NOTES ON "GROUPOIDS, IMAGINARIES AND INTERNAL COVERS"

OLEG CHTERENTAL

April 30, 2009

These notes are an exposition of the beginning of Ehud Hrushovski's paper "Groupoids, Imaginaries and Internal Covers". They assume a background consisting of an introductory level course in Model Theory. I would like to thank NSERC for the USRA that provided funding for the term and my supervisor Professor Rahim Moosa for his time and guidance and also Moshe Kamensky for some helpful discussions.

1. Definable Groupoids and Functors

A *category* C is a certain 2-sorted structure with two sorts O and M denoting objects and arrows or maps respectively. There are two maps $\mathrm{dom}, \mathrm{cod} : M \to O$ representing domain and codomain of elements in M. If $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$ we write $f : a \to b$. There is a partial composition $\circ : M \times_{\mathrm{dom,cod}} M \to M : (f, g) \mapsto f \circ g$, where $M \times_{\mathrm{dom,cod}} M = \{(f, g) \in M \times M : \mathrm{dom}(f) = \mathrm{cod}(g)\}$ and an identity map $\mathrm{Id} : O \to M$ satisfying the associativity and identity axioms. We write $\mathrm{Ob}C$ for the collection of objects of C and $\mathrm{Mor}C$ for the collection of all maps in C or simply O and M if the category is clear. Also, $\mathrm{Mor}_C(a, b)$ will be the collection of maps with domain a and codomain b and $C_a = \mathrm{Mor}_C(a, a)$. We write Id_a for the identity map on the object a. Sometimes we omit the $\circ$ symbol when writing compositions, so $f \circ g$ is written as fg.

A *groupoid* $\mathcal{G}$ is a category with the property that for every $f : a \to b$ there is $g : b \to a$ such that $f \circ g = \mathrm{Id}_b$ and $g \circ f = \mathrm{Id}_a$. Such a g is unique and we denote it f^{-1}. A groupoid is *connected* if each set $\mathrm{Mor}_{\mathcal{G}}(a, b)$ is not empty. We will work with connected groupoids.

Let T be a theory with a universal domain $\mathbb{U}$. Let $\mathrm{Def}(\mathbb{U})$ be the category whose objects are $\mathbb{U}$-definable sets and $\mathbb{U}$-definable maps between them. Note that "definable" by itself will mean $\emptyset$-definable.

A *definable groupoid* $\mathcal{G}$ (in $\mathbb{U}$) is intuitively a groupoid whose sets of objects and morphisms are definable sets in $\mathbb{U}$ and the domain, codomain, composition and identity maps are also definable. To be precise, we need two definable sets O, M and four definable maps (that is, having definable graphs) $\mathrm{dom}, \mathrm{cod} : M \to O$, $\circ : M \times_{\mathrm{dom},\mathrm{cod}} M \to M$ and $\mathrm{Id} : O \to M$. As well we need the groupoid axioms to hold for these maps.

Note that in the above, the definability of M, dom and cod means that $M \times_{\mathrm{dom},\mathrm{cod}} M$ is definable.

By a definable family of definable subsets of some fixed sort, we mean a collection of nonempty sets S_a indexed by some definable set I of tuples a from $\mathbb{U}$, and an L-formula $\phi(x, y)$ such that $\phi(a, y)$ defines S_a. Each set S_a is then $\{a\}$-definable and the set $S = \coprod_{a \in I} \{a\} \times S_a$ is defined by $(x \in I) \wedge \phi(x, y)$. The projection $\pi : \coprod_{a \in I} \{a\} \times S_a \to I$ onto the first coordinate is a definable map and the S_a's are canonically and definably isomorphic to the fibres, the $\{a\} \times S_a$'s, of this map.

Conversely, if S and I are definable sets and $\pi : S \to I$ is a surjection and its graph (after swapping positions)

$$\{(b, a) : a \in I, b \in S, \pi(b) = a\},$$

is defined by $\phi(x, y)$, then the fibre S_a over $a \in I$ is defined by $\phi(a, y)$. Thus definable families are just the fibres of a definable map. Note also that if $\mathcal{F} = \{S_a : a \in I\}$ is a definable family of definable subsets then there is a definable equivalence relation $\sim$ on S such that $\mathcal{F}$ is the set of $\sim$-equivalence classes.

There are some definable families of definable sets at work in our definition of a definable groupoid $\mathcal{G}$:

(1) The family $\{G_a : a \in O\}$ is a definable family of definable subsets of M. Indeed if $G \subseteq M$ is defined by the formula

$$(f \in M) \wedge (\mathrm{dom}(f) = \mathrm{cod}(f)),$$

then the G_a's are the fibres of the definable map $\pi : G \to O : f \mapsto \mathrm{dom}(f)$.

(2) There is the definable map $+ : G \times_\pi G \to G$ which is the restriction of $\circ$ to $G \times_\pi G$. This makes $\{G_a : a \in O\}$ into a definable family of definable groups.

(3) More generally we have the definable map $\mathrm{dom} \times \mathrm{cod} : M \to O \times O$. For each $(a, b) \in O \times O$ the fibre $(\mathrm{dom} \times \mathrm{cod})^{-1}(a, b) = \mathrm{Mor}_{\mathcal{G}}(a, b)$ is $\{a, b\}$-definable. Each $h \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ induces an $\{a, b, h\}$-definable isomorphism from G_a to G_b given by $m \mapsto hmh^{-1}$. Its graph is

$$\{(m, n) : m \in G_a, n \in G_b, n = hmh^{-1}\}.$$

This gives us a definable family of definable isomorphisms among the G_a's.

In this definition the objects of a definable groupoid are elements of a definable set and the maps are also elements of a definable set. We will see that indeed any definable groupoid can be represented in such a way that its objects are in fact definable sets and the maps are definable maps. First we need the notion of a definable functor, from a definable groupoid $\mathcal{G}$ to $\mathrm{Def}(\mathbb{U})$.

Assume that $\mathcal{G}$ is a definable groupoid in $\mathbb{U}$. A definable functor $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is a functor where

funX
$$X = \coprod_{a \in O} \{a\} \times F(a) = \{(a, d) : a \in O, d \in F(a)\}, \tag{1.1}$$

and

funY
$$\begin{aligned} (1.2)\ Y &= \{(a, b, c, d, e) : a, b \in O, c \in \mathrm{Mor}_{\mathcal{G}}(a, b), \\ (1.3)\qquad & \quad d \in F(a), e \in F(b), F(c)(d) = e\} \\ (1.4)\qquad &= \{\{(a, b, c)\} \times \mathrm{Graph}(F(c)) : a, b \in O, c \in \mathrm{Mor}_{\mathcal{G}}(a, b)\}, \end{aligned}$$

are definable. Let's unpack this definition. For $a \in O$ we have $F(a) \in \mathrm{ObDef}(\mathbb{U})$ so $F(a)$ is a $\mathbb{U}$-definable set. If the functor is definable, the set X is definable and then the projection $\pi : X \to O$, on the first coordinate is definable. Each $F(a)$ is the fibre $\pi^{-1}(a)$ and is $\{a\}$-definable, so the $F(a)$'s form a definable family of definable subsets.

If $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is a definable functor, then by identifying the elements of $\{F(G_a) : a \in O\}$ (which are morphisms in $\mathrm{Def}(\mathbb{U})$) with their graphs, we get a definable family of definable groups such that the natural action of $F(G_a)$ on $F(a)$ is definable. This follows from the fact that Y in Equation 1.2 is definable.

