An Introduction to Equations and Equational Theories

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Abstract

One of the primary objects of study in model theory is definable sets. In many common theories, such as the theory of R-modules, the theory of algebraically closed fields, and the theory of differentially closed fields, we can often find a well behaved collection of "closed" definable sets, with respect to which all definable sets can be expressed as finite boolean combinations. Equations and equational theories, introduced by Anand Pillay and Gabriel Srour in 1984 are an abstraction of this useful concept.

1 Introduction

It is often the case in model theory that we can find a collection of "nice" definable sets, from which we can construct all the other definable sets of the theory. Examples are the Zariski closed sets in ACF, the Kolchin closed sets in DCF and positive primitive definable sets in the theory of R-modules. Equational theories, largely due to Anand Pillay and Gabriel Srour in 1984, is one of the earliest attempts to isolate in what sense these sets are "nice" and then abstract to a general model theoretic setting.

Loosely speaking, a formula is an "equation" if any collection of instances is equivalent to a finite subset of those instances. A set will be "closed" if it is defined by an instance of an equation. And a theory, T, will be called equational if every definable set can be expressed as a boolean combination of these "closed" sets.

There are three primary sources for this paper. The first is *Closed Sets and Chain Conditions in Stable Theories*, the joint work by A. Pillay and G. Srour [5]. Their paper deals with many of the closure conditions we'll discuss later on, and shows where equational theories lie in terms of concepts like ω-stable theories and one-based theories. Their work is considered as the primary reference for any study of equations and equational theories. M. Junker's *A Note on Equational Theories* contains many useful properties of equations and equational theories [3]. It also clarifies the different notions of an "equational theory" that had been circulating until its publication. Lastly, G. Srour's *The notion of Independence In Categories Of Algebraic Structures, Part II: S-Minimal Extensions* takes a non model theory approach to the notion of equations [9]. As such it is the most abstract of this paper's references. Together with Part I and Part III, [8] and [10], Srour explores the concept of an equation in broad terms touching on category theory and universal algebra.

The layout for this paper is as follows. In Section 2 we will define equations, following up with many examples. We will then investigate the basic properties of equations, and generalise the equation concept to sets of formulas, rather than one formula at a time. Section 3 will deal with of the topological structure that equations are endowed with. Section 4 will introduce the notion of an equational theory, and will explore a few examples as well as some interesting results, culminating in a notion of independence based on "equational freeness".

Model Theory Background

As is to be expected, a basic understanding of model theory is required to make sense of this paper. An excellent reference is D. Marker's *Model Theory: An Introduction*, [4]. For the majority of this paper, the first four chapters and appendix A of Marker's book will suffice as background knowledge.

By L, we mean a first order language. Per model theoretic convention, $\mathcal{M}, \mathcal{N}, \cdots$ will denote models of a given theory. Correspondingly, $M, N, \cdots$ will denote the domains of these models. Capital letters from the beginning of the alphabet, like $A, B, C, \cdots$, will represent subsets of domains of models.

Tuples of variables will be represented by later letters in the alphabet, like $\overline{x} = (x_1, \cdots, x_n)$ and $\overline{y}$. Tuples of elements in a domain will use earlier letters, $\overline{a}, \overline{b}, \cdots$. Ultimately, the context will make clear which is which. By $|\overline{x}|$ we mean the length of the tuple. Although not often considered in mathematics, this paper will make use of the unique 0-tuple. That is, the tuple of variables or elements which has length 0 (no variables or elements in it).

When it comes to formulas we use greek letters, as is standard, like $\phi, \psi, \chi, \cdots$. When we write a formula with its tuple of variables we will often want to distinguish a particular partition of the variables with a semicolon, $\phi(\overline{x}; \overline{y})$. As M. Junker describes it, this is to distinguish between the so called "special variables", $\overline{x}$, and the "parameter variables", $\overline{y}$. More often than not, the special variables will be denoted by $\overline{x}$, or some variation thereof.

Consider linear equations, $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$, where the a_is are particular constants. If we wanted to generalize this equation to a formula we would make something like, $\phi(\overline{x}; \overline{y}) = (y_1 x_1 + y_2 x_2 + \cdots + y_n x_n = 0)$. Here we keep the "special" nature of the x_i's, in that they are still the variables we are going to determine solutions for. But now, rather than looking at one particular instance of a linear equation we can consider all of them. By plugging in different tuples of elements for the $\overline{y}$ we can get many different linear equations and deal with them all at once. The $\overline{y}$ tuple is the parameter variable. When we start to substitute in tuples of elements we will drop the semicolon in favour of a comma to signal the distinction between the raw form of the formula and an instance.

As we mentioned just now, and before in the abstract, the properties of equations we are going to investigate come from the solution sets to the formula. If we are working in a model, $\mathcal{M}$, of some theory, we denote by $\phi(M, \overline{a})$, where $\overline{a} \in M^{|\overline{y}|}$, the set of all tuples in $M^{|\overline{x}|}$ that satisfy $\phi(\overline{x}, \overline{a})$ in $\mathcal{M}$. The L_M-formula $\phi(\overline{x}, \overline{a})$ is called an **instance** of $\phi(\overline{x}; \overline{y})$ while, $\phi(M, \overline{a}) = \{\overline{b} \in M^n \mid \mathcal{M} \models \phi(\overline{b}, \overline{a})\}$ is called the **solution set** of this instance.

One of the aims of this paper is to present equations and equational theories as a natural object to look at in mathematics. While equational theories are related to stable theories, which is a very important notion in model theory, we wish our discussion to be more or less independent of stability theory; indeed we hope to present equational theories as a warm-up toward stability. To this end no stability theory will be assumed on the part of the reader.

Some Chain Conditions

To begin our investigation we introduce a few conditions involving set intersections and chains of sets ordered by the $\subseteq$ relation.

Definition 1.1. *A family of sets, $\mathcal{F}$, is said to have the* **descending intersection condition** *if every intersection of sets in $\mathcal{F}$ can be expressed as the intersection of a finite subset of those sets. That is, for any collection*

$\{A_i \in \mathcal{F} \mid i \in I\}$, *we can find a finite subset* $I_0 \subseteq I$ *so that* $\cap_{i \in I} A_i = \cap_{i \in I_0} A_i$. *We often abbreviate "descending intersection condition" to* **DIC**.

A slightly stronger notion than the descending intersection condition is given by placing an upperbound on the size of the finite subcollection our intersection reduces to.

Definition 1.2. *Fix a natural number,* $n \in \omega$. *A family of sets,* $\mathcal{F}$, *is said to have the* **bounded descending intersection condition with an upper bound of n** *if every intersection of sets in* $\mathcal{F}$ *can be expressed as the intersection of a finite subset of those sets of size less than or equal to* n. *That is, for any collection* $\{A_i \in \mathcal{F} \mid i \in I\}$, *we can find a subset* $I_0 \subseteq I$, *with* $|I_0| \leq n$ *so that* $\cap_{i \in I} A_i = \cap_{i \in I_0} A_i$. *We often abbreviate "bounded descending intersection condition with an upper bound of* n*" to* **n-DIC**.

It is rather clear, when comparing Defintion 1.1 and Defintion 1.2 that for any $m, n \in \omega$ with n < m, $n - DIC \implies m - DIC \implies DIC$.

A closely related concept to the descending intersection condition is the descending chain condition.

Definition 1.3. *A family of sets,* $\mathcal{F}$, *is said to have the* **descending chain condition** *if it contains no infinite proper descending chain of sets. That is, we cannot find a collection of sets,* $\{A_i\}_{i \in \omega}$, *with each* $A_i \in \mathcal{F}$, *with the property that* $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq A_3 \supsetneq \cdots$. *We often abbreviate "descending chain condition" to* **DCC**.

Just as restricting the size of the intersection allowed us to strengthen the descending intersection condition, we now restrict the length of chains to strengthen the descending chain condition.

Definition 1.4. *Fix a natural number,* $n \in \omega$. *A family of sets,* $\mathcal{F}$, *is said to have the* **bounded descending chain condition with an upper bound of n** *if it contains no proper descending chain of sets of size n+1. That is, we cannot find a collection of sets,* $\{A_0, A_1, \cdots, A_n\}$, *with each* $A_i \in \mathcal{F}$, *with the property that* $A_0 \supsetneq A_1 \supsetneq \cdots \supsetneq A_n$. *We often abbreviate "bounded descending chain condition with an upper bound of* n*" to* **n-DCC**.

Again, it is clear, when comparing Defintion 1.3 and Defintion 1.4 that for any $m, n \in \omega$ with n < m, $n - DCC \implies m - DCC \implies DCC$. As well, one can see from the definitions, that $DIC \implies DCC$.

Now that we have our conditions defined, we note a very simple fact. It is rather apparent from our definitions and as such does not require a proof.

Fact 1.5. *Suppose that* $\mathcal{F}$ *and* $\mathcal{G}$ *are families of sets with* $\mathcal{G} \subseteq \mathcal{F}$. *If* $\mathcal{F}$ *satisfies the DCC, DIC, n-DCC or n-DIC then* $\mathcal{G}$ *satisfies the DCC, DIC, n-DCC or n-DIC respectively. This means that our conditions are preserved when we take subfamilies.*

Definition 1.6. *Given a family of sets,* $\mathcal{F}$, *we define* **the closure of** $\mathcal{F}$ **under finite intersections** *to be the collection of all sets that can be formed from* $\mathcal{F}$ *using intersections,* $cl(\mathcal{F}) = \{\cap_{i=0}^{n} A_i \mid n \in \omega,\ A_i \in \mathcal{F} \text{ for all } i < n\}$.

The following fact is simple to see from the definition.

Fact 1.7. *Let* $\mathcal{F}$ *be any collection of sets.*
(i) $cl(cl(\mathcal{F})) = cl(\mathcal{F})$
(ii) Suppose that $\mathcal{G} \subseteq \mathcal{F}$. *Then* $cl(\mathcal{G}) \subseteq cl(\mathcal{F})$.
(iii) Suppose that $X \in cl(\mathcal{F})$, *then there exists a finite subset* $\mathcal{G} \subset \mathcal{F}$ *so that* $X \in cl(\mathcal{G})$.

Using the closure we can relate the DCC and DIC.

Proposition 1.8. *A family of sets, $\mathcal{F}$, satisfies the DIC if and only if $cl(\mathcal{F})$ satisfies the DCC.*

Proof. ($\Leftarrow$) Let $\{a_i \mid i \in I\}$ be a collection of sets in $\mathcal{F}$. By the Well Ordering Property we may assume that I is an ordinal number. For each $i \in I$ define B_i to be $\cap_{j \le i} A_j$. Notice that each $B_i \in cl(\mathcal{F})$ and that we have formed a descending chain, $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$. By the descending chain condition we can see that there are only finitely many $i \in I$ so that $B_{i-1} \supsetneq B_i$. Label them $i_1 < i_2 < \cdots < i_n$. Additionly, let $i_0 = 0$. We claim that $\cap_{i \in I} A_i = \cap_{j=0}^{n} A_{i_j}$.

Indeed, we start with $A_{i_0} = A_0 = B_0$ and by definition of the i_j, for any $0 < k < i_1$ we know that $B_{k-1} = B_k$. Hence $A_0 = A_0 \cap A_1 = \cdots = \cap_{k < i_1} A_k$. It follows that $B_{i_1} = B_{i_1 - 1} \cap A_{i_1} = B_0 \cap A_{i_1} = A_{i_0} \cap A_{i_1}$.

Now, suppose that $B_{i_k} = \cap_{j \le k} A_{i_j}$. By definition of the i_j, for any $i_k < \ell < i_{k+1}$ we know that $B_{\ell-1} = B_\ell$. Hence $B_{i_k} = B_{i_k} \cap A_{i_k+1} = \cdots = B_{i_k} \bigcap (\cap_{i_k < j < i_{k+1}} A_j)$. Thus, $B_{i_{k+1}} = B_{i_{k+1}-1} \cap A_{i_{k+1}} = B_{i_k} \cap A_{i_{k+1}} = A_{i_0} \cap \cdots \cap A_{i_{k+1}}$.

By mathematical induction we have shown that $B_{i_n} = \cap_{j=0}^{n} A_{i_j}$. Finally, since i_n was the last index for which $B_{i-1} \supsetneq B_i$ we can see that for all $k > i_n$, we have $B_{i_n} = B_k$.

Thus, $\cap_{i \in I} A_i = B_{i_n} \bigcap (\cap_{i \in I, i > i_n} A_i) \supseteq B_{i_n} \bigcap (\cap_{i \in I, i > i_n} B_i) = B_{i_n} \bigcap (\cap_{i \in I, i > i_n} B_{i_n}) = B_{i_n} = \cap_{j=0}^{n} A_{i_j} \supseteq \cap_{i \in I} A_i$ By squeezing the result we have demonstrated that $\cap_{i \in I} A_i = \cap_{j=0}^{n} A_{i_j}$. So DCC for $cl(\mathcal{F}) \Rightarrow DIC$ for $\mathcal{F}$.

($\Rightarrow$) Let $\{A_i\}_{i \in \omega}$ be a descending chain of sets in $cl(\mathcal{F})$, $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$. Since each $A_i \in cl(\mathcal{F})$ we can find a finite number, n_i, and sets, A_{ij}, in $\mathcal{F}$ so that $A_i = A_{i0} \cap A_{i1} \cap \cdots \cap A_{in_i}$. Consider the collection $\{A_{ij} \mid i \in \omega, j \le n_i\}$.

By the descending intersection condition we can find finitely many i, j, say $i_0, i_1, \cdots, i_n$ and $j_0, j_1, \cdots, j_n$, so that $\cap_{\substack{i \in \omega \\ j \le n_i}} A_{ij} = \cap_{k=0}^{n} A_{i_k j_k}$. Let $\ell = max\{i_0, i_1, \cdots, i_n\}$. We claim that this descending chain has length at most ℓ.

Indeed, it suffices to prove that $A_\ell = A_m$ for all $m \ge \ell$. Because our sequence is a descending chain we can see that for any $k \in \omega$, $A_k = \cap_{j \le k} A_j$. By our maximal choice of ℓ we can see that $\cap_{i \in \omega} A_i = \cap_{\substack{i \in \omega \\ j \le n_i}} A_{ij} = \cap_{k=0}^{n} A_{i_k j_k} = A_\ell$. Thus, for any $m \ge \ell$ we have $A_\ell \supseteq A_m \supseteq \cap_{i \in \omega} A_i = A_\ell$. and hence $A_\ell = A_m$ as desired. So DIC for $\mathcal{F} \Rightarrow DCC$ for $cl(\mathcal{F})$. □

Proposition 1.9. *Let $\mathcal{F}$ be a family of sets. If $cl(\mathcal{F})$ satisfies the n-DCC, for some $n \in \omega$, then $\mathcal{F}$ satisfies the n-DIC.*

Proof. Let $\{a_i \mid i \in I\}$ be a collection of sets in $\mathcal{F}$. By the Well Ordering Property we may assume that I is an ordinal number. For each $i \in I$ define B_i to be $\cap_{j \le i} A_j$. Notice that each $B_i \in cl(\mathcal{F})$ and that we have formed a descending chain, $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$. By the descending chain condition with upper bound n we can see that there are only m many $i \in I$ so that $B_{i-1} \supsetneq B_i$. Label them $i_1 < i_2 < \cdots < i_m$, with $m < n$. Additionly, let $i_0 = 0$. We claim that $\cap_{i \in I} A_i = \cap_{j=0}^{m} A_{i_j}$.

Indeed, we start with $A_{i_0} = A_0 = B_0$ and by definition of the i_j, for any $0 < k < i_1$ we know that $B_{k-1} = B_k$. Hence $A_0 = A_0 \cap A_1 = \cdots = \cap_{k < i_1} A_k$. It follows that $B_{i_1} = B_{i_1 - 1} \cap A_{i_1} = B_0 \cap A_{i_1} = A_{i_0} \cap A_{i_1}$.

