THE MODEL THEORY OF COMPACT COMPLEX SPACES

RAHIM N. MOOSA†

The usual model-theoretic approach to complex algebraic geometry is to view complex algebraic varieties as living definably in the structure $(\mathbb{C}, +, \times)$. Various model-theoretic properties of algebraically closed fields (such as quantifier elimination and strong minimality) are then used to obtain geometric information about the varieties. This approach extends to other geometric contexts by considering expansions of algebraically closed fields to which the methods of stability or simplicity apply. For example, differential algebraic varieties live in differentially closed fields, and difference algebraic varieties in algebraically closed fields equipped with a generic automorphism. Another approach would be to consider the variety as a structure in its own right, equipped with the algebraic (respectively differential or difference algebraic) subsets of its cartesian powers. This point of view is compatible with the theory of Zariski-type structures, developed by Hrushovski and Zilber (see [21] and [41]). While the two approaches are equivalent (i.e., bi-interpretable) in the case of complex algebraic varieties, the latter point of view extends to certain fragments of complex *analytic* geometry in a manner that does not seem accessible by the former.

Zilber showed in [41] that a compact complex analytic space with the structure induced by the analytic subsets of its cartesian powers is of finite Morley rank and admits quantifier elimination. Since then, there have been a number of papers investigating various aspects of the model theory of such structures, as well as possible applications to complex analytic geometry. The most notable achievement has been a kind of Chevalley Theorem due to Pillay and Scanlon [35] that classifies meromorphic groups in terms of complex tori and linear algebraic groups. It has become clear that compact complex spaces serve as a particularly rich setting in which many of the more advanced phenomena of stability theory are witnessed. For example, the full trichotomy for strongly minimal sets (trivial, not trivial but locally modular, and not locally modular) occurs in this category. It also seems reasonable to expect that a greater model-theoretic understanding of compact complex spaces can contribute to complex analytic geometry, particularly around issues concerning the bimeromorphic classification of Kähler-type spaces.

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In this article, I hope to give both an introduction to, and a survey of, the model theory of compact complex spaces. Notions from complex analytic geometry that are of particular interest will be explained in some detail, and this is partially responsible for the length of the article. Formal proofs will generally be omitted, but I will make some attempt to motivate the results and provide information about the techniques employed in obtaining them. While most of the material I have included has already appeared (or is to appear) elsewhere, I have also taken this oppurtunity to describe some of the results obtained in my thesis [25] which is still being prepared for publication.[1]

Here are some of the topics that are covered. I begin with a brief introduction to the basic objects of study in Section 1, followed by a summary of the first results in the model theory of compact complex spaces (Section 2). In Section 3, I discuss how the work of Hrushovski and Zilber on Zariski geometries applies to this category in order to establish a dichotomy for strongly minimal sets in terms of their relationship to projective space.[2] The notion of a meromorphic group and the classification theorem of Pillay and Scanlon are presented in Section 4, together with the results of Kowalski and Pillay on the socle of a commutative meromorphic group. A somewhat detailed discussion of Douady spaces and their relationship to issues of saturation for compact complex spaces, as it appears in my thesis, is given in Section 5. In Section 6, I explain how the model theory of elementary extensions of a compact complex space can be related to various notions about families of analytic sets from complex analytic geometry. Finally, in Section 7, I give an analogue of the Riemann Existence Theorem for elementary extensions (also from my thesis). Several examples and open questions are dicussed along the way.

The emphasis given to the various topics in this article may very well have more to do with my own knowledge of them rather than their relative worth. Moreover, this is an active area of research, and I have probably not discussed all the work that has been done. One obvious omission is the work of Peterzil and Starchenko on complex analysis in o-minimal structures ([27] and [28]), whereby—despite my comments at the beginning of this introduction—complex analytic spaces can be viewed as living in an ambient enriched field ($\mathbb{R}_{an}$), though outside the stable/simple context.

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[1] Please see *Note added in proof* at the end of the article.