Here is an example, for any definable groupoid $\mathcal{G}$. Consider the functor F taking a to G_a and $f : a \to b$ to $F(f) : G_a \to G_b : m \mapsto f \circ m \circ f^{-1}$. One sees this is indeed a functor. It is in fact a definable functor. The set X is just

$$\{(a, d) \in O \times M : \mathrm{dom}(d) = \mathrm{cod}(d) = a\},$$

and the set Y is

$$\begin{aligned} \{(a, b, c, d, e) \in O^2 \times M^3 \ &: \ \mathrm{dom}(c) = \mathrm{dom}(d) = \mathrm{cod}(d) = a, \\ & \quad\ \mathrm{cod}(c) = \mathrm{dom}(e) = \mathrm{cod}(e) = b, \\ & \quad\ cdc^{-1} = e\}. \end{aligned}$$

Recall that a functor is faithful if whenever $f, g : a \to b$ are maps and $F(f) = F(g)$ then $f = g$, that is, F induces an injection $\mathrm{Mor}(a, b) \to \mathrm{Mor}(F(a), F(b))$ for all objects $a, b \in \mathrm{Ob}(\mathcal{G})$. The functor in the above example is faithful if all the G_a have trivial center, since if $f, g : a \to b$

and $fmf^{-1} = gmg^{-1}$ for all $m \in G_a$ then $(g^{-1}f)m = m(g^{-1}f)^{-1}$ for all $m \in G_a$ thus $g^{-1}f = \mathrm{Id}_a$.

For any definable groupoid however, there is a faithful definable functor into $\mathrm{Def}(\mathbb{U})$ as shown in [1]. Indeed one may take the functor δ taking an object $a \in O$ to the $\{a\}$-definable set

$$\delta(a) = \{g \in M : \mathrm{dom}(f) = a\},$$

and taking a morphism $f : a \to b$ to the morphism

$$\delta(f) : \delta(a) \to \delta(b) : g \mapsto gf^{-1}.$$

A *concrete* definable category is a pair $(\mathcal{G}, \delta_{\mathcal{G}})$ where $\mathcal{G}$ is a definable category and $\delta_{\mathcal{G}} : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is a faithful definable functor. By the above paragraph, any definable groupoid can be made concrete with the appropriate functor.

2. Two special cases

Let us now work out in detail [2, Example 1.1]. Let $\mathcal{G}$ be a definable groupoid. Note that if $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is a definable functor, each of the sets $F(a)$ are definably isomorphic over parameters, since $\mathcal{G}$ is connected. However there is usually no canonical choice of isomorphisms. The following examples consider cases where there is a canonical choice.

eg1 **Example 2.1.** Suppose each G_a is trivial. Then for each $a, b \in O$, $\mathrm{Mor}_{\mathcal{G}}(a, b)$ consists of a unique morphism. In this case if $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is a definable functor, one can interpret without parameters a set S, definably isomorphic to each $F(a)$.

Proof. If $f, g \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ then $g^{-1}f \in G_a$ and since it is trivial, $g^{-1}f = \mathrm{Id}_a$ so $f = g$. Let $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ be a definable functor. The set S is constructed as follows. Let

$$E_S \;=\; \{(a, b, a', b') \in X^2 : \exists c \in \mathrm{Mor}_{\mathcal{G}}(a, a')F(c)(b) = b'\},$$

and let $S = X/E_S$ where X is as in equation 1.1. One may check that E_S is in fact an equivalence relation. For example it is symmetric because if there is a $c \in \mathrm{Mor}_{\mathcal{G}}(a, a')$ with $F(c)(b) = b'$ then there is automatically $c^{-1} \in \mathrm{Mor}_{\mathcal{G}}(a', a)$ with $F(c^{-1})(b') = b$ by functorality. The relation is definable since it is given by

$$\exists c (a, a', c, b, b') \in Y,$$

where Y is the set described in equation 1.2.

We are taking the quotient of a definable set by a definable equivalence relation and claiming it is definably isomorphic to each $F(a)$. The quotient is in $\mathbb{U}^{eq}$ but using the discussion in the appendix we can pullback to $\mathbb{U}$ to witness this isomorphism.

The definable set $X = \{(a, d) : a \in O, d \in F(a)\}$ of equation 1.1 is a subset of some $\mathbb{U}^k$ so extend E_S to a definable equivalence relation on $\mathbb{U}^k$. This can be done by making all elements not in X to be equivalent. Then we wish to show that X/E_S is definably isomorphic to $F(q)$ for any fixed $q \in O$.

Consider the map $r : X/E_S \to F(q)$ given by $r([a, b]) = F(c)(b)$ where $c \in \mathrm{Mor}_{\mathcal{G}}(a, q)$ is the unique morphism from a to q. If $(a, b) \sim (a', b')$ then $F(d)(b) = b'$ for the unique $d \in \mathrm{Mor}_{\mathcal{G}}(a, a')$. Thus if $c \in \mathrm{Mor}_{\mathcal{G}}(a, q)$ is the unique morphism from a to q, then cd^{-1} is the unique morphism from a' to q, so

$$r([a', b']) = F(cd^{-1})(b') = F(c)F(d^{-1})(b') = F(c)(b) = r([a, b]),$$

so r is well-defined as a function.

It can be checked that r is indeed a bijection. Now we check that the function $r \circ f_{E_S} : X \to F(q)$ is definable, albeit with parameters, since $F(q)$ is only definable with parameters. This suffices by Proposition 6.2. Its graph is

$$\{(a, b, u) : (a, b) \in X, u \in F(q), \exists c(\mathrm{dom}(c) = a \wedge \mathrm{cod}(c) = q \wedge F(c)(b) = u)\}.$$

□

Now we work out [2, Example 1.2].

Example 2.2. If $\mathcal{G}$ is abelian, then the G_a are all canonically isomorphic, and one can interpret without parameters a single group, isomorphic to all G_a.

Proof. Here an abelian groupoid is one in which each G_a is abelian. Consider again the functor $F(a) = G_a$ and $F(f) : G_a \to G_b : m \mapsto fmf^{-1}$. If $g, h \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ then $F(g) = F(h)$. Indeed we have $h^{-1}g \in G_a$ and by commutativity we get $(h^{-1}g)m = m(h^{-1}g)$ for all $m \in G_a$ which expands to $gmg^{-1} = hmh^{-1}$ for all $m \in G_a$.

This means that the groupoid $\mathcal{H}$ which is the image of the functor F has the property that any morphism set $\mathrm{Mor}_{\mathcal{H}}(G_a, G_b)$ has a single element (the objects of $\mathcal{H}$ are the groups G_a for $a \in \mathrm{Ob}_{\mathcal{G}}$ and the morphisms are the induced conjugation maps between these groups). Thus by Example 2.1 with the identity functor Id on $\mathcal{H}$, we get a set $S = X/E_S$ constructed from $\mathcal{H}$ and we get an isomorphism $r : S \to G_q$ for any $q \in \mathrm{Ob}\mathcal{G}$. Note that in this case S consists of pairs (G_a, b) where $b \in G_a$, under the equivalence $(G_a, b) \sim (G_{a'}, b')$ if $F(c)(b) = b'$ for some element c of $\mathrm{Mor}_{\mathcal{G}}(a, a')$. We have $r([G_a, b]) = F(c)(b)$ where c is an element of $\mathrm{Mor}_{\mathcal{G}}(a, q)$.

