THE DEGREE OF NONMINIMALITY IS AT MOST 2

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Abstract. It is shown that if $p \in S(A)$ is a complete type of Lascar rank at least 2, in the theory of differentially closed fields of characteristic zero, then there exists a pair of realisations a_1, a_2 such that p has a nonalgebraic forking extension over Aa_1a_2. Moreover, if A is contained in the field of constants then p already has a nonalgebraic forking extension over Aa_1. The results are also formulated in a more general setting.

1. Introduction

In [4], motivated by the search for general techniques that might aid in proving strong minimality for certain algebraic differential equations, the first and third authors introduced *degree of nonminimality* as a measure of how many parameters are needed to witness that a type is *not* minimal. Working in a sufficiently saturated model of a stable theory eliminating imaginaries, here is a precise formulation:

Definition 1.1. Suppose $p \in S(A)$ is a stationary type with $U(p) > 1$. The *degree of nonminimality of* p, denoted by $\mathrm{nmdeg}(p)$, is the least positive integer d such that there exist realisations $a_1, \dots, a_d$ of p and a nonalgebraic forking extension of p over $Aa_1, \dots, a_d$. If $U(p) \leq 1$ then we set $\mathrm{nmdeg}(p) = 0$ by convention.

Using an analysis of the multiple transitivity of binding group actions, it was shown in [4] that $\mathrm{nmdeg}(p) \leq U(p) + 1$ in the theory of differentially closed fields of characteristic zero (DCF_0). Bounds on the degree of nonminimality have played a significant role in recent proofs of strong minimality; of the generic differential equation in [2] and of the differential equations satisfied by the Schwarz triangle functions in [1]. Based on a maturing of the techniques used in [4], and informed by the approach taken in [3] to a related problem, we give in this note a short proof of a dramatic improvement to that bound:

Theorem. *Suppose $T = \mathrm{DCF}_0$ and p is a complete stationary type of finite rank. Then* $\mathrm{nmdeg}(p) \leq 2$. *Moreover, if p is over constant parameters then* $\mathrm{nmdeg}(p) \leq 1$.

The bound is sharp; see [4, Example 4.2] for types of nonminimality degree 2.

The argument we give for the main clause, namely that $\mathrm{nmdeg}(p) \leq 2$, works equally well in $\mathrm{DCF}_{0,m}$, the theory of differentially closed fields in m commuting derivations, and in CCM, the theory of compact complex manifolds. All one needs is

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that T be totally transcendental, eliminate imaginaries, eliminate the "there exists infinitely many" quantifier, and admit a 0-definable pure algebraically closed field to which every non locally modular minimal type is nonorthogonal. In $\mathrm{DCF}_{0,m}$ that pure algebraically closed field is the field of absolute constants and in CCM it is the (interpretation in $\mathcal{U}$ of the) complex field living on the projective line.

The "moreover" clause of the theorem, however, does make use of the fact that, in DCF_0, the binding group of a type over the constants and internal to the constants cannot be centerless.

The most general setting for the results is articulated, for the record, in Section 3.

Remark 1.2. A corollary of our theorem is a significant improvement to the main result of [2], where it was shown that generic algebraic differential equations of order $h \geq 2$ and degree at least $2(h+2)$ are strongly minimal. The proof in [2] used that $\mathrm{nmdeg}(p) \leq U(p)+1$. The same proof, but using the improved bound of $\mathrm{nmdeg}(p) \leq 2$ obtained here, allows one to replace $2(h+2)$ by 6 in that result.

2. The proof

We work in a fixed sufficiently saturated model $\mathcal{U}$ of a complete totally transcendental theory T eliminating imaginaries and the "there exists infinitely many" quantifier, with $\mathcal{C}$ a 0-definable pure algebraically closed field such that every non locally modular minimal type is nonorthogonal to $\mathcal{C}$.

Maybe the first thing to observe is that the degree of nonminimality is invariant under interalgebraicity. Here we use the following, possibly nonstandard but unambigious, terminology:

Definition 2.1. Complete types $p, q \in S(A)$ are said to be *interalgebraic* if for each (equivalently some) $a \models p$ there exists $b \models q$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Ab)$.

That $\mathrm{nmdeg}(p) = \mathrm{nmdeg}(q)$ when p and q are interalgebraic is more or less immediate from the definitions; see for example [4, Lemma 3.1(c)].

The following consequences of $\mathrm{nmdeg} > 1$ were observed in [4], but we include some details here for the sake of completeness:

Fact 2.2. *Suppose $p \in S(A)$ is stationary of finite rank with* $\mathrm{nmdeg}(p) > 1$*. Then p is interalgebraic with a stationary type $q \in S(A)$ such that q is $\mathcal{C}$-internal and $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal.*

Proof. Note, first of all, that

(∗) if $a \models p$ and $b \in \mathrm{acl}(Aa) \setminus \mathrm{acl}(A)$ then $a \in \mathrm{acl}(Ab)$.

