

Model theory and Kähler geometry

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Summary

We survey and explain some recent work at the intersection of model theory and bimeromorphic geometry (classification of compact complex manifolds). Included here are the essential saturation of the many-sorted structure \mathcal{C} of Kähler manifolds, the conjectural role of hyperkähler manifolds in the description of strongly minimal sets in \mathcal{C} , and Campana's work on the isotriviality of hyperkähler families and its connection with the nonmultidimensionality conjecture.

1 Introduction

The aim of this paper is to discuss in some detail the relationship between ideas from model theory (classification theory, geometric stability theory) and those from bimeromorphic geometry (classification of compact complex manifolds), with reference to current research. Earlier work along these lines is in [17], [18], [21], [22], [23], [14], [15], [16], [1]. We will also take the opportunity here to describe for model-theorists some of the basic tools of complex differential geometry, as well as summarise important notions, facts and theorems such as the Hodge decomposition, and local Torelli.

Zilber [26] observed some time ago that if a compact complex manifold M is considered naturally as a first order structure (with predicates for analytic subsets of M , $M \times M$, etc.) then $\text{Th}(M)$ has finite Morley rank. The same holds if we consider the category \mathcal{A} of compact complex (possibly singular) spaces as a many-sorted first order structure. This

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observation of Zilber was closely related, historically, to the work on Zariski structures and geometries by Hrushovski and Zilber [11].

There is a rich general theory of theories of finite Morley rank, encompassing both Shelah's work on classification theory (classifying first order theories and their models) as well as the more self-consciously geometric theory of 1-basedness (modularity), definable groups, definable automorphism groups, etc . . . It turns out that $\text{Th}(\mathcal{A})$ witnesses most of the richness of this theory. Among the main points of the current article is that notions belonging to the Shelah theory such as nonorthogonality and nonmultidimensionality, have a very clear geometric content, and are connected with things such as "variation of Hodge structure".

The class of compact Kähler manifolds has been identified as an important rather well-behaved class of compact complex manifolds, where there is a better chance of classification. The first author [16] observed that such manifolds can to all intents and purposes be treated as saturated structures (inside which one can apply the compactness theorem). We give some more details in section 3, explaining the role of the Kähler condition.

The category of compact Kähler manifolds (or rather compact complex analytic spaces that are holomorphic images of compact Kähler manifolds) is a "full reduct" of the many-sorted structure \mathcal{A} . We call it \mathcal{C} . In [21] it was pointed out how, from work of Lieberman, one can see that $\text{Th}(\mathcal{A})$ is about as complicated as it can be from the point of view of Shelah's theory (it has the DOP). We have conjectured on the other hand that $\text{Th}(\mathcal{C})$ is rather tame. $\text{Th}(\mathcal{C})$ could not be uncountably categorical (unidimensional), but we believe it to be the next best thing, nonmultidimensional. The description of U -rank 1 types (equivalently *simple* compact complex manifolds) in $\text{Th}(\mathcal{C})$ which are *trivial* is still open, and it is conjectured that they are closely related to so-called irreducible hyperkähler manifolds. As we explain in section 5, an isotriviality result for families of hyperkähler manifolds in \mathcal{C} , due to Campana, represents some confirmation of the nonmultidimensionality of $\text{Th}(\mathcal{C})$.

We now give a brief survey of the model theory of compact complex manifolds, continuing in a sense [19]. There are several published survey-type articles, such as [15] and [17], to which the interested reader is referred for more details. We assume familiarity with the notion of a complex manifold M . An *analytic* subset X of M is a subset X such that for each $a \in M$ there is an open neighbourhood U of a in M such that $X \cap U$ is the common zero-set of a finite set of holomorphic functions on U . A compact complex manifold M is viewed as a first or-

der structure by equipping it with predicates for all the analytic subsets of M and its cartesian powers. The fundamental fact observed by Zilber is that the theory of this first order structure has quantifier elimination and has finite Morley rank. We can of course consider the collection of *all* compact complex manifolds (up to biholomorphism) and view it as a many-sorted first order structure (predicates for all analytic subsets of cartesian products of sorts) which again has QE and finite Morley rank (sort by sort). The same holds for the larger class of *compact complex analytic spaces*. A compact complex analytic space is (a compact topological space) locally modelled on zero-sets of finitely many holomorphic functions on open domains in \mathbb{C}^n with of course biholomorphic transition maps. We have the notion of an analytic subset of X and its cartesian powers, and we obtain thereby a first order structure as before. As above we let \mathcal{A} denote the many-sorted structure of compact complex analytic spaces, and we let \mathcal{L} denote its language. (A complex analytic space is usually presented as a ringed space, where the rings may be nonreduced. We refer to [15] for more discussion of this. In any case by a *complex variety* we will mean a reduced and irreducible complex analytic space.)

If X is a compact complex variety and $a \in X$, then $\{a\}$ is an analytic subset of X , and hence is essentially named by a constant. So really \mathcal{A} has names for all elements (of all sorts). Let \mathcal{A}' be a saturated elementary extension of \mathcal{A} . If S is a sort in \mathcal{A} , we let S' be the corresponding sort in \mathcal{A}' . For example, $(\mathbb{P}_1)'$ denotes the projective line over a suitable elementary extension \mathbb{C}' of \mathbb{C} .

Among the basic facts connecting definability and geometry are:

(i) For X, Y sorts in \mathcal{A} the definable maps from X to Y are precisely the piecewise meromorphic maps.

(ii) If $p(x)$ is a complete type of $\text{Th}(\mathcal{A})$ then p is the generic type (over \mathcal{A}) of a unique compact complex variety X . That is, $p(x)$ is axiomatised by “ $x \in X'$ but $x \notin Y'$ for any proper analytic subset $Y \subset X$ ”.

(iii) If a, b are tuples from \mathcal{A}' with $tp(a)$ the generic type of X and $tp(b)$ the generic type of Y , then $dcl(a) = dcl(b)$ iff X and Y are bimeromorphic.

(iv) Suppose a, b are tuples from \mathcal{A}' , and $tp(a/b)$ is stationary. Then there are compact complex varieties X, Y and a meromorphic dominant map $f : X \rightarrow Y$ whose fibres over a non-empty Zariski open subset of Y are irreducible, and such that: ab is a generic point (realizes the generic type) of X , b is a generic point of Y , and (in \mathcal{A}') $f(ab) = b$. So $tp(a/b)$ is the “generic type” of the “generic fibre” X_b of $f : X \rightarrow Y$. We consider definable sets such as X_b as “nonstandard” analytic subsets of X ”.

Algebraic geometry lives in \mathcal{A} on the sort \mathbb{P}_1 . Any irreducible complex quasi-projective *algebraic* variety V has a compactification \bar{V} which will be a compact complex variety living as a sort in \mathcal{A} , and biholomorphic with a closed subvariety of \mathbb{P}_1^n for some $n > 0$. The variety V will be a Zariski open, hence definable, subset of \bar{V} .

A compact complex variety X is said to be *Moishezon* if X is bimeromorphic with a complex projective algebraic variety. This is equivalent to X being internal to the sort \mathbb{P}_1 , and also equivalent to a generic point a of X being in the definable closure of elements from $(\mathbb{P}_1)'$. The expression ‘‘algebraic’’ is sometimes used in place of Moishezon. We extend naturally this notion to nonstandard analytic sets as well as definable sets and stationary types in \mathcal{A}' . The ‘‘strong conjecture’’ from [19] then holds in \mathcal{A}' in the more explicit form: if $(Y_z : z \in Z)$ is a normalized family of definable subsets of a definable set X , then for $a \in X$, $Z_a = \{z \in Z : a \text{ is generic on } Y_z\}$ is Moishezon (namely generically internal to $(\mathbb{P}_1)'$). This result was derived in [18] from a theorem due independently to Campana and Fujiki.

In [21] it was shown that any strongly minimal modular group definable in \mathcal{A} is definably isomorphic to a complex torus. An appropriate generalization to strongly minimal modular groups in \mathcal{A}' was obtained in [1].

More details on the classification of strongly minimal sets (or more generally types of U -rank 1) in $\text{Th}(\mathcal{C})$ will appear in section 5.

2 Preliminaries on complex forms

We give in this section a brief review of the basic theory of complex-valued differential forms. The reader may consult [8] or [25] for a more detailed treatment of this material.

Suppose X is an n -dimensional complex manifold. By a *coordinate system* (z, U) on X we mean an open set $U \subset X$ and a homeomorphism z from U to an open ball in \mathbb{C}^n . Composing with the coordinate projections we obtain *complex coordinates* $z_i : U \rightarrow \mathbb{C}$ for $i = 1, \dots, n$, which we decompose into real and imaginary parts as $z_i = x_i + iy_i$.

