group  $\mathcal{O}_K$ , consider the semidirect product  $\Gamma = U \ltimes \mathcal{O}_K$ . Let  $m = s + t$  and consider the action of  $\Gamma$  on  $\mathbb{C}^m$  given by,

$$(a, x)(z_1, \dots, z_m) := (\sigma_1(ax) + \sigma_1(a)z_1, \dots, \sigma_m(ax) + \sigma_m(a)z_m).$$

As  $U < \mathcal{O}_K^{*,+}$ , this action leaves  $\mathbb{H}^s \times \mathbb{C}^t$  invariant, and the admissibility condition is equivalent to the action being proper and discontinuous. The original OT-manifold, denoted by  $X(K, U)$ , is the quotient of  $\mathbb{H}^s \times \mathbb{C}^t$  by this action. In the sequel we will denote these manifolds by  $X(\mathcal{O}_K, U)$  in order to distinguish them from their generalisations.

The above construction is generalised by replacing the role of  $\mathcal{O}_K$  in  $\Gamma$  by any rank  $n$  additive subgroup  $M \leq \mathcal{O}_K$  that is stable under the action of  $U$ . We say then that  $U$  is admissible for  $M$ . Taking  $\Gamma = U \ltimes M$ , we again get a proper and discontinuous action on  $\mathbb{H}^s \times \mathbb{C}^t$ , and the quotient is denoted by  $X(M, U)$ . We will continue to call these compact complex manifolds OT-manifolds. To avoid confusing them with the previous construction we will occasionally say that they are of type  $X(M, U)$  (otherwise of type  $X(\mathcal{O}_K, U)$ ). Note that the possibility of generalising the original construction by replacing  $\mathcal{O}_K$  with an order of  $K$  is already mentioned in [7]. However only the  $\mathbb{Z}$ -submodule structure of  $M$  and the stability under the  $U$ -action are necessary to make the construction work.

The universal cover of  $X(M, U)$  is  $\mathbb{H}^s \times \mathbb{C}^t$  and the fundamental group is  $U \ltimes M$ . As the latter is of finite index in  $U \ltimes \mathcal{O}_K$ , we see that  $X(M, U)$  is a finite unramified covering of  $X(\mathcal{O}_K, U)$ . In fact, all finite unramified covers are of this form:

**Lemma 5.** *The class of OT-manifolds of type  $X(M, U)$  is closed under finite unramified coverings.*

**Proof:** Given  $X(M, U)$ , such a covering would correspond to a finite index subgroup  $\Gamma_1 \leq U \ltimes M$ . Taking  $U_1$  to be the image of  $\Gamma_1$  in  $U$ , and setting  $M_1 := \Gamma_1 \cap M$ , it is not hard to check that  $U_1$  is admissible for  $M_1$  and that the covering is nothing other than  $X(M_1, U_1)$ .  $\square$

Much of the theory of OT-manifolds developed in [7] goes through in this more general setting. In particular,

**Lemma 6.** *If  $X = X(M, U)$  is an OT-manifold then  $H^0(X, T_X) = 0$ .*

**Proof:** For OT-manifolds of type  $X(\mathcal{O}_K, U)$  this is Proposition 2.5 of [7]. Imitating that argument, it suffices to prove for  $M$  a rank  $n$  additive subgroup of  $\mathcal{O}_K$ , that the image of  $M$  in  $\mathbb{R}^s$  under  $(\sigma_1, \dots, \sigma_s)$  is dense. But this is the case because  $M$  has finite index in  $\mathcal{O}_K$  and the latter does have dense image (see the proof of Lemma 2.4 of [7]).  $\square$

The following remarks serve as further evidence that the above extension of the definition of OT-manifolds is natural.

**Remark 7.** Any OT-manifold of type  $X(M, U)$  admits a finite unramified cover of type  $X(\mathcal{O}_K, U)$ .

**Proof:** Indeed, since  $M$  is of maximal rank in  $\mathcal{O}_K$ , there exists a positive integer  $l$  such that  $l\mathcal{O}_K \subset M$ . Thus  $X(l\mathcal{O}_K, U)$  is a finite unramified cover of  $X(M, U)$ . But the multiplication by  $l$  at the level of  $\mathbb{H}^s \times \mathbb{C}^t$  conjugates the actions of  $U \times \mathcal{O}_K$  and of  $U \times l\mathcal{O}_K$  and thus induces an isomorphism between  $X(\mathcal{O}_K, U)$  and  $X(l\mathcal{O}_K, U)$ .  $\square$

**Remark 8.** When  $s = t = 1$  the class OT-manifolds of type  $X(M, U)$  coincides with the class of Inoue surfaces of type  $S_M$  defined in [2].