The map $r = r_q$ depends on q. The set S is canonical because it has only one induced group structure from all the maps r_q. That is, if you pullback the group structure of G_q to S via r_q and of $G_{q'}$ to S via $r_{q'}$, then the two group multiplications on S agree. $\square$

3. Binding Groupoids

We turn our attention to [2, Proposition 1.5]. A type definable set is the solution set of a partial type. In [2] this is also called ∞-definable. If the type is allowed to have potentially infinitely many but a small cardinality of variables we call the solution set $\star$-definable. By restricting to finitely many variables and increasing the variables at each stage, a $\star$-definable set can be seen as a projective system of type definable sets.

Let $\mathbb{V}$ be a collection of sorts of $\mathbb{U}$, closed under cartesian products. We work throughout in $\mathbb{U}^{eq}$ and $\mathbb{V}^{eq}$ when appropriate.

We assume that $\mathbb{V}$ is stably embedded: If S is a sort of $\mathbb{V}^{eq}$ and P is a $\mathbb{U}$-definable subset of S^n, then in fact P is definable with parameters from $\mathbb{V}$.

A definable set $Q \subset \mathbb{U}$ is said to be internal to $\mathbb{V}$ if there is a tuple $c \in \mathbb{U}$ such that $Q \subset \mathrm{dcl}(\mathbb{V} \cup c)$. We then have the following proposition which is [2, Proposition 1.5].

`groupoid` **Proposition 3.1.** *Assume $\mathbb{V}$ is stably embedded in $\mathbb{U}$ and Q is a definable set of $\mathbb{U}$ that is internal to $\mathbb{V}$. There exist connected $\star$-definable groupoids $\mathcal{G}$ in $\mathbb{U}^{eq}$ and $\mathcal{G}_{\mathbb{V}}$ in $\mathbb{V}^{eq}$, and relatively definable functors $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U}^{eq})$ and $F_{\mathbb{V}} : \mathcal{G}_{\mathbb{V}} \to \mathrm{Def}(\mathbb{V}^{eq})$ such that $\mathrm{Ob}\mathcal{G} = \mathrm{Ob}\mathcal{G}_{\mathbb{V}} \cup \{*\}$, the groupoid $\mathcal{G}_{\mathbb{V}}$ is a full subgroupoid of $\mathcal{G}$, the functor $F_{\mathbb{V}}$ is the restriction of F to $\mathcal{G}_{\mathbb{V}}$, and $F(*) = Q$. Moreover $F(G_*)$ with its action on Q is type-definable and isomorphic to $\mathrm{Aut}(Q/\mathbb{V})$.*

First some remarks about the proposition. The group $\mathrm{Aut}(Q/\mathbb{V})$ is the group of permutations of Q that arise as restrictions of automorphisms of $\mathbb{U}$ fixing $\mathbb{V}$. Thus $\mathrm{Aut}(Q/\mathbb{V}) = \{f_{|Q} : f \in \mathrm{Aut}(\mathbb{U}/\mathbb{V})\}$. One may equivalently think of this as $\mathrm{Aut}(\mathbb{U}/\mathbb{V})/\mathrm{Aut}(\mathbb{U}/(Q \cup \mathbb{V}))$. Note that $\mathrm{Aut}(\mathbb{U}/(Q \cup \mathbb{V}))$ is indeed a normal subgroup of $\mathrm{Aut}(\mathbb{U}/\mathbb{V})$.

One consequence of this theorem is that $F(G_*) = \mathrm{Aut}(Q/\mathbb{V})$ is interpretable in $\mathbb{U}$. Since $\mathcal{G}$ is connected, all the $F(G_a)$ for $a \in \mathrm{Ob}\mathcal{G}_{\mathbb{V}}$ are definably isomorphic to $F(G_*)$ and hence $\mathrm{Aut}(Q/\mathbb{V})$, but not canonically. In particular $\mathrm{Aut}(Q/\mathbb{V})$ is type definable without parameters in $\mathbb{U}^{eq}$ but with parameters in $\mathbb{V}^{eq}$.

Let us work out the proof of this proposition. The following lemmas will be needed.

`stable` **Lemma 3.2.** *If $\mathrm{tp}(a/\mathbb{V}) = \mathrm{tp}(b/\mathbb{V})$ then there exists $\sigma \in \mathrm{Aut}(\mathbb{U}/\mathbb{V})$ such that $\sigma(a) = b$.*

Proof. This is essentially [3, Lemma 0.8] and is a consequence of the stable embeddedness of $\mathbb{V}$ in $\mathbb{U}$. Since $\mathrm{tp}(a/\mathbb{V}) = \mathrm{tp}(b/\mathbb{V})$ we have $\mathrm{tp}(a) = \mathrm{tp}(b)$. Thus by saturation there is $\gamma \in \mathrm{Aut}(\mathbb{U})$ with $\gamma(a) = b$. Note that extending the language by a constant symbol c_a and interpreting it as a in $\mathbb{U}$ we get a new structure $(\mathbb{U}, a)$. Furthermore $\mathbb{V}$ is still stably embedded in $(\mathbb{U}, a)$, indeed any definable set D contained in a sort from $\mathbb{V}$ is now definable possibly with the extra parameter a, and by the stable embeddedness of $\mathbb{V}$ in $\mathbb{U}$ it is still definable with parameters from $\mathbb{V}$. Consider the automorphism $\gamma_{|\mathbb{V}}$ of $\mathbb{V}$. By [4, Appendix, Lemma 1], we can extend this to an automorphism τ of $(\mathbb{U}, a)$, so $\tau(a) = a$. Now let $\sigma = \gamma \circ \tau^{-1}$. Then σ fixes $\mathbb{V}$ pointwise and $\sigma(a) = b$ as required. □

Let c be a tuple from $\mathbb{U}$ witnessing the internality of Q in $\mathbb{V}$.

cata **Lemma 3.3.** *There is a tuple $b \in \mathbb{V}$, a b-definable set $Q_b \subset \mathbb{V}^{eq}$ and a bc-definable bijection $f_c : Q \to Q_b$.*

Proof. First we show that there is a c-definable set $X \subset \mathbb{V}$ and a c-definable surjection $g : X \to Q$. Let $a \in Q$. Then there is a tuple d with elements from some sort S (depending on a) in $\mathbb{V}$ and $\phi_a(x, y, c)$ an L-formula with parameter c such that $\{a\}$ is defined by $\phi_a(x, d, c)$. Consider the set

$$X_a = \{d' \in S : \phi_a(x, d', c) \text{ defines a singleton from } Q\}.$$

This is nonempty as it contains d. It is $\{c\}$-definable by

$$\begin{aligned} &\forall u(\phi_a(u, x, c) \implies u \in Q) \\ \wedge \quad &\forall u \forall v((\phi_a(u, x, c) \wedge \phi_a(v, x, c)) \implies u = v), \end{aligned}$$

where x is of the sort S. Consider the function $g_a : X_a \to Q$ taking d' to the singleton element in Q that is defined by $\phi_a(x, d', c)$. One sees that it is also $\{c\}$-definable and its image $g_a(X_a)$ is thus $\{c\}$-definable. Now the set $g_a(X_a)$ contains a, and so as a ranges over all elements in Q, we see that the collection $\{g_a(X_a)\}_{a \in Q}$ is a cover of Q.