Indeed, if a' realises the nonforking extension of p to Aab then $\mathrm{tp}(a'/Aa)$ is a forking extension of p. Since $\mathrm{nmdeg}(p) > 1$ we must have that $a' \in \mathrm{acl}(Aa)$, from which it follows that $a' \in \mathrm{acl}(Ab)$, and hence $a \in \mathrm{acl}(Ab)$.

In the finite rank setting, condition (∗), which is a weak form of exchange, implies that either p is interalgebraic with a locally modular minimal type, or p is almost internal to a non locally modular minimal type – see [6, Proposition 2.3]. The former is impossible as $U(p) > 1$, and by assumption on T the latter implies p is almost $\mathcal{C}$-internal. We thus find a stationary $\mathcal{C}$-internal $q \in S(A)$ that is interalgebraic with p. Note that $\mathrm{nmdeg}(q) > 1$ as well.

Suppose that q is not weakly $\mathcal{C}$-orthogonal. Since the induced structure on $\mathcal{C}$, namely that of a pure algebraically closed field, eliminates imaginaries, this failure of weak $\mathcal{C}$-orthogonality will be witnessed by some $b \models q$ and $c \in \mathcal{C}$ such that $c \in \operatorname{dcl}(Ab) \setminus \operatorname{acl}(A)$. By $(*)$ applied to q this would force $b \in \operatorname{acl}(Ac)$, contradicting $U(q) > 1$. So q is weakly $\mathcal{C}$-orthogonal. In particular, as it is $\mathcal{C}$-internal, q is isolated. We let Ω be the definable set $q(\mathcal{U})$.

Now suppose that $q^{(2)}$ is not weakly $\mathcal{C}$-orthogonal. Then there are independent b_1, b_2 realising q and $c \in \mathcal{C}$ such that $c \in \operatorname{dcl}(Ab_1b_2) \setminus \operatorname{acl}(A)$. Note that $b_2 \notin \operatorname{acl}(Ab_1c)$ as $U(b_2/Ab_1) = U(q) > 1$. So there is a partial Ab_1-definable function $f : \Omega \to \mathcal{C}$ with infinite image and infinite generic fibre. It follows, by elimination of the "there exists infinitely many" quantifier, that all but finitely many of the fibres are infinite. As $\mathcal{C} \cap \operatorname{acl}(A)$ is infinite (it is an algebraically closed subfield of $\mathcal{C}$), there exists $b \in \Omega \setminus \operatorname{acl}(Ab_1)$ such that $f(b) \in \operatorname{acl}(A)$. If $b \mathop{\smile\!\!\!\!\!\mid}_A b_1$ then $\operatorname{tp}(b/Ab_1) = \operatorname{tp}(b_2/Ab_1)$ contradicting the fact that $f(b_2) = c \notin \operatorname{acl}(A)$. So $b \not\mathop{\smile\!\!\!\!\!\mid}_A b_1$. That is, $\operatorname{tp}(b/Ab_1)$ is a nonalgebraic forking extension of q. But this contradicts $\operatorname{nmdeg}(q) > 1$. Hence $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal. □

The following improvement to Fact 2.2 was *not* remarked in [4].

Lemma 2.3. *Suppose $p \in S(A)$ is stationary of finite rank with* $\operatorname{nmdeg}(p) > 1$. *Then p is interalgebraic with some stationary $q \in S(A)$ such that*

(a) *q is $\mathcal{C}$-internal,*
(b) *$q^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and,*
(c) *any two distinct realisations of q are independent over A.*

Proof. Suppose a, b are realisations of p such that $a \not\mathop{\smile\!\!\!\!\!\mid}_A b$. If $a \notin \operatorname{acl}(Ab)$ then $\operatorname{tp}(a/Ab)$ is a nonalgebraic forking extension of p, contradicting $\operatorname{nmdeg}(p) > 1$. Similarly, we must have $b \in \operatorname{acl}(Aa)$. In other words, $a \not\mathop{\smile\!\!\!\!\!\mid}_A b$ if and only if $\operatorname{acl}(Aa) = \operatorname{acl}(Ab)$. In particular, $a \not\mathop{\smile\!\!\!\!\!\mid}_A b$ is an equivalence relation on $p(\mathcal{U})$, which we now denote by E.

Applying Fact 2.2, we may assume that p is $\mathcal{C}$-internal and $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal. In particular, both p and $p^{(2)}$ are isolated, say by the L_A-formulae $\phi(x)$ and $\psi(x, y)$, respectively. Note then, that $\phi(x) \wedge \phi(y) \wedge \neg\psi(x, y)$ defines the forking relation E. So E is an A-definable equivalence relation.