Now, suppose that $B_{i_k} = \cap_{j \le k} A_{i_j}$. By definition of the i_j, for any $i_k < \ell < i_{k+1}$ we know that $B_{\ell-1} = B_\ell$. Hence $B_{i_k} = B_{i_k} \cap A_{i_k+1} = \cdots = B_{i_k} \bigcap (\cap_{i_k < j < i_{k+1}} A_j)$. Thus, $B_{i_{k+1}} = B_{i_{k+1}-1} \cap A_{i_{k+1}} = B_{i_k} \cap A_{i_{k+1}} = A_{i_0} \cap \cdots \cap A_{i_{k+1}}$.

By mathematical induction we have shown that $B_{i_m} = \cap_{j=0}^{m} A_{i_j}$. Finally, since i_m was the last index for which $B_{i-1} \supsetneq B_i$ we can see that for all $k > i_m$, we have $B_{i_{n-1}} = B_k$.

Thus, $\cap_{i \in I} A_i = B_{i_m} \bigcap (\cap_{i \in I, i > i_m} A_i) \supseteq B_{i_m} \bigcap (\cap_{i \in I, i > i_m} B_i) = B_{i_m} \bigcap (\cap_{i \in I, i > i_m} B_{i_m}) = B_{i_m} = \cap_{j=0}^{m} A_{i_j} \supseteq \cap_{i \in I} A_i$ By squeezing the result we have demonstrated that $\cap_{i \in I} A_i = \cap_{j=0}^{m} A_{i_j}$. And since m < n, this is the intersection of at most n many sets, we conclude that $n - DCC$ for $cl(\mathcal{F}) \Rightarrow n - DIC$ for $\mathcal{F}$. □

Sadly, this property is highly nonreversible. We will now give an example of a family of sets which satisfies 1-DIC but which doesn't satisfy n-DCC for any given n.

Example 1.10. *Fix a natural number n. Let* $\mathcal{F} = \{\ \{0,1,\cdots,i\}\ \mid i \leq n\}$*. Observe that since this collection is totally ordered by containment, any intersection of a collection of sets in* $\mathcal{F}$ *is another set in* $\mathcal{F}$*. Thus,* $cl\,(\mathcal{F}) = \mathcal{F}$*.*

As was just remarked, $\mathcal{F}$ *satisfies the bounded descending interesection condition with an upper bound of 1. Indeed, take a collection of m sets in* $\mathcal{F}$ *,* $\{0,1,\cdots,j_1\}$*,* $\{0,1,\cdots,j_2\}$*,* $\cdots$ $\{0,1,\cdots,j_m\}$ *and look at their intersection. It is not hard to see that if* $j = min\,(j_1, j_2, \cdots j_m)$ *then,* $\cap_{i=1}^{m}\{0,1,\cdots,j_i\} = \{0,1,\cdots,j\}$*. So for any collection of sets in* $\mathcal{F}$*, their intersection can be expressed as the intersection of a subcollection of size 1.*

Also note that $\mathcal{F}$ *has a chain of length n+1. Namely,* $\{0,1,\cdots,n\} \supsetneq \{0,1,\cdots,n-1\} \supsetneq \cdots \supsetneq \{0,1\} \supsetneq \{0\}$*. So it follows that* $\mathcal{F}$ *does not satisfy the bounded descending intersection condition with an upper bound of n.*

Let us briefly summarize the implications we have demonstrated. For any fixed $n \in \omega$,

$$n-DCC \text{ for } cl(\mathcal{F}) \implies n-DIC \text{ for } \mathcal{F} \implies DIC \text{ for } \mathcal{F} \iff DCC \text{ for } cl(\mathcal{F}),$$

and for $n < m \in \omega$,

$$n-DCC \text{ for } cl(\mathcal{F}) \implies m-DCC \text{ for } cl(\mathcal{F}) \qquad \text{and} \qquad n-DIC \text{ for } \mathcal{F} \implies m-DIC \text{ for } \mathcal{F}$$

2 Equations

In this section we will introduce the notion of an equation in model theory. In addition to introducing the concept, we will display a wide range of examples, both equations and nonequations, to help cement the idea. From there we will introduce some simple properties of equations to ease us into working with chain conditions and to help us later on.

Fix a language, L, and a complete L-theory, T.

Definition 2.1. *An L-formula,* $\phi(\overline{x};\overline{y})$*, where* $\overline{x} = (x_1,\cdots,x_n)$ *and* $\overline{y} = (y_1,\cdots,y_m)$*, is called an* **equation in** $\overline{\mathbf{x}}$ *if for all models* $\mathcal{M}$ *of T, the family of sets,* $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ *satisfies the DIC.*

Let us state some rather obvious results from the definition. First, observe that if our complete theory, T, has a finite model then every L-formula is an equation, for any partition of its variables. This is because every model of T must have a finite domain (since T is complete), and so each solution set of an instance of a formula must be of finite size. The descending intersection condition is clear when each set in our collection is finite.

As a consequence, we only need to consider complete theories with infinite models, as the case for finite models is uninteresting (everything is an equation!). We will proceed with this assumption unspoken.

Note as well, that if either $\overline{x}$ or $\overline{y}$ is the 0-tuple, then $\phi(\overline{x};\overline{y})$ is an equation in $\overline{x}$. In the former case, all solution sets are either the emptyset or the set containing the 0-tuple (we are working in M^0). In the latter case, there is only one instance of the equation. In particular all sentences are, trivially, equations.

A Litany Of Examples

In this subsection we will give some natural examples (and nonexamples) of equations.

Example 2.2. *Consider any formula* $\phi(x;y)$ *with the property that for any model* $\mathcal{M} \models T$ *and arbitrary* $\overline{a} \in M^{|\overline{y}|}$*, the set* $\phi(M,\overline{a})$ *is finite. Then* $\phi(\overline{x};\overline{y})$ *is an equation in* $\overline{x}$*. Indeed,* $\{\phi(M,\overline{a}) \mid \overline{a} \in M^{|\overline{y}|}\}$ *is a collection of finite sets and the DIC is clear.*

In particular, this shows that "$x = y$*" is always an equation in x, as one would expect.*

We now show that any equivalence relation is an equation.

Example 2.3. *Let $\phi(\overline{x};\overline{y})$ be an L-formula. Suppose that $\phi(\overline{x};\overline{y})$ defines an equivalence relation. That is, T implies, $\forall\overline{x}\phi(\overline{x},\overline{x})$, $\forall\overline{x}\forall\overline{y}\phi(\overline{x},\overline{y}) \to \phi(\overline{y},\overline{x})$ and $\forall\overline{x}\forall\overline{y}\forall\overline{z}\,(\phi(\overline{x},\overline{y}) \wedge \phi(\overline{y},\overline{z})) \to \phi(\overline{x},\overline{z})$.*

Let $\mathcal{M} \models T$. We see for an arbitrary $\overline{a}$, $\phi(M,\overline{a})$ is just the equivalence class of $\overline{a}$ in $M^{|\overline{x}|}$. For any two tuples $\overline{a}$ and $\overline{b}$ we know that either $\phi(M,\overline{a}) = \phi(M,\overline{b})$ or $\phi(M,\overline{a}) \cap \phi(M,\overline{b}) = \varnothing$. From this it is clear that $\{\phi(M,\overline{a}) \mid \overline{a} \in M^{|\overline{x}|}\}$ satisfies the DIC.

For our next example, we need to recall a very important result in mathematics, Hilbert's Basis Theorem.

Fact 2.4. *Let R be a ring. If every ideal in R is finitely generated, then every ideal in R[x] is finitely generated.*

Example 2.5. *Let T be the theory of algebraically closed fields, in the language of rings, for some fixed characteristic. That is, $T = ACF_p$ where p = 0 or p is a prime. Consider the formula $\phi(\overline{x};\overline{y})$ with $\overline{x} = (x_1,\cdots,x_n)$ and $\overline{y} = (y_1,\cdots y_m)$, defined as $p(x_1,\cdots,x_n,y_1,\cdots y_m) = 0$, where p is a polynomial in $x_1,\cdots x_n,y_1,\cdots y_m$ over the prime field ($\mathbb{Q}$ in characteristic 0 and $\mathbb{F}_p$ in characteristic $p > 0$).*

Let $\mathcal{K}$ be any algebraically closed field of characteristic p. For a given $\overline{a} \in K^m$ one can think of $\phi(\overline{x},\overline{a})$ as a polynomial equation in $x_1,\cdots,x_n$, where $\overline{a}$ is determining the coefficients of the polynomial. So $\phi(K,\overline{a})$ is just the zero set of this polynomial in $x_1,\cdots,x_n$.

Consider $\{\overline{a}_i\}_{i\in I}$ in K^m. Then $\cap_{i\in I}\phi(K,\overline{a}_i)$ is just the intersection of all those zero sets. To prove that ϕ is an equation it suffices to prove that any intersection of the zero sets of polynomials in $K[x_1,\cdots x_n]$ can be represented as the intersection of only finitely many of those zero sets.

So now let us consider a collection of polynomials in $K[x_1,\cdots x_n]$, $\{p_i(\overline{x})\}_{i\in I}$. For a set of polynomials, S, by V(S) we mean the intersection of all the zero sets of every polynomial in S. We first claim that $V(\{p_i(\overline{x})\}_{i\in I}) = V(\langle\{p_i(\overline{x})\}_{i\in I}\rangle)$, where $< S >$ represents the ideal in $K[x_1,\cdots x_n]$ generated by S.

For each $i \in I$, $p_i \in \langle\{p_i(\overline{x})\}_{i\in I}\rangle$ so we see $V(\{p_i(\overline{x})\}_{i\in I}) \supseteq V(\langle\{p_i(\overline{x})\}_{i\in I}\rangle)$. Let $q \in \langle\{p_i(\overline{x})\}_{i\in I}\rangle$ be arbitrary. Then we can find $p_{i_1},\cdots p_{i_k}$ and $f_1,\cdots f_k \in K[x_1,\cdots x_n]$ so that $q = \Sigma_{j=1}^k f_j p_{i_j}$. Suppose that $\overline{t} \in V(\{p_i(\overline{x})\}_{i\in I})$. Then $q(\overline{t}) = \Sigma_{j=1}^k f_j(\overline{t})p_{i_j}(\overline{t}) = \Sigma_{j=1}^k f_j(\overline{t})0 = 0$. It follows that since q was arbitrary, $V(\{p_i(\overline{x})\}_{i\in I}) \subseteq V(\langle\{p_i(\overline{x})\}_{i\in I}\rangle)$ and hence $V(\{p_i(\overline{x})\}_{i\in I}) = V(\langle\{p_i(\overline{x})\}_{i\in I}\rangle)$.

Now since $\langle\{p_i(\overline{x})\}_{i\in I}\rangle$ is an ideal in $K[x_1,\cdots x_n]$ it is finitely generated. This is by Hilbert's Basis Theorem, as K is a field (it only has K=<1> and {0} = <0> as ideals). So we can find $q_1,\cdots,q_k \in \langle\{p_i(\overline{x})\}_{i\in I}\rangle$ so that $\langle\{p_i(\overline{x})\}_{i\in I}\rangle = \langle\{q_1,\cdots,q_k\}\rangle$.

Now, for each q_j we can also find $p_{i_{j,1}},\cdots,p_{i_{j,k_j}}$ and $f_{j,1},\cdots,f_{k,k_j} \in K[x_1,\cdots x_n]$ so that $q_j = \Sigma_{\ell=1}^{k_j} f_{j,\ell}p_{i_{j,\ell}}$. Thus $\{q_1,\cdots,q_k\} \subseteq \langle\{p_{i_{j,\ell}} \mid 1 \le j \le k, 1 \le \ell \le k_j\}\rangle$. So $\langle\{q_1,\cdots,q_k\}\rangle = \langle\{p_{i_{j,\ell}} \mid 1 \le j \le k, 1 \le \ell \le k_j\}\rangle$ and hence,

$$V(\{p_i(\overline{x})\}_{i\in I}) = V(\langle\{p_i(\overline{x})\}_{i\in I}\rangle) = V(\langle\{q_1,\cdots,q_k\}\rangle) = V(\langle\{p_{i_{j,\ell}} \mid 1 \le j \le k, 1 \le \ell \le k_j\}\rangle)$$

which is a finite subset of the $\{p_i\}_{i\in I}$ as desired. Thus, $\phi(\overline{x};\overline{y})$ is an equation in $\overline{x}$.

Our next example comes from the model theory of R-modules and was part of the original motivation for equational theories. We will begin by introducing the first order theory of modules.

Example 2.6. *Let R be a ring with identity, 1_R. Let L be the language of left R-modules. That is, the lanugage, $L = \{0,+,-\} \cup \{\lambda_r \mid r \in R\}$. Let T be any theory that extends the theory of left R-modules. That is, T contains the following sentences,*

$$\{\forall a\,(a+0=a)\,,\forall a\,(-a+a=0)\,,\forall a\forall b\,(a+b=b+a)\,,\forall a\forall b\forall c\,((a+b)+c=a+(b+c))\,,\forall a\,(\lambda_{1_R}(a)=a)\}$$
$$\cup\{\forall a\forall b\,(\lambda_r(a+b)=\lambda_r(a)+\lambda_r(b)) \mid r \in R\} \cup \{\forall a\,(\lambda_{r+s}(a)=\lambda_r(a)+\lambda_s(a)) \mid r,s \in R\}$$
$$\cup\{\forall a\,(\lambda_{r\times s}(a)=\lambda_r(\lambda_s(a))) \mid r,s \in R\}$$

It is not hard to see that all the atomic formulas in this language are equivalent, modulo T, to a formula of the form $\lambda_{r_1}(x_1) + \lambda_{r_2}(x_2) + \cdots \lambda_{r_n}(x_n) = 0$. A formula $\phi(\overline{z})$ is called **positive primitive** *if it is of the form,*

$$\exists \overline{w} \left(\wedge_{j=1}^{n} \gamma_j(\overline{w}, \overline{z}) \right)$$

with each $\gamma_j(\overline{w}, \overline{z})$ an atomic formula. We claim that for any partition $(\overline{x}; \overline{y})$ of $\overline{z}$, $\phi(\overline{x}; \overline{y})$ is an equation.

The proof relies on M. Ziegler's paper on the model theory of modules, [11]. As he points out, each $\phi(M, \overline{a})$ is either empty or a coset of the subgroup of $M^{|\overline{x}|}$ defined by $\phi(M, 0)$. So for any two tuples, $\overline{a}$ and $\overline{b}$, the sets $\phi(\mathcal{M}, \overline{a})$ and $\phi(\mathcal{M}, \overline{b})$ are either disjoint or coincident.

Like Example 2.3 we can now conclude that $\phi(\overline{x}; \overline{y})$ is an equation in $\overline{x}$.

In addition to showing that equations do exist, it is also important to demonstrate that there are formulas which are not equations. Otherwise there would be no reason to investigate the properties of equations.

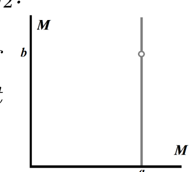
Example 2.7. *Consider any formula $\phi(\overline{x}; \overline{y})$ with the property that for any model $\mathcal{M} \models T$ and arbitrary $\overline{a} \in M^{|\overline{y}|}$, the set $\phi(M, \overline{a})$ is cofinite. Additionally assume that $\bigcap_{\overline{a} \in M^{|\overline{y}|}} \phi(M, \overline{a}) = \varnothing$. Then $\phi(\overline{x}; \overline{y})$ is not an equation in $\overline{x}$. Indeed any finite subintersection of $\{\phi(M, \overline{a}) \mid \overline{a} \in M^{|\overline{y}|}\}$ will be cofinite and hence nonempty.*

In particular, the formula "$x \neq y$" is never an equation in x.

While $x \neq y$ is not an equation, it is, rather obviously, the negation of an equation. As such it satisfies the ascending union condition, which is the opposite of the descending intersection condition. That is, for any $\{\overline{a}_i\}_{i \in I}$ we can find a finite subset $\{i_1, \cdots i_n\}$ so that $\cup_{i \in I} \phi(M, \overline{a}_i) = \cup_{j=1}^{n} \phi(M, \overline{a}_{i_j})$. Had we wanted to, we could have started with the ascending union condition in place of the descending intersection condition. These new "equations" would essentially be representing the same information. That is, we would still have the basic property that an infinite amount of information can be represented by a finite amount of information. The reason we started with the DIC is that it was the condition that fell in line with equations that we already know (polynomial equations, linear equations, etc).