[2] Recent work of Pillay [32], based on results of Campana [2] and Fujiki [11], gives a direct proof of the dichotomy that does not involve Zariski geometries. I have included a brief appendix on these methods.

§1. Complex analytic spaces. In this section I will introduce complex analytic spaces and some of the basic notions associated with them. While I will sometimes use the language of sheaves and ringed spaces, only a rudimentary understanding of these objects will be assumed. More detailed introductions to the theory of complex analytic spaces can be found in [16] (for a classical treatment) and [38] (for a modern treatment).

Just as algebraic geometry is essentially concerned with the zero sets of polynomials, the fundamental objects of study in complex analytic geometry are controlled by the zero sets of holomorphic functions. Let D be a domain in $\mathbb{C}^n$ (some $n \geq 1$), and suppose that $f_1, f_2, \dots, f_m$ are holomorphic functions on D. Then the set of common zeros, $V = V(f_1, \dots, f_m)$, is called an *analytic set in D*. If we let $\mathcal{O}_D$ denote the sheaf of germs of holomorphic functions on D, then the quotient of $\mathcal{O}_D$ by the ideal sheaf generated by $f_1, \dots, f_m$ equips V with a structure sheaf. Note that this sheaf may not be reduced—it may contain nilpotent elements. However, if we let $\mathcal{I}_V \subset \mathcal{O}_D$ denote the ideal sheaf of *all* holomorphic functions vanishing on V, then the quotient, $\mathcal{O}_V = \mathcal{O}_D/\mathcal{I}_V$, gives rise to a reduced sheaf structure on V. Each section of $\mathcal{O}_V$ can be naturally identified with a unique continuous $\mathbb{C}$-valued function on V, and is called a *holomorphic function* on V. If W is another analytic set in some domain, then a *holomorphic map* (respectively *biholomorphic map*) from V to W is a morphism (respectively isomorphism) between the ringed spaces $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$.

Complex analytic spaces are ringed spaces that are locally modeled after analytic sets in domains of $\mathbb{C}^n$ (various n). This can be expressed as follows: a second countable Hausdorff topological space X is a (reduced) *complex analytic space* if there exists an open cover $\{X_\alpha\}$ of X, such that for each α there is a homeomorphism $\phi_\alpha \colon X_\alpha \to V_\alpha$, where V_α is an analytic set in a domain of $\mathbb{C}^{n_\alpha}$; and such that for all α and β for which $X_\alpha \cap X_\beta \neq \emptyset$, the induced transition function

$$\phi_\beta \phi_\alpha^{-1} \colon \phi_\alpha(X_\alpha \cap X_\beta) \to \phi_\beta(X_\alpha \cap X_\beta)$$

is a biholomorphic map between analytic sets in domains of $\mathbb{C}^{n_\alpha}$ and $\mathbb{C}^{n_\beta}$ respectively. We obtain a reduced sheaf structure on X by pulling back the $\mathcal{O}_{V_\alpha}$'s and using the transition functions to glue them together. This structure sheaf is denoted by $\mathcal{O}_X$, and its sections are the *holomorphic functions* on X. If we allowed nonreduced sheaf structures on the V_α's (as described in the above paragraph), then we would obtain the general notion of a complex analytic space (i.e., not necessarily reduced). However, since the model-theoretic perspective is essentially set-theoretic, and not sheaf-theoretic, I will usually only consider reduced complex analytic spaces and deal explicitly with nonreduced spaces as they appear. Again, if X and Y are complex analytic spaces, then a *holomorphic map* (respectively *biholomorphic map* or *isomorphism*) between X and Y is a morphism (respectively isomorphism) of the ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

Suppose X is a complex analytic space, and $x \in X$. The (complex) *dimension of X at x*, denoted $\dim_x X$, is the least $d \geq 0$ such that there exists a finite-to-one holomorphic map $f: U \to D$, where U is a neighbourhood around x in X, and D is a domain in $\mathbb{C}^d$. If X is connected then $\dim_x X$ is constant for all $x \in X$, and we let the *dimension of X* (denoted by $\dim X$) be this quantity. We say that X is *smooth at x* if there is a neighbourhood about x that is biholomorphic with a domain in $\mathbb{C}^d$, in which case $d = \dim_x X$. A *complex manifold* can then be described as a complex analytic space that is smooth at every point.