Each class of E is finite. Indeed, if $a \models p$ has an infinite E-class then there is $b \in p(\mathcal{U}) \setminus \operatorname{acl}(Aa)$ with aEb. But that means that $\operatorname{tp}(b/Aa)$ is a nonalgebraic forking extension of p, contradicting $\operatorname{nmdeg}(p) > 1$.

Fixing $a \models p$, let $e := a/E$ and $q := \operatorname{tp}(e/A)$. Note that $e \in \operatorname{dcl}(Aa)$, and so we still have that q is $\mathcal{C}$-internal and $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal. Also, as the E-classes are finite, p and q are interalgebraic. So it remains to show that any two distinct realisations of q are independent. Suppose $e' \models q$ with $e' \neq e$. Then $e' = a'/E$ for some $a' \models p$ such that $\neg(aEa')$. That is $a \mathop{\smile\!\!\!\!\!\mid}_A a'$. As $\operatorname{acl}(Aa) = \operatorname{acl}(Ae)$ and $\operatorname{acl}(Aa') = \operatorname{acl}(Ae')$, we have that $e \mathop{\smile\!\!\!\!\!\mid}_A e'$, as desired. □

We now work toward a proof of the main clause of the Theorem. That is, fixing a finite rank stationary type $p \in S(A)$, we wish to show that $\operatorname{nmdeg}(p) \leq 2$. Let $\overline{p}$ denote the unique extension of p to $\operatorname{acl}(A)$. It is immediate from the definition that $\operatorname{nmdeg}(\overline{p}) = \operatorname{nmdeg}(p)$. We may therefore assume that $A = \operatorname{acl}(A)$. Let $k := A \cap \mathcal{C}$, it is an algebraically closed subfield of $\mathcal{C}$.

In order to prove that $\operatorname{nmdeg}(p) \leq 2$ we may of course assume that $\operatorname{nmdeg}(p) > 1$. Hence, by Lemma 2.3, we can further reduce to the case that p is $\mathcal{C}$-internal, $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and any two distinct realisations of p are independent over A.

Let $\Omega := p(\mathcal{U})$ and let $G := \mathrm{Aut}(p/\mathcal{C})$ be the binding group of p relative to $\mathcal{C}$. So (G, Ω) is an A-definable faithful group action. The action is transitive because p is weakly $\mathcal{C}$-orthogonal. Weak $\mathcal{C}$-orthogonality of p also implies, along with $A = \mathrm{acl}(A)$, that G is connected. The fact that $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal implies that G acts transitively on $p^{(2)}(\mathcal{U})$. But $p^{(2)}(\mathcal{U}) = \Omega^2 \setminus \Delta$ where Δ is the diagonal, because any two distinct realisations of p are independent over A. So (G, Ω) is a 2-transitive connected A-definable homogeneous space.

Now, the binding group action of any $\mathcal{C}$-internal type is isomorphic to the $\mathcal{C}$-points of an algebraic group action, though possibly over additional parameters. More precisely, let $M \preceq \mathcal{U}$ be a prime model over A. Note that $M \cap \mathcal{C} = k$. There exists an algebraic homogeneous space $(\overline{G}, \overline{\Omega})$ defined over k, and an M-definable isomorphism $\alpha : (G, \Omega) \to (\overline{G}(\mathcal{C}), \overline{\Omega}(\mathcal{C}))$.

In particular, $(\overline{G}, \overline{\Omega})$ is a 2-transitive connected algebraic homogeneous space. This is a very restrictive condition; a theorem of Knop [5] tells us that $(\overline{G}, \overline{\Omega})$ is either isomorphic to the action of PGL_{n+1} on $\mathbb{P}^n$, or is isomorphic to the action of an algebraic subgroup of the group of affine transformations on $\mathbb{A}^n$, for some $n > 1$. In either case we have a notion of *collinearity* which is preserved by the group action. That is, given distinct $u, v \in \overline{\Omega}(\mathcal{C})$ we can talk about the line $\ell_{u,v} \subseteq \overline{\Omega}(\mathcal{C})$ connecting u and v, and for all $g \in \overline{G}(\mathcal{C})$ we have that $g\ell_{u,v} = \ell_{gu,gv}$.

Fix distinct $a, b \in \Omega$, and consider the set $X := \alpha^{-1}(\ell_{\alpha(a),\alpha(b)})$. Then X is a rank 1 Mab-definable subset of Ω.