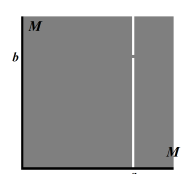
The following example is of a formula which is neither an equation, nor the negation of an equation.

Example 2.8. *Let us consider the formula $\phi(\overline{x}; \overline{y}) = \phi(x_1, x_2; y_1, y_2)$ defined as $x_1 = y_1 \wedge x_2 \neq y_2$.*

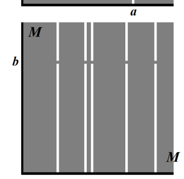


Let $\mathcal{M} \models T$. This formula has a nice visualisation. If we place all the elements of our domain, M, onto axes, we can view the set $\phi(M, a, b)$ as the line $x_1 = a$ removing the point (a, b).

Fix an element $a \in M$. Then we can see that our intersection $\cap_{b \in M} \phi(M, a, b)$, is just the empty set. However, for any finite subset $B \subseteq M$, $\cap_{b \in B} \phi(M, a, b)$ is just the line $x_1 = a$ after removing only finitely many points. Since $|M|$ is infinite we can see that ϕ cannot be an equation in (x_1, x_2).



Now we will show that $\neg\phi$ is not an equation in (x_1, x_2). A visualisation of $\neg\phi(M, a, b)$ is pictured, it is the whole plane, removing the line $x_1 = a$ and adding the point (a, b) back in.



Fix an element $b \in M$. Then we can see that our intersection $\cap_{a \in M} \neg\phi(M, a, b)$, is just the line $x_2 = b$. However, for any finite subset $A \subseteq M$, $\cap_{a \in A} \neg\phi(M, a, b)$ will contain points which are not on $x_2 = b$ (those whose x_1 coordinate is not in A). Since A is arbitrary we can see that $\neg\phi$ cannot be an equation in (x_1, x_2).

This example is a boolean combination of equations, namely $x_1 = y_1$ and $x_2 = y_2$. The following example gives us a formula which is not even a boolean combination of equations, though we will not prove this until later.

Example 2.9. *Let us consider the theory of dense linear orderings without endpoints, that is,* $T = Th((\mathbb{Q},<))$ *and let us work in the model* $\mathbb{Q}$*. Let* $\phi(x;y)$ *be defined as* $x < y$*.*

Consider the sequence $\{a_i\}_{i\in\omega}$ *where* $a_i = \frac{1}{i+1}$*. Notice that* $\phi(\mathbb{Q}, a_i) = \{k \mid k < \frac{1}{i+1}\}$*. So it is not hard to see that,* $\cap_{i\in\omega}\phi(\mathbb{Q}, a_i) = \{k \mid k \leq 0\}$*, which has no positive numbers in it. Suppose that we can find* $i_1, \cdots, i_n$ *so that* $\cap_{i\in\omega}\phi(\mathbb{Q}, a_i) = \cap_{j=1}^{n}\phi(\mathbb{Q}, a_{i_j})$*. Let* $i_* = max(i_1, \cdots, i_n)$*. Then* $\cap_{j=1}^{n}\phi(\mathbb{Q}, a_{i_j}) = \{k \mid k < \frac{1}{i_*+1}\}$ *which contains a positive number, namely* $\frac{1}{i_*+2}$*. This is a contradiction. Thus,* $x < y$ *is not an equation in* x*.*

As well, we can see that $\neg\phi(x;y)$*, given by* $x \geq y$*, is also not an equation in* x*. Consider the sequence* $\{b_i\}_{i\in\omega}$ *where* $b_i = i$*. Notice that* $\phi(\mathbb{Q}, b_i) = \{k \mid k \geq i\}$*. So it is not hard to see that,* $\cap_{i\in\omega}\phi(\mathbb{Q}, b_i) = \varnothing$*. Suppose that we can find* $i_1, \cdots, i_n$ *so that* $\cap_{i\in\omega}\phi(\mathbb{Q}, b_i) = \cap_{j=1}^{n}\phi(\mathbb{Q}, b_{i_j})$*. Let* $i_* = max(i_1, \cdots, i_n)$*. Then* $\cap_{j=1}^{n}\phi(\mathbb{Q}, b_{i_j}) = \{k \mid k \geq i_*\}$ *which is nonempty, namely it contains* i_**. This is a contradiction. Thus,* $x \geq y$ *is not an equation in* x*.*

For the moment, we have not developed enough of the theory to show why $x < y$ is not equivalent to a boolean combination of equations. However, by the end of the following subsection we will be able to demonstrate this.

Basic Properties

This first proposition provides us with a useful equivalent characterisation of equations.

Proposition 2.10. *Fix an L-formula* $\phi(\overline{x};\overline{y})$*, where* $\overline{x} = (x_1, \cdots, x_n)$ *and* $\overline{y} = (y_1, \cdots, y_m)$*. The following are equivalent,*

(i) $\phi(\overline{x};\overline{y})$ *is an equation in* $\overline{x}$ *in* T

(ii) For any infinite cardinal κ*, for all* κ *saturated, strongly* κ *homogeneous models* $\mathcal{M}$ *of* T*, the family of sets,* $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ *satisfies the DIC.*

Proof. $(i) \Rightarrow (ii)$ Clear, since all κ saturated, strongly κ homogeneous models $\mathcal{M}$ of T are still models of T.
$(ii) \Rightarrow (i)$ Fix $\mathcal{N}$, a model of T. Let $\mathcal{M}$ be an elementary extension of $\mathcal{N}$ which is $|N|$ saturated and strongly $|N|$ homogeneous. Then the family of sets, $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ satisfies the descending intersection condition.

Let $J \subseteq N^m \subseteq M^m$. Then by the descending intersection condition, we can find a finite subset $J_0 \subseteq J$ such that, $\cap_{\overline{a}\in J}\phi(M,\overline{a}) = \cap_{\overline{a}\in J_0}\phi(M,\overline{a})$. Now, since $J \subseteq N^m$ and $\mathcal{N} \preceq \mathcal{M}$ then for any $\overline{a} \in J$ and $\overline{b} \in N^n$, $\mathcal{N} \models \phi(\overline{b},\overline{a})$ if and only if $\mathcal{M} \models \phi(\overline{b},\overline{a})$. Hence for any $\overline{b} \in N^n$,

$$\overline{b} \in \cap_{\overline{a}\in J}\phi(N,\overline{a}) \iff \forall\overline{a} \in J, \mathcal{N} \models \phi(\overline{b};\overline{a}) \iff \forall\overline{a} \in J, \mathcal{M} \models \phi(\overline{b},\overline{a}) \iff \overline{b} \in \cap_{\overline{a}\in J}\phi(M,\overline{a})$$

and so $\cap_{\overline{a}\in J}\phi(N,\overline{a}) = \cap_{\overline{a}\in J}\phi(M,\overline{a}) \cap N^n$. Likewise $\cap_{\overline{a}\in J_0}\phi(N,\overline{a}) = \cap_{\overline{a}\in J_0}\phi(M,\overline{a}) \cap N^n$.

Finally, since $\cap_{\overline{a}\in J}\phi(M,\overline{a}) = \cap_{\overline{a}\in J_0}\phi(M,\overline{a})$, we deduce that $\cap_{\overline{a}\in J}\phi(N,\overline{a}) = \cap_{\overline{a}\in J_0}\phi(N,\overline{a})$. Since J was arbitrary, we have shown that the family of sets, $\{\phi(N,\overline{a}) \mid \overline{a} \in N^m\}$ satisfies the descending intersection condition.

Since $\mathcal{N}$ was arbitrary, this concludes the proof. □

The implication of this equivalence in definition means that we may now proceed to work in a κ saturated, strongly κ homogeneous model, $\overline{\mathcal{M}}$, for "large enough" infinite cardinal κ. All parameter sets will be assumed to be subsets of $\overline{M}$ of size $< \kappa$, and all models of T will be assumed to be elementary submodels of $\overline{\mathcal{M}}$ of size $\leq \kappa$. What we mean by" large enough" is that no matter what cardinals we run into later on in this paper, we may just assume that κ is larger. (Recall that, since T is complete, a κ saturated and strongly κ homogenous model contains elementary embeddings of all models of T whose domains are of size less than or equal to κ.)

The next property shows the advantage of working in a κ saturated and strongly κ homogenous model.

Proposition 2.11. *Let $\phi(\overline{x};\overline{y})$ be an equation.*
(i) $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ satisfies the DIC if and only if for some $n \in \omega$, $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ satisfies the n-DIC.
(ii) $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ satisfies the DCC if and only if for some $n \in \omega$, $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ satisfies the n-DCC.

Proof. We have previously demonstrated the ($\Leftarrow$) direction, and so we only need to show the ($\Rightarrow$) direction.

(i) Since $\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\}$ satisfies the DIC if and only if $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ satisfies the DCC, and we know that $n - DCC \Rightarrow n - DIC \Rightarrow DIC$, it suffices to prove (ii).

(ii) Since $n - DCC \Rightarrow DCC$ we just need to show that $DCC \Rightarrow n - DCC$ for some $n \in \omega$. Suppose that for all $n \in \omega$, $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ does not satisfy the n-DCC. Then we can find a sequence $\{\overline{a}_{n,i}\}_{i=0}^{n}$, so that, $\phi(\overline{M},\overline{a}_{n,0}) \supsetneq \phi(\overline{M},\overline{a}_{n,0}) \cap \phi(\overline{M},\overline{a}_{n,1}) \supsetneq \cdots \supsetneq \cap_{i=0}^{n}\phi(\overline{M},\overline{a}_{n,i})$ is a descending chain of length n+1.

Consider, for any $k \in \omega$, the formula,

$$\psi_{0,k}(\overline{y}_0) = \exists y_1 \cdots \exists y_k \bigwedge_{i=0}^{k-1} \neg\forall\overline{u}\left(\wedge_{j=0}^{i}\phi(\overline{u},\overline{y}_j) \to \wedge_{j=0}^{i+1}\phi(\overline{u},\overline{y}_j)\right)$$

For $\psi_{0,k}$ to be satisfied $\overline{y}_0$ must be some element $\overline{b}_0$ in some sequence $\{\overline{b}_i\}_{i\le k}$ with the property that $\phi(\overline{M},\overline{b}_0) \supsetneq \phi(\overline{M},\overline{b}_0) \cap \phi(\overline{M},\overline{b}_1) \supsetneq \cdots \supsetneq \cap_{i=0}^{n}\phi(\overline{M},\overline{b}_i)$. Observe that if we let $\overline{b}_i = \overline{a}_{k,i}$ then $\psi_{0,k}$ can be satisfied. What's more is that if $\psi_{0,k}$ is satisfied by $\overline{b}_0$ then we can see that $\psi_{0,\ell}$ is also satisfied by $\overline{b}_0$ for any $\ell \le k$.

Let us consider the collection of formulas, $p_0(\overline{y}_0) = \{\psi_{0,0}(\overline{y}_0), \psi_{0,1}(\overline{y}_0), \psi_{0,2}(\overline{y}_0), \psi_{0,3}(\overline{y}_0), \psi_{0,4}(\overline{y}_0), \cdots\}$. As we just explained any finite subset of formulas of p_0 is satisfiable, so p_0 is a type over $\varnothing$. By sufficient saturation of $\overline{\mathcal{M}}$ we can find $\overline{c}_0$ so that $\overline{\mathcal{M}} \models p_0(\overline{c}_0)$. In this way we begin to construct our sequence $\{\overline{c}_i\}_{i\in\omega}$ to make a descending chain.

Suppose now that we have defined the first m terms in the sequence, ie $\{\overline{c}_i\}_{i<m}$. Let us consider the formula,

$$\psi_{m,k}(\overline{y}_m) = \exists\overline{y}_{m+1}\cdots\exists\overline{y}_{m+k} \bigwedge_{i=0}^{m+k-1} \neg\forall\overline{u}\left(\bigwedge_{j=0}^{min(m,i)} \phi(\overline{u},\overline{c}_j) \wedge \bigwedge_{j=m}^{i} \phi(\overline{u},\overline{y}_j)\right) \to \left(\bigwedge_{j=0}^{min(m,i+1)} \phi(\overline{u},\overline{c}_j) \wedge \bigwedge_{j=m}^{i+1} \phi(\overline{u},\overline{y}_j)\right)$$

which is a formula over $\{\overline{c}_i\}_{i<m}$. First notice that, like before, $\psi_{m,k}$ being satisfied means that $\psi_{m,\ell}$ is also satisfied for any $\ell \le k$. In order to satisfy $\psi_{m,k}$, $\overline{y}_m$ must be an element $\overline{b}_m$ in some sequences $\{\overline{b}_i\}_{i\le m+k}$ where $\phi(\overline{M},\overline{b}_0) \supsetneq \phi(\overline{M},\overline{b}_0) \cap \phi(\overline{M},\overline{b}_1) \supsetneq \cdots \supsetneq \cap_{i=0}^{m+k}\phi(\overline{M},\overline{b}_i)$ and $\overline{b}_i = \overline{c}_i$ for all i $<$ m. But we know such a sequence exists, since by assumption $\overline{c}_{m-1}$ satisfies $\psi_{m-1,k+1}$.

Consider the set of fomulas, $p_m(\overline{y}_m) = \{\psi_{m,0}(\overline{y}_m), \psi_{m,1}(\overline{y}_m), \psi_{m,2}(\overline{y}_m), \psi_{m,3}(\overline{y}_m), \psi_{m,4}(\overline{y}_m), \cdots\}$. Since $\psi_{m,k} \Rightarrow \psi_{m,k-1}$, any finite subset of formulas of p_m is satisfiable, so p_m is a type over $\{\overline{c}_i\}_{i<m}$. By sufficient saturation of $\overline{\mathcal{M}}$ we can find $\overline{c}_m$ so that $\overline{\mathcal{M}} \models p_m(\overline{c}_m)$.

By induction, we can find a sequence $\{\overline{c}_i\}_{i\in\omega}$ so that for each $k \in \omega$, $\cap_{i=0}^{k}\phi(\overline{M},\overline{c}_i) \supsetneq \cap_{i=0}^{k+1}\phi(\overline{M},\overline{c}_i)$. Thus, $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ does not satisfy the DCC. But this is a contradiction! So we can see that if $cl(\{\phi(M,\overline{a}) \mid \overline{a} \in M^m\})$ satisfies the DCC, then is satisfies the n-DCC for some $n \in \omega$. □

Proposition 2.12. *A formula $\phi(\overline{x};\overline{y})$ is an equation in $\overline{x}$ if and only if it is an equation in $\overline{y}$.*

Proof. First let $n = |\overline{x}|$ and $m = |\overline{y}|$. By the symmetry of this statement, it suffices to prove that if $\phi(\overline{x};\overline{y})$ is not an equation in $\overline{y}$, then it is not an equation in $\overline{x}$. If $\phi(\overline{x};\overline{y})$ is not an equation in $\overline{y}$, then, by Propositions 1.8 and 2.11, we can find an infinite sequence of n-tuples $\{\overline{a}_i\}_{i\in\omega}$ in $\overline{M}^n$ so that,

$$\phi(\overline{a}_0, \overline{M}) \supsetneq \phi(\overline{a}_0, \overline{M}) \cap \phi(\overline{a}_1, \overline{M}) \supsetneq \phi(\overline{a}_0, \overline{M}) \cap \phi(\overline{a}_1, \overline{M}) \cap \phi(\overline{a}_2, \overline{M}) \supsetneq \cdots$$

So we can find a set of m-tuples $\{\bar{b}_i\}_{i\in\omega}$ in $\overline{M}^m$ such that for each $i \in \omega$, $\bar{b}_i \in \left(\cap_{j=0}^{i}\phi(\overline{a}_j, \overline{M})\right) \setminus \left(\cap_{j=0}^{i+1}\phi(\overline{a}_j, \overline{M})\right)$. It follows that $\overline{a}_j \in \phi(\overline{M}, \bar{b}_i)$ for all j < i, and $\overline{a}_{i+1} \notin \phi(\overline{M}, \bar{b}_i)$. So for each $i \in \omega$, we can construct a descending chain of length i,

$$\phi(\overline{M}, \bar{b}_i) \supsetneq \phi(\overline{M}, \bar{b}_i) \cap \phi(\overline{M}, \bar{b}_{i-1}) \supsetneq \phi(\overline{M}, \bar{b}_i) \cap \phi(\overline{M}, \bar{b}_{i-1}) \cap \phi(\overline{M}, \bar{b}_{i-2}) \supsetneq \cdots \supsetneq \cap_{j=0}^{i}\phi(\overline{M}, \bar{b}_j)$$

Since $\overline{\mathcal{M}}$ models arbitrarily long descending chains, it follows that for any $k \in \omega$, $cl\left(\{\phi(\overline{M}, \bar{b}) \mid \bar{b} \in \overline{M}^m\}\right)$ cannot satisfy the k-DCC. But then, by Proposition 2.11 this means that $cl\left(\{\phi(\overline{M}, \bar{b}) \mid \bar{b} \in \overline{M}^m\}\right)$ cannot satisfy the DCC. So, by Proposition 1.8, $\{\phi(\overline{M}, \bar{b}) \mid \bar{b} \in \overline{M}^m\}$ cannot satisfy the DIC. Hence ϕ is not an equation in $\overline{x}$. □

What this shows us is that the property of a formula being an equation has only to do with the partition of our variables, not our choice of $\overline{x}$ over $\overline{y}$. In the same vein, the following proposition also deals with the presentation of an equation in terms of its variables. Specifically, an equation is still an equation if you add dummy variables to its presentation.