Consider the set $\{x \in Q \wedge x \notin g_a(X_a)\}_{a \in Q}$ of formulas with parameter c. This set is not realised by any tuple in $\mathbb{U}$ thus by saturation some finite subset is not realized. Hence there are $a_1, \dots, a_n \in Q$ such that Q is covered by $\{g_{a_i}(X_{a_i})\}_{1 \le i \le n}$.

Each X_{a_i} is contained in a sort from $\mathbb{V}$, we can take the product of these sorts to get another sort from $\mathbb{V}$ (since it is closed under cartesian products) and find a set X, in this new sort, that is definably isomorphic to $\coprod_i X_{a_i}$. Note that to construct the product and embed our sets into it, we need $\emptyset$-definable elements which may have to be named if not already available. This set X is c-definable and we get a c-definable surjection $g : X \to Q$ by letting $g_{|X_{a_i}} = g_{a_i}$.

From this $g : X \to Q$ we construct the desired f_c. For $a \in Q$ let $X_a \subset X$ be the fibre of g over a. Then X_a is ca-definable. Let $\phi(x, d)$ with $d \in \mathbb{U}^{eq}$ be a code for X_a. Since $X_a \subset \mathbb{V}$, by stable embeddability it is b-definable by some formula $\psi(x, b)$ for $b \in \mathbb{V}$. Now then $d \in \mathrm{dcl}(b)$, as it is defined in the variable y by

$$\forall x\, (\phi(x, y) \leftrightarrow \psi(x, b))\,,$$

thus we have $d \in \mathbb{V}^{eq}$.

Let S be the sort of $\mathbb{V}^{eq}$ containing d. Since d is also in $\mathrm{dcl}(ca)$, let $\varphi(x, a, c)$ define $\{d\}$. Consider the following c-definable subset of $Q \times S$:

$$\{(a', d') : \varphi(x, a', c) \text{ defines } \{d'\} \text{ and } \phi(x, d') \text{ is a code for } X_{a'}\}.$$

Note that for any $a' \in Q$ there is at most one possible d', since it would have to be a code with $\phi(x, d')$ for $X_{a'}$. Thus the relation is actually a c-definable function $f_a : U_a \to C_a$ for c-definable sets $U_a \subset Q$ and $C_a \subset S$ containing a and d respectively. We can safely assume that it is surjective and note that it is also injective, for if $f_a(a') = d' = f_a(a'')$ then $\phi(x, d')$ is a code for both $X_{a'}$ and $X_{a''}$. The fibres are then equal and so $a' = a''$. Thus f_a is a bijection.

Now the U_a's cover Q so as before we may assume that some finite collection $U_{a_1}, \ldots, U_{a_n}$ covers Q. Furthermore by taking appropriate boolean operations we may assume the U_{a_i}'s are disjoint. We consider the sorts that contain the C_{a_i}'s and take a product of them and find a new sort S of $\mathbb{V}^{eq}$ with $C = \coprod_i C_{a_i} \subset S$. Then by gluing the functions f_{a_i} together we have a single c-definable bijection $f_c : Q \to C$. Since $C \subset \mathbb{V}^{eq}$ is c-definable, by stable embeddablility it is b-definable for some finite tuple $b \in \mathbb{V}$ and we are done. □

Now let $f_c : Q \to Q_b$ be as in the above lemma. Since $\mathbb{V}$ is stably embedded, by [4, Appendix, Lemma 1] we have $\mathrm{tp}(c/(\mathrm{dcl}(c) \cap \mathrm{dcl}(\mathbb{V}))) \vdash \mathrm{tp}(c/\mathbb{V})$. We have $\mathrm{dcl}(\mathbb{V}) = \mathbb{V}^{eq}$. Thus by extending b, we may assume that $b = \mathrm{dcl}(c) \cap \mathbb{V}^{eq}$ so that

imply (3.1)
$$\mathrm{tp}(c/b) \vdash \mathrm{tp}(c/\mathbb{V})$$

and b is an infinite tuple of elements of $\mathbb{V}^{eq}$. Note that the cardinality of b depends on the size of the language and so can be taken to be small. The original finite tuple b we will refer to as $\bar{b}$.

auto *Remark* 3.4. Note that if $b' \in \mathbb{V}$ and $c' \in \mathbb{U}$ are tuples such that $\mathrm{tp}(bc) = \mathrm{tp}(b'c')$ then by saturation there is an automorphism σ of $\mathbb{U}$ such that $\sigma(b) = b'$ and $\sigma(c) = c'$. Then the bijection $f_c : Q \to Q_b$

discussed earlier gives rise to a bijection $f_{c'} : Q \to Q_{b'}$ through σ. If $\phi(x, y, b, c)$ defines $f_c(x) = y$ then $\phi(x, y, b', c')$ defines $f_{c'}(x) = y$. Thus for all $x \in Q$ we have $\mathbb{U} \models \phi(x, f_c(x), b, c)$ and so $\mathbb{U} \models \phi(\sigma(x), \sigma(f_c(x)), b', c')$. This gives the identity $\sigma_{|Q_b} \circ f_c = f_{c'} \circ \sigma_{|Q} = f_{\sigma(c)} \circ \sigma_{|Q}$. Sometimes we refer to $f_{c'}$ without mentioning the implicit b' if it is clear from the context.

Let $\mathrm{Ob}\mathcal{G}_{\mathbb{V}}$ be the set of solutions of $\mathrm{tp}(b)$ and let $\mathrm{Ob}\mathcal{G} = \mathrm{Ob}\mathcal{G}_{\mathbb{V}} \cup \{*\}$ where $*$ is any $\emptyset$-definable element in $\mathbb{U}$. Then $\mathrm{Ob}\mathcal{G}_{\mathbb{V}}$ is $\star$-definable since it is just the set defined by the type in infinitely many variables determined by b and $\mathrm{Ob}\mathcal{G}$ is defined by $\{\phi \vee (x = *) : \phi \in \mathrm{tp}(b)\}$. We will first define the morphisms from $*$ to all $b' \neq *$ and use this to define the set of morphisms between arbitrary elements.

For $b' \in \mathrm{Ob}\mathcal{G}$ with $b' \neq *$ one should think of the set $\mathrm{Mor}_{\mathcal{G}}(*, b')$ as consisting of those functions $f_{c'} : Q \to Q_{b'}$ where $\mathrm{tp}(bc) = \mathrm{tp}(b'c')$. To show that this is $\star$-definable first let

$$M(*, b') = \{(*, b', c') : \mathrm{tp}(bc) = \mathrm{tp}(b'c')\}.$$

Now this is a $\star$-definable set since it is essentially the realizations of $\mathrm{tp}(bc)$. We put an equivalence relation on $M(*, b')$ by saying $(*, b', c') \sim (*, b', d')$ if they induce the same functions, that is if $f_{c'} = f_{d'}$. Since $f_{c'}$ is definable by a formula in b' and c', the equivalence is definable. Then we think of $\mathrm{Mor}_{\mathcal{G}}(*, b')$ as $M(*, b')/\sim$, a $\star$-definable set in $\mathbb{U}^{eq}$. The morphisms $\mathrm{Mor}_{\mathcal{G}}(b', *)$ are treated in much the same way: as the set

$$M(b', *) = \{(b', *, c') : \mathrm{tp}(bc) = \mathrm{tp}(b'c')\},$$

modulo the appropriate equivalence. We think of $\mathrm{Mor}_{\mathcal{G}}(b', *)$ as consisting of the maps $f_{c'}^{-1} : Q_{b'} \to Q$ for $\mathrm{tp}(bc) = \mathrm{tp}(b'c')$.