Fix a coordinate system (z, U) on X and a point $x \in U$. Let $T_{X,x}$ denote the (real) tangent space of X at x . Viewed as the space of \mathbb{R} -linear derivations on real-valued smooth functions at x , we have that

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$$

forms an \mathbb{R} -basis for $T_{X,x}$. But the complex manifold structure on X gives $T_{X,x}$ also an n -dimensional *complex* vector space structure, which can be described as follows. Let $T_{X,\mathbb{C},x} := T_{X,x} \otimes \mathbb{C}$ denote the complexification of the real tangent space. We have a decomposition $T_{X,\mathbb{C},x} = T_{X,x}^{1,0} \oplus T_{X,x}^{0,1}$ where $T_{X,x}^{1,0}$ is the complex subspace generated by

$$\left\{ \frac{\partial}{\partial z_i} := \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \mid i = 1, \dots, n \right\}$$

and $T_{X,x}^{0,1}$ is generated by

$$\left\{ \frac{\partial}{\partial \bar{z}_i} := \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \mid i = 1, \dots, n \right\}.$$

If we view $T_{X,\mathbb{C},x}$ as the space of \mathbb{C} -linear derivations on *complex*-valued smooth functions at x , then $T_{X,x}^{1,0}$ corresponds to those that vanish on all the anti-holomorphic functions (functions whose complex conjugates are holomorphic at x). In any case, $T_{X,x}^{1,0}$ is called the *holomorphic tangent space* of X at x . The natural inclusion $T_{X,x} \subset T_{X,\mathbb{C},x}$ followed by the projection $T_{X,\mathbb{C},x} \rightarrow T_{X,x}^{1,0}$ produces an \mathbb{R} -linear isomorphism between the real tangent space and the holomorphic tangent space. This isomorphism makes $T_{X,x}$ canonically into a complex vector space.

Despite our presentation, the above constructions do not depend on the coordinates and extend globally: We have the complexification of the (real) tangent bundle $T_{X,\mathbb{C}} := T_X \otimes \mathbb{C}$, and a decomposition into complex vector subbundles, $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$, whereby the holomorphic tangent bundle $T_X^{1,0}$ is naturally isomorphic as a real vector bundle with T_X . It is with respect to this isomorphism that we treat T_X as a complex vector bundle.

A *complex-valued differential k -form* (or just a *k -form*) at $x \in X$ is an alternating k -ary \mathbb{R} -multilinear map $\phi : T_{X,x} \times \dots \times T_{X,x} \rightarrow \mathbb{C}$. The complex vector space of all k -forms at x is denoted by $F_{X,\mathbb{C},x}^k$. The *real* differential k -forms, $F_{X,\mathbb{R},x}^k$, are exactly those forms in $F_{X,\mathbb{C},x}^k$ that are real-valued. So $F_{X,\mathbb{C},x}^k = F_{X,\mathbb{R},x}^k \otimes \mathbb{C}$. In particular, $F_{X,\mathbb{C},x}^1 = \text{Hom}_{\mathbb{R}}(T_{X,x}, \mathbb{R}) \otimes \mathbb{C}$ is the complexification of the real cotangent space at x . Hence, in a coordinate system (z, U) about x , if we let

$$\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\}$$

be the dual basis to $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$ for $\text{Hom}_{\mathbb{R}}(T_{X,x}, \mathbb{R})$, then

$$dz_i := dx_i + idy_i,$$

$$d\bar{z}_i := dx_i - idy_i$$

for $i = 1, \dots, n$, form a \mathbb{C} -basis for $F_{X,\mathbb{C},x}^1$.

Now $F_{X,\mathbb{C},x}^k$ is the k th exterior power of $F_{X,\mathbb{C},x}^1$. Given $I = (i_1, \dots, i_p)$ an increasing sequence of numbers between 1 and n , let dz_I be the p -form $dz_{i_1} \wedge \dots \wedge dz_{i_p}$. Similarly, let $d\bar{z}_I := d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_p}$. Then

$$\{dz_I \wedge d\bar{z}_J \mid I = (i_1, \dots, i_p), J = (j_1, \dots, j_q), p + q = k\}$$

is a \mathbb{C} -basis for $F_{X,\mathbb{C},x}^k$. This gives us a natural decomposition

$$F_{X,\mathbb{C},x}^k = \bigoplus_{p+q=k} F_{X,x}^{p,q}$$

where $F_{X,x}^{p,q}$ is generated by the forms $dz_I \wedge d\bar{z}_J$ where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are increasing sequences of numbers between 1 and n . The complex vector subspaces $F_{X,x}^{p,q}$ can also be more intrinsically described as made up of those k -forms ϕ such that

$$\phi(cv_1, \dots, cv_n) = c^p \bar{c}^q \phi(v_1, \dots, v_n)$$

for all $v_1, \dots, v_k \in T_{X,x}$ and $c \in \mathbb{C}$. Such forms are said to be of *type* (p, q) .

Once again, these constructions extend globally to X and we have complex vector bundles $F_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} F_{X,\mathbb{C}}^{p,q}$. For $U \subseteq X$ an open set, by a complex k form *on* U we mean a smooth section to the bundle $F_{X,\mathbb{C}}^k$ over the set U . Similarly for forms of *type* (p, q) *on* U . We denote by \mathcal{A}^k and $\mathcal{A}^{p,q}$ the sheaves on X of smooth sections to $F_{X,\mathbb{C}}^k$ and $F_{X,\mathbb{C}}^{p,q}$ respectively. So $\mathcal{A}^k(U)$ is the space of all complex k -forms on U while $\mathcal{A}^{p,q}(U)$ is the space of all complex forms of type (p, q) on U . Given a coordinate system (z, U) on X , a k -form $\omega \in \mathcal{A}^k(U)$ can be expressed as $\omega = \sum_{|I|+|J|=k} f_{IJ} dz_I \wedge d\bar{z}_J$ where $f_{IJ} : U \rightarrow \mathbb{C}$ are smooth. Note that dz_i and $d\bar{z}_i$ are being viewed here as 1-forms on U . By convention, \mathcal{A}^0 is the sheaf of \mathbb{C} -valued smooth functions.

The *exterior derivative* map $d : \mathcal{A}^k(U) \rightarrow \mathcal{A}^{k+1}(U)$ is defined by

$$d \left(\sum_{|I|+|J|=k} f_{IJ} dz_I \wedge d\bar{z}_J \right) = \sum_{|I|+|J|=k} df_{IJ} \wedge dz_I \wedge d\bar{z}_J$$

where for any smooth function $f : U \rightarrow \mathbb{C}$, $df \in \mathcal{A}^1(U)$ is given by

$$df := \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

If we define $\partial f := \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$ and $\bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$, and then extend these maps so that $\partial : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p+1,q}(U)$ is given by

$$\partial \left(\sum f_{IJ} dz_I \wedge d\bar{z}_J \right) = \sum \partial f_{IJ} \wedge dz_I \wedge d\bar{z}_J$$

and $\bar{\partial} : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p,q+1}(U)$ is given by

$$\bar{\partial} \left(\sum f_{IJ} dz_I \wedge d\bar{z}_J \right) = \sum \bar{\partial} f_{IJ} \wedge dz_I \wedge d\bar{z}_J;$$

then we see that $d = \partial + \bar{\partial}$.

One can show that d , ∂ , and $\bar{\partial}$ are all independent of the coordinate system and extend to sheaf maps on \mathcal{A}^k and $\mathcal{A}^{p,q}$ as the case may be. Moreover,

- $d, \partial, \bar{\partial}$ are \mathbb{C} -linear;
- $d \circ d = 0$, $\partial \circ \partial = 0$, and $\bar{\partial} \circ \bar{\partial} = 0$; and,
- $d(\bar{\phi}) = \bar{d}\bar{\phi}$, $\partial(\bar{\phi}) = \bar{\partial}\bar{\phi}$, and $\bar{\partial}(\bar{\phi}) = \bar{\partial}\bar{\phi}$.

We say that $\omega \in \mathcal{A}^k(X)$ is *d-closed* if $d\omega = 0$, and *d-exact* if $\omega = d\phi$ for some $\phi \in \mathcal{A}^{k-1}(X)$. Since $d \circ d = 0$ the exact forms are closed. The *De Rham cohomology groups* are the complex vector spaces

$$H_{\text{DR}}^k(X) := \frac{\{d\text{-closed } k\text{-forms}\}}{\{d\text{-exact } k\text{-forms}\}}.$$

We can relate this cohomology to the classical singular cohomology (which we assume the reader is familiar with) by integration: Given a complex form $\omega \in \mathcal{A}^k(X)$ and a *k-simplex*

$$\phi : \Delta_k := \left\{ (t_1, \dots, t_{k+1}) \in [0, 1]^{k+1} \mid \sum_{i=1}^{k+1} t_i = 1 \right\} \longrightarrow X,$$

it makes sense to consider $\int_{\phi} \omega := \int_{\Delta_k} \phi^* \omega \in \mathbb{C}$. Every complex k -form thereby determines a homomorphism from the free abelian group generated by the k -simplices (i.e., the group of *singular k-chains*) to the complex numbers. We restrict this homomorphism to the *singular k-cycles* (those chains whose boundary is zero), and denote it by

$$\int \omega : \{k\text{-cycles on } X\} \longrightarrow \mathbb{C}.$$

If ω is *d-closed* then $\int \omega$ vanishes on boundaries by Stokes' theorem, and hence $\int \omega$ induces a complex-valued homomorphism on the *singular homology* group $H_k(X) = \frac{k\text{-cycles in } X}{k\text{-boundaries in } X}$. Moreover, by Stokes' theorem

again, $\int \omega = 0$ if ω is d -exact. So $\omega \mapsto \int \omega$ induces a homomorphism

$$H_{\text{DR}}^k(X) \longrightarrow \text{Hom}_{\mathbb{Z}}(H_k(U), \mathbb{C}) = H_{\text{sing}}^k(X, \mathbb{C})$$

where $H_{\text{sing}}^k(X, \mathbb{C})$ is the *singular cohomology* group with complex coefficients.