Proposition 2.13. *Suppose that $\phi(\overline{x}; \overline{y})$ is an equation in $\overline{x}$. Suppose that $\overline{x}'$ and $\overline{y}'$ are tuples such that $\overline{x} \subseteq \overline{x}'$ and $\overline{y} \subseteq \overline{y}'$. Then considering ϕ as a formula in $\overline{x}'$, $\overline{y}'$, $\phi(\overline{x}'; \overline{y}')$ is an equation in $\overline{x}'$.*

Proof. To make this proof easier, let us invent a notation where $\overline{a}' \upharpoonright_{\overline{x}}$ means the restriction of the tuple to only those elements corresponding to variables in $\overline{x}$. That is, if $\overline{x} = (x_2)$ and $\overline{x}' = (x_1, x_2, x_3)$, then $(a_1, a_2, a_3) \upharpoonright_{\overline{x}} = (a_2)$. Without loss of generality we will assume that $\overline{x}$ is an initial segment of $\overline{x}'$.

Take any collection of tuples $\{\bar{b}_i\}_{i\in I}$ in $\overline{M}^{|\overline{y}'|}$. Look at $\cap_{i\in I}\phi(\overline{M}, \bar{b}_i)$. Since the extra variables in $\overline{x}'$ have no affect on ϕ it is not hard to see that $\phi(\overline{M}, \bar{b}_i) = \phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}}) \times \overline{M}^{|\overline{x}'|-|\overline{x}|}$. It follows that,

$$\cap_{i\in I}\phi(\overline{M}, \bar{b}_i) = \cap_{i\in I}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}}) \times \overline{M}^{|\overline{x}'|-|\overline{x}|} = \left(\cap_{i\in I}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}})\right) \times \overline{M}^{|\overline{x}'|-|\overline{x}|}$$

And since ϕ is an equation in $\overline{x}$ we can find a finite subset J so that $\cap_{i\in I}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}}) = \cap_{i\in J}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}})$. But then,

$$\left(\cap_{i\in I}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}})\right) \times \overline{M}^{|\overline{x}'|-|\overline{x}|} = \left(\cap_{i\in J}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}})\right) \times \overline{M}^{|\overline{x}'|-|\overline{x}|} = \cap_{i\in J}\phi(\overline{M}, \bar{b}_i \upharpoonright_{\overline{y}}) \times \overline{M}^{|\overline{x}'|-|\overline{x}|} = \cap_{i\in J}\phi(\overline{M}, \bar{b}_i)$$

And so we have found a finite subset J, so that $\cap_{i\in I}\phi(\overline{M}, \bar{b}_i) = \cap_{i\in J}\phi(\overline{M}, \bar{b}_i)$. Hence, ϕ is an equation in $\overline{x}'$. □

We now show that equations do not have the "order property", which we first define.

Definition 2.14. *Let $\psi(\overline{x}; \overline{y})$ be an L-formula. We say that ψ satisfies the order property if there exists $\mathcal{M} \models T$ and sequences $\{\overline{a}_i\}_{i\in\omega}$, $\{\bar{b}_i\}_{i\in\omega}$ so that $\mathcal{M} \models \psi(\overline{a}_i, \bar{b}_j)$ if and only if $i > j$.*

Proposition 2.15. *Suppose that $\phi(\overline{x}; \overline{y})$ is an equation. Then ϕ does not satisfy the order property.*

Proof. Suppose, for a contradiction, that $\phi(\overline{x}; \overline{y})$ satisfies the order property. This means that we can find sequences $\{\overline{a}_i\}_{i\in\omega}$ and $\{\bar{b}_i\}_{i\in\omega}$ so that $\overline{\mathcal{M}} \models \phi(\overline{a}_i, \bar{b}_j)$ if and only if i > j. We can see that $a_i \in \phi(\overline{M}, b_j)$ if and only if i > j. Let $J \subseteq \omega$ be a finite subset, then $a_i \in \cap_{j\in J}\phi(\overline{M}, b_j)$ if and only if $i > max(J)$.

Since ϕ is an equation in $\overline{x}$ we can find a finite subset $J \subseteq \omega$ so that $\cap_{i\in\omega}\phi(\overline{M}, b_i) = \cap_{j\in J}\phi(\overline{M}, b_j)$. Let $i_* = max(J) + 1$. Then $a_{i_*} \in \cap_{j\in J}\phi(\overline{M}, b_j) \setminus \cap_{i\in\omega}\phi(\overline{M}, b_i)$ which is a contradiction. □

Not only do equations not satisfy the order property, but no boolean combination of equations does either. Indeed this follows from Proposition 2.15 once we know the following fact (see Lemma 2.1 of [6]).

Fact 2.16. *If $\psi(\overline{x}; \overline{y})$ and $\chi(\overline{x}; \overline{z})$ don't satisfy the order property, then neither do $\psi \wedge \chi(\overline{x}; \overline{yz})$, $\psi \vee \chi(\overline{x}; \overline{yz})$, and $\neg\psi(\overline{x}; \overline{y})$.*

This allows us to conclude Example 2.9; that $x < y$ in DLO is not equivalent to a boolean combination of equations. Indeed any model of $Th((\mathbb{Q},<))$ must contain a copy of $(\mathbb{Q},<)$ and so it is not hard to see by taking $a_i = b_i = i$, that the formula $x < y$ satisfies the order property.

However, not all formulas which aren't boolean combinations of formulas satisfy the order property. The best example of which can be found in [2], where O. Beyersdorff demonstrates a theory which is stable but not equational (to be defined later).

The last property we will exhibit in this subsection we will make use of later.

Proposition 2.17. *Suppose that $\phi(x_1, x_2, \cdots, x_n; \overline{y})$ is an equation. Then $\phi(x_2, x_2, \cdots, x_n; \overline{y})$ is also an equation.*

Proof. By Proposition 2.12 it suffices to prove that $\phi(x_2, x_2, \cdots, x_n; \overline{y})$ is an equation in $\overline{y}$. Likewise we know that $\phi(x_1, x_2, \cdots, x_n; \overline{y})$ is an equation in $\overline{y}$. So then $\{\phi(a_1, a_2, \cdots, a_n, \overline{M}) \mid \overline{a} \in \overline{M}^n\}$ satisfies the DIC.

But since $\{\phi(b_1, b_1, \cdots, b_{n-1}, \overline{M}) \mid \overline{b} \in \overline{M}^{n-1}\} \subseteq \{\phi(a_1, a_2, \cdots, a_n, \overline{M}) \mid \overline{a} \in \overline{M}^n\}$, it immediately follows that $\{\phi(b_1, b_1, \cdots, b_{n-1}, \overline{M}) \mid \overline{b} \in \overline{M}^{n-1}\}$ must also satisfy the DIC. Thus $\phi(x_2, x_2, \cdots, x_n; \overline{y})$ is an equation in $\overline{y}$. □

Equationality for Sets of Formulas

Now we take the time to generalise the notion of equationality from a property of a single formula, to a property over a collection of formulas. To do this, we will go back to the basic definition involving the descending intersection condition.

We begin by defining closures on collections of formulas. Just as *cl* was a closure of sets with respect to the $\cap$ operation, the following three closures will be based on operations on formulas. Our first closure, $cl^{\wedge}$ will be the analog of *cl* to formulas. From there we will build up two other closures, each encompassing the previous one(s).

Definition 2.18. *Let $F = \{\phi_i(\overline{x}, \overline{y}_i) \mid i \in I\}$ be a collection of L-formulas which have exactly the tuple of variables $\overline{x}$ in common.*

We define **the closure of F under finite conjunctions** *to be exactly what the name suggests,*

$cl^{\wedge}(F) = \{\phi_{i_1}(\overline{x}, \overline{y}_{i_1}) \wedge \cdots \wedge \phi_{i_n}(\overline{x}, \overline{y}_{i_n}) \mid n \in \omega, i_j \in I, \phi_{i_j} \in F\}$

We define **the closure of F under finite positive boolean combinations** *to be the set of all formulas that can be formed from F using conjunctions and disjunctions,*

$cl^{+}(F) = \{\bigvee_{j=1}^{m} \left(\bigwedge_{k=1}^{n_j} \phi_{i_{j,k}}(\overline{x}, \overline{y}_{i_{j,k}})\right) \mid m, n_j \in \omega, i_{j,k} \in I, \phi_{i_{j,k}} \in F\}$

cl^{+} gets its name because it doesn't allow negation, only "positive" boolean operations. We can go further, extending these boolean combinations to those that use any of the three operations, $\wedge, \vee$ and $\neg$. We define **the closure of F under finite boolean combinations** *to be,*

$cl^{\mathcal{B}}(F) = \{\bigvee_{j=1}^{m} \left(\bigwedge_{k=1}^{n_j} \phi_{i_{j,k}}(\overline{x}, \overline{y}_{i_{j,k}}) \wedge \bigwedge_{k=1}^{n'_j} \neg\phi_{i'_{j,k}}(\overline{x}, \overline{y}_{i'_{j,k}})\right) \mid m, n_j, n'_j \in \omega, i_{j,k}, i'_{j,k} \in I, \phi_{i_{j,k}}, \phi_{i'_{j,k}} \in F\}$

Like *cl* from the first section, we can see from the definitions above that $cl^{\wedge}$, cl^{+} and $cl^{\mathcal{B}}$ are closure operators.

Fact 2.19. *Let F be any collection of formulas in a common tuple of variables, $\overline{x}$.*

(i) $cl^{\wedge}(cl^{\wedge}(F)) = cl^{\wedge}(F)$, $cl^{+}(cl^{+}(F)) = cl^{+}(F)$ and $cl^{\mathcal{B}}(cl^{\mathcal{B}}(F)) = cl^{\mathcal{B}}(F)$

(ii) Suppose that $G \subseteq F$. Then $cl^{\wedge}(G) \subseteq cl^{\wedge}(F)$, $cl^{+}(G) \subseteq cl^{+}(F)$ and $cl^{\mathcal{B}}(G) \subseteq cl^{\mathcal{B}}(F)$.

(iii) Suppose that $\phi \in cl^{\wedge}(F)$ (respectively $\in cl^{+}(F)$, $\in cl^{\mathcal{B}}(F)$), then there exists a finite subset $G \subset F$ so that $\phi \in cl^{\wedge}(G)$ (respectively $\in cl^{+}(G)$, $\in cl^{\mathcal{B}}(G)$).

Additionally, we can observe that $F \subseteq cl^{\wedge}(F) \subseteq cl^{+}(F) \subseteq cl^{\mathcal{B}}(F)$. We can relate these new closures to our previous cl and the chain conditions in the natural way, by extending these conditions to formulas, based on the instances of those formulas.

Definition 2.20. *Let $F = \{\phi_i(\overline{x}, \overline{y}_i) \mid i \in I\}$ be a set of formulas with a $\overline{x}$ common to each formula. F is said to satisfy the* **descending intersection condition** *(respectively* **descending chain condition**, **descending intersection condition with an upper bound of n**, **descending chain condition with an upper bound of n***) if the set $\{\phi_i(\overline{M}, \overline{b}_i) \mid i \in I, \overline{b}_i \in \overline{M}^{|\overline{y}_i|}\}$ satisfies the descending intersection condition (respectively descending chain condition, descending intersection condition with an upper bound of n, descending chain condition with an upper bound of n). As usual, we may choose to abbreviate to DIC (respectively DCC, n-DIC, n-DCC).*

We now define the notion of a set of formulas behaving "like an equation".

Definition 2.21. *A set of L-formulas, E, is said to be* **equational** *if E satisfies the DIC.*

It is plain to see that "equational" is not a misnomer. Indeed, $\{\phi(\overline{x};\overline{y})\}$ is equational as a set of formulas in $\overline{x}$ if and only if $\phi(\overline{x};\overline{y})$ is an equation in $\overline{x}$. This next proposition is the analogue of Proposition 1.8.

Proposition 2.22. *Let F be a collection of formulas in $\overline{x}$. F is equational if and only if $cl^{\wedge}(F)$ satisfies the DCC.*

Proof. F is equational if and only if $\{\phi(\overline{M}, \overline{b}) \mid \phi(\overline{x};\overline{y}) \in F, \overline{b} \in \overline{M}^{|\overline{y}|}\}$ satisfies the DIC.

We know by Proposition 1.8 that $\{\phi(\overline{M}, \overline{b}) \mid \phi(\overline{x};\overline{y}) \in F,\ \overline{b} \in \overline{M}^{|\overline{y}|}\}$ satisfies the DIC if and only if $cl\left(\{\phi(\overline{M}, \overline{b}) \mid \phi(\overline{x};\overline{y}) \in F, \overline{b} \in \overline{M}^{|\overline{y}|}\}\right)$ satisfies the DCC. Expanding this out, we see that F is equational if and only if $\{\cap_{i=0}^{n}\phi_i(\overline{M}, \overline{b}_i) \mid n \in \omega, \phi_i(\overline{x};\overline{y}_i) \in F, \overline{b}_i \in \overline{M}^{|\overline{y}_i|}$ for $0 \le i \le n\}$ satisfies the DCC. But this is just the definition of $cl^{\wedge}(F)$ satisfying the DCC.

So F is equational if and only if $cl^{\wedge}(F)$ satisfies the DCC. □

Proposition 2.23. *Suppose that E is an equational set of L-formulas, then each element in E is an equation.*

Proof. For any $\phi \in E$, notice that $\{\phi\} \subseteq E$. Since E has the DIC, then it follows that $\{\phi\}$ also has the DIC (since any infinite intersection of instances from $\{\phi\}$ is also contained in E. As we noted before, $\{\phi\}$ satisfies the DIC if and only if ϕ is an equation. Thus, ϕ must be an equation. □

The converse is not true, in general, but it is true for a finite sets of formulas.

Proposition 2.24. *Suppose that E is a finite collection of equations in $\overline{x}$. Then E is equational.*

Proof. Since E is finite, we have $E = \{\phi_1(\overline{x};\overline{y}_1), \cdots, \phi_n(\overline{x};\overline{y}_n)\}$. Let us take a collection $\{\phi_i(\overline{M}, \overline{a}_i)\}_{i \in I}$ of instances. Each ϕ_i is one of the ϕ_j for some $1 \le j \le n$. So let us partition our collection into n disjoint subcollections, $\{\phi_j(\overline{M}, \overline{a}_i)\}_{i \in I_j}$ where $i \in I_j$ if $\phi_i = \phi_j$.