Similarly we define $\mathrm{Mor}_{\mathcal{G}}(b', b'')$ for $b', b'' \neq *$. One should think of this as the set of maps of the form $f_{c''} \circ f_{c'}^{-1}$ where $\mathrm{tp}(bc) = \mathrm{tp}(b'c') = \mathrm{tp}(b''c'')$. We define it by first letting

$$M(b', b'') = M(b', *) \times M(*, b'').$$

This, as essentially seen before, is a $\star$-definable set. Then we take the equivalence relation that says $(b', *, c', *, b'', c'') \sim (b', *, d', *, b'', d'')$ if

$$f_{c''} \circ f_{c'}^{-1} = f_{d''} \circ f_{d'}^{-1},$$

which is definable. In this way we again see $\mathrm{Mor}_{\mathcal{G}}(b', b'')$ as a $\star$-definable set in $\mathbb{V}^{eq}$.

The composition in $\mathcal{G}$ should be nothing more than a reflection of composing the induced $f_{c'}$'s, $f_{c''}^{-1} \circ f_{c'}$'s or $f_{c''} \circ f_{c'}^{-1}$'s as regular functions. For example consider an element $[(b', *, c')]$ of $\mathrm{Mor}_{\mathcal{G}}(b', *)$ and

$[(*, b'', c'')]$ of $\mathrm{Mor}_{\mathcal{G}}(*, b'')$. Then define

$$[(*, b'', c'')] \circ [(b', *, c')] = [(b', *, c', *, b'', c'')],$$

which is an element of $\mathrm{Mor}_{\mathcal{G}}(b', b'')$. That this is well-defined follows from the identities imposed by the equivalence relations. Indeed assume that $[(b', *, c')] = [(b', *, d')]$ and $[(*, b'', c'')] = [(*, b'', d'')]$, which implies $f_{c'} = f_{d'}$ and $f_{c''} = f_{d''}$. Then we expect that

$$[(b', *, c', *, b'', c'')] = [(b', *, d', *, b'', d'')],$$

which is the case as $f_{c''} \circ f_{c'}^{-1} = f_{d''} \circ f_{d'}^{-1}$.

Let us now define $\mathrm{Mor}_{\mathcal{G}}(*, *)$ and its composition. One should essentially think of $\mathrm{Mor}_{\mathcal{G}}(*, *)$ as the set of functions of the form $f_{c''}^{-1} \circ f_{c'}$. The definition of composition in $\mathrm{Mor}_{\mathcal{G}}(*, *)$ also serves as a template for defining composition in the remaining cases in $\mathcal{G}$.

To define $\mathrm{Mor}_{\mathcal{G}}(*, *)$ let

$$M(*, *) = \coprod_{b'} M(*, b') \times M(b', *),$$

where b' ranges over elements satisfying $\mathrm{tp}(b') = \mathrm{tp}(b)$. This is the set of tuples $(*, b', c', b', *, c'')$ with $\mathrm{tp}(bc) = \mathrm{tp}(b'c') = \mathrm{tp}(b'c'')$. We mod out by the equivalence $(c', c'') \sim (d', d'')$ if

$$f_{c''}^{-1} \circ f_{c'} = f_{d''}^{-1} \circ f_{d'}.$$

Note that $M(*, *)$ is $\star$-definable as it consists of pairs $(b'c'), (b'c'')$ of realizations of $\mathrm{tp}(bc)$. The equivalence, as before, is definable. Thus we get that $\mathrm{Mor}_{\mathcal{G}}(*, *)$ is a $\star$-definable set in $\mathbb{U}^{eq}$.

Let us consider composition in the case of $\mathrm{Mor}_{\mathcal{G}}(*, *)$. Define a ternary relation R on $M(*, *)$ as follows. Let

$$(*, b', d', b', *, d''), (*, b'', c', b'', *, c''), (*, b''', e', b''', *, e'') \in R,$$

if

$$f_{d''}^{-1} \circ f_{d'} \circ f_{c''}^{-1} \circ f_{c'} = f_{e''}^{-1} \circ f_{e'}.$$

This is not a binary operation on $M(*, *)$ since there can be many candidates (e', e'') for the right hand side. However once we mod out by the equivalence relation it becomes a binary operation on $\mathrm{Mor}_{\mathcal{G}}(*, *)$. To be specific, modding out by the equivalence $\sim$ means to create a ternary relation $R/\sim$ on $\mathrm{Mor}_{\mathcal{G}}(*, *)$ where a triple $([u], [v], [w])$ is in $R/\sim$ if (u, v, w) is in R. One can see that this is a well-defined condition.

Let us see how each element of $\mathrm{Mor}_{\mathcal{G}}(*, *)$ induces an element of $\mathrm{Aut}(Q/\mathbb{V})$. An element of $\mathrm{Mor}_{\mathcal{G}}(*, *)$ is an equivalence class

$$[(*, b', c', *, b', c'')],$$

representing a function of the form $f_{c''}^{-1} \circ f_{c'}$ where $\mathrm{tp}(bc) = \mathrm{tp}(b'c') = \mathrm{tp}(b'c'')$.

Claim 3.5. *The function*

$$h : \mathrm{Mor}_{\mathcal{G}}(*, *) \to \mathrm{Aut}(Q/\mathbb{V}) : [(*, b', c', *, b', c'')] \mapsto f_{c''}^{-1} \circ f_{c'},$$

is a composition preserving bijection from $\mathrm{Mor}_{\mathcal{G}}(*, *)$ *to* $\mathrm{Aut}(Q/\mathbb{V})$.

Note that through this bijection $\mathrm{Mor}_{\mathcal{G}}(*, *)$ inherits the obvious action on Q. The bijection is essentially the restriction to $\mathrm{Mor}_{\mathcal{G}}$ of the as yet to be defined functor $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$.

Proof. First let us show that $f_{c''}^{-1} \circ f_{c'} \in \mathrm{Aut}(Q/\mathbb{V})$. Equation 3.1 says that $\mathrm{tp}(c/b) \vdash \mathrm{tp}(c/\mathbb{V})$ and since $\mathrm{tp}(bc) = \mathrm{tp}(b'c') = \mathrm{tp}(b'c'')$ we get that $\mathrm{tp}(c'/\mathbb{V}) = \mathrm{tp}(c''/\mathbb{V})$. Then by Lemma 3.2 there is $\sigma \in \mathrm{Aut}(\mathbb{U}/\mathbb{V})$ such that $\sigma(c') = c''$. The last part of Remark 3.4 essentially gives us that

$$\sigma_{|Q_{b'}} \circ f_{c'} = f_{c''} \circ \sigma_{|Q}.$$

Note that σ extends to $\mathbb{V}^{eq}$ in a unique way and since it fixes $\mathbb{V}$ pointwise it fixes $\mathbb{V}^{eq}$ and in particular $Q_{b'}$ pointwise. Thus we get

$$f_{c'} = f_{c''} \circ \sigma_{|Q},$$

from which it follows that $f_{c''}^{-1} \circ f_{c'} \in \mathrm{Aut}(Q/\mathbb{V})$.