Given a complex analytic space X, we will be interested in those subsets of X and its cartesian powers (which are again complex analytic spaces) that are given locally by holomorphic functions. A subset $A \subset X$ is *analytic* if for all $x \in X$ there is a neighbourhood U of x in X, and finitely many holomorphic functions on U, $f_1, \ldots, f_m$, such that $A \cap U$ is the set of common zeros of $f_1, \ldots, f_m$ in U. Note that an analytic subset of X is closed and inherits from X the structure of a complex analytic space in its own right. The analytic subsets form the closed sets of another topology on X that is coarser than the underlying complex topolgy. I will refer to this as the *Zariski topology*, and use the terms "Zariski closed set" and "analytic set" interchangeably. We say that X is *irreducible* to mean that it is irreducible in the Zariski topology; it cannot be written as the union of two proper analytic subsets. In fact, if X is irreducible and $A \subset X$ is a proper analytic subset, then A is nowhere dense. If X is irreducible and P is a property of points in X, then I will say that *P holds for general $x \in X$* if it holds in some nonempty Zariski open subset.

Suppose X and Y are irreducible complex analytic spaces. A holomorphic map, $f: X \to Y$, is a *modification* if it is proper, surjective, and there exist proper analytic subsets $A \subset X$ and $B \subset Y$ such that f restricts to a biholomorphic map from $(X \setminus A)$ to $(Y \setminus B)$. By a *meromorphic map* from X to Y (written $g: X \to Y$) I will mean a multivalued map whose graph, $\Gamma(g) \subset X \times Y$ is an irreducible analytic set, such that the first coordinate projection map $\Gamma(g) \to X$ is a modification. Off a proper analytic subset of X, $\Gamma(g)$ is the graph of a well-defined holomorphic map to Y. Note that a meromorphic map $g: X \to Y$ is a holomorphic map (everywhere) exactly when $\Gamma(g) \to X$ is an isomorphism. For any $y \in Y$, X_y will denote the set-theoretic fibre of g above y—that is, the analytic set given by $\{x \in X : (x, y) \in \Gamma(g)\}$. In the case when g is holomorphic, this is just the pre-image $g^{-1}(y)$. If the second coordinate projection $\Gamma(g) \to Y$ is surjective, then the meromorphic map g is called *surjective*. If $\Gamma(g) \to Y$ is also a modification, then $g: X \to Y$ is called a *bimeromorphic map*. (Bi)meromorphic maps are the analogue in complex analytic geometry of (bi)rational maps in algebraic geometry.

For ease of notation, by a *complex variety* I will mean a reduced irreducible complex analytic space.[3] For much of this article I will restrict my attention to *compact* complex varieties. Part of what makes compact complex varieties accessible to model-theoretic analysis is that in this case the Zariski topology is rather well behaved. If X is a compact complex variety then every Zariski closed set in X can be written uniquely as an irredundant union of finitely many irreducible Zariski closed subsets, which are called its *irreducible components*. We can then define the *dimension* of a Zariski closed set in X to be the maximum of the dimensions of its irreducible components. Moreover, if $A \subset X$ is a complex subvariety and $B \subset A$ is a Zariski closed set, then $\dim B = \dim A$ if and only if $B = A$. This yields a descending chain condition on Zariski closed sets, showing that the Zariski topology on X is noetherian. It would not be too inaccurate to describe the model-theoretic point of view as being that of ignoring the underlying complex topology on X and considering only the structure induced by the Zariski topology.