We see that $\cap_{i \in I}\phi_i(\overline{M}, \overline{a}_i) = \cap_{j=1}^{n} \cap_{i \in I_j} \phi_j(\overline{M}, \overline{a}_i)$. Now, since each ϕ_j is an equation, we can find a finite subset $J_j \subseteq I_j$ so that $\cap_{i \in I_j}\phi_j(\overline{M}, \overline{a}_i) = \cap_{i \in J_j}\phi_j(\overline{M}, \overline{a}_i)$.

So then $\cap_{i \in I}\phi_i(\overline{M}, \overline{a}_i) = \cap_{j=1}^{n} \cap_{i \in I_j} \phi_j(\overline{M}, \overline{a}_i) = \cap_{j=1}^{n} \cap_{i \in J_j} \phi_j(\overline{M}, \overline{a}_i) = \cap_{i \in \cup_{j=1}^{n} J_j}\phi_i(\overline{M}, \overline{a}_i)$. And since $\cup_{j=1}^{n} J_j$ is finite and I was arbitrary we can see that E satisfies the DIC. Thus, E is equational. □

Hence, for a finite set of formulas, E, E is equational if and only if each formula in E is an equation. Here we exhibit a counterexample where E is infinite.

Example 2.25. *Let $L = \{R_n \mid n \in \omega\}$ and let T be the L-theory that says "R_n is an equivalence relation with infinitely many equivalence classes, each of infinite size" and that "R_j splits each equivalence class of R_i into two distinct equivalence classes of infinite size each whenever $j > i$". By Example 2.3 we know that each R_i is an equation. We claim that $\{R_i(x; y_i)\}_{i\in\omega}$ is not equational.*

Pick an element $a \in \overline{M}$. Consider the sequence of sets $\{R_i(\overline{M}, a)\}_{i\in\omega}$. Since each R_j strictly refines R_i for $j > i$, and a is constant, we can see that $R_i(\overline{M}, a) \supsetneq R_j(\overline{M}, a)$ for all $j > i$. Thus we see that this is an infinite descending chain. So $\{R_i(x; y_i)\}_{i\in\omega}$ does not satisfy the DCC.

Since $\{R_i(x; y_i)\}_{i\in\omega} \subseteq cl^\wedge(\{R_i(x; y_i)\}_{i\in\omega})$ we conclude that $cl^\wedge(\{R_i(x; y_i)\}_{i\in\omega})$ does not satisfy the DCC and hence, by Proposition 2.22, $\{R_i(x; y_i)\}_{i\in\omega}$ cannot be equational.

Proposition 2.26. *Suppose that E is an equational set of L-formulas, then $cl^+(E)$ is also equational.*

Proof. It is clear from the definitions of $cl^\wedge$ and cl^+ that $cl^\wedge(cl^+(E)) = cl^+(E)$. So we just need to show that $cl^+(E)$ satisfies the DCC. Suppose not, then we can find a sequence $\{\phi_i(\overline{x}; \overline{y}_i)\}_{i\in\omega}$ of equations in $cl^+(E)$ and $\{\overline{a}_i\}_{i\in\omega}$ so that $\phi_0(\overline{M}, \overline{a}_0) \supsetneq \phi_1(\overline{M}, \overline{a}_1) \supsetneq \phi_2(\overline{M}, \overline{a}_2) \supsetneq \cdots$. Our goal is now to construct an infinite descending chain of $cl^\wedge(E)$-definable sets.

To make this proof easier, let us recall our notation where $\overline{a} \restriction_{\overline{x}}$ means the restriction of the tuple to only those elements corresponding to variables in $\overline{x}$ (see Proposition 2.13).

Since $\phi_0 \in cl^+(E)$ we can write it as a disjunction of equations in $cl^\wedge(E)$. So we can express ϕ_0 as, $\phi_0(\overline{x}; \overline{y}_0) = \psi_0(\overline{x}; \overline{y}_{00}) \vee \cdots \vee \psi_{n_0}(\overline{x}; \overline{y}_{0n_0})$, where each $\psi_k(\overline{x}; \overline{y}_{0k}) \in cl^\wedge(E)$.

It is not hard to see that we can find $0 \le j \le n_0$ so that there exists a sequence, $\{i_0 < i_1 < i_2 < \cdots\} \subseteq \omega$ with $\psi_j(\overline{M}, \overline{a}_{0j}) \supsetneq \psi_j(\overline{M}, \overline{a}_{0j}) \cap \phi_{i_0}(\overline{M}, \overline{a}_{i_0}) \supsetneq \psi_j(\overline{M}, \overline{a}_{0j}) \cap \phi_{i_1}(\overline{M}, \overline{a}_{i_1}) \supsetneq \psi_j(\overline{M}, \overline{a}_{0j}) \cap \phi_{i_2}(\overline{M}, \overline{a}_{i_2}) \supsetneq \cdots$, where $\overline{a}_{0j} = \overline{a}_0 \restriction_{\overline{y}_{0j}}$. Observe, also, that each $\psi_j(\overline{x}; \overline{y}_{0j}) \wedge \phi_{i_k}(\overline{x}; \overline{y}_{i_k})$ belongs to $cl^+(E)$.

Define new equations, $\phi'_k(\overline{x}; \overline{y}'_k) = \psi_j(\overline{x}; \overline{y}_{0j}) \wedge \phi_{i_k}(\overline{x}; \overline{y}_{i_k})$ (which are in $cl^+(E)$), where $\overline{y}'_k = \overline{y}_{0j}\overline{y}_{i_k}$. We can see that $\phi'_0(\overline{M}, \overline{a}_{0j}\overline{a}_{i_0}) \supsetneq \phi'_1(\overline{M}, \overline{a}_{0j}\overline{a}_{i_1}) \supsetneq \phi'_2(\overline{M}, \overline{a}_{0j}\overline{a}_{i_2}) \supsetneq \cdots$, forms an infinite descending chain.

Since $\phi'_0 \in cl^+(E)$ we can write it as a disjunction of equations in $cl^\wedge(E)$. So we can express ϕ'_0 as, $\phi'_0(\overline{x}; \overline{y}'_0) = \psi'_0(\overline{x}; \overline{y}'_{00}) \vee \cdots \vee \psi'_{n'_0}(\overline{x}; \overline{y}'_{0n'_0})$, where each $\psi'_k(\overline{x}; \overline{y}'_{0k}) \in cl^\wedge(E)$.

It is not hard to see that we can find $0 \le j' \le n'_0$ so that there exist $\{i'_0 < i'_1 < i'_2 < \cdots\} \subseteq \{i_k\}_{k\in\omega}$ with $\psi'_{j'}(\overline{M}, \overline{a}_{0j'}) \supsetneq \psi'_{j'}(\overline{M}, \overline{a}_{0j'}) \cap \phi'_{i'_0}(\overline{M}, \overline{a}_{0j}\overline{a}_{i'_0}) \supsetneq \psi'_{j'}(\overline{M}, \overline{a}_{0j'}) \cap \phi'_{i'_1}(\overline{M}, \overline{a}_{0j}\overline{a}_{i'_1}) \supsetneq \psi'_{j'}(\overline{M}, \overline{a}_{0j'}) \cap \phi'_{i'_2}(\overline{M}, \overline{a}_{0j}\overline{a}_{i'_2}) \supsetneq \cdots$, where $\overline{a}_{0j'} = (\overline{a}_0\overline{a}_{i_0}) \restriction_{\overline{y}'_{0j'}}$. Observe, also, that each $\psi_{j'}(\overline{x}; \overline{y}'_{0j'}) \wedge \phi'_{i'_k}(\overline{x}; \overline{y}'_{i'_k})$ belongs to $cl^+(E)$.

Define new equations, $\phi''_k(\overline{x}; \overline{y}''_k) = \psi'_{j'}(\overline{x}; \overline{y}'_{0j'}) \wedge \phi_{i'_k}(\overline{x}; \overline{y}'_{i'_k})$ (which are in $cl^+(E)$). We can repeat this process indefinitely. In the end what we'll get is a descending chain of sets defined by formulas which belong to $cl^\wedge(E)$.

$$\psi_j(\overline{M}, \overline{a}_{0j}) \supsetneq \psi'_{j'}(\overline{M}, \overline{a}_{0j'}) \supsetneq \psi''_{j''}(\overline{M}, \overline{a}_{0j''}) \supsetneq \cdots$$

Which of course is a contradiction, since E is equational. Thus $cl^+(E)$ is equational. □

Corollary 2.27. *Suppose that E is a collection of equations, then each formula in $cl^+(E)$ is an equation.*

Proof. Let $\phi(\overline{x}; \overline{y}) \in cl^+(E)$ be arbitrary. By Fact 2.19 (iii) we can find finitely many equations in E, $\{\phi_1, \cdots, \phi_n\}$ so that $\phi \in cl^+(\{\phi_1, \cdots, \phi_n\})$. By Proposition 2.24, $\{\phi_1, \cdots, \phi_n\}$ is equational. By Proposition 2.26 it follows that $cl^+(\{\phi_1, \cdots, \phi_n\})$ is equational. By Proposition 2.23 it follows that each element of $cl^+(\{\phi_1, \cdots, \phi_n\})$ is an equation in $\overline{x}$. Thus ϕ is an equation in $\overline{x}$. □

Let us look at few examples of equational sets of equations.

Example 2.28. *In ACF_p (p=0 or a prime), the collection of all polynomial equations is equational. Indeed, in Example 2.5 we have shown that for any collection of polynomials, S, by Hilbert's Basis Theorem, we can find a finite subcollection $S_0 \subseteq S$ so that $V(S) = V(S_0)$. That is, every intersection of solution sets of polynomials can be expressed by a finite subintersection.*

So the set of all polynomials satisfies the DIC, and hence is equational.

Example 2.29. *Fix a ring R (with identity) and let T be any complete theory extending the theory of R-modules. Let us consider E the set of all positive primilitve formulas. Since a conjunction of positive primitive formulas is again positive primitive, this boils down to whether E satisfies the DCC. That is, can we find a descending chain of cosets of subgroups defined by positive primitive formulas?*

Suppose that we have such an infinite descending chain. Notice that the subgroups of which they are cosets must also form an infinite descending chain. So now we just need to determine whether or not there is an infinite descending sequence of subgroups which are defined by positive primitive formulas.

As remarked by Ziegler, there is no infinite descending sequence of positive primitive definable subgroups if and only if T is totally transcendental. (Theorem 2.1 in [11]). (Recall that a totally transcendental theory is one in which every definable set has bounded Morley rank; see [4].) So the set of all positive primite formulas is equational if and only if T is totally transcendental.

3 Srour Closed Sets

As before, we work with a fixed language, L, a complete L-theory, T, and a sufficiently κ saturated, strongly κ homogeneous model, $\overline{\mathcal{M}} \models T$.

Definition 3.1. *Let E be a collection of equations. We say that a definable set, X, is* **E-closed** *if it is definable by an instance of an equation in E. That is, there exists $\phi(\overline{x};\overline{y}) \in E$, and $\overline{a} \in \overline{M}^{|\overline{y}|}$ so that $X = \phi(\overline{M}, \overline{a})$. When E is the set of all equations we will call these sets* **Srour closed***.*

We say that a set is **E-constructible** *if it can be written as a boolean combination of E-closed sets. Again, if E is the set of all equations, we will say* **Srour-constructible***.*

Note that every $\varnothing$-definable set is, vaccuously, Srour-closed.

Proposition 3.2. *Let E be a collection of equations. Fix a natural number, $n \in \omega$. The topology on $\overline{M}^n$ generated by the E-closed sets is the same as the topology generated by the $cl^+(E)$-closed sets.*

Proof. To prove this, it suffices to show that every $cl^+(E)$-closed set is closed in the topology given by E. Let X be an $cl^+(E)$-closed set. Then we can find an equation $\phi \in cl^+(E)$ and $\overline{a}$, so that $X = \phi(\overline{M}, \overline{a})$.

Now, since $\phi \in cl^+(E)$ we can express it as a positive boolean combination of equations in E. That is, we can find $\{\psi_{ij}(\overline{x};\overline{y}_{ij})\}_{i=1,j=1}^{n,n_i}$ in E so that $\phi(\overline{x};\overline{y}) = \bigvee_{i=1}^n \left(\wedge_{j=1}^{n_i} \psi_{ij}(\overline{x};\overline{y}_{ij})\right)$. Hence, we can find tuples, $\overline{a}_{ij}$ so that $X = \bigvee_{i=1}^n \left(\wedge_{j=1}^{n_i} \psi_{ij}(\overline{M}, \overline{a}_{ij})\right) = \bigcup_{i=1}^n \left(\cap_{j=1}^{n_i} \psi_{ij}(\overline{M}, \overline{a}_{ij})\right)$.

Now, each $\psi_{ij}(\overline{M}, \overline{a}_{ij})$ is closed set in the topology given by E. Since a finite intersection of closed sets is still closed, then $\cap_{j=1}^{n_i} \psi_{ij}(\overline{M}, \overline{a}_{ij})$ is also closed. Since a finite union of closed sets is still closed, then $\bigcup_{i=1}^n \left(\cap_{j=1}^{n_i} \psi_{ij}(\overline{M}, \overline{a}_{ij})\right)$ is also closed. So it follows that X is closed.

Since X was arbitrary, we see the topology on $\overline{M}^n$ given by E is the same as the one made by $cl^+(E)$. □

Proposition 3.3. *Let E be equational. A nonempty definable set $X \subsetneq \overline{M}^n$ is closed in the topology generated by the E-closed sets if and only if X is $cl^+(E)$-closed.*

Proof. By Proposition 3.2 and the fact that $cl^+(E)$ is closed under finite disjunctions, it suffices to show that any intersection of $cl^+(E)$-closed sets is also $cl^+(E)$-closed.

Let us take an intersection of $cl^+(E)$-closed sets, $\cap_{i \in I} \phi_i(\overline{M}, \overline{a}_i)$. Since E is equational then $cl^+(E)$ is also equational, so we know that $\{\phi_i(\overline{M}, \overline{a}_i) \mid i \in I\}$ must satisfy the DIC. That is, we can find $\phi_1, \cdots, \phi_n$ so that $\cap_{i \in I} \phi_i(\overline{M}, \overline{a}_i) = \cap_{i=1}^n \phi_i(\overline{M}, \overline{a}_i)$.

But we can see that $\phi_1 \wedge \cdots \wedge \phi_n \in cl^+(E)$, so $\cap_{i=1}^n \phi_i(\overline{M}, \overline{a}_i)$ is $cl^+(E)$-closed. Hence $\cap_{i \in I} \phi_i(\overline{M}, \overline{a}_i)$ is $cl^+(E)$-closed. □

Now, let us exhibit a few examples of these topologies. To start off, let us look at the topology which $x = y$ generates, as it is the bare minimum topology we can generate.

Example 3.4. *Let $E = cl^+(\{x = y\})$. Take any $\phi(x_1, \cdots x_n; y_1, \cdots, y_m)$. We can view it in disjuntive normal form, $\phi(x_1, \cdots x_n; y_1, \cdots, y_m) = \bigvee_{j=1}^k (x_{i_{j,1}} = y_{\ell_{j,1}} \wedge \cdots \wedge x_{i_{j,k_j}} = y_{\ell_{j,k_j}})$. Now, given a tuple, $\overline{a} \in \overline{M}^{|\overline{y}|}$ we can see that each clause $(x_{i_{j,1}} = a_{\ell_{j,1}} \wedge \cdots \wedge x_{i_{j,k_j}} = a_{\ell_{j,k_j}})$ defines a fibre of a coordinate projection on $\overline{M}^{|\overline{x}|}$. So $\phi(\overline{M}, \overline{a})$ is a finite union of such coordinate subspaces.*

It is not hard to see that each coordinate subspace can be defined by an instance of some formula in E. So the topology generated by "$x = y$" is just the topology whose closed sets are finite unions of fibres of coordinate projections.