By the nature of the equivalence on $M(*, *)$, the map h is indeed a function and does not depend on the representative of the equivalence class. Thus the map is well-defined. One can check also that it preserves composition. Also from the equivalence we see it is injective. It remains only to show it is surjective. If we have $\sigma \in \mathrm{Aut}(Q/\mathbb{V})$, then σ fixes $\mathbb{V}$ and $\mathbb{V}^{eq}$, thus we have $\sigma_{|Q_b} = \mathrm{Id}$ so by Remark 3.4 we have $f_{\sigma(c)} \circ \sigma_{|Q} = f_c$. Hence, $\sigma_{|Q} = f_{\sigma(c)}^{-1} \circ f_c$, proving surjectivity of h. ☐

Now via h, we get an action of $\mathrm{Mor}_{\mathcal{G}}(*, *)$ on Q. Indeed for $q \in Q$ and we have

$$[(*, b', c', *, b', c'')] \cdot q = f_{c''}^{-1} \circ f_{c'}(q).$$

This action is definable. Indeed consider the type definable subset R of $M(*, *) \times Q^2$ containing tuples

$$(*, b', c', *, b', c''), q, f_{c''}^{-1}(f_{c'}(q)),$$

where $\mathrm{tp}(bc) = \mathrm{tp}(b'c') = \mathrm{tp}(b'c'')$ and $q \in Q$. Consider the equivalence relation on this set by taking $\sim$ from before on the first coordinate, equality on the second and third coordinates. Then $R/\sim$ gives the graph of the action of $\mathrm{Mor}_{\mathcal{G}}(*, *)$ on Q.

The rest of the cases for defining composition in $\mathcal{G}$ are handled in similar ways as presented above.

The functor $F : \mathcal{G} \to \mathrm{Def}(\mathbb{U})$ is defined by $F(*) = Q$ and $F(b') = Q_{b'}$ on objects and it takes any morphism to the corresponding function $f_{c'}$, $f_{c''}^{-1} \circ f_{c'}$ or $f_{c''} \circ f_{c'}^{-1}$.

4. Generalized Imaginaries

Let T' be an extension of the complete theory T in a language containing the language of T and one additional sort S. Every model M' of T' is of the form $(M, S_{M'})$ where M is a model of T and $S_{M'}$ is the domain of the sort S. Note that if M' is a universal domain for T' then is M one for T.

equiv **Proposition 4.1.** *Assume that $M' = (M, S_{M'})$ is a model of T'. The following are equivalent:*

1. $S_{M'} \subseteq \mathrm{dcl}(M)$
2. *The restriction map* $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ *is an isomorphism.*
3. *S is a sort of M^{eq}.*

First we need a lemma:

nostruc **Lemma 4.2.** *The set M is stably embedded in M' iff M' induces no new structure on M. That is, if $X \subseteq M^n$ is definable in M' (with parameters in M'), then it is definable in M (with parameters in M).*

Proof. Clearly if M' induces no new structure then M is stably embedded. Assume that M is stably embedded. First we show that for any small $A \subset M$ and $a \in M$ we have

$$\mathrm{tp}_M(a/A) \vdash \mathrm{tp}_{M'}(a/A).$$

Indeed if $b \in M$ is a realization of $\mathrm{tp}_M(a/A)$ then there is an automorphism σ of M that fixes A pointwise and $\sigma(a) = b$. By [4, Appendix, Lemma 1] σ extends to an automorphism σ of M'. Clearly then σ witnesses the fact that $\mathrm{tp}_{M'}(b/A) = \mathrm{tp}_{M'}(a/A)$ as desired.

Now we show that M' induces no new structure on M. Assume $X \subset M^n$ is M'-definable via the formula $\phi(x, m)$. By stable embeddedness of M we may assume $m \in M$. Let

$$\Gamma = \{\psi(x, m) \in L(M) : M' \models \forall x(\phi(x, m) \to \psi(x, m))\},$$

where $\psi \in L(M)$ means ψ is a formula in the language of the sorts of M. Consider the set $\Gamma(x, m) \cup \{\neg\phi(x, m)\}$. Assume it is realized by some tuple $b \in M'$. We show that $\mathrm{tp}_M(b/m) \cup \{\phi(x, m)\}$ has a realization. Assume not, then some finite subset has no realization. By taking conjunctions we may assume that $\varphi(x, m) \wedge \phi(x, m)$ has no realization for some $\varphi \in \mathrm{tp}_M(b/m)$. Then $M' \models \forall x(\phi(x, m) \to \neg\varphi(x, m))$. Thus $\neg\varphi \in \Gamma \subset \mathrm{tp}_M(b/m)$. But this is a contradiction since $\varphi \in \mathrm{tp}_M(b/m)$.

Thus there must be $a \in M$ such that a realizes $\mathrm{tp}_M(b/m) \cup \{\phi(x,m)\}$. By the discussion at the beginning of the proof, we must have that a realizes $\mathrm{tp}_{M'}(b/m)$, but this includes $\neg\phi(x,m)$ which contradicts the fact that a realizes $\phi(x,m)$.

Thus $\Gamma(x,m) \cup \{\neg\phi(x,m)\}$ is not realized in M' and by saturation some finite subset is not realized in M'. Let $\Sigma \subset \Gamma$ be finite such that $\Sigma(x,m) \cup \{\neg\phi(x,m)\}$ is not realized in M'. Let $\varphi(x,m)$ be the conjunction of all formulas in Σ. Then

$$M' \models \forall x(\varphi(x,m) \to \phi(x,m)),$$

and based on the definition of Γ we see that in fact

$$M' \models \forall x(\varphi(x,m) \leftrightarrow \phi(x,m)).$$

Thus X is defined by φ and so M' induces no new structure on M. □

Now we can proceed with the proof of Proposition 4.1.

Proof. ($1 \implies 2$) Note that the restriction map is a group homomorphism. First we show it is one to one. Assume that $\sigma_{|M} = \mathrm{Id}_M$. Let $a \in S_{M'}$. By assumption there is a formula $\phi(x,m)$ with $m \in M$ defining a. Then $\sigma(a)$ is defined by $\phi(x,\sigma(m)) = \phi(x,m)$ so $\sigma(a) = a$ and $\sigma = \mathrm{Id}_{M'}$. To show that the map is surjective note that by [4, Appendix, Lemma 1] it is enough to show that M is stably embedded in M'. Assume $X \subset M$ is defined by a formula $\phi(x,m,a)$ for $m \in M$ and $a \in S_{M'}$. Then by assumption each co-ordinate of a is defined by a formula with parameters from M. From this one sees that we may define X using only parameters from M and so M is stably embedded.

($2 \implies 1$) Since the restriction map is surjective, every automorphism of M extends to one of M', so by [4, Appendix, Lemma 1] M is stably embeddeded in M'. Now let $a \in S_{M'}$. Again by [4, Appendix, Lemma 1] there is a small set $M_0 \subset M$ such that $\mathrm{tp}(a/M_0) \vdash \mathrm{tp}(a/M)$. If b is a realization of $\mathrm{tp}(a/M_0)$ then $\mathrm{tp}(b/M_0) = \mathrm{tp}(a/M_0)$ and so $\mathrm{tp}(b/M) = \mathrm{tp}(a/M)$. Thus by Lemma 3.2 there is an automorphism σ of M' fixing M pointwise and $\sigma(a) = b$. But since the restriction map is injective, we must have that σ is the identity on M' so $a = b$. Thus by saturation $a \in \mathrm{dcl}(M_0) \subseteq \mathrm{dcl}(M)$.