Claim 2.4. *There is a finite tuple c from $\mathcal{C}$ such that X is $Aabc$-definable.*

Proof. It suffices to show that if $\sigma \in \mathrm{Aut}_{Aab}(\mathcal{U}/\mathcal{C})$, that is, if σ is an automorphism of $\mathcal{U}$ that fixes $A \cup \{a, b\} \cup \mathcal{C}$ point-wise, then $\sigma(X) = X$. Now, the restriction of σ to Ω is an element of the binding group, say $g_\sigma \in G$, which fixes a and b. Hence $\alpha(g_\sigma) \in \overline{G}(\mathcal{C})$ fixes $\alpha(a)$ and $\alpha(b)$, and hence preserves the line $\ell_{\alpha(a),\alpha(b)}$. It follows that

$$\begin{aligned} \alpha(\sigma(X)) &= \alpha\big(g_\sigma(\alpha^{-1}(\ell_{\alpha(a),\alpha(b)}))\big) \\ &= \alpha(g_\sigma)(\ell_{\alpha(a),\alpha(b)}) \\ &= \ell_{\alpha(a),\alpha(b)}. \end{aligned}$$

Applying α^{-1} to both sides we obtain that $\sigma(X) = X$, as desired. $\square$

Let $\theta(x, y)$ be an L_{Aab}-formula such that $X = \theta(\mathcal{U}, c)$. If, in addition, we chose $a, b \in \Omega(M)$, then X and $\theta(x, y)$ are over M, and it follows that there is $c' \in M \cap \mathcal{C}$ such that $X = \theta(\mathcal{U}, c')$. But $M \cap \mathcal{C} = k \subseteq A$, so that this witnesses the definability of X over Aab.

We have thus found $a, b \in \Omega$ and an Aab-definable subset $X \subseteq \Omega$ of rank 1. Since $U(p) > 1$, the generic type of X over Aab is a nonalgebraic forking extension of p. Since a and b realise p, this witnesses that $\mathrm{nmdeg}(p) = 2$.

This completes the proof of the main clause of the Theorem.

For the "moreover" clause, we return to the particular setting of $T = \mathrm{DCF}_0$ and $\mathcal{C}$ the field of constants. We make the additional assumption that $A \subseteq \mathcal{C}$ and show that $\mathrm{nmdeg}(p) > 1$ leads to a contradiction. Indeed, that (G, Ω) is 2-transitive forces G to be centerless; see for example the elementary argument at the beginning of the proof of Satz 2 in [5]. But, in DCF_0, the binding group of a type that is $\mathcal{C}$-internal and over constant parameters cannot be centerless; see for example the proof of Theorem 3.9 in [3]. This contradiction proves that $\mathrm{nmdeg}(p) \leq 1$. $\square$

3. Some remarks on the assumptions

We carried out the above proof under assumptions on T that were suitable for generalisation to both $\mathrm{DCF}_{0,m}$ and CCM. But it may be worth recording the minimal hypotheses on T required for the proofs to go through. We leave it to the reader to inspect those proofs and verify that what is actually proved are the following two statements:

Theorem 3.1. *Suppose T is a complete totally transcendental theory eliminating imaginaries and the "there exists infinitely many" quantifier. Let $\mathcal{U} \models T$ be a sufficiently saturated model and $A \subseteq \mathcal{U}$ a parameter set.*

(a) *Suppose each non locally modular minimal type in T is nonorthogonal to some A-definable pure algebraically closed field. Then $\mathrm{nmdeg}(p) \leq 2$ for all stationary $p \in S(A)$ of finite rank.*

(b) *Suppose there exists a collection $\{\mathcal{C}_i : i \in I\}$ of A-definable non locally modular strongly minimal sets such that each non locally modular minimal type in T is nonorthogonal to $\mathcal{C}_i$ for some $i \in I$, and such that for all $i \in I$,*
  (i) *$\mathcal{C}_i \cap \mathrm{acl}(A)$ is infinite, and,*
  (ii) *for all weakly $\mathcal{C}_i$-orthogonal $\mathcal{C}_i$-internal $q \in S(\mathrm{acl}(A))$, the binding group $\mathrm{Aut}(q/\mathcal{C}_i)$ has a nontrivial center.*

  Then $\mathrm{nmdeg}(p) \leq 1$ for all stationary $p \in S(A)$ of finite rank.

References

[1] Guy Casale, Matthew Devilbiss, James Freitag, and Joel Nagloo. Strong minimality of triangle functions. Preprint, arXiv:2205.08684.

[2] Matthew Devilbiss and James Freitag. Generic differential equations are strongly minimal. Preprint, arXiv:2106.02627v2.

[3] James Freitag, Rémi Jaoui, and Rahim Moosa. When any three solutions are independent. *Invent. Math.*, 2022. DOI 10.1007/s00222-022-01143-8.

[4] James Freitag and Rahim Moosa. Bounding nonminimality and a conjecture of Borovik-Cherlin. Preprint, arXiv:2106.02537.

[5] Friedrich Knop. Mehrfach transitive operationen algebraischer gruppen. *Archiv der Mathematik*, 41(5):438–446, 1983.

[6] Rahim Moosa and Anand Pillay. Some model theory of fibrations and algebraic reductions. *Selecta Mathematica*, 20(4):1067–1082, 2014.

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