The "= topology" is actually a subtopology of the following, Zariski topology, from algebraic geometry. The Zariski topology is the topology generated by polynomial equations.

Example 3.5. *Let T be ACF_p (p = 0 or prime). Let E be the set of all polynomial equations as described in Example 2.5. As we remarked in Example 2.28, E is equational. Applying Proposition 3.3 we see that the closed sets in the topology generated by the E-closed sets are exactly the $cl^+(E)$-closed sets. Since $E = cl^+(E)$ we see that these are exactly the Zariski closed sets.*

Example 3.6. *Let T be any complete theory extending the theory of R-modules, for a fixed ring R (with identity). Let E be the set of all positive primitive formulas. As we have noted previously (in Example 2.6), the solution sets of a positive primitive formula are just cosets of the subgroup of $\overline{M}^n$ defined by that same positive primitive formula ($\phi(\overline{M}, 0)$). So rather obviously, the topology generated by the positive primitive formulas on $\overline{M}^n$ is just the topology generated by all the cosets of the positive primitive subgroups of $\overline{M}^n$.*

As we noted in Example 2.29, we know that there exists no infinite descending chain of positive primitive definable cosets if and only if T is totally transcendental. So we can conclude that the topology on $\overline{M}^n$ is Noetherian if and only if T is totally transcendental.

The following is a rather remarkable proposition, which says that if a set is $\overline{a}$-definable and is defined by an instance of an equation, then it can be defined as a $\overline{a}$-instance of an equation.

Proposition 3.7. *Let $\psi(\overline{x}, \overline{y})$ be an arbitrary formula, and $\overline{a}$ be an arbitrary tuple with $|\overline{a}| = |\overline{y}|$. Suppose that $X = \psi(\overline{M}, \overline{a})$ and $\{f(X) \mid f \in Aut(\overline{M})\}$ satisfies the DIC. Then we can find an equation, $\phi(\overline{x}, \overline{y})$, so that $X = \phi(\overline{M}, \overline{a})$. In particular, any A-definable Srour-closed set is an A-instance of an equation.*

Proof. Since we know that $\{f(X) \mid f \in Aut(\overline{M})\}$ satisfies the DIC we can find some $n \in \omega$ so that $cl\left(\{f(X) \mid f \in Aut(\overline{M})\}\right)$ satisfies the n-DCC. Observe, by saturation, that $Y = \psi(\overline{M}, \overline{b})$ is a conjugate of X if and only if $\overline{b}$ and $\overline{a}$ have the same type over $\varnothing$.

Let $p(\overline{y}) = tp(\overline{a})$. Since we cannot have a descending chain of intersections of conjugates of length n+1, we see the set,

$$p(\overline{y}_0) \cup p(\overline{y}_1) \cup \cdots \cup p(\overline{y}_n) \cup \{\neg \left(\forall \overline{u} \wedge_{j=0}^{i-1} \psi(\overline{u}, \overline{y}_j) \to \wedge_{j=0}^{i} \psi(\overline{u}, \overline{y}_j)\right) \mid 0 < i \leq n\}$$

is inconsistent. So it follows that we can find a finite subset $\Phi(\overline{y})$ of $p(\overline{y})$ such that $\Phi(\overline{y}_0) \cup \cdots \cup \Phi(\overline{y}_n) \models \neg \bigwedge_{i=1}^{n} \left(\neg \left(\forall \overline{u} \wedge_{j=0}^{i-1} \psi(\overline{u}, \overline{y}_j) \to \wedge_{j=0}^{i} \psi(\overline{u}, \overline{y}_j)\right)\right)$. Since $\Phi(\overline{y})$ is finite we can define the L-formula, $\delta(\overline{y}) := \bigwedge\{\chi(\overline{y}) \mid \chi(\overline{y}) \in \Phi(\overline{y})\}$. We claim that $\phi(\overline{x}, \overline{y}) := \psi(\overline{x}, \overline{y}) \wedge \delta(\overline{y})$ has the desired properties.

Observe that since $\delta(\overline{y}) \in tp(\overline{a})$, $X = \phi(\overline{M}, \overline{a})$. Indeed, for any $\overline{b}$ with $\overline{\mathcal{M}} \models \delta(\overline{b})$, then $\phi(\overline{M}, \overline{b}) = \psi(\overline{M}, \overline{b})$. For the rest, $\phi(\overline{M}, \overline{b}) = \varnothing$. So it remains to show that ϕ is an equation in $\overline{x}$. If not, then we could find $\overline{b}_0, \cdots, \overline{b}_n$ so that $\phi(\overline{M}, \overline{b}_0) \supsetneq \phi(\overline{M}, \overline{b}_0) \cap \phi(\overline{M}, \overline{b}_1) \supsetneq \cdots \supsetneq \bigcap_{i=0}^{n} \phi(\overline{M}, \overline{b}_i) \neq \varnothing$. Note that $\overline{\mathcal{M}} \models \delta(\overline{b}_i)$ for each $0 \leq i \leq n$, so this can also be seen as a descending chain of ψ-definable sets, $\psi(\overline{M}, \overline{b}_0) \supsetneq \psi(\overline{M}, \overline{b}_0) \cap \psi(\overline{M}, \overline{b}_1) \supsetneq \cdots \supsetneq \bigcap_{i=0}^{n} \psi(\overline{M}, \overline{b}_i) \neq \varnothing$. By construction, though, $\{\delta(\overline{y}_i)\}_{i=0}^{n} \models \neg \bigwedge_{i=1}^{n} \left(\neg \left(\forall \overline{u} \wedge_{j=0}^{i-1} \psi(\overline{u}, \overline{y}_j) \to \wedge_{j=0}^{i} \psi(\overline{u}, \overline{y}_j)\right)\right)$, which is a contradiction.

So we conclude that ϕ is indeed an equation, and this finishes our proof. □

Corollary 3.8. *A definable set $X \subseteq \overline{M}^n$ is Srour-closed if and only if $\{f(X) \mid f \in Aut(\overline{\mathcal{M}})\}$ satisfies the descending intersection condition.*

Proof. (⇒) Since X is Srour-closed, we can find an equation $\phi(\overline{x}; \overline{y})$ in $\overline{x}$, with $|\overline{x}| = n$, and $\overline{a} \in \overline{M}^{|\overline{y}|}$ so that $X = \phi(\overline{M}, \overline{a})$. Let $f \in Aut(\overline{\mathcal{M}})$. Then $f(X) = \phi(\overline{M}, f(\overline{a}))$. We see that $f(\overline{a}) \in M^{|\overline{y}|}$ and so $\{f(X) \mid f \in Aut(\overline{\mathcal{M}})\}$ is a subset of $\{\phi(\overline{M}, \overline{b}) \mid \overline{b} \in M^{|\overline{y}|}\}$. It follows that since ϕ is an equation in $\overline{x}$ then $\{f(X) \mid f \in Aut(\overline{\mathcal{M}})\}$ satisfies the DIC.

(⇐) Suppose that $\{f(X) \mid f \in Aut(\overline{\mathcal{M}})\}$ satisfies the DIC. Since X is definable we can find a formula $\psi(\overline{x}; \overline{y})$, with $|\overline{x}| = n$, and $\overline{a} \in \overline{M}^{|\overline{y}|}$ so that $X = \psi(\overline{M}, \overline{a})$. By Proposition 3.7 it follows that we can find an equation $\phi(\overline{x}, \overline{y})$ in $\overline{x}$, so that $X = \phi(\overline{M}, \overline{a})$. Thus X is Srour closed. □

We conclude this section with a useful lemma on E-constructible sets.

Lemma 3.9. *Fix a set of equations, E, such that $cl^+(E)$. Let P be a property of definable sets satisfying:*
(i) If $X \subseteq Y$ are definable sets satisfying property P, then $Y \backslash X$ satisfies P as well.
(ii) If $X \subseteq Y$ are definable sets, with X satisfying P and Y is an E-closed set, then there is an E-closed set Z satisfying P so that $X \subseteq Z \subseteq Y$.

Then any E-constructible set with property P is a boolean combination of E-closed sets that also satisfy property P.

Proof. We will make use of the following fact, a proof of which can be found in [3],

(i) A **difference chain** is a sequence of sets $C_0, C_1, \cdots, C_h$ written as $C_0 \backslash C_1 \backslash \cdots \backslash C_h$, which abbbreviates the set defined as $C_0 \backslash (C_1 \backslash (\cdots \backslash (C_{h-1} \backslash C_h)))$. We call the number h the **length** of the difference chain.

(ii) Let $\mathcal{F}$ be a collection of sets which is closed under finite unions and finite intersections. Then any boolean combination of sets in $\mathcal{F}$ can be expressed as a difference chain of sets in $\mathcal{F}$.

Suppose that X is an E-constructible set satisfying P. Since $E = cl^+(E)$ then it is not hard to see that the collection of all E-closed sets is closed under finite unions and finite intersections. So we can represent X by a difference chain. Let $X = C_0 \backslash C_1 \backslash \cdots \backslash C_h$ for E-closed sets $C_0, \cdots, C_h$. We prove by induction on h that X can be written as a boolean combination of E-closed sets satisfying P. Notice that $X \subseteq C_0$, so by condition (ii) we can find an E-closed set C_0' which satisfies property P and $X \subseteq C_0' \subseteq C_0$.

If h = 0, then we see by squeezing that $X = C'_0$ and so X can be written as a boolean combination of E-closed sets satisfying P.

If h > 0, then by letting $C'_i = C'_0 \cap C_i$ we can see that $X = C'_0 \backslash C'_1 \backslash \cdots \backslash C'_h$. Since X and C'_0 satisfy P we can see by condition (i) that $C'_0 \backslash X = C'_1 \backslash \cdots \backslash C'_h$ satisfies P. $C'_1 \backslash \cdots \backslash C'_h$ is an E-constructible set satisfying P, so by induction we can write $C'_1 \backslash \cdots \backslash C'_h$ as a boolean combination of E-closed sets satisfying P, and hence we can write X as a boolean combination of E-closed sets satisfying P. ☐

Incidently, one might expect a more natural second condition, like,

$(ii)'$ If X is a definable set satisfying P, then the E-closure of X satisfies P.

where the E-closure of X is the intersection of all the E-closed sets containing X. The problem with this, however, is that the "E-closure" of X, may not be an E-closed set!

That said, when E is equational and $E = cl^+(E)$ then it is not hard to see that the intersection of a collection E-closed sets is also E-closed. In that case, (ii) and $(ii)'$ are now equivalent statements. Indeed, if we have $(ii)'$, then for any E-closed Y containing X, we know that the E-closure of X lies between X and Y and satisfies P. So $(ii)' \Rightarrow (ii)$. Conversely, since E is equational, the E-closure of X is E-closed. So then (ii) tell us there is an E-closed set between X and the E-closure of X which satisfies P. But the only E-closed set between X and its E-closure is the E-closure of X. Thus the E-closure of X must satisfy P. Hence $(ii) \Rightarrow (ii)'$.

The main use of the previous lemma will be in the following subsection. However, we take the time now to illustrate another use of Lemma 3.9. This result deals with $\varnothing$-definable equivalence relations.

Proposition 3.10. *Let R be a $\varnothing$-definable equivalence relation on $\overline{M}^n$. Any R-saturated Srour-constructible set in $\overline{M}^n$ can be written as a boolean combination of R-saturated Srour-closed sets in $\overline{M}^n$.*

Proof. Recall that a set is **R-saturated** if it is a union of R-equivalence classes.

Once we apply Lemma 3.9 we are done. So it remains to show that the property P = "is R-saturated" satisfies conditions (i) and (ii) of the lemma.

(i) If $X \subseteq Y$ are definable sets and both are R-saturated, then it is fairly evident that $Y \backslash X$ is also R-saturated. Indeed, $Y \backslash X$ must be the union of the R-equivalence classes that are in Y but not in X.

(ii) Suppose that X is R-saturated. Let Y be a Srour-closed set containing X. Let Z be the union of all the R-equivalence classes which are fully contained in Y. Clearly $X \subseteq Z \subseteq Y$, so it remains to show that Z is Srour-closed. By Corollary 3.8 it suffices to show that the conjugates of Z satisfy the descending intersection condition.

First, let us define a new function, ^{R}, on definable sets, where W^R is the union of all the R-equivalence classes which are fully contained in W. Thus, $Z = Y^R$. It is not difficult to see that ^{R} commutes with automorphisms and intersections, just by definition.

Look at $\{f_i\}_{i \in I}$ with each $f_i \in Aut(\overline{\mathcal{M}})$. Then, $\cap_{i \in I} f_i(Z) = \cap_{i \in I} f_i(Y^R) = \cap_{i \in I} f_i(Y)^R = (\cap_{i \in I} f_i(Y))^R$. Since Y is Srour-closed we know that we can find a finite subset J of I so that $\cap_{i \in I} f_i(Y) = \cap_{i \in J} f_i(Y)$. Thus $(\cap_{i \in I} f_i(Y))^R = (\cap_{i \in J} f_i(Y))^R = \cap_{i \in J} f_i(Y)^R = \cap_{i \in J} f_i(Y^R)$. So we can see that $\cap_{i \in I} f_i(Z) = \cap_{i \in J} f_i(Z)$ and Z is Srour-closed.

This concludes the proof. ☐

4 Equational Theories

Equational Theories

Definition 4.1. *For a set of equations, E, we say that T is* **n-E-equational** *if every L-formula $\psi(\overline{x};\overline{y})$ with $|\overline{x}| = n$ is equivalent, in T, to a boolean combination of equations in $\overline{x}$ from E. That is, for each $\psi(\overline{x};\overline{y})$ we can find collections $\{\phi_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n_i}$ and $\{\phi'_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n'_i}$, of equations in $\overline{x}$ belonging to E, so that*

$$\overline{\mathcal{M}} \models \forall\overline{x}\forall\overline{y}\left(\psi(\overline{x};\overline{y}) \leftrightarrow \bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i}\phi_{ij}(\overline{x};\overline{y}) \wedge \bigwedge_{j=1}^{n'_i}\neg\phi'_{ij}(\overline{x};\overline{y})\right)\right)$$

We say T is **E-equational** *if it is n-E-equational for all $n \in \omega$. When E is the set of all equations in T, we drop the prefix and just say* **n-equational** *and* **equational**.

Notice that vaccuously every theory is 0-equational. That is because every formula $\phi(\overline{x};\overline{y})$ with $|\overline{x}| = 0$ is an equation. Let us take some time now to illustrate examples of theories that are equational. Our first example takes us back to equivalence relations.

Example 4.2. *Let $L = \{E\}$ and let T be any complete L-theory that says E is an equivalence relation with infinitely many equivalence classes and infinitely many elements in each class. We claim that this theory is equational.*

It is well-known that T admits quantifier elimination. As such, every formula is equivalent, modulo T, to a boolean combination of formulas of the form $E(x,y)$ and $x = y$.

We have previously shown (Example 2.2 and 2.3) that each of these atomic formulas are equations, so we conclude that any formula is equivalent to a boolean combination of equations. Thus T is equational.

The next example shows that the theory of R-modules is equational. To make our lives easier and to avoid a very lengthy side proof, we now quote the positive primitive elimination of quantifiers result, proven by W. Baur in [1].

Fact 4.3. *Fix a ring R (with identity). Every formula in the language of R-modules is equivalent, relative to the theory of R-modules, to a boolean combination of positive primitive formulas.*

Example 4.4. *Fix a ring R (with identity), and let T be any complete theory extending the theory of R-modules. Then T is equational.*

Let $\phi(\overline{x};\overline{y})$ be an arbitrary formula in the language of R-modules. By Baur's positive primitive elimination of quantifiers we can find a finite number of positive primitive formulas so that $\phi(\overline{x};\overline{y})$ is logically equivalent to a boolean combination of them. But we showed in Example 2.6 that positive primitive formulas are all equations.

Since ϕ was arbitrary we conclude that T is equational.