I will conclude this section with a few familiar (and not so familiar) examples of compact complex varieties. First of all, consider the algebraic case. I will use $\mathbb{P}_n(\mathbb{C})$, or just $\mathbb{P}_n$, to denote projective n-space viewed as a compact complex variety. Chow's Theorem states that every analytic set in projective space is algebraic (i.e., given by homogeneous polynomials). Hence, for projective space, the analytic Zariski topology defined above coincides with the usual algebraic Zariski topology. A *projective variety* is a compact complex variety that is biholomorphic to a closed subvariety of $\mathbb{P}_n$, for some $n \geq 0$.

More generally, a *Moishezon variety* is a compact complex variety that is bimeromorphic to a projective variety. In classifying compact complex varieties one is often only interested in bimeromorphic equivalence classes. From this point of view, Moishezon varieties are part of the "algebraic universe". It is a nontrivial fact that a Moishezon variety can also be characterised as a compact complex variety that is the holomorphic image of a projective variety.

Recall that a real $2n$-dimensional lattice of $\mathbb{C}^n$ is an additive subgroup of the form $\Delta = \{m_1\alpha_1 + \cdots + m_{2n}\alpha_{2n} \mid m_1, \dots, m_{2n} \in \mathbb{Z}\}$, where $(\alpha_1, \dots, \alpha_{2n})$ is an $\mathbb{R}$-basis for $\mathbb{C}^n$. An n-dimensional *complex torus* is a quotient of $\mathbb{C}^n$ by a real $2n$-dimensional lattice, equipped with the induced analytic structure. Complex tori are compact complex manifolds, and, depending on the choice of the lattice, may give rise to non-Moishezon varieties. The group structure induced on a complex torus from $\mathbb{C}^n$ is holomorphic.

Finally, let me briefly mention a class of compact complex varieties that will eventually play an important role in this article. A *Kähler manifold* is a compact complex manifold that admits a Hermitian metric whose associated

[3] This terminology differs from the one I used in my thesis [25], where a "complex variety" was assumed to also be compact. In doing so I was following Ueno [40], but now agree with those who objected to that convention.

differential 2-form of type $(1, 1)$ is closed. A compact complex variety is said to be of *Kähler-type* if it is the holomorphic image of a Kähler manifold. While these definitions may not be particularly enlightening, certain relevant properties of the class of all Kähler-type varieties, referred to as the class $\mathcal{C}$, will become clear later. For now, I only mention that the class $\mathcal{C}$ contains all Moishezon varieties and complex tori, and is closed under taking cartesian products and bimeromorphic equivalence. Kähler-type varieties have been extensively studied (see, for example, Fujiki [12]), mainly because many methods from algebraic geometry are applicable to them.

§**2. Basic model-theoretic properties.** A compact complex variety can be viewed as a structure in the sense of model theory by taking the Zariski closed subsets of all its (finite) cartesian powers as the basic relations. In order to deal with several compact complex varieties at once, it is convenient to consider the many-sorted structure $\mathcal{A}$ where there is a sort for each compact complex variety, and where the relations are now the Zariski closed subsets of the cartesian products of the sorts. I let $\mathcal{L}$ denote the corresponding many-sorted language consisting of predicates for these relations. In this section I will survey some of the first results on the model theory of $\mathcal{A}$.

Clearly, a quantifier-free definable set in $\mathcal{A}$ will be a (finite) boolean combination of Zariski closed sets. *A priori*, arbitrary definable sets are obtained from Zariski closed sets using the usual boolean operations together with the coordinate projection maps. That projection maps turn out to be unnecessary is the first indication that $\mathrm{Th}(\mathcal{A})$ is well-behaved:

THEOREM 2.1 (Łojasiewicz [24], Zilber [41]). $\mathrm{Th}(\mathcal{A})$ *admits quantifier elimination*; *every definable set is a boolean combination of Zariski closed sets.*

Two classical results of Remmert are responsible for quantifier elimination. The first is Remmert's Proper Mapping Theorem which states that the image of an analytic set under a proper holomorphic map is again analytic.[4] In the compact case properness comes for free, and so we have that the holomorphic image of a Zariski closed set is Zariski closed. Quantifier elimination says something slightly different, that the holomorphic image of a Zariski constructible set is Zariski constructible. Nevertheless, Theorem 2.1 can be deduced from the Proper Mapping Theorem using an inductive argument that involves the following fact (also due to Remmert) about how the dimension of an analytic set varies in families:[5]

FACT 2.2 (Dimension Formula). Suppose X and Y are compact complex varieties and $f : X \to Y$ is a surjective holomorphic map. Then there exists a nonempty Zariski open set $U \subset Y$, such that for all $y \in U$ and $x \in X_y$,

[4] A proof of this can be found on page 162 of [16].