De Rham's Theorem (cf. Section 4.3.2 of [25]). *The above homomorphism is an isomorphism: $H_{\text{DR}}^k(X) \cong H_{\text{sing}}^k(X, \mathbb{C})$.*

3 Saturation and Kähler manifolds

One obstacle to the application of model-theoretic methods to compact complex manifolds is that the structure \mathcal{A} is not saturated; for example, every element of every sort of \mathcal{A} is \emptyset -definable. However, for some sorts this can be seen to be an accident of the language of analytic sets in which we are working: Suppose V is a complex projective algebraic variety viewed as a compact complex manifold and consider the structure

$$\mathcal{V} := (V; P_A \mid A \subset V^n \text{ is a subvariety over } \mathbb{Q}, n \geq 0)$$

where there is a predicate for every subvariety of every cartesian power of V *defined over the rationals*. It is not hard to see that \mathcal{V} is saturated (it is ω_1 -compact in a countable language). Moreover, Chow's theorem says that every complex analytic subset of projective space is complex algebraic, and hence a subset of a cartesian power of V is definable in \mathcal{A} if and only if it is definable (with parameters) in \mathcal{V} . That is, with respect to the sort V , the lack of saturation in \mathcal{A} is a result of working in too large (and redundant) a language. This property was formalised in [16] as follows.

Definition 3.1 A compact complex variety X is *essentially saturated* if there exists a countable sublanguage of the language of \mathcal{A} , \mathcal{L}_0 , such that every subset of a cartesian power of X that is definable in \mathcal{A} is already definable (with parameters) in the reduct of \mathcal{A} to \mathcal{L}_0 .

The structure induced on X by such a reduct will be saturated.

It turns out that essential saturation, while motivated by internal model-theoretic considerations, has significant geometric content. The purpose of this section is to describe this geometric significance and to show in particular that compact Kähler manifolds are essentially saturated.

We will make use of Barlet’s construction of the space of compact cycles of a complex variety. For X any complex variety and n a natural number, a (*holomorphic*) n -cycle of X is a (formal) finite linear combination $Z = \sum_i n_i Z_i$ where the Z_i ’s are distinct n -dimensional irreducible compact analytic subsets of X , and each n_i is a positive integer called the *multiplicity* of the component Z_i .¹ By $|Z|$ we mean the underlying set or *support* of Z , namely $\bigcup_i Z_i$. We denote the set of all n -cycles of X by $\mathcal{B}_n(X)$, and the set of all cycles of X by $\mathcal{B}(X) := \bigcup_n \mathcal{B}_n(X)$. In [2] Barlet endowed $\mathcal{B}(X)$ with a natural structure of a reduced complex analytic space. If for $s \in \mathcal{B}_n(X)$ we let Z_s denote the cycle represented by s , then the set $\{(s, x) : s \in \mathcal{B}_n(X), x \in |Z_s|\}$ is an analytic subset of $\mathcal{B}_n(X) \times X$. Equipped with this complex structure, $\mathcal{B}(X)$ is called the *Barlet space of X* . When X is a projective variety the Barlet space coincides with the Chow scheme.

An cycle is called *irreducible* if it has only one component and that component is of multiplicity 1. In [6] it is shown that

$$\mathcal{B}^*(X) := \{s \in \mathcal{B}(X) : Z_s \text{ is irreducible}\}$$

is a Zariski open subset of $\mathcal{B}(X)$. An irreducible component of $\mathcal{B}(X)$ is *prime* if it has nonempty intersection with $\mathcal{B}^*(X)$. Suppose S is a prime component of the Barlet space and set

$$G_S := \{(s, x) : s \in S, x \in |Z_s|\}.$$

Then G_S is an irreducible analytic subset of $S \times X$ and, if $\pi : G_S \rightarrow S$ denotes the projection map, the general fibres of π are reduced and irreducible. We call G_S the *graph* of (the family of cycles parametrised) by S .

Fact 3.2 (cf. Theorem 3.3 of [16]). *A compact complex variety X is essentially saturated if and only if every prime component of $\mathcal{B}(X^m)$ is compact for all $m \geq 0$.*

One direction of 3.2 is straightforward: If every prime component of $\mathcal{B}(X^m)$ is compact, then they are all sorts in \mathcal{A} and their graphs are definable in \mathcal{A} . Consider the sublanguage \mathcal{L}_0 of the language of \mathcal{A} made up of predicates for the graphs G_S as S ranges over all prime components of $\mathcal{B}(X^m)$ for all $m \geq 0$. Then \mathcal{L}_0 is countable because the Barlet space has countably many irreducible components (this actually follows from

¹ We hope the context will ensure that holomorphic n -cycles will not be confused with the singular n -cycles of singular homology discussed in the previous section.

Lieberman's Theorem 3.6 below). Every irreducible analytic subset of X^m , as it forms an irreducible cycle, is a fibre of G_S for some such S , and hence is \mathcal{L}_0 -definable. By quantifier elimination for \mathcal{A} , it follows that every \mathcal{A} -definable subset of every cartesian power of X is \mathcal{L}_0 -definable. So X is essentially saturated. The converse makes use of Hironaka's flattening theorem and the universal property of the Barlet space.²

It is in determining whether a given component of the Barlet space is compact that Kähler geometry intervenes. We review the fundamentals of this theory now, and suggest [25] for further details.

Suppose X is a complex manifold. A *hermitian metric* h on X assigns to each point $x \in X$ a positive definite *hermitian form* h_x on the tangent space $T_{X,x}$. That is, $h_x : T_{X,x} \times T_{X,x} \rightarrow \mathbb{C}$ satisfies:

- (i) $h_x(-, w)$ is \mathbb{C} -linear for all $w \in T_{X,x}$,
- (ii) $h_x(v, w) = \overline{h_x(w, v)}$ for all $v, w \in T_{X,x}$, and
- (iii) $h_x(v, v) > 0$ for all nonzero $v \in T_{X,x}$.

Note that h_x is \mathbb{R} -bilinear and that $h_x(v, -)$ is \mathbb{C} -antilinear for all $v \in T_{X,x}$. Moreover this assignment should be smooth: Given a coordinate system (z, U) on X , a hermitian metric is represented on U by

$$h = \sum_{i,j=1}^n h_{ij} dz_i \otimes d\bar{z}_j$$

where $h_{ij} : U \rightarrow \mathbb{C}$ are smooth functions and $dz_i \otimes d\bar{z}_j$ is the map taking a pair of tangent vectors (v, w) to the complex number $dz_i(v)d\bar{z}_j(w)$.

A hermitian metric encodes both a riemannian and a symplectic structure on X . The real part of h , $\operatorname{Re}(h_x) : T_{X,x} \times T_{X,x} \rightarrow \mathbb{R}$, is positive definite, symmetric, and \mathbb{R} -bilinear. That is, $\operatorname{Re}(h)$ is a riemannian metric on X . On the other hand, the imaginary part, $\operatorname{Im}(h_x) : T_{X,x} \times T_{X,x} \rightarrow \mathbb{R}$, is an alternating \mathbb{R} -bilinear map. So $\operatorname{Im}(h)$ is a real 2-form on X . Moreover, if in a coordinate system (z, U) we have $h = \sum_{i,j=1}^n h_{ij} dz_i \otimes d\bar{z}_j$, then a straightforward calculation shows that $\operatorname{Im}(h) = -\frac{i}{2} \sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar{z}_j$. So as a complex 2-form on X , $\operatorname{Im}(h)$ is of type $(1, 1)$.

The assignment $h \mapsto -\operatorname{Im}(h)$ is a bijection between hermitian metrics and positive real 2-forms of type $(1, 1)$ on X . We call $\omega := -\operatorname{Im}(h)$ the *Kähler form* associated to h . A hermitian metric is a *Kähler metric* if

² Actually, this is done in [16] with restricted *Douady spaces* (the complex analytic analogue of the Hilbert scheme) rather than Barlet spaces. However, it is routine to see that the argument works for Barlet spaces as well.

its Kähler form is d -closed. A complex manifold is a *Kähler manifold* if it admits a Kähler metric.

Example 3.3 The *standard* Kähler metric on \mathbb{C}^n is $\sum_{i=1}^n dz_i \otimes d\bar{z}_i$.

Example 3.4 Every complex manifold admits a hermitian metric (but not necessarily a Kähler one). Indeed, given any complex manifold X , a cover $\mathcal{U} = (z^t, U_t)_{t \in I}$ by coordinate systems, and a partition of unity $\rho = (\rho_t)_{t \in I}$ subordinate to \mathcal{U} , $h := \sum_{t \in I} \rho_t (\sum_{i=1}^n dz_i^t \otimes d\bar{z}_i^t)$ is a hermitian metric on X .

Example 3.5 (Fubini-Study) Let $[z_0; \dots; z_n]$ be complex homogeneous coordinates for \mathbb{P}_n . For each $i = 0, \dots, n$ let U_i be the affine open set defined by $z_i \neq 0$. Let $F_i : U_i \rightarrow \mathbb{R}$ be the smooth function given by $\log \left(\frac{|z_0|^2 + \dots + |z_n|^2}{|z_i|^2} \right)$. Then $i\partial\bar{\partial}F_i \in F_X^{1,1}(U_i)$ is real-valued. Moreover, for all $j = 0, \dots, n$, $i\partial\bar{\partial}F_i$ agrees with $i\partial\bar{\partial}F_j$ on $U_i \cap U_j$. Hence, the locally defined forms $i\partial\bar{\partial}F_0, \dots, i\partial\bar{\partial}F_n$ patch together and determine a global complex 2-form on \mathbb{P}_n . It is real-valued, of type $(1, 1)$, and d -closed. The associated Kähler metric is called the *Fubini-Study* metric on \mathbb{P}_n .