There are several other examples of equational theories. In [8], G. Srour shows that the theory of algebraically closed fields (for a fixed characteristic) as well as differentially closed fields of characteristic 0 are both equational. For a prime, p, Srour goes on to show that the theory of radical differentially closed fields of characteristic p is also equational.

As before, we can use the theory of dense linear orderings without endpoints as a counterexample. In this case, we can see that $Th(\mathbb{Q},<)$ is not equational.

Example 4.5. *Let $T = Th(\mathbb{Q},<)$ and let n be a fixed positive natural number. Consider the formula $\phi(\overline{x};\overline{y})$ where $|\overline{x}| = n$ and $|\overline{y}| = 1$, defined as $x_1 < y_1$. By taking sequences $\{\overline{a}_i\}_{i\in\omega}$ and $\{\overline{b}_i\}_{i\in\omega}$, given by $\overline{a}_i = (i,0,0,\cdots,0)$, and*

$\overline{b}_i = (i)$, we can see that ϕ satisfies the order property. But then we know that ϕ cannot be expressed as a boolean combination of equations.

Since n was arbitrary, we can see that T is not n-equational for any positive n.

From the definition we can immediately note two things. If T is n-E-equational and $E \subseteq E'$, then T is n-E′-equational. As well, by simply applying Proposition 2.17 we see that if T is n-equational for some natural number $n \in \omega$, then for any natural number $m < n$, T is m-equational.

Many properties in model theory, like saturation and stability, while appearing to be a property concerning n-tuples, can actually be checked by only considering 1-tuples. One would expect equationality to have this property. However, this is currently an open question: for $0 < m < n$ does m-equational imply n-equational?

This next theorem, which appears as Proposition 2.9 in [3], restates equationality in terms of Srour-closed sets.

Theorem 4.6. *Fix $n \in \omega$. Then T is n-equational if and only if every definable set in $\overline{M}^n$ is Srour constructible.*

Proof. ($\Rightarrow$) Suppose T is n-equational. Let X be a definable set in $\overline{M}^n$. Then we can find an L-formula $\psi(\overline{x};\overline{y})$, with $|\overline{x}| = n$ and $\overline{a}$ so that $X = \phi(\overline{M},\overline{a})$. But since T is n-equational, we can find equations, $\{\phi_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n_i}$ and $\{\phi'_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n'_i}$, in $\overline{x}$ so that $\psi(\overline{x};\overline{y})$ is logically equivalent, in T, to $\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i}\phi_{ij}(\overline{x},\overline{y}) \wedge \bigwedge_{j=1}^{n'_i}\neg\phi'_{ij}(\overline{x},\overline{y})\right)$.

It follows that,

$$X = \bigcup_{i=1}^{n}\left(\bigcap_{j=1}^{n_i}\phi_{ij}(\overline{M},\overline{a}) \cap \bigcap_{j=1}^{n'_i}\left(\overline{M}\backslash\phi'_{ij}(\overline{M},\overline{a})\right)\right)$$

But we observe that each of the $\phi^{*}_{**}(\overline{M},\overline{a})$ sets is Srour-closed, hence X is Srour-constructible as desired.

($\Leftarrow$) First, fix a tuple, $\overline{a}$ in $\overline{\mathcal{M}}$, and let us consider the property P = "to be an $\overline{a}$-definable set". A definable set, X, has this property if we can find an L-formula, $\psi(\overline{x};\overline{y})$ so that $X = \psi(\overline{M},\overline{a})$. We claim that P satisfies the conditions of Lemma 3.9. Suppose the Y and X are both $\overline{a}$-definable, by, say, formulas ψ and χ respectively. Then we can see that $Y\backslash X$ is $\overline{a}$-defined by the formula $\phi(\overline{x};\overline{y}) \coloneqq \psi(\overline{x};\overline{y}) \wedge \neg\chi(\overline{x};\overline{y})$.

Next suppose that X is $\overline{a}$-definable and $X \subseteq Y$, with Y a Srour-closed set, say defined by an instance of equation $\phi(\overline{x};\overline{y})$. Let $Z = \cap\{\phi(\overline{M},\overline{b}) \mid X \subseteq \phi(\overline{M},\overline{b})\}$. Observe that $X \subseteq Z \subseteq Y$. By the DIC, we can see that Z is defined by a conjunction of instances of ϕ. By Proposition 2.23, this conjunction must be an equation, and so Z is Srour-closed. In addition, we can see that Z is $\overline{a}$-invariant. That is, X, and hence Z by definition, is fixed by any automorphism that fixes $\overline{a}$, and so, by saturation, Z is $\overline{a}$-definable.

So by Lemma 3.9 it follows that if X is $\overline{a}$-definable and Srour-constructible then X can be written as a boolean combination of $\overline{a}$-definable Srour-closed sets. Furthermore, by Proposition 3.7, we may assume that these sets are given by $\overline{a}$-instances of equations (that is, of the form $\phi(\overline{M},\overline{a})$ for some equation ϕ).

It follows from our assumption that all definable sets are Srour-closed, if $\phi(\overline{x};\overline{y})$ is an arbitrary L-formula with $|\overline{x}| = n$ then for each $\overline{a} \in \overline{M}^{|\overline{y}|}$ we can find an L-formula, $\beta_{\overline{a}}(\overline{x};\overline{y})$, which is a boolean combination of equations, and $\phi(\overline{M},\overline{a}) = \beta_{\overline{a}}(\overline{M},\overline{a})$.

Let us consider the collection of L-formulas, $p(\overline{z}) = \{\neg(\forall\overline{x}\phi(\overline{x},\overline{z}) \leftrightarrow \beta_{\overline{a}}(\overline{x},\overline{z})) \mid \overline{a} \in \overline{M}^{|\overline{y}|}\}$. Suppose that it were satisfiable. Then we could find $\overline{b} \in \overline{M}^{|\overline{y}|}$ so that for all $\overline{a} \in \overline{M}^{|\overline{y}|}$, $\phi(\overline{M},\overline{b}) \neq \beta_{\overline{a}}(\overline{M},\overline{b})$. But by definition, $\phi(\overline{M},\overline{b}) = \beta_{\overline{b}}(\overline{M},\overline{b})$! This is a contradiction, so it follows that we can find some finite collection of the $\{\beta_{\overline{a}}\}_{\overline{a}\in\overline{M}^{|\overline{y}|}}$, which we rename, $\beta_1, \beta_2, \cdots, \beta_m$, such that for all $\overline{a}$, $\phi(\overline{M},\overline{a})$ is equal to one of the $\beta_i(\overline{M},\overline{a})$.

Consider the formulas, $\delta_i(\overline{x};\overline{y}) \coloneqq \forall\overline{u}\,(\phi(\overline{u};\overline{y}) \leftrightarrow \beta_i(\overline{u};\overline{y}))$. We can see that $\delta_i(\overline{x};\overline{y})$ is trivially an equation in $\overline{x}$. One can easily see that $\overline{\mathcal{M}} \models \forall\overline{x}\forall\overline{y}\,(\phi(\overline{x};\overline{y}) \leftrightarrow (\bigvee_{i=1}^{m}(\beta_i(\overline{x};\overline{y}) \wedge \delta_i(\overline{x};\overline{y}))))$. So ϕ is logically equivalent to a boolean combination of equations.

Since ϕ was arbitrary this concludes the proof. □

Corollary 4.7. *T is equational if and only if every definable set is Srour constructible.*

Definition 4.8. *Let E be a set of equations. Let $p(\overline{x})$ be a type over A. We define the* **E-part** *of p to be the type over A defined, $p^E(\overline{x}) = \{\psi(\overline{x},\overline{a}) \in p \mid$ for some $\phi(\overline{x};\overline{y}) \in E$, and $\overline{m}$ from $\overline{M}, \overline{\mathcal{M}} \models \forall \overline{x} \phi(\overline{x},\overline{m}) \leftrightarrow \psi(\overline{x},\overline{a})\}$. That is, it is the collection of all formulas in p that are equivalent (modulo T) to an instance of an equation in E.*

If $p = tp(\overline{a}/A)$ we may choose to write $tp^E(\overline{a}/A)$ instead of $p^E(\overline{x})$.

It is not hard to see some immediate facts from this definition, such as ^{E} commutes with automorphisms ($f(p)^E = f(p^E)$) and ^{E} commutes with restrictions ($(q \upharpoonright A)^E = q^E \upharpoonright A$). Also, $p^E = \cup_{\phi \in E} p^{\{\phi\}}$.

Proposition 4.9. *If E is the set of all equations, then $\{\phi(\overline{x},\overline{a}) \in p \mid \phi(\overline{x};\overline{y}) \in E\} \models p^E(\overline{x})$.*

Proof. Let $\psi(\overline{x},\overline{a}) \in p^E(\overline{x})$. Then we can find some equation $\phi(\overline{x};\overline{z})$ and some $\overline{m}$ so that $\psi(\overline{M},\overline{a}) = \phi(\overline{M},\overline{m})$. We can see that $\psi(\overline{M},\overline{a})$ is Srour-closed, and so by applying Proposition 3.7 we can find an equation $\phi'(\overline{x};\overline{y}) \in E$ so that $\psi(\overline{M},\overline{a}) = \phi'(\overline{M},\overline{a})$. Thus $\phi'(\overline{x};\overline{a}) \in p$ and $\phi'(\overline{x};\overline{a}) \models \psi(\overline{x},\overline{a})$. Moreover, $\{\phi(\overline{x},\overline{a}) \in p \mid \phi(\overline{x};\overline{y}) \in E\} \models \psi(\overline{x},\overline{a})$

Since $\psi(\overline{x},\overline{a})$ was arbitrary, we conclude that $\{\phi(\overline{x},\overline{a}) \in p \mid \phi(\overline{x};\overline{y}) \in E\} \models p^E(\overline{x})$. □

Proposition 4.10. *Suppose that E is equational and $E = cl^{\wedge}(E)$. Then for any complete type p there is a formula $\psi_p(\overline{x}) \in p^E(\overline{x})$ such that $\psi_p(\overline{x}) \models p^E(\overline{x})$*

Proof. Since E is equational, E satisfies the DIC. By Definition 2.20 we know that E satisfies the DIC if and only if $\{\phi(\overline{M},\overline{a}) \mid \phi(\overline{x};\overline{y}) \in E, \overline{a} \in \overline{M}^{|\overline{y}|}\}$ satisfies the DIC.

By definition of p^E, for each $\psi(\overline{x}) \in p^E$ we can an equation $\phi_\psi(\overline{x},\overline{y}_\psi) \in E$ and $\overline{m}_\psi$ so that $\psi(\overline{x})$ is equivalent in $\overline{\mathcal{M}}$ to $\phi_\psi(\overline{x},\overline{m}_\psi)$.

So let us consider $\cap_\psi \phi_\psi(\overline{M},\overline{m}_\psi)$. By the descending intersection condition, we can find $\psi_1, \cdots, \psi_k$ so that $\cap_\psi \phi_\psi(\overline{M},\overline{m}_\psi) = \cap_{j=1}^k \phi_{\psi_j}(\overline{M},\overline{m}_{\psi_j})$.

It follows that for any $\overline{a} \in \overline{M}^n$, $\overline{\mathcal{M}} \models \wedge_{j=1}^k \phi_{\psi_j}(\overline{a},\overline{m}_{\psi_j})$ if and only if $\overline{\mathcal{M}} \models \phi_\psi(\overline{a},\overline{m}_\psi)$ for all $\psi \in p^E$.

Let $\psi_p(\overline{x}) = \psi_1(\overline{x}) \wedge \cdots \wedge \psi_k(\overline{x})$. By completeness of p, $\psi_p(\overline{x}) \in p$. We can see that $\psi_p(\overline{x})$ is equivalent in $\overline{\mathcal{M}}$ to $\wedge_{j=1}^k \phi_{\psi_j}(\overline{x},\overline{m}_{\psi_j})$. So by the definition of each ϕ_ψ, for any $\overline{a} \in \overline{M}^n$, $\overline{\mathcal{M}} \models \psi_p(\overline{a})$ if and only if $\overline{\mathcal{M}} \models \psi(\overline{a})$ for all $\psi \in p^E$.

Since $\psi_p(\overline{x})$ is equivalent in $\overline{\mathcal{M}}$ to $\wedge_{j=1}^k \phi_{\psi_j}(\overline{x},\overline{m}_{\psi_j})$ and $\wedge_{j=1}^k \phi_{\psi_j}(\overline{x},\overline{y}_{\psi_j}) \in E$ (since $E = cl^{\wedge}(E)$) it follows that $\psi_p(\overline{x}) \in p^E(\overline{x})$. □

Proposition 4.11. *Suppose that our theory, T, is E-equational. Then for any two complete types $p, q \in S(A)$, $p^E = q^E$ if and only if $p = q$.*

Proof. ($\Leftarrow$) Is clear.

($\Rightarrow$) By symmetry it suffices to show that $p \subseteq q$. Take $\psi(\overline{x},\overline{a}) \in p$, where $\overline{a} \in A$. Since T is equational we can find equations $\{\phi_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n_i}$ and $\{\phi'_{ij}(\overline{x};\overline{y})\}_{i=1,j=1}^{n,n'_i}$, in $\overline{x}$ in E so that $\psi(\overline{x};\overline{y})$ is logically equivalent in T to $\bigvee_{i=1}^n \left(\bigwedge_{j=1}^{n_i} \phi_{ij}(\overline{x},\overline{y}) \wedge \bigwedge_{j=1}^{n'_i} \neg\phi'_{ij}(\overline{x},\overline{y})\right)$.

Since $\psi(\overline{x},\overline{a}) \in p$ by completeness of p we have that for some i and all j, $\phi_{ij}(\overline{x},\overline{a}) \in p$ and $\phi'_{ij}(\overline{x},\overline{a}) \notin p$. As $\phi_{ij}(\overline{x},\overline{a})$ is an A-instance of an equation, $\phi_{ij}(\overline{x},\overline{a}) \in p^E = q^E \subseteq q$.

Suppose that for some j, $\phi'_{ij}(\overline{x},\overline{a}) \in q$. Then, since it is equivalent to an instance of an equation in E (namely itself), it follows that $\phi'_{ij}(\overline{x},\overline{a}) \in q^E$. But $\phi'_{ij}(\overline{x},\overline{a}) \notin p \supseteq p^E = q^E$. Contradiction! So $\phi'_{ij}(\overline{x},\overline{a}) \notin q$.

Then we can see that $\bigwedge_{j=1}^{n_i} \phi_{ij}(\overline{x},\overline{a}) \wedge \bigwedge_{j=1}^{n'_i} \neg\phi'_{ij}(\overline{x},\overline{a}) \in q$ and so $\bigvee_{i=1}^n \left(\bigwedge_{j=1}^{n_i} \phi_{ij}(\overline{x},\overline{a}) \wedge \bigwedge_{j=1}^{n'_i} \neg\phi'_{ij}(\overline{x},\overline{a})\right) \in q$. By completeness of q we conclude that $\psi(\overline{x},\overline{a}) \in q$. □

Equationally Free Extensions

The bulk of these next two subsections is due to G. Srour in his papers, [9], [10]. While Srour's papers take the viewpoint of categories and universal algebra, many of his proofs and concepts become much simpler in the realm of first order model theory. Readers are urged to read Srour's paper if they wish to see the broadest possible exploration of what an equation is.

Definition 4.12. *Let E be a set of equations and $A \subseteq B$. Let $p \in S(A)$ and $q \in S(B)$ with $p \subseteq q$. Then q is an* **E-equationally free extension** *of p to B if for any $q' \in S(B)$ extending p, $q^E \supseteq (q')^E$ implies $q^E = (q')^E$.*

E-equationally free extensions are extensions which have a minimal E-part.