($3 \implies 1$) An element of $S_{M'}$ is of the form a/E for some $a \in M$ and $E(x,y)$ a $\emptyset$-definable equivalence relation on M. Then a/E is defined by the a-formula in y that says there is $x \in M$ such that $E(x,a)$ and $f_E(x) = y$ where f_E is the natural projection $M \to S_{M'} = M/E$.

($1 \implies 3$) Using a similar argument to that in Lemma 3.3, we get a $\emptyset$-definable surjection $f : X \to S_{M'}$ where $X \subset M^n$ is a $\emptyset$-definable set. The surjection is in the language of M'. The induced equivalence relation E on X is $\emptyset$-definable and it is a subset of M^{2n}. Since $1 \implies 2$, the restriction map $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ is surjective so M is stably embedded in M'. By Lemma 4.2 E can be defined by a formula in the language of M. Thus the natural projection $f_E : X \to X/E$ which is defined entirely in M, has the same fibres as f and through this we may identify $S_{M'}$ with X/E. □

The above result gives us some alternate characterizations of a sort in M^{eq}. As a generalization, say that the sort S is a *finite generalized imaginary* sort, if the restriction map $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ has finite kernel and is surjective. Note that the kernel is the binding group of S. The following extends Proposition 4.1:

fingen **Proposition 4.3.** *Assume that M is stably embedded. Then the following are equivalent:*

(1) $S \subseteq \mathrm{dcl}(aM)$ *where* $a \in \mathrm{acl}(M)$.
(2) $S_{M'}$ *is a finite generalized imaginary sort.*

Proof. ($1 \implies 2$) We must show that the kernel of the restriction map is finite. Assume $\sigma \in \mathrm{Aut}_M(M')$. Note that since $S \subseteq \mathrm{dcl}(aM)$, the map σ depends entirely on its action on a, that is, if $\tau \in \mathrm{Aut}_M(M')$ and $\sigma(a) = \tau(a)$ then $\sigma = \tau$. Since $a \in \mathrm{acl}(M)$, there only finitely choices for $\sigma(a)$ and thus only finitely many such maps σ.

($2 \implies 1$) It suffices to find a tuple $a \in M'$ such that $S \subseteq \mathrm{dcl}(aM)$. Indeed by stable embeddability, $\mathrm{tp}(a/M)^{M'}$ is the orbit of a under $\mathrm{Aut}(S/M)$ which is finite by (2). So $\mathrm{tp}(a/M)^{M'}$ is finite and by a stable embeddedness argument, $a \in \mathrm{acl}(M)$.

Assume there is a finite tuple $a \in S$ such that for every $\sigma \in \mathrm{Aut}(S/M)$ if $\sigma(a) = a$ then $\sigma = \mathrm{Id}_S$. We show that then $S \subseteq \mathrm{dcl}(aM)$. Indeed let $b \in S$. By [4, Appendix, Lemma 1] there is a small $M_0 \subset M$ such that $\mathrm{tp}(a/M_0) \vdash \mathrm{tp}(a/M)$ and $\mathrm{tp}(b/M_0) \vdash \mathrm{tp}(b/M)$. We show that $b \in \mathrm{dcl}(aM_0)$. Indeed let $f \in \mathrm{Aut}_{aM_0}(M')$. Since f fixes aM_0 we have $\mathrm{tp}(f(b)/aM_0) = \mathrm{tp}(b/aM_0)$ and thus $\mathrm{tp}(f(b)/aM) = \mathrm{tp}(b/aM)$. By a slight variant of the proof of Lemma 3.2 there is $\sigma \in \mathrm{Aut}_{aM}(M')$ such that $\sigma(b) = f(b)$. But by our initial assumption since σ fixes both a and M pointwise we must have $\sigma(b) = b$ and so $f(b) = b$.

To construct the tuple a, consider an enumeration $f_0, \ldots, f_n$ of $\mathrm{Aut}(S/M)$ with $f_0 = \mathrm{Id}_S$. For $1 \leq i \leq n$ let $a_i \in S$ be such that $f_i(a_i) \neq a_i$. Then $a = a_1 a_2 \ldots a_n$ satisfies our condition since if $f \in \mathrm{Aut}(S/M)$ is such that $f(a) = a$ then f can only be the identity. □

A finite generalized imaginary sort is a special case of the more general notion of an *internal generalized imaginary sort.* An internal generalized imaginary sort S is a sort where we require the kernel $\mathrm{Aut}(S/M)$

of the restriction map $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ to have small cardinality. The following is an extension of Proposition 4.3 to internal generalized imaginaries based on [5, Proposition 15].

Proposition 4.4. *Suppose M is stably embedded in M'. Then the following are equivalent:*

(1) *S is internal to M.*
(2) *S is an internal generalized imaginary sort.*
(3) *The binding group $\mathrm{Aut}(S/M)$ with its action on S is definable in M'.*

Proof. ($1 \implies 3$) This is the content of Proposition 3.1.

($3 \implies 2$) If the binding group is definable then it can be assumed small.

($2 \implies 1$) We proceed essentially as in the proof of Proposition 4.3 by finding a small set A such that for every $f \in \mathrm{Aut}(S/M)$, if f fixes A pointwise then $f_{|S} = \mathrm{Id}_S$. Then we show that $S \subseteq \mathrm{dcl}(AM)$ and we finish by showing that one can then find a finite tuple $a \subset A$ such that $S \subseteq \mathrm{dcl}(aM)$.

Let $\{f_\alpha\}_{\alpha<\kappa}$ be an enumeration of $\mathrm{Aut}(S/M)$ with $f_0 = \mathrm{Id}_S$. Choose as before $a_\alpha \in S$ such that $f_\alpha(a_\alpha) \neq a_\alpha$ for $0 < \alpha < \kappa$. Let A be the set of all a_α's.

First we show that $S \subseteq \mathrm{dcl}(AM)$. Let $s \in S$. Then by stable embeddedness there is a small $M_0 \subset M$ such that $\mathrm{tp}(s/M_0) \vdash \mathrm{tp}(s/M)$. We show that $s \in \mathrm{dcl}(AM_0)$. Assume $f \in \mathrm{Aut}_{AM_0}(M')$. Then $\mathrm{tp}(s/M_0) = \mathrm{tp}(f(s)/M_0)$ so $\mathrm{tp}(s/M) = \mathrm{tp}(f(s)/M)$. Now add all of A as constant symbols to M'. Then M is still stably embedded in this new structure. Since f fixes A pointwise one sees that it is an automorphism of this new structure as well. Thus $\mathrm{tp}(s/M) = \mathrm{tp}(f(s)/M)$ holds in the new structure. By stable embeddedness there is an automorphism σ of M' fixing M and A pointwise such that $\sigma(s) = f(s)$. But by construction of A we must have that σ is the identity and so $f(s) = \sigma(s) = s$ as required.

Now using an argument similar to that in Lemma 3.3 we can find an A-definable surjection from a power of M to S. Since this formula will use only finitely many elements from A, we can find a finite tuple $a \in A$ such that $S \subseteq \mathrm{dcl}(aM)$ as required. □

5. Finite Internal Covers

We are now in a position to define a *finite internal cover*. Let N be a structure and M a union of some of the sorts in N. We say N is a finite internal cover of M if M is stably embedded in N and $\mathrm{Aut}(N/M)$ is finite.