Suppose h is a hermitian metric on X . There is strong interaction between the Kähler form ω and the riemannian metric $\text{Re}(h)$. This is encapsulated in Wirtinger’s formula for the volume of compact submanifolds of X . When we speak of the *volume* of a submanifold of X with respect to h , denoted by vol_h , we actually mean the riemannian volume with respect to $\text{Re}(h)$.

Wirtinger’s Formula (cf. Section 3.1 of [25]). *If $Z \subset X$ is a compact complex submanifold of dimension k , then*

$$\text{vol}_h(Z) = \int_Z \omega^k$$

where $\omega^k = \omega \wedge \dots \wedge \omega$ is the k th exterior power of ω .

Note that Z is of real-dimension $2k$ and ω^k is a real $2k$ -form on X , and hence it makes sense to integrate ω^k along Z . If $X = \mathbb{P}_n$ and h is the Fubini-Study metric of Example 3.5, then for any algebraic subvariety $Z \subseteq \mathbb{P}_n$, $\text{vol}_h(Z)$ is the degree of Z .

For possibly singular complex analytic subsets $Z \subset X$ (irreducible, compact, dimension k), Wirtinger’s formula can serve as the *definition*

of volume; it agrees with the volume of the regular locus of Z . More generally, if $Z = \sum_i n_i Z_i$ is a k -cycle of X , then the volume of Z with respect to h is $\text{vol}_h(Z) := \sum_i n_i \text{vol}_h(Z_i)$.

Taking volumes of cycles induces a function $\text{vol}_h : \mathcal{B}(X) \rightarrow \mathbb{R}$ given by

$$\text{vol}_h(s) := \text{vol}_h(Z_s).$$

The link between hermitian geometry and saturation comes from the following striking fact.

Theorem 3.6 (Lieberman [13]) *Suppose X is a compact complex manifold equipped with a hermitian metric h , and W is a subset of $\mathcal{B}_k(X)$. Then W is relatively compact in $\mathcal{B}_k(X)$ if and only if vol_h is bounded on W .*

Sketch of proof. Wirtinger's formula tells us that vol_h is computed by integrating ω^k over the fibres of a morphism.³ It follows that vol_h is continuous on $\mathcal{B}_k(X)$ and hence is bounded on any relatively compact subset.

The converse relies on Barlet's original method of constructing the cycle space. We content ourselves with a sketch of the ideas involved. First, let $K(X)$ denote the space of closed subsets of X equipped with the Hausdorff metric topology. So given closed subsets $A, B \subset X$,

$$\text{dist}(A, B) := \frac{1}{2} [\max\{\text{dist}_h(A, b) : b \in B\} + \max\{\text{dist}_h(a, B) : a \in A\}].$$

Now suppose $W \subset \mathcal{B}(X)$ is a subset on which vol_h is bounded. Given a sequence $(s_i : i \in \mathbb{N})$ of points in W we need to find a convergent subsequence. Consider the sequence $(|Z_{s_i}| : i \in \mathbb{N})$ of points in $K(X)$. Since X is compact, so is $K(X)$, and hence there exists a subsequence $(|Z_{s_i}| : i \in I)$ which converges in the Hausdorff metric topology to a closed set $A \subset X$. Since $\text{vol}_h(|Z_{s_i}|)$ is bounded on this sequence, a theorem of Bishop's [5] implies that A is in fact complex analytic. Now, by Barlet's construction, the topology on $\mathcal{B}(X)$ is closely related to the Hausdorff topology on $K(X)$. In particular, it follows from the fact that $(|Z_{s_i}| : i \in I)$ converges in the Hausdorff topology to a complex analytic subset $A \subset X$, that some subsequence of $(s_i : i \in I)$ converges to a point $t \in \mathcal{B}(X)$ with $|Z_t| = A$. In particular, $(s_i : i \in \mathbb{N})$ has a convergent subsequence. Hence W is relatively compact. \square

³ To be more precise, given a component S of $\mathcal{B}_k(X)$, one considers the *universal cycle* Z_S on $S \times X$ whose fibre at $s \in S$ is the cycle Z_s . Then integrating ω^k over the fibres of $Z_S \rightarrow S$ gives us vol_h on S .

Corollary 3.7 (Lieberman [13]) *If X is a compact Kähler manifold then the prime components of $\mathcal{B}(X)$ are compact.*

Sketch of proof. Let h be a Kähler metric on X . The d -closedness of the Kähler form $\omega = -\text{Im}(h)$ will imply by Wirtinger’s formula that vol_h is constant on the components of the Barlet space. We sketch the argument for this here, following Proposition 4.1 of Fujiki [9]. Fix a prime component S of $\mathcal{B}_k(X)$. By continuity of vol_h , we need only show that for sufficiently general points $s, t \in S$, $\text{vol}_h(Z_s) = \text{vol}_h(Z_t)$. Let $G_S \subset S \times X$ be the graph of the cycles parametrised by S , and let $\pi_X : G_S \rightarrow X$ and $\pi_S : G_S \rightarrow S$ be the natural projections. We work with a prime component so that for general $s, t \in S$, the fibres of G_S over s and t are the reduced and irreducible complex analytic subsets Z_s and Z_t . Now let I be a piecewise real analytic curve in S connecting s and t . For the sake of convenience, let us assume that there is only one piece: so we have a real analytic embedding $h : [0, 1] \rightarrow S$ with $h(0) = s$, $h(1) = t$ and $I = h([0, 1])$. Consider the semianalytic set $R := \pi_S^{-1}(I) \subset G_S$. Given the appropriate orientation we see that the boundary ∂R of R in G_S is $\pi_S^{-1}(s) - \pi_S^{-1}(t)$. Note that π_X restricts to an isomorphism between $\pi_S^{-1}(s)$ and Z_s (and similarly for $\pi_S^{-1}(t)$ and Z_t). Also, if $\pi_X^*(\omega^k)$ is the pull-back of ω^k to G_S , then $d\pi_X^*(\omega^k) = 0$ since $d\omega = 0$. Using a semianalytic version of Stokes’ theorem (see, for example, Herrera [10]), we compute

$$\begin{aligned} 0 &= \int_R d\pi_X^*(\omega^k) \\ &= \int_{\partial R} \pi_X^*(\omega^k) \\ &= \int_{\pi_S^{-1}(s)} \pi_X^*(\omega^k) - \int_{\pi_S^{-1}(t)} \pi_X^*(\omega^k) \\ &= \int_{Z_s} \omega^k - \int_{Z_t} \omega^k \\ &= \text{vol}_h(Z_s) - \text{vol}_h(Z_t), \end{aligned}$$

We have shown that vol_h is constant on S , and hence S is compact by Theorem 3.6. \square

If X is Kähler then so is X^m for all $m > 0$. Hence, from Corollary 3.7 and Fact 3.2 we obtain:

Corollary 3.8 *Every compact Kähler manifold is essentially saturated.*

A complex variety is said to be of *Kähler-type* if it is the holomorphic image of a compact Kähler manifold. The class of all complex varieties of Kähler-type is denoted by \mathcal{C} , and was introduced by Fujiki [9]. This class is preserved under cartesian products and bimeromorphic equivalence. Many of the results for compact Kähler manifolds discussed above extend to complex varieties of Kähler-type. In particular, Kähler-type varieties are essentially saturated (see Lemma 2.5 of [16] for how this follows from Corollary 3.8 above), and their Barlet spaces have compact components which are themselves again of Kähler-type.

Model-theoretically we can therefore view \mathcal{C} as a many-sorted structure in the language where there is a predicate for each G_S as S ranges over all prime components of the Barlet space of each cartesian product of sorts. We call this the *Barlet language*.⁴ Note that every analytic subset of every cartesian product of Kähler-type varieties is definable (with parameters) in this language, and so we are really looking at the full induced structure on \mathcal{C} from \mathcal{A} . Moreover, when studying the models of $\text{Th}(\mathcal{C})$, we may treat \mathcal{C} as a “universal domain” in the sense that we may restrict ourselves to definable sets and types in \mathcal{C} itself. This is for the following reason: Fix some definable set F in an elementary extension \mathcal{C}' . So there will be some sort X of \mathcal{C} such that F is a definable subset of the nonstandard X' . Essential saturation implies that there is a countable sublanguage \mathcal{L}_0 such that $X|_{\mathcal{L}_0}$ is saturated and every definable subset of X^n in \mathcal{A} is already definable in $X|_{\mathcal{L}_0}$ (with parameters). In particular, F is definable in $X|_{\mathcal{L}_0}$ over some parameters, say b , in X' . The \mathcal{L}_0 -type of b is realised in X , by say b_0 . Let F_0 be defined in $X'|_{\mathcal{L}_0}$ over b_0 in the same way as F is defined over b . Then in $X'|_{\mathcal{L}_0}$ there is an automorphism taking F to F_0 . So in so far as any structural properties of F are concerned we may assume it is \mathcal{L}_0 -definable over X . But as $X|_{\mathcal{L}_0}$ is a saturated elementary substructure of $X'|_{\mathcal{L}_0}$, the first order properties of F are then witnessed by $F \cap X$. The latter is now a definable set in \mathcal{C} . The same kind of argument works also with types.