Theorem 4.13. *Suppose that $E = cl^+(E)$ is a set of equations. Let p be a complete type over A. Let $B \supseteq A$ and let q be a complete type over B with $q^E \supseteq p^E$. Then we can find an E-equationally free extension q' of p to B, so that $q^E \supseteq (q')^E$.*

Proof. The first step is to find an extension of p to B, r', so that $q^E \supseteq (r')^E$. Let us consider the set,

$$r_* := p \cup \{\neg\psi(\overline{x},\overline{b}) \mid \psi(\overline{x};\overline{y}) \text{ L-formula}, \overline{b} \in B, \psi(\overline{x},\overline{b}) \notin q^E, \text{ but equivalent to an instance of an equation in E}\}$$

Suppose r_* is inconsistent. Then we can find $\psi_1(\overline{x},\overline{b}_1),\cdots,\psi_n(\overline{x},\overline{b}_n) \notin q^E$ so that $p \models \bigvee_{i=1}^n \psi_i(\overline{x},\overline{b}_i)$. We have $\phi_{\psi_i}(\overline{x},\overline{y}_{\psi_i}) \in E$ and $\overline{m}_{\psi_i}$ so that $\psi_i(\overline{x},\overline{b}_i)$ is equivalent in $\overline{\mathcal{M}}$ to $\phi_{\psi_i}(\overline{x},\overline{m}_{\psi_i})$. Then $\bigvee_{i=1}^n \phi_{\psi_i}(\overline{x},\overline{y}_{\psi_i})$ is an equation in $\overline{x}$ belonging to E.

For any automorphism $f \in Aut_A(\overline{M})$, $p \models \bigvee_{i=1}^n \phi_{\psi_i}(\overline{x}, f(\overline{m}_{\psi_i}))$. Since $\bigvee_{i=1}^n \phi_{\psi_i}(\overline{x},\overline{y}_{\psi_i})$ is an equation, we can find a finite set $\{\overline{m}_{i,j}\}_{i=1,j=1}^{n,k}$ so that $\bigcap_{j=1}^k \left(\bigcup_{i=1}^n \phi_{\psi_i}(\overline{M},\overline{m}_{i,j})\right) = \bigcap_{f\in Aut_A(\overline{M})} \left(\bigcup_{i=1}^n \phi_{\psi_i}(\overline{M}, f(\overline{m}_{\psi_i}))\right)$. In particular, $\bigcap_{j=1}^k \left(\bigcup_{i=1}^n \phi_{\psi_i}(\overline{M},\overline{m}_{i,j})\right)$ is invariant under any automorphism that fixes A.

By the saturation of our model, $\overline{\mathcal{M}}$, we can find an L-formula $\psi(\overline{x};\overline{y})$ and a tuple $\overline{a}$ from A so that $\bigcap_{j=1}^k \left(\bigcup_{i=1}^n \phi_{\psi_i}(\overline{M},\overline{m}_{i,j})\right) = \psi(\overline{M},\overline{a})$. By completeness of p, $\psi(\overline{x},\overline{a}) \in p$, and hence $\psi(\overline{x},\overline{a}) \in p^E \subseteq q^E \subseteq q$. So then we see that $q \models \bigvee_{i=1}^n \psi_i(\overline{x},\overline{b}_i)$ and by completeness of q, we can find k so that $\psi_k(\overline{x},\overline{b}_k) \in q$. But this contradicts our assumption that $\psi_k(\overline{x},\overline{b}_k) \notin q^E$.

Thus r_* is consistent, and we can extend it to some $r' \in S(B)$. Note that by construction, $r' \supseteq p$ and $q^E \supseteq (r')^E$.

Next, let us consider the set $R = \{r \in S(B) \mid r \supseteq p, q^E \supseteq r^E\}$. R contains r', so we know it is nonempty. We can put a natural partial order on R by saying $r_1 \leq_R r_2$ if and only if $r_1^E \subseteq r_2^E$. Let $\{r_\beta\}_{\beta<\alpha}$ be a descending chain in $(R,\leq_R)$ (where α is an ordinal). That is, for all $\gamma < \beta < \alpha$ we have $r_\gamma^E \leq_R r_\beta^E$. Define the collection of formulas,

$$r_{**} := p \cup \{\neg\psi(\overline{x},\overline{b}) \mid \psi \text{ an L-formula}, \overline{b} \in B, \psi(\overline{x},\overline{b}) \notin \cap_{\beta<\alpha} r_\beta^E, \text{ but equivalent to an instance of an equation in E}\}$$

We claim that r_{**} is consistent. It suffices to show for any finite set $\{\neg\psi_1(\overline{x},\overline{b}_1),\cdots,\neg\psi_n(\overline{x},\overline{b}_n)\}$, that $p \cup \{\neg\psi_1(\overline{x},\overline{b}_1),\cdots,\neg\psi_n(\overline{x},\overline{b}_n)\}$ is consistent.

Since there are only finitely many of the ψ_is, we can find $\beta < \alpha$ so that each $\psi_i(\overline{x},\overline{b}_i) \notin r_\beta$. But by completeness of r_β is follows that $\neg\psi_i(\overline{x},\overline{b}_i) \in r_\beta$. And since each of the r_γ extends p it follows that $p \cup \{\neg\psi_1(\overline{x},\overline{b}_1),\cdots,\neg\psi_n(\overline{x},\overline{b}_n)\}$ is consistent. So r_{**} is consistent.

Extend r_{**} to a complete type over B, say r. We claim that $r \leq_R r_\beta$ for any $\beta < \alpha$. Indeed, suppose that $\psi(\overline{x},\overline{b}) \notin r_\beta^E$.Then $\neg\psi(\overline{x},\overline{b}) \in r_{**} \subseteq r$. Since r is consistent, we can see that $\psi(\overline{x},\overline{b}) \notin r$ and hence $\psi(\overline{x},\overline{b}) \notin r^E$. Thus r is a lower bound of $\{r_\beta\}_{\beta<\alpha}$.

Since every descending chain has a lower bound in R, by Zorn's Lemma, R contains a minimal element, say q'. Since $(q')^E$ is a minimal E-part then q' is an equationally free extension of p, and $q^E \supseteq (q')^E$ as desired. □

We now move on to a very important result. The following theorem shows that E-equationally freeness is transitive.

Theorem 4.14. *Suppose $E = cl^+(E)$ is a set of equations. Let $A \subseteq B \subseteq C$. Suppose $p \in S(A)$, $q \in S(B)$ and $r \in S(C)$ and $p \subseteq q \subseteq r$. Then r is an E-equationally free extension of p if and only if q is an E-equationally free extension of p and r is an E-equationally free extension of q.*

Proof. ($\Leftarrow$) Suppose that there is $r_1 \in S(C)$ where $r_1 \supseteq p$ and $r^E \supseteq r_1^E$. We can see $q^E = (r \upharpoonright B)^E \supseteq (r_1 \upharpoonright B)^E$. Since $r_1 \supseteq p$ we can see that $r_1 \upharpoonright B \supseteq p$. Since q is an E-equationally free extension of p it follows that $q^E = (r_1 \upharpoonright B)^E$.

Now, since $r_1^E \supseteq q^E$, by Theorem 4.13, we can find $r_2 \in S(C)$ so that $r_2 \supseteq q$ and $r_1^E \supseteq r_2^E$. But then $r^E \supseteq r_1^E \supseteq r_2^E$, and by minimality of r^E with regards to q^E it follows that $r^E = r_2^E$. So by squeezing, we conclude that $r^E = r_1^E$.

($\Rightarrow$) First we show that r is an E-equationally free extension of q. Suppose that there is a complete type $r_1 \in S(C)$ so that $r_1 \supseteq q$ and $r^E \supseteq r_1^E$. Then $r_1 \supseteq p$ and so by E-equational freeness of r we can see that $r_1^E = r^E$. So r is an E-equationally free extension of q.

Now we demonstrate that q is an E-equationally free extension of p. Suppose that there is a complete type $q_1 \in S(B)$ so that $q_1 \supseteq p$ and $q^E \supseteq q_1^E$. Now, since $r^E \supseteq q^E \supseteq q_1^E$, by Theorem 4.13 we can find an E-equationally free extension, r_1, of q_1 to C such that $r^E \supseteq r_1^E$. Since, $p \subseteq q_1 \subseteq r_1$, by minimality of r^E we can see that $r^E = r_1^E$. Then $q^E = (r \upharpoonright B)^E = (r_1 \upharpoonright B)^E = q_1^E$. So it follows that q is an E-equationally free extension of p. □

We can define a notion of independence based on equationally free extensions.

Definition 4.15. *Let E be a set of equations. Given a tuple, $\overline{a}$ and sets A and B, we say that $\overline{a}$ is* **E-equationally independent** *of B* **over** *A, if $tp(\overline{a}/A \cup B)$ is an E-equationally free extension of $tp(\overline{a}/A)$. We will use the notation, $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$, to denote E-equational independence.*

This next result summarizes some of the first properties of this notion of independence.

Theorem 4.16. *Let E be a set of equations closed under finite conjunctions and disjunctions. Then E-equational independence, $\mathop{\perp\!\!\!\perp}^E$, satisfies the following five properties,*

- *(i)* (Existence) *For any $\overline{a}$, A and B, there exists $\overline{a}'$ such that $tp(\overline{a}'/A) = tp(\overline{a}/A)$ and $\overline{a}' \mathop{\perp\!\!\!\perp}^E_A B$.*
- *(ii)* (Transitivity) *If $A \subseteq B \subseteq C$ are sets, then $\overline{a} \mathop{\perp\!\!\!\perp}^E_A C$ if and only if $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$ and $\overline{a} \mathop{\perp\!\!\!\perp}^E_B C$.*
- *(iii)* (Finite Character) *$\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$ if and only if $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B_0$ for all finite subsets $B_0 \subseteq B$.*
- *(iv)* (Invariance) *If $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$ then for any automorphism of $\overline{M}$, f, $f(\overline{a}) \mathop{\perp\!\!\!\perp}^E_{f(A)} f(B)$.*
- *(v)* (Local Character) *For any $\overline{a}$ and B, there exists $A \subseteq B$ such that $|A| \le |T|$ and $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$*

Proof of Existence. This is just Theorem 4.13. □

Proof of Transitivity. This is just Theorem 4.14. □

Proof of Finite Character. One direction is clear from Transitivity. So it remains to show that if $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B_0$ for all finite subsets $B_0 \subseteq B$ then $\overline{a} \mathop{\perp\!\!\!\perp}^E_A B$.

Suppose that p is a complete type over B extending $tp(\overline{a}/A)$ such that $tp^E(\overline{a}/B) \supseteq p^E$. By definition, we need to show that $tp^E(\overline{a}/B) = p^E$. Take an arbitrary $\psi(\overline{x}) \in tp^E(\overline{a}/B)$. $\psi(\overline{x})$ is an L_B-formula, and we can see that $\psi(\overline{x})$ must be an L_{B_0}-formula for some finite subset $B_0 \subseteq B$. We can restrict to $A \cup B_0$ and see that $tp^E(\overline{a}/A \cup B_0) \supseteq (p \upharpoonright A \cup B_0)^E$. By our assumption that $\overline{a} \mathop{\downarrow}^E_A B_0$, it follows that $tp^E(\overline{a}/A \cup B_0) = (p \upharpoonright A \cup B_0)^E$. By definition of B_0, $\psi(\overline{x}) \in tp^E(\overline{a}/A \cup B_0) = (p \upharpoonright A \cup B_0)^E \subseteq p^E$.

So it follows that $tp^E(\overline{a}/B) \subseteq p^E$. Thus $tp^E(\overline{a}/B) = p^E$ and since p was arbitrary we conclude that $\overline{a} \mathop{\downarrow}^E_A B$. □

Proof of Invariance. Suppose that $\overline{a} \mathop{\downarrow}^E_A B$, but $f(\overline{a}) \not\mathop{\downarrow}^E_{f(A)} f(B)$ for some automorphism, f. Then we can find a complete type $p \in S(f(A \cup B))$ with $tp(f(\overline{a})/f(A)) \subseteq p$ and $p^E \subsetneq tp^E(f(\overline{a})/f(A \cup B))$.

Since f is an automorphism, $f^{-1}(p)$ is a complete type over $A \cup B$ and $f^{-1}(p)^E = f^{-1}\left(p^E\right) \subsetneq tp^E(\overline{a}/A \cup B)$. But then $tp(\overline{a}/A \cup B)$ is not an E-equationally free extension of $tp(\overline{a}/A)$ (as its E-part is not minimal). This contradicts our assumption that $\overline{a} \mathop{\downarrow}^E_A B$.

Thus for any automorphism, f, we have $f(\overline{a}) \mathop{\downarrow}^E_{f(A)} f(B)$. □

Proof of Local Character. Let $\phi(\overline{x};\overline{y}) \in E$ be an arbitrary equation in $\overline{x}$. Let us consider $tp^{cl^+(\{\phi\})}(\overline{a}/B)$. For each $\phi \in E$ we know that $cl^+(\{\phi\})$ is equational, so we can apply Proposition 4.10. Thus, we can find an L_B-formula, $\psi^\phi(\overline{x}) \in tp^{cl^+(\{\phi\})}(\overline{a}/B)$ so that $\psi^\phi(\overline{x}) \models tp^{cl^+(\{\phi\})}(\overline{a}/B)$.

Since ψ^ϕ is an L_B-formula we can find a finite subset $B_\phi \subseteq B$ so that ψ^ϕ is an L_{B_ϕ}-formula. Let $A = \cup_{\phi \in E} B_\phi$ be our candidate for $\overline{a} \mathop{\downarrow}^E_A B$. We need to show that for any complete type over B, p, extending $tp(\overline{a}/A)$, with $tp^E(\overline{a}/B) \supseteq p^E$, that $tp^E(\overline{a}/B) = p^E$.

Let p be any complete type in B, extending $tp(\overline{a}/A)$ with $p^E \subseteq tp^E(\overline{a}/B)$. By our construction of A, we know for each $\phi \in E$, that $\psi^\phi(\overline{x}) \in tp(\overline{a}/A)$ and hence $\psi^\phi(\overline{x}) \in p$. We know that $\psi^\phi(\overline{x}) \models tp^{cl^+(\{\phi\})}(\overline{a}/B)$, so by completeness, $tp^{cl^+(\{\phi\})}(\overline{a}/B) \subseteq p$. By definition of the E-part we can see that $tp^E(\overline{a}/B) = \cup_{\phi \in E} tp^{cl^+(\{\phi\})}(\overline{a}/B) \subseteq p$. So we conclude that $tp^E(\overline{a}/B) \subseteq p^E$ and hence, $tp^E(\overline{a}/B) = p^E$. Thus $tp(\overline{a}/B)$ is an E-equationally free extension of $tp(\overline{a}/A)$.

It remains to show that $|A| \leq |T|$. Note that T is infinite, and since T is complete we know that for each equation $\phi \in E$ we can find either $\exists\overline{x}\exists\overline{y}\phi(\overline{x};\overline{y})$ or $\neg\exists\overline{x}\exists\overline{y}\phi(\overline{x};\overline{y})$ in T. Thus, $|E| \leq |T|$. Since $A = \cup_{\phi \in E} B_\phi$ and each B_ϕ was finite, we can see that $|A| \leq |E| \times \aleph_0 \leq |T| \times \aleph_0 = |T|$, as required. □

If T is E-equational one can show that $\mathop{\downarrow}^E$ satisfies these two additional properties:

(Symmetry) For all $\overline{a}$, $\overline{b}$, A, $\overline{a} \mathop{\downarrow}^E_A \overline{b} \Rightarrow \overline{b} \mathop{\downarrow}^E_A \overline{a}$.

(Stationarity Over Models) If $\mathcal{M} \models T$ and $B \supseteq M$ then every type over M has a unique E-equationally free extension to B.

These two properties are much harder to prove. One way to do so is to show that $\mathop{\downarrow}^E$ agrees with Shelah's nonforking independence in stable theories [7]. This is the route taken by Pillay and Srour in their paper, [5], however such a proof is beyond the scope of this paper.

5 Finale and Acknowledgements

Equational theories still have several unanswered questions. One which we've mentioned before is "Does being 1-equational imply being equational?" Other questions that M. Junker asks in [3], which we have not covered are, "Is every theory interpretable in an equational theory itself equational?" and "Is the reduct of an equational theory also equational?" Thus, there is still a good deal of work and research left for equational theories.

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