Assume $\mathcal{G}_1$ and $\mathcal{G}_2$ are concrete definable categories in $\mathbb{U}$. We have previously defined a definable functor from a definable category to $\mathrm{Def}(\mathbb{U})$. A functor $F : \mathcal{G}_1 \to \mathcal{G}_2$ between concrete definable categories will be called definable if the functor $\delta_2 \circ F : \mathcal{G}_1 \to \mathrm{Def}(\mathbb{U})$ is a definable functor.

6. Appendix

The following is used in Example 2.1.

We need some facts about M^{eq} given an L-structure M. Note that if $\phi(\bar{x})$ is an L-formula and $\bar{a} \in M$ then $M \models \phi(\bar{a})$ iff $M^{eq} \models \phi(\bar{a})$. We need also the following fact:

reduce **Proposition 6.1.** *Let $\phi(x_1, \ldots, x_k)$ be an L^{eq}-formula where x_i is of the sort S_{E_i}. Then there is an L-formula $\psi(y_1, \ldots, y_k)$ such that $M^{eq} \models \psi(y_1, \ldots, y_k) \leftrightarrow \phi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k))$.*

Proof. The proof is by induction on the complexity of ϕ. First we need to show the following claim about terms. If $t(x_1, \ldots, x_k)$ is an L^{eq}-term with x_i of sort S_{E_i} and the term maps to the sort S_E, then there is an L-term $t'(y_1, \ldots, y_k)$ such that $t(f_{E_1}(y_1), \ldots f_{E_k}(y_k)) = f_E(t'(y_1, \ldots, y_k))$. Clearly this holds if t is a constant, since then it is an L-term already. If $t(x_1, \ldots, x_k) = x_i$ then take $t'(y_1, \ldots, y_k) = y_i$. By assumption $E = E_i$ and so the result holds in this case. We also must check the function case so assume g is an n-ary function symbol in L. Then its input variables each have sort $S_=$ so consider for $1 \leq j \leq n$ terms $t_j(x_1, \ldots, x_k)$ which map to the sort $S_=$ and x_i is of sort S_{E_i}. Then by induction there are L-terms $t'_j(y_1, \ldots, y_k)$ satisfying $t_j(f_{E_i}(y_1), \ldots, f_{E_k}(y_k)) = t'_j(y_1, \ldots, y_k)$. Thus

$$\begin{aligned} g(\ldots, t_j(f_{E_i}(y_1), \ldots, f_{E_k}(y_k)), \ldots) &= g(\ldots, t'_j(y_1, \ldots, y_k), \ldots) \\ &= f_=(g(\ldots, t'_j(y_1, \ldots, y_k), \ldots)), \end{aligned}$$

so $g(\ldots, t'_j(y_1, \ldots, y_k), \ldots)$ is the desired L-term.

Now we can proceed with the proof. We illustrate just the case where ϕ is the equality of two L^{eq}-terms:

$$t_1(x_1, \ldots, x_k) = t_2(x_1, \ldots, x_k),$$

where the terms are of sort S_E and x_i is of sort S_{E_i}. Let $t'_j(y_1, \ldots, y_k)$ for $j = 1, 2$ be as described above. Then $M^{eq} \models \phi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k))$ iff $f_E(t'_1(y_1, \ldots, y_k)) = f_E(t'_2(y_1, \ldots, y_k))$, that is, iff

$$M \models E(t'_1(y_1, \ldots, y_k), t'_2(y_1, \ldots, y_k)),$$

and iff

$$M^{eq} \models E(t'_1(y_1, \ldots, y_k), t'_2(y_1, \ldots, y_k)),$$

where $E(x, y)$ is the L-formula defining the equivalence relation E. Thus we can take ψ to be $E(t_1'(y_1, \ldots, y_k), t_2'(y_1, \ldots, y_k))$. □

Suppose $A \subseteq M^k$ and $A' \subset M^l$ are definable sets defined by $\phi(\bar{x})$ and $\phi'(\bar{u})$ respectively. Also let $E(\bar{x}, \bar{y})$ and $E'(\bar{u}, \bar{v})$ be definable equivalence relations on M^k and M^l respectively. We have that $A/E \subseteq M^k/E$ and $A'/E' \subseteq M^l/E'$. Indeed A/E and A'/E' are definable sets in M^{eq} defined by

$$\exists \bar{x}(\bar{y} = f_E(\bar{x}) \wedge \phi(\bar{x})),$$

and

$$\exists \bar{u}(\bar{v} = f_{E'}(\bar{u}) \wedge \phi'(\bar{u})),$$

respectively in the variables $\bar{y}$ and $\bar{v}$. In these formulas $\bar{x}$ and $\bar{u}$ are of the sort $S_=$, that is to say each of the k and l variables are of this sort. Note that the expression $f_E(\bar{x})$ means that f_E is applied to each component of $\bar{x}$. If we wish to show that some map between A/E and A'/E' is definable, we may do so in M^{eq} but there is a condition that one may check strictly in M.

pull **Proposition 6.2.** *Let A, A' be definable sets and E, E' definable equivalence relations on them as above and assume $g : A/E \to A'/E'$ is any map. Then g is a definable map in M^{eq} iff there is a definable set $S \subseteq A \times A'$ in M such that $(r, s) \in S$ iff $g(r/E) = s/E'$.*

Proof. First assume that g is definable in M^{eq}. Let $\varphi(\bar{x}_1, \bar{x}_2)$ define g where $\bar{x}_1$ has sort S_E and $\bar{x}_2$ has sort $S_{E'}$. Then by Proposition 6.1 there is an L-formula $\psi(\bar{y}_1, \bar{y}_2)$ such that $M^{eq} \models \psi(\bar{y}_1, \bar{y}_2) \leftrightarrow \varphi(f_E(\bar{y}_1), f_{E'}(\bar{y}_2))$. Then $\psi(\bar{y}_1, \bar{y}_2)$ defines the appropriate set $S \subseteq A \times A'$.

Conversely assume that $S \subseteq A \times A'$ is definable and satisfies the condition. Then we can define g by $g(x) = y$ for $x \in A/E$ and $y \in A'/E'$ iff there is $(r, s) \in S$ such that $x = r/E$ and $y = s/E'$. □

As a special case, if E' is equality and we wish to check that $g : A/E \to A'$ is a definable isomorphism in M^{eq}, we need only check that the map $g \circ f_E$ is definable in M and that g is bijective.

References

GK [1] J. Goodrick and A. Kolesnikov, Groupoids, Covers, and 3-Uniqueness in Stable Theories, *http://www.math.umd.edu/ goodrick/research.html* 31 Oct 2008.

H [2] E. Hrushovski, Groupoids, Imaginaries and Internal Covers, *arXiv: math/0603413 v1* 16 Mar 2006.

H2 [3] E. Hrushovski, Notes for Model Theory 2, http://math.huji.ac.il/ ehud/MT2/

CH [4] Z. Chatzidakis, E. Hrushovski, Model Theory of Difference Fields, Trans. Am. Math. Soc. 351 No. 8, 2997-3071, 8 April 1999.

K [5] M. Kamensky, Definable groups of partial automorphisms, *arXiv: math/0607718 v3* 15 Mar 2009.

M [6] R. Moosa, Some Elementary Facts About M^{eq}.