We can also work, somewhat more canonically, as follows: In any given situation we will be interested in at most countably many Kähler-type varieties, $(X_i : i \in \mathbb{N})$, at once. We then consider the smallest (countable) subcollection \mathcal{X} of sorts from \mathcal{C} containing the X_i 's and closed under taking prime components of Barlet spaces of cartesian products of sorts in \mathcal{X} . We view \mathcal{X} as a multi-sorted structure in the language where there is a predicate for each G_S as S ranges over all such prime

⁴ This is in analogy with the *Douady language* from Definition 4.3 of [16].

components of the Barlet spaces. We call this the *Barlet language of* $(X_i : i \in \mathbb{N})$. Then \mathcal{X} is saturated (2^ω -saturated and of cardinality 2^ω), ω -stable, and every analytic subset of every cartesian product of sorts in \mathcal{X} is definable in \mathcal{X} . Moreover, after possibly naming countably many constants, \mathcal{X} has elimination of imaginaries (cf. Lemma 4.5 of [16]). When working with Kähler-type varieties we will in general pass to such countable reducts of \mathcal{C} without saying so explicitly, and it is in this way that we treat \mathcal{C} as a universal domain for $\text{Th}(\mathcal{C})$.

4 Holomorphic forms on Kähler manifolds

In section 2 we defined the De Rham cohomology groups on any complex manifold X by

$$H_{\text{DR}}^k(X) := \frac{\{d\text{-closed } k\text{-forms on } X\}}{\{d\text{-exact } k\text{-forms on } X\}}.$$

There are also cohomology groups of forms coming from the operators $\bar{\partial} : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X)$. Since $\bar{\partial} \circ \bar{\partial} = 0$, the $\bar{\partial}$ -exact forms are $\bar{\partial}$ -closed. The *Dolbeault cohomology groups* are the complex vector spaces

$$H^{p,q}(X) := \frac{\{\bar{\partial}\text{-closed forms of type } (p, q)\}}{\{\bar{\partial}\text{-exact forms of type } (p, q)\}}.$$

We denoted by $h^{p,q}(X)$ the dimension of $H^{p,q}(X)$.

A fundamental result about Kähler manifolds is the following fact:

Hodge decomposition (cf. Section 6.1 of [25]). *If X is a compact Kähler manifold then $H^{p,q}(X)$ is isomorphic to the subspace of $H_{\text{DR}}^{p+q}(X)$ made up of those classes that are represented by d -closed forms of type (p, q) . Moreover, under these isomorphisms,*

$$H_{\text{DR}}^k(X) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

A consequence of Hodge decomposition is that complex conjugation, which takes forms of type (p, q) to forms of type (q, p) , induces an isomorphism between $H^{p,q}(X)$ and $H^{q,p}(X)$. In particular, $h^{p,q}(X) = h^{q,p}(X)$. So for X compact Kähler and k odd, $\dim_{\mathbb{C}} H_{\text{DR}}^k(X)$ – which is called the *kth Betti number* of X – is always even.

For any complex manifold X , given an open set $U \subset X$, a form $\omega \in \mathcal{A}^{p,0}(U)$ is called a *holomorphic p -form* on U if $\bar{\partial}\omega = 0$. The sheaf

of holomorphic p -forms on X is denoted Ω^p . Since there can be no non-trivial $\bar{\partial}$ -exact forms of type $(p, 0)$, the holomorphic p -forms make up the $(p, 0)$ th Dolbeaut cohomology group – that is, $H^{p,0}(X) = \Omega^p(X)$. In terms of local coordinates a holomorphic p -form is just a form $\omega = \sum_{|I|=p} f_I dz_I$ where each $f_I : U \rightarrow \mathbb{C}$ is holomorphic. A straightforward calculation shows that holomorphic forms are also d -closed.

If X has dimension n , then Ω^n is locally of rank 1. The corresponding complex line bundle is called the *canonical bundle* of X , denoted by K_X . The triviality of K_X is then equivalent to the existence of a nowhere zero global holomorphic n -form on X . For X Kähler, this is precisely the condition for X to be a *Calabi-Yau* manifold.

Holomorphic 2-forms will play an important role for us. Being of type $(2, 0)$, a holomorphic 2-form $\omega \in \Omega^2(X)$ determines a \mathbb{C} -bilinear map $\omega_x : T_{X,x} \times T_{X,x} \rightarrow \mathbb{C}$ for each $x \in X$. Hence it induces a \mathbb{C} -linear map from $T_{X,x}$ to $\text{Hom}_{\mathbb{C}}(T_{X,x}, \mathbb{C}) = \Omega_x^1$. To say that ω is *non-degenerate* at x is to say that this map is an isomorphism.

Definition 4.1 An *irreducible hyperkähler* manifold (also called *irreducible symplectic*) is a compact Kähler manifold X such that (i) X is simply connected and (ii) $\Omega^2(X)$ is spanned by an everywhere non-degenerate holomorphic 2-form.

The basic properties of irreducible hyperkähler manifolds can be found in Section 1 of [12]. Such properties include: $\dim(X)$ is even, $h^{2,0}(X) = h^{0,2}(X) = 1$, and K_X is trivial. (The latter is because, if ϕ is a holomorphic 2-form on X witnessing the hyperkähler condition, and $\dim(X) = 2r$ then ϕ^r is an everywhere nonzero holomorphic $2r$ -form on X .) For surfaces, condition (ii) in Definition 4.1 is equivalent to the triviality of K_X . The so-called *K3 surfaces* are precisely the irreducible hyperkähler manifolds which have dimension 2. *K3* surfaces have been widely studied since their introduction by Weil. A considerable amount of information on them can be found in [3]. Irreducible hyperkähler manifolds are now widely studied as higher-dimensional generalizations of *K3* surfaces. We will see in the next section the (conjectured) role of *K3* surfaces and higher dimensional irreducible hyperkählers in the model theory of Kähler manifolds.

Given a complex manifold X , a cohomology class $[\omega] \in H_{\text{DR}}^k(X)$ is called *integral* if under the identification

$$H_{\text{DR}}^k(X) = H_{\text{sing}}^k(X, \mathbb{C}) = H_{\text{sing}}^k(X, \mathbb{Z}) \otimes \mathbb{C},$$

$[\omega]$ is contained in $H_{\text{sing}}^k(X, \mathbb{Z}) \otimes 1$. Equivalently, the map $\gamma \mapsto \int_{\gamma} \omega$ on real k -cycles is integer-valued. Similarly, $[\omega]$ is *rational* if it is contained in $H_{\text{sing}}^k(X, \mathbb{Z}) \otimes \mathbb{Q}$ under the above identification.⁵

Definition 4.2 A *Hodge manifold* is a compact complex manifold which admits a hermitian metric h whose associated Kähler form ω – which recall is a real 2-form of type $(1, 1)$ – is d -closed and $[\omega]$ is integral.

In particular, a Hodge manifold is Kähler.

For example, $\mathbb{P}_n(\mathbb{C})$ is a Hodge manifold. Indeed, if $\omega \in \mathcal{A}^{1,1}(\mathbb{P}_n)$ is the Kähler form associated to the Fubini-Study metric (see Example 3.5), and we view \mathbb{P}_1 as a real 2-cycle in \mathbb{P}_n , then $\int_{\mathbb{P}_1} \omega = \pi$. Since the (class of) \mathbb{P}_1 generates $H_2(\mathbb{P}_n)$, $[\frac{1}{\pi}\omega] \in H_{\text{DR}}^2(\mathbb{P}_n)$ is integral. It follows that every projective algebraic manifold is Hodge. A famous theorem of Kodaira (sometimes called Kodaira’s embedding theorem) says the converse:

Kodaira’s Embedding Theorem. *Every Hodge manifold is (biholomorphic to) a projective algebraic manifold.*

A consequence of Kodaira’s theorem relevant for us is:

Corollary 4.3 *Any compact Kähler manifold with no nonzero global holomorphic 2-forms is projective.*

Sketch of proof. If $0 = \Omega^2(X) = H^{2,0}(X)$ then also $H^{0,2}(X) = 0$. By Hodge decomposition it follows that $H_{\text{DR}}^2(X) = H^{1,1}(X)$. Now let

$$H^{1,1}(X, \mathbb{R}) := \{[\omega] : \omega \text{ is real, } d\text{-closed, type}(1, 1)\}.$$

Then the set

$$C := \{[\omega] \in H^{1,1}(X, \mathbb{R}) : \omega \text{ corresponds to a Kähler metric}\}$$

is *open* in $H^{1,1}(X, \mathbb{R})$ – the argument being that a small deformation of a Kähler metric is Kähler. As X is Kähler, $C \neq \emptyset$. On the other hand, as $H^{1,1}(X) = H_{\text{DR}}^2(X)$, we have that $H^{1,1}(X, \mathbb{R}) = H_{\text{sing}}^2(X, \mathbb{Z}) \otimes \mathbb{R}$, and hence C must contain an element of $H_{\text{sing}}^2(X, \mathbb{Z}) \otimes \mathbb{Q}$. Taking a suitable