[5] See Chapter 3 of [7]

$\dim_x X_y = \dim X - \dim Y$. In particular, each irreducible component of a general fibre of f is of dimension $\dim X - \dim Y$. Moreover, this is the least dimension of any fibre of f.

The above fact is of independent interest and implies the definability of dimension; if S is a definable set and $\{F_s : s \in S\}$ is a definable family of Zariski closed sets parametrised by S, then for each $n \geq 0$, the set

$$U(n) = \{s \in S : \dim F_s = n\}$$

is a definable subset of S. As we will see later, it is the definability of dimension in $\mathcal{A}$ that allows for a well-behaved notion of dimension for definable sets in elementary extensions of $\mathcal{A}$.

Suppose F is a definable subset of some compact complex variety X. By quantifier elimination, F has a unique irredundant expression of the form:

$$F = (S_1 \setminus T_1) \cup (S_2 \setminus T_2) \cup \cdots \cup (S_k \setminus T_k)$$

where each S_i is an irreducible Zariski closed subset of X and T_i is a proper Zariski closed subset of S_i. The Zariski closure of F, denoted by $\overline{F}$, is then $S_1 \cup \cdots \cup S_k$. I say that F is *irreducible* if $k = 1$, which is equivalent to $\overline{F}$ being irreducible. For each i, $S_i \setminus T_i$ in the above expression is called an *irreducible component of* F. By the *dimension of* F I mean the dimension of $\overline{F}$, and denote it by $\dim F$. Note that the dimension of a definable set is equal to the maximum of the dimensions of its irreducible components.

A natural consequence of quantifier elimination is that every definable map is "piecewise meromorphic":

COROLLARY 2.3. *Suppose $f : A \to B$ is a definable map between definable sets. Then $A = A_1 \cup \cdots \cup A_m$, where each A_i is a Zariski open subset of a compact complex variety X_i, and such that the restriction of f to each A_i is holomorphic on A_i and meromorphic on X_i.*

This may require some explanation. Suppose X and Y are compact complex varieties, U is a nonempty Zariski open subset of X, and $f : U \to Y$ is a holomorphic map. I will say that f is *meromorphic on X* to mean that there exists a meromorphic map from X to Y that agrees with f on U. It should be clear that in this case f is definable. The above corollary says that all definable maps are of this form, after a finite decomposition of the domain.

Quantifier elimination also yields a correspondence between complete types and irreducible Zariski closed sets. Suppose x is a (finite) tuple of variables corresponding to a certain cartesian product of sorts from $\mathcal{A}$, say X, and $p(x)$ is a complete type over the empty set in these variables.[6] By quantifier elimination and the noetherianity of the Zariski topology, there exists a unique irreducible Zariski closed subset $F \subset X$, such that $p(x)$ is determined by the

[6]Notice that this is equivalently to $p(x)$ being over $\mathcal{A}$, since every point of a compact complex variety, being itself a Zariski closed set, is named in $\mathcal{L}$.

formulae stating that "$x \in F$ and $x \notin G$ for any proper Zariski closed subset $G \subset F$". Conversely, if F is any irreducible Zariski closed subset of X, then the collection of formulae stating that "$x \in F$ and $x \notin G$ for any proper Zariski closed subset $G \subset F$" is consistent and determines a complete type. This type is often referred to as the *generic type of F over the empty set*. If F is an irreducible definable set, then by the *generic type of F* I will mean the generic type of its Zariski closure $\overline{F}$.