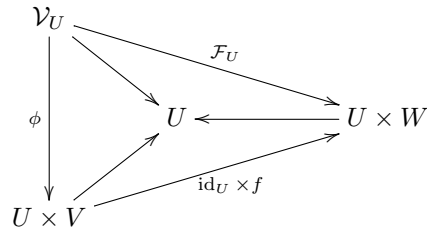
**Proof:** Indeed, if one starts with the manifold  $X(M, U)$ , then choosing a generator  $a$  of  $U$  with  $\sigma_1(a) > 1$  and a base  $(\alpha_1, \alpha_2, \alpha_3)$  of  $M$  over  $\mathbb{Z}$  one obtains a matrix  $A(a) \in GL(3, \mathbb{Z})$  which represents the action of  $a$  on  $M$  with respect to this basis. Applying the embedding  $\sigma_k$  to the relation  $a(\alpha_1, \alpha_2, \alpha_3)^\top = A(a)(\alpha_1, \alpha_2, \alpha_3)^\top$  shows that  $(\sigma_k(\alpha_1), \sigma_k(\alpha_2), \sigma_k(\alpha_3))^\top$  is an eigenvector of  $A(a)$  associated to the eigenvalue  $\sigma_k(a)$ . In particular this implies  $A(a) \in SL(3, \mathbb{Z})$  since  $\sigma_1(a) > 0$ . At this point one sees that  $X(M, U)$  coincides with the surface  $S_{A(a)}$  as defined in [2].

Conversely, starting with any matrix  $A \in SL(3, \mathbb{Z})$ , with one real eigenvalue larger than 1 and two complex non-real eigenvalues, we denote by  $K$  the splitting field of the characteristic polynomial  $\chi_A$  of  $A$  over  $\mathbb{Q}$ . Then there exists an element  $a \in \mathcal{O}_K^{*,+}$  such that the eigenvalues of  $A$  (i.e the roots of  $\chi_A$ ) are precisely  $\sigma_1(a), \sigma_2(a), \sigma_3(a)$ . We find now an eigenvector  $v \in \mathbb{Z}[\sigma_1(a)]^3$  associated to  $\sigma_1(A)$  by solving the system  $(A - \sigma_1(a)I_3)v^\top = 0$  over  $K$ . There exist now elements  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}_K$  such that  $v = (\sigma_1(\alpha_1), \sigma_1(\alpha_2), \sigma_1(\alpha_3))$ . Moreover  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent over  $\mathbb{Q}$  since a linear relation would entail a linear relation between the components  $v_1, v_2, v_3$  of  $v$  over  $\mathbb{Q}$ , which combined with the equations  $(A - \sigma_1(a)I_3)v^\top = 0$  would show that  $\sigma_1(a)$  is quadratic over  $\mathbb{Q}$ . Now choosing  $M$  to be the  $\mathbb{Z}$ -submodule of  $K$  generated by  $\alpha_1, \alpha_2, \alpha_3$  and  $U$  the multiplicative group generated by  $a$  we get again  $X(M, U) = S_{A(a)}$ .  $\square$

### 3 The Proof

As in the case of Inoue surfaces of type  $S_M$  studied in [4], we will make use of some deformation theory to prove the main theorem. But we will need a bit more than was used in [4]. We say that a holomorphic map  $f : V \rightarrow W$  between compact complex manifolds is *rigid over*

$W$  if there are no nontrivial deformations of  $f$  that keep  $W$  fixed. More precisely: Whenever  $\mathcal{V} \rightarrow D$  is a proper and flat holomorphic map of compact complex varieties with  $V = \mathcal{V}_d$  for some  $d \in D$ , and  $\mathcal{F} : \mathcal{V} \rightarrow D \times W$  is a holomorphic map over  $D$  with  $\mathcal{F}_d = f$ , then there is an open neighbourhood  $U$  of  $d$  in  $D$  and a diagram



where  $\phi$  is a biholomorphism. In particular  $\mathcal{F}_s(\mathcal{V}_s) = f(V)$  for all  $s \in U$ .

**Fact 9** (Section 3.6 of [6]). *Suppose  $f : V \rightarrow W$  is a holomorphic map between compact complex manifolds such that*

- $H^0(V, f^*T_W) = 0$ , and
- $f_* : H^1(V, T_V) \rightarrow H^1(V, f^*T_W)$  is injective.

*Then  $f$  is rigid over  $W$ .*

**Lemma 10.** *Suppose  $X$  and  $Y$  are compact complex manifolds,  $H^0(Y, T_Y) = 0$ , and  $f : Y \rightarrow X^n$  is a holomorphic map such that  $\text{pr}_i \circ f : Y \rightarrow X$  is a finite unramified cover for each  $i = 1, \dots, n$ . Then  $f$  is rigid over  $X^n$ .*

**Proof:** Note that here  $\text{pr}_i : X^n \rightarrow X$  is the projection onto the  $i$ th co-ordinate. Let  $f_i := \text{pr}_i \circ f : Y \rightarrow X$ . As each  $f_i$  is unramified, we have that

$$f^*T_{X^n} = f^* \left( \bigoplus_{i=1}^n \text{pr}_i^* T_X \right) = \bigoplus_{i=1}^n f_i^* T_X = \bigoplus_{i=1}^n T_Y$$

Hence,  $H^0(Y, f^*T_{X^n}) = \bigoplus_{i=1}^n H^0(Y, T_Y) = 0$ . On the other hand, the isomorphism  $(f_1)_* : H^1(Y, T_Y) \rightarrow H^1(Y, f_1^*T_X)$  factors through  $f_* : H^1(Y, T_Y) \rightarrow H^1(Y, f^*T_{X^n})$ , and hence the latter is injective. So  $f : Y \rightarrow X^n$  is rigid over  $X^n$  by Fact 9.  $\square$

We can now prove the main theorem.