⁵ We have chosen not to go through the definitions of *sheaf cohomology*, but for those familiar with it, $H_{\text{sing}}^k(X, \mathbb{Z})$ coincides with $H^k(X, \mathbb{Z})$, the k th sheaf cohomology group of X with coefficients in the constant sheaf \mathbb{Z} . Likewise for \mathbb{Q} , \mathbb{R} , or \mathbb{C} in place of \mathbb{Z} . In fact for Kähler manifolds, the Dolbeaut cohomology group $H^{p,q}(X)$ coincides with $H^q(X, \Omega^p)$. In any case, the integral classes can be described as those in $H^k(X, \mathbb{Z})$ and the rational ones as those in $H^k(X, \mathbb{Q})$.

integral multiple, we obtain an integral class in C . Thus X is a Hodge manifold, and so projective by Kodaira's embedding theorem. \square

In fact we will require rather a *relative* version, proved in a similar fashion, and attributed in [7] to Claire Voisin:

Corollary 4.4 *Suppose that $f : X \rightarrow S$ is a fibration in \mathcal{C} , and the generic fibre of f is not projective, then there exists a global holomorphic 2-form $\omega \in \Omega^2(X)$ whose restriction ω_a to a generic fibre X_a is a nonzero global holomorphic 2-form on X_a .*

5 Stability theory and Kähler manifolds

In this section we will discuss some outstanding problems concerning the model theory (or rather stability theory) of $\text{Th}(\mathcal{C})$. One concerns identifying (up to nonorthogonality, or even some finer equivalence relation), the trivial U -rank 1 types. The second is the conjecture that $\text{Th}(\mathcal{C})$ is *nonmultidimensional*. As we shall see the problems are closely related.

Because of the results in section 3, and as discussed at the end of that section, we may treat \mathcal{C} as a universal domain for $\text{Th}(\mathcal{C})$. The main use of this is the existence of *generic* points: given countably many parameters A from \mathcal{C} , and a Kähler-type variety X , there exist points in X that are not contained in any proper analytic subsets of X *defined over* A in the Barlet language of X . The fact that we need not pass to elementary extensions in order to find such generic points makes the model-theoretic study of $\text{Th}(\mathcal{C})$ much more accessible than that of $\text{Th}(\mathcal{A})$.

In [19] strongly minimal sets were discussed as “building blocks” for structures of finite Morley rank. In fact one needs a slightly more general notion, that of a *stationary type of U -rank 1*, sometimes also called a *minimal type*. Let us assume for now that T is a stable theory, and we work in a saturated model \bar{M} . A complete (nonalgebraic) type $p(x) \in S(A)$ is *minimal* or *stationary of U -rank 1* if for any $B \supseteq A$, p has a unique extension to a nonalgebraic complete type over B . Equivalently any (relatively) definable subset of the set of realizations of p is finite or cofinite. If X is a strongly minimal set defined over A , and $p(x) \in S(A)$ is the “generic” type of X , then $p(x)$ is minimal. However not every minimal type comes from a strongly minimal set. For example take T to be the theory with infinitely many disjoint infinite unary predicates P_i (and nothing else), and take p to be the complete type over \emptyset axiomatized by $\{\neg P_i(x) : i < \omega\}$. We discussed the notion of *modularity* of a definable

set X in [19]. The original definition was that X is modular if for all tuples a, b of elements of X , a is independent from b over $\text{acl}(a) \cap \text{acl}(b)$ (together with a fixed set of parameters over which X is defined), where acl is computed in \bar{M}^{eq} . The same definition makes sense with a type-definable set (such as the set of realizations of a complete type) in place of X . So we obtain in particular the notion of a modular minimal type. The minimal type $p(x) \in S(A)$ is said to be *trivial* if whenever a, b_1, \dots, b_n are realizations of p and $a \in \text{acl}(A, b_1, \dots, b_n)$ then $a \in \text{acl}(A, b_i)$ for some i . Triviality implies modularity (for minimal types). On the other hand, if p is a modular nontrivial type then $p(x)$ is nonorthogonal (see below or [19]) to a minimal type q which is the generic type of a definable group G . Assuming the ambient theory to be totally transcendental, G will be strongly minimal, so p will also come from a strongly minimal set. Under the same assumption (ambient theory is totally transcendental), any nonmodular minimal type will come from a strongly minimal set. So a divergence between strongly minimal sets and minimal types is only possible for trivial types.

Let us apply these notions to $\text{Th}(\mathcal{C})$. Those compact complex varieties in \mathcal{C} whose generic type is minimal are precisely the so-called *simple* complex varieties. The formal definition is that a compact complex variety X in \mathcal{C} is *simple* if it is irreducible and if a is a generic point of X (over some set of definition) then there is no analytic subvariety Y of X containing a with $0 < \dim(Y) < \dim(X)$. (There is an appropriate definition not mentioning generic points, and hence also applicable to all compact complex varieties.) We are allowing the possibility that $\dim(X) = 1$, although sometimes this case is formally excluded in the definition of simplicity. In fact, all compact complex curves are simple. Moreover, a projective algebraic variety is simple if and only if it is of dimension 1. If X is simple we may sometimes say “ X is modular, trivial, etc.” if its generic type has that property.

Example 5.1 Given a $2n$ -dimensional lattice $\Lambda \leq \mathbb{C}^n$, the quotient $T = \mathbb{C}^n/\Lambda$ inherits the structure of an n -dimensional compact Kähler manifold. Such manifolds are called *complex tori*. The additive group structure on \mathbb{C}^n induces a compact complex Lie group structure on T . If the lattice is chosen sufficiently generally – namely the real and imaginary parts of a \mathbb{Z} -basis for Λ form an algebraically independent set over \mathbb{Q} – then it is a fact that T has no proper infinite complex analytic subsets, and hence is strongly minimal.

A complex torus which is *algebraic* (bimeromorphic with an algebraic variety) is a certain kind of complex algebraic group: an abelian variety. So the only strongly minimal algebraic complex tori are the elliptic curves, that is the 1-dimensional abelian varieties.

If p and q are the generic types of X and Y respectively, then p is nonorthogonal to q (we might say X is nonorthogonal to Y) if and only if there is a proper analytic subvariety $Z \subset X \times Y$ projecting onto both X and Y . Note that if p and q are minimal then Z must be a *correspondence*: the projections $Z \rightarrow X$ and $Z \rightarrow Y$ are both generically finite-to-one.

Fact 5.2 *Let $p(x)$ be a minimal type over \mathcal{C} . Let X be the compact complex variety whose generic type is p . Then either:*

- (i) p is nonmodular in which case X is an algebraic curve,
- (ii) p is modular, nontrivial, in which case X is nonorthogonal to (i.e. in correspondence with) a strongly minimal complex nonalgebraic torus (necessarily of dimension > 1), or
- (iii) p is trivial, and $\dim(X) > 1$.

Proof. This is proved in [22] for the more general case of \mathcal{A} . We give a slightly different argument here.

From the truth of the *strong conjecture* (cf. [19]) for \mathcal{A} one deduces that if p is nonmodular then X is nonorthogonal to a simple algebraic variety Y . Y has to be of dimension 1. Simplicity implies that X is also of dimension 1, and so, by the Riemann existence theorem, an algebraic curve.

If p is modular and nontrivial, then as remarked above, up to non-orthogonality p is the generic type of a strongly minimal (modular) group G . It is proved in [22] that any such group is definably isomorphic to a (strongly minimal) complex torus T . If T had dimension 1 then by the Riemann existence theorem it would be algebraic, so not modular.

Likewise in the trivial case, X could not be an algebraic curve so has dimension > 1 . □

So the classification or description of simple trivial compact complex varieties in \mathcal{C} remains. Various model-theoretic conjectures have been made in earlier papers: for example that they are strongly minimal, or even that they must be ω -categorical when equipped with their canonical Barlet language (see section 3).

To understand the simple trivial compact *surfaces* we look to the

classification of compact complex surfaces carried out by Kodaira in a series of papers in the 1960's, extending the Enriques classification of algebraic surfaces. An account of Kodaira's work appears in [3]. In particular Table 10 in Chapter VI there is rather useful. From it we can deduce:

Proposition 5.3 *Let X be a simple trivial compact complex variety of dimension 2 which is in the class \mathcal{C} . Then X is bimeromorphic to a K3 surface. Expressed otherwise, a stationary trivial minimal type in \mathcal{C} of dimension 2 is, up to interdefinability, the generic type of a K3 surface.*

Proof. The classification of Kodaira gives a certain finite collection of (abstractly defined) classes, such that every compact surface has a “minimal model” in exactly one of the classes, in particular is bimeromorphic to something in one of the classes. Suppose X is a simple trivial surface in \mathcal{C} . Then X has algebraic dimension 0 (namely X does not map holomorphically onto any algebraic variety of dimension > 0), X is not a complex torus, and X has first Betti number even. Moreover these properties also hold of any Y bimeromorphic to X . By looking at Table 10, Chapter VI of [3], the only possibility for a minimal model of X is to be a K3 surface. \square

Among K3 surfaces are (i) smooth surfaces of degree 4 in \mathbb{P}_3 , and (ii) *Kummer surfaces*. A Kummer surface is something obtained from a 2-dimensional complex torus by first quotienting by the map $x \mapsto -x$ and then taking a minimal resolution. See Chapter VIII of [3] for more details. In particular there are algebraic K3 surfaces, and there are simple K3 surfaces which are not trivial. However there do exist K3 surfaces of algebraic dimension 0 (that is, which do not map onto any algebraic variety) and which are not Kummer, and these *will be* simple and trivial (see [17]). On the other hand all K3 surfaces are *diffeomorphic* (that is, isomorphic as real differentiable manifolds), and in fact they were first defined by Weil precisely as compact complex analytic surfaces diffeomorphic to a smooth quartic surface in \mathbb{P}_3 .