Every element of $\mathcal{A}$ is named in $\mathcal{L}$. In particular, $\mathcal{A}$ is not even ω-saturated. Hence in order to evaluate complete types, one is forced to pass to elementary extensions of $\mathcal{A}$. However, using quantifier elimination and the fact that no complex variety can be written as a countable union of proper analytic subsets (this is a consequence of the Baire Category Theorem), the following weakening of saturation can be obtained:

THEOREM 2.4 (Zilber [41]). *$\mathcal{A}$ is ω_1-compact; the intersection of any countable collection of definable sets is nonempty as long as all of its finite subcollections have nonempty intersection.*

For an ω_1-compact structure it is not necessary to consider elementary extensions in order to decide whether the structure is of finite Morley rank. Moreover, if this is the case, the Morley rank of the definable sets can be computed directly inside the given model.[7] That is, the following definitions are valid for an ω_1-compact structure $\mathcal{M}$:

(a) For F a definable set and $n \in \omega$, $\mathrm{RM}(F) \geq n$ can be defined inductiveley as follows:
   - $\mathrm{RM}(F) \geq 0$ iff F is nonempty;
   - $\mathrm{RM}(F) \geq n+1$ iff there are disjoint definable subsets $(F_i)_{i \in \omega}$ of F with $\mathrm{RM}(F_i) \geq n$ for all $i \in \omega$;

(b) $\mathcal{M}$ is of *finite Morley rank* if for any definable set F, there exists $n \in \omega$, such that $\mathrm{RM}(F)$ is not greater than or equal to $n+1$; in which case the minimal such number is the *Morley rank* of F and is denoted by $\mathrm{RM}(F)$.

Using quantifier elimination and Theorem 2.4, Zilber shows:

THEOREM 2.5 (Zilber [41]). *$\mathrm{Th}(\mathcal{A})$ is of finite Morley rank. Moreover, for definable sets in $\mathcal{A}$, Morley rank is bounded by dimension.*

The methods of stability theory are therefore applicable to the structure $\mathcal{A}$. One useful consequence is *uniform definability of types*.[8] This allows us to assume, in a uniform manner, that the parameters for a given definable set come from the sort in which the definable set itself lives. More precisely, suppose $\phi(x, y)$ is an $\mathcal{L}$-formula where x is an n-tuple of variables belonging to a sort X and y is an m-tuple of variables belonging to a sort Y. Then there exists an $\mathcal{L}$-formula $\psi(x, z)$, where z is now an m'-tuple of variables also

[7] For example, see Proposition 0.16 in [25].

[8] See III.1.24 of [1].

belonging to the sort X, such that for all $c \in Y^m$ there is a $d \in X^{m'}$, such that $\phi(x, c)$ and $\psi(x, d)$ define the same subset of X^n.

REMARK 2.6. The notion of a *Zariski-type* structure, as described by Zilber in [41], provides a framework in which to study compact complex varieties. Indeed, all the basic model-theoretic properties that I have discussed thus far can be deduced from the fact that the sorts of $\mathcal{A}$ are Zariski-type structures. As I will not be adopting this point of view, I omit a discussion of these axioms here, and instead refer the interested reader to the aforementioned article.

Every compact complex variety can be obtained as the holomorphic image of a compact complex manifold. Indeed, by resolution of singularities, every complex variety is the image of a complex manifold under a modification. It follows that a compact complex variety is interpretable in a compact complex manifold; it can be obtained as the quotient of a compact complex manifold by a definable equivalence relation. Hence, even if we had begun by only considering the smooth case, arbitrary compact complex varieties would have appeared naturally as imaginary sorts. Moreover, having allowed all compact complex varieties as sorts of $\mathcal{A}$, the process of taking quotients of definable sets by definable equivalence relations does not produce anything more:

THEOREM 2.7. $\mathrm{Th}(\mathcal{A})$ *admits elimination of imaginaries.*

In [31] Pillay describes how a result of Grauert can, in principle, be used to obtain Theorem 2.7. Grauert's result concerns definable equivalence relations that satisfy certain geometric conditions—what are called "meromorphic equivalence relations". In [14] Grauert shows that the quotient of a compact complex manifold by a meromorphic equivalence relation exists (at least generically) in the category of compact complex varieties. In my thesis [25], I have provided the details of how quantifier elimination and the dimension formula can be used to show that all definable equivalence relations are essentially meromorphic, and then to conclude from this that $\mathrm{Th}(\mathcal{A})$ admits elimination of imaginaries.