[**Proof of Theorem 1**] Suppose  $X$  is an OT-manifold of type  $X(M, U)$ . As in [4], in order to show that  $\text{Corr}_n(X)$  is discrete we let  $S \in \text{Corr}_n(X)$  be arbitrary, consider the irreducible component  $D$  of the Douady space of  $X^n$  in which  $S$  lives, and show that  $D$  is zero-dimensional. This suffices as it proves that each element of  $\text{Corr}_n(X)$  is isolated in the Douady space.

Let  $Z \subset D \times X^n$  be the restriction of the universal family to  $D$ . By the flatness of  $Z \rightarrow D$ , for general  $d \in D$ ,  $Z_d \in \text{Corr}_n(X)$  also. Let  $\tilde{Z} \rightarrow Z$  be a normalisation and denote by  $f : \tilde{Z} \rightarrow D \times X^n$  the composition of the normalisation with the inclusion of  $Z$  in  $D \times X^n$ . Then for general  $d \in D$  we have that  $f_d : \tilde{Z}_d \rightarrow X^n$  is such that each projection  $\text{pr}_i \circ f_d : \tilde{Z}_d \rightarrow X$  is a finite surjective map. In [1] it is shown that OT-manifolds of type  $X(\mathcal{O}_K, U)$ , and hence also OT-manifolds of type  $X(M, U)$ , have no divisors. So the purity of branch locus theorem (which applies as  $\tilde{Z}_d$  is normal and  $X$  is smooth) implies that  $\text{pr}_i \circ f_d$  is a finite unramified covering. In particular,  $\tilde{Z}_d$  is a generalised OT-manifold by Lemma 5, and so  $H^0(\tilde{Z}_d, T_{\tilde{Z}_d}) = 0$  by Lemma 6. But moreover, by Lemma 10,  $f_d$  is rigid over  $X^n$ . It follows that for some open neighbourhood  $U$  of  $d$  in  $D$ ,  $f_U : \tilde{Z}_U \rightarrow U \times X^n$  is biholomorphic over  $U \times X^n$  with  $\text{id}_U \times f_d : U \times \tilde{Z}_d \rightarrow U \times X^n$ . In particular, for all  $s \in U$ ,  $Z_s = f_s(\tilde{Z}_s) = f_d(\tilde{Z}_d) = Z_d$ . The universality of the Douady space now implies that  $U = \{d\}$ , so that in fact  $D = \{d\}$ , as desired.

## References

- [1] L. BATTISTI AND K. OELJEKLAUS, Holomorphic line bundles over domains in Cousin groups and the algebraic dimension of OT-manifolds, Preprint, arXiv:1306.3047.
- [2] M. INOUE, On surfaces of class VII<sub>0</sub>, *Invent. Math.* 24 (1974), 269-310.
- [3] R. MOOSA, On saturation and the model theory of compact Kähler manifolds, *Journal für die reine und angewandte Mathematik*, 586:1–20, 2005.
- [4] R. MOOSA, R. MORARU, AND M. TOMA, An essentially saturated surface not of Kähler-type. *Bulletin of the London Mathematical Society*, 40(5):845–854, 2008.
- [5] R. MOOSA AND A. PILLAY,  $\aleph_0$ -categorical strongly minimal compact complex manifolds, *Proc. Amer. Math. Soc.*, 140(5):1785–1801, 2012.
- [6] M. NAMBA, *Families of meromorphic functions on compact Riemann surfaces*, volume 767 of *Lecture Notes in Mathematics*, Springer, Berlin, 1979.
- [7] K. OELJEKLAUS AND M. TOMA, Non-Kähler compact complex manifolds associated to number fields, *Ann. Inst. Fourier (Grenoble)*, 55(1):161–171, 2005.
- [8] L. ORNEA AND M. VERBITSKY, Oeljeklaus-Toma manifolds admitting no complex subvarieties, *Math. Res. Lett.*, 18(4):747–754, 2011.
- [9] A. PILLAY AND T. SCANLON, Meromorphic groups, *Transactions of the American Mathematical Society*, 355(10):3843–3859, 2003.

Received: 25.11.2014

Accepted: 24.01.2015

<sup>(1)</sup>University of Waterloo  
 Department of Pure Mathematics  
 200 University Avenue West  
 Waterloo, Ontario N2L 3G1  
 Canada.  
 E-mail: rmoosa@uwaterloo.ca

<sup>(2)</sup>Institut de Mathématiques Elie Cartan  
 Université de Lorraine, B.P. 70239  
 54506 Vandoeuvre-lès-Nancy Cedex  
 France.  
 E-mail: Matei.Toma@univ-lorraine.fr