It is conceivable, and consistent with the examples, that the natural analogue of Proposition 5.1 holds for higher dimensions:

Conjecture I. Any simple trivial compact complex variety in \mathcal{C} which is bimeromorphic to (or at least in correspondence with) an irreducible hyperkähler manifold. Equivalently any trivial minimal type in \mathcal{C} is

nonorthogonal to the generic type of some irreducible hyperkähler manifold.

Note that any irreducible hyperkähler manifold has even dimension. Also, as with the special case of $K3$ surfaces, there are (irreducible) hyperkählers of any even dimension which are algebraic (and hence not trivial).

Let us now pass to the stability-theoretic notion of *nonmultidimensionality*. We start with an arbitrary complete (possibly many-sorted) stable theory T , and work in a saturated model \bar{M} of T . Let $p(x) \in S(A)$, $q(y) \in S(B)$ be stationary types (over small subsets A, B of \bar{M}). Then p is said to be *nonorthogonal* to q if there is $C \supseteq A \cup B$ and realizations a of p , and b of q , such that: (i) a is independent from C over A , and b is independent from C over B , and (ii) a forks with b over C .

A stationary type $p(x) \in S(A)$ is said to be nonorthogonal to a set of parameters B if p is nonorthogonal to some complete type over $\text{acl}(B)$. The theory T is said to be *nonmultidimensional* if every stationary nonalgebraic type $p(x) \in S(A)$, is nonorthogonal to \emptyset . An equivalent characterization is:

- (\star) Whenever $p(x, a)$ is a stationary nonalgebraic type (with domain enumerated by the possibly infinite tuple a), and $\text{stp}(a') = \text{stp}(a)$, then $p(x, a)$ is nonorthogonal to $p(x, a')$.

Remark 5.4 *If T happens to be superstable, then it suffices that (\star) holds for $p(x, a)$ regular, and moreover we may assume that a is a finite tuple. If moreover T has finite rank (meaning every finitary type has finite U -rank), then it suffices for (\star) to hold for types $p(x, a)$ of U -rank 1.*

A stronger condition than nonmultidimensionality is *unidimensionality* which says that any two stationary nonalgebraic types are nonorthogonal. This is equivalent to T having exactly one model of cardinality κ for all $\kappa > |T|$. Nonmultidimensionality was also introduced by Shelah [24] in connection with classifying and counting models. For totally transcendental T (namely every formula has ordinal valued Morley rank), T is nonmultidimensional if and only if there is some fixed cardinal μ_0 (which will be at most $|T|$) such that *essentially* the models of T are naturally in one-one correspondence with sequences $(\kappa_\alpha : \alpha < \mu_0)$

of cardinals. When μ_0 is finite, T is called *finite-dimensional*. Alternatively (for T superstable of finite rank) this means that there are only finitely many stationary U -rank 1 types up to nonorthogonality.

Remark 5.5 (cf. [20]) *Suppose that T is superstable of finite rank and nonmultidimensional. Suppose moreover that T is one-sorted and that every stationary type of U -rank 1 is nonorthogonal to a type of Morley rank 1. Then T is finite-dimensional.*

The following conjecture was formulated (by Thomas Scanlon and the second author) around 2000-2001. They also pointed out (in [21]) that it fails for $\text{Th}(\mathcal{A})$.

Conjecture II. $\text{Th}(\mathcal{C})$ is nonmultidimensional.

Let $p(x, a)$ be a stationary type in \mathcal{C} realized by b say (where a is a finite tuple). Then $stp(a, b)$ is the generic type of a compact complex variety X , $stp(a)$ is the generic type of a compact complex variety S and the map $(x, y) \rightarrow x$ gives a dominant meromorphic map f from X to S , and $tp(a/b)$ is the generic type of the irreducible fibre X_a . Without changing $p(x, a)$ we may assume that X and S are manifolds and that f is a holomorphic submersion (so that the generic fibre of f is a manifold also). The requirement that for another realization a' of $stp(a)$, $p(x, a)$ and $p(x, a')$ are nonorthogonal, becomes: for a' another generic point of S , there is some proper analytic subset Z of $X_a \times X_{a'}$ which projects onto both X_a and $X_{a'}$. So by (\star) above, we obtain the following reasonably geometric account or interpretation of the nonmultidimensionality of $\text{Th}(\mathcal{C})$: for any fibration $f : X \rightarrow S$ in \mathcal{C} , any two generic fibres have the feature that there is a proper analytic subset of their product, projecting onto each factor. By Remark 5.4, we may restrict to the case where the generic fibre X_a is *simple*. So we obtain:

Remark 5.6 *Conjecture II is equivalent to:* Whenever $f : X \rightarrow S$ is a fibration in \mathcal{C} with generic fibre a simple compact complex manifold, then f is *weakly isotrivial* in the sense that for any generic fibres $X_s, X_{s'}$, there is a *correspondence* between X_s and $X_{s'}$.

Of course there are other stronger conditions than weak isotriviality which a fibration $f : X \rightarrow S$ may satisfy, for example that any two generic fibres are bimeromorphic or even that any two generic fibres

are biholomorphic. If the latter is satisfied we will call the fibration *isotrivial*.

Let us begin a discussion of Conjecture II. Let $f : X \rightarrow S$ be a fibration in \mathcal{C} with simple generic fibre X_s . By 5.2, X_s is either (i) an algebraic curve, (ii) a simple nonalgebraic complex torus, or (iii) has trivial generic type. In case (i), we obtain weak isotriviality (as any two algebraic curves project generically finite-to-one onto \mathbb{P}_1). So we are reduced to cases (ii) and (iii). Special cases of case (ii) are proved by Campana [7]. Assuming the truth of Conjecture I, case (iii) is also proved in [7]. An exposition of this work is one of the purposes of this paper and appears in the next section.

For now, we end this section with a few additional remarks on isotriviality.

Remark 5.7 *Let $f : X \rightarrow S$ be a fibration in \mathcal{C} . If f is locally trivial in the sense that for some nonempty open subset U of S , X_U is biholomorphic to $U \times Y$ over U for some compact complex variety Y , then f is isotrivial.*

Proof. By Baire category, we can find $s_1, s_2 \in U$ which are mutually generic. So X_{s_1} is isomorphic to X_{s_2} by assumption. But $tp(s_1, s_2)$ is uniquely determined by the mutually genericity of s_1, s_2 . Hence for any mutually generic $s_1, s_2 \in S$, X_{s_1} is isomorphic to X_{s_2} . Now given generic $s_1, s_2 \in S$, choose $s \in S$ generic over $\{s_1, s_2\}$. So X_s is isomorphic to each of X_{s_1}, X_{s_2} . \square

Remark 5.8 *Suppose that $f : X \rightarrow S$ is a fibration in \mathcal{C} whose generic fibre X_s is a simple nonalgebraic complex torus. Suppose moreover that any (some) two mutually generic fibres $X_s, X_{s'}$ are nonorthogonal. Then any two generic fibres are isomorphic (as complex tori).*

Proof. Fix two mutually generic fibres X_s and $X_{s'}$. These are both locally modular strongly minimal groups. Hence nonorthogonality implies that there is a strongly minimal subgroup C of $X_s \times X_{s'}$ projecting onto both factors, and this induces an isogeny from $X_{s'}$ onto X_s , and thus an isomorphism (of complex tori) between $X_{s'}/A_{s'}$ and X_s for some finite subgroup $A_{s'}$ of $X_{s'}$. Note that $A_{s'}$ is $\text{acl}(s')$ -definable. Now let s_1, s_2 be generic points of S . Let $s' \in S$ be generic over $\{s_1, s_2\}$. So there is an isomorphism f_1 between $X_{s'}/A_{s'}$ and X_{s_1} (for some finite, so $\text{acl}(s')$ -definable subgroup of $X_{s'}$) with X_{s_1} . As s_1 and s_2 have the same type

over $\text{acl}(s')$, we obtain an isomorphism f_2 between $X_{s'}/A_{s'}$ and X_{s_2} . Thus X_{s_1} and X_{s_2} are isomorphic. \square

6 Local Torelli and the isotriviality theorem

In this section we state and sketch the proof of a recent result of Campana [7], which was motivated by and partially resolves the nonmultidimensionality conjecture for \mathcal{C} discussed in the previous section.