The projective line over $\mathbb{C}$ is a sort in $\mathcal{A}$. I will use $\mathbb{P}^n(\mathbb{C})$, or just $\mathbb{P}^n$, to denote the nth cartesian power of $\mathbb{P}(\mathbb{C})$ (to be distinguished from projective n-space which is denoted by $\mathbb{P}_n$). Note that $\mathbb{P}^n$ is a projective variety (this is witnessed by the Segre embedding). By quantifier elimination for $\mathcal{A}$ together with Chow's theorem, every definable subset of $\mathbb{P}^n$ is given by a boolean combination of algebraic subsets. Since every algebraic set is naturally interpretable in $(\mathbb{C}, +, \times)$, the full structure induced on $\mathbb{P}(\mathbb{C})$ by $\mathcal{A}$, is interpretable in $(\mathbb{C}, +, \times)$. On other hand, after fixing an identification of $\mathbb{C}$ with a Zariski open subset of $\mathbb{P}(\mathbb{C})$, the complex field $(\mathbb{C}, +, \times)$ is definable in $\mathbb{P}(\mathbb{C})$. Hence the full structure induced on the sort $\mathbb{P}(\mathbb{C})$ by $\mathcal{A}$ is bi-interpretable with the pure algebraically closed field $(\mathbb{C}, +, \times)$.

§3. Strongly minimal sets and Zariski geometries. Recall that in an ω_1-compact structure, a definable set is *strongly minimal* if all of its definable subsets are either finite or cofinite.[9] In $\mathcal{A}$, the first examples of strongly minimal sets are the compact complex curves (i.e., irreducible 1-dimensional Zariski closed sets). The Riemann Existence Theorem says that these are exactly the projective curves. However, part of what makes the model theory of $\mathcal{A}$ interesting is that there are higher dimensional examples of strongly minimal sets. Recall that a complex torus of dimension $n \geq 0$ is a compact complex manifold that can be obtained as the quotient of $\mathbb{C}^n$ by a $2n$-dimensional latttice $\Lambda \subset \mathbb{C}^n$. I will say that the complex torus $M = \mathbb{C}^n/\Lambda$ is *generic*, if Λ is generated by complex numbers $(a_1 + b_1 i), \dots, (a_{2n} + b_{2n} i)$, where $\{a_1, \dots, a_{2n}, b_1, \dots, b_{2n}\} \subset \mathbb{R}$ are algebraically independent over $\mathbb{Q}$. Generic complex tori have no proper infinite analytic subsets, and are therefore strongly minimal in $\mathcal{A}$. Note that this also yields a family of examples where Morley rank and dimension differ—the Morley rank of a strongly minimal set is 1 while the dimension of M above is n.

Recall the notion of (model-theoretic) algebraic closure: a tuple b is in the *algebraic closure* of a set A (denoted $b \in \operatorname{acl} A$) if b belongs to a finite definable set with parameters from A. The algebraic closure relation on a strongly minimal set gives rise to a pregeometry with an associated notion of independence and dimension, that I will refer to here as *acl-independence* and *acl-dimension*, respectively. Strongly minimal sets can then be studied in terms of the behaviour of this relation. For example, there is a natural dividing line among the strongly minimal sets that serves to isolate those on which acl-independence behaves in an essentially "linear" fashion. More precisely, recall that a strongly minimal set X is *locally modular* if the following condition holds after passing to a sufficiently saturated elementary extension of the ambient structure, and possibly adding names for finitely many parameters: for all algebraically closed sets