Suppose Y is a compact Kähler manifold and consider a *deformation* $f : X \rightarrow S$ – that is, f is a proper holomorphic submersion between complex manifolds X and S and there is a point $o \in S$ such that $X_o = Y$. For s near o , X_s will be a compact Kähler manifold (see Theorem 9.23 of [25]). Diffeomorphically, f is locally trivial: there exists an open neighbourhood $U \subseteq S$ of o such that X_U is diffeomorphic to $U \times Y$ over U . Letting u be this diffeomorphism we have the commuting diagram:

$$\begin{array}{ccc}
 U \times Y & \xrightarrow[u]{\text{diffeo}} & X_U \\
 & \searrow & \swarrow f_U \\
 & & U
 \end{array}$$

For each $s \in U$, the diffeomorphism $u_s : Y \rightarrow X_s$ induces an isomorphism of singular cohomology groups, and hence by De Rham’s Theorem, of the De Rham cohomology groups. In particular, we obtain a group isomorphism, $\hat{u}_s : H_{\text{DR}}^2(X_s) \rightarrow H_{\text{DR}}^2(Y)$. From our discussion of De Rham’s theorem in Section 2 it is not hard to see that, under the identification

$$H_{\text{DR}}^2(Y) = H_{\text{sing}}^2(Y, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(H_2(Y), \mathbb{C}),$$

the isomorphism $\hat{u}_s : H_{\text{DR}}^2(X_s) \rightarrow H_{\text{DR}}^2(Y)$ is given by

$$[\omega] \mapsto \left([\gamma] \mapsto \int_{\gamma} u_s^* \omega \right)$$

where ω is a d -closed 2-form on X_s and γ is a real 2-cycle on Y .

Since u may not be biholomorphic, \hat{u}_s does not necessarily respect the Hodge decomposition of $H_{\text{DR}}^2(X_s)$ and $H_{\text{DR}}^2(Y)$. Indeed, one measure of how far u is from being a biholomorphic trivialisation is the *period map* of Y for holomorphic 2-forms (with respect to the deformation f):

$$p : U \rightarrow \text{Grass}(H_{\text{DR}}^2(Y))$$

which assigns to each $s \in U$ the subspace $\hat{u}_s(H^{2,0}(X_s))$. Recall that for

any complex manifold M , $H^{2,0}(M)$ is just the space $\Omega^2(M)$ of global holomorphic 2-forms on M .

Note that if u is a biholomorphism then the period map is constant on U since for all $s \in U$ $\hat{u}_s(H^{2,0}(X_s)) = H^{2,0}(Y)$.

Definition 6.1 Suppose Y is a compact Kähler manifold. We say that Y satisfies *local Torelli for holomorphic 2-forms* if the following holds: given any deformation $f : X \rightarrow S$ of Y with a local diffeomorphic trivialisation $u : U \times Y \rightarrow X_U$, if the corresponding period map is constant on U then u is in fact a biholomorphic trivialisation.

Example 6.2 Complex tori and irreducible hyperkähler manifolds all satisfy local Torelli for holomorphic 2-forms. (See Theorem 5(b) of [4] for the case of irreducible hyperkähler manifolds.)

We can now state the isotriviality theorem we are interested in.

Theorem 6.3 (Campana [7]) *Suppose $f : X \rightarrow S$ is a fibration where X and S are compact Kähler manifolds. Assume that for $a \in S$ generic, (i) X_a is not projective, (ii) $\dim_{\mathbb{C}} \Omega^2(X_a) = 1$, and (iii) X_a satisfies local Torelli for holomorphic 2-forms. Then f is isotrivial.*

Sketch of proof. From condition (i) and Corollary 4.4 there exists a global holomorphic 2-form $\omega \in \Omega^2(X)$ whose restriction ω_a to the generic fibre X_a is a nonzero global holomorphic 2-form on X_a . Moreover by condition (ii), ω_a spans $\Omega^2(X_a)$.

Let U be an open neighbourhood of a such that there is a diffeomorphic trivialisation $u : U \times X_a \rightarrow X_U$ over U . We show that the corresponding period map $p : U \rightarrow \text{Grass}(H_{\text{DR}}^2(X_a))$ is constant on U . By local Torelli this will imply that u is biholomorphic and hence, by Remark 5.7, f is isotrivial.

For any $s \in U$ let $\hat{u}_s : H_{\text{DR}}^2(X_s) \rightarrow H_{\text{DR}}^2(X_a)$ be the isomorphism induced by u and discussed above. We need to show that $\hat{u}_s(H^{2,0}(X_s)) = \hat{u}_t(H^{2,0}(X_t))$ for all $s, t \in U$. But, shrinking U if necessary, $H^{2,0}(X_s) = \Omega^2(X_s)$ is spanned by the restriction ω_s of ω to X_s , for all $s \in U$. Hence it suffices to show that $\hat{u}_s(\omega_s) = \hat{u}_t(\omega_t)$ for all $s, t \in U$.

Now fix a real 2-cycle γ on X_a . Viewing $\hat{u}_s(\omega_s)$ and $\hat{u}_t(\omega_t)$ as elements of $\text{Hom}_{\mathbb{C}}(H_2(X_a), \mathbb{C}) = H_{\text{DR}}^2(X_a)$ we compute

$$(\hat{u}_s(\omega_s) - \hat{u}_t(\omega_t))[\gamma] = \int_{\gamma} u_s^* \omega_s - \int_{\gamma} u_t^* \omega_t = \int_{u_s \circ \gamma} \omega_s - \int_{u_t \circ \gamma} \omega_t.$$

Here $u_s \circ \gamma$ and $u_t \circ \gamma$ are 2-cycles on X_s and X_t respectively. Viewed as 2-cycles on X we have

$$\int_{u_s \circ \gamma} \omega_s - \int_{u_t \circ \gamma} \omega_t = \int_{(u_s \circ \gamma - u_t \circ \gamma)} \omega.$$

But $(u_s \circ \gamma - u_t \circ \gamma)$ is the boundary of some 3-cycle λ on X . By Stokes', $\int_{(u_s \circ \gamma - u_t \circ \gamma)} \omega = \int_{\lambda} d\omega$. Since holomorphic forms are d -closed it follows that

$$(\hat{u}_s(\omega_s) - \hat{u}_t(\omega_t))[\gamma] = 0$$

for all 2-cycles γ on X_a . That is, $\hat{u}_s(\omega_s) = \hat{u}_t(\omega_t)$ for all $s, t \in U$. So the period map is constant on U and f is isotrivial. \square

Remark 6.4 *The hypotheses of Theorem 6.3 are valid in the following cases:*

- (a) *The generic fibre X_a is irreducible hyperkähler and nonprojective,*
- (b) *The generic fibre X_a is a simple complex torus of dimension 2.*

Proof. We have already mentioned that complex tori and irreducible hyperkähler satisfy local Torelli for holomorphic 2-forms. Nonprojectivity is assumed in (a) and follows for (b) by the fact that the only simple projective varieties are curves. Finally, $\dim_{\mathbb{C}} \Omega^2(X_a) = 1$ is true of irreducible hyperkähler manifolds by definition, and true of simple complex tori of dimension 2 by the fact that dimension 2 forces $\dim_{\mathbb{C}} \Omega^2(X_a)$ to be at most 1 while nonprojectivity forces it to be at least 1. \square

Let us return to the nonmultidimensionality conjecture (Conjecture II) from section 5, bearing in mind the equivalence stated in Remark 5.6.

Corollary 6.5 *The nonmultidimensionality conjecture holds in $\text{Th}(\mathcal{C})$ for surfaces. In other words if $p(x)$ is a minimal type of dimension 1 or 2 over some model of $\text{Th}(\mathcal{C})$ then p is nonorthogonal to \emptyset .*

Proof. As discussed at the end of section 3 we may work in \mathcal{C} itself. Let $p(x) = tp(b/a)$ for a, b from \mathcal{C} and b a generic point of an a -definable simple compact complex manifold X_a of dimension 1 or 2. We have already pointed out that in the case of dimension 1 (i.e. of projective curves), X_a is nonorthogonal to $X_{a'}$ whenever $stp(a) = stp(a')$. So assume X_a is a simple compact complex surface. It is then not projective.

By Fact 5.2 and Proposition 5.3, we may assume that X_a is either a 2-dimensional simple complex torus, or a nonprojective K3 surface. So by Theorem 6.3 and Remark 6.4, X_a is biholomorphic to $X_{a'}$ whenever $stp(a) = stp(a')$. Hence $tp(b/a)$ is nonorthogonal to \emptyset . \square

Condition (ii) of Theorem 6.3 seems rather strong, and indeed, Campana works with the following weaker condition: A *rational Hodge substructure* of $H_{\text{DR}}^2(X_a)$ is a \mathbb{C} -vector subspace V such that $V^{i,j} = \overline{V^{j,i}}$ where $V^{i,j} := V \cap H^{i,j}(X_a)$, and $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ where

$$V_{\mathbb{Q}} := V \cap H_{\text{sing}}^2(X_a, \mathbb{Q}).$$

Campana says that X_a is *irreducible in weight 2* if for any rational Hodge substructure $V \subseteq H_{\text{DR}}^2(X_a)$, either $V^{2,0} = 0$ or $V^{2,0} = H^{2,0}(X_a)$. By a theorem of Deligne, the image of $H_{\text{DR}}^2(X)$ in $H_{\text{DR}}^2(X_a)$ under the restriction map is a rational Hodge substructure. Hence, the above proof of Theorem 6.3 works if condition (ii) is replaced by the irreducibility of X_a in weight 2. Campana proves that the “general” torus of dimension ≥ 3 is irreducible in weight 2. Apparently it is open whether any simple nonalgebraic torus is irreducible in weight 2. This together with Conjecture I (that any simple trivial compact Kähler manifold is nonorthogonal to an irreducible hyperkahler manifold) are the remaining obstacles to the nonmultidimensionality conjecture for \mathcal{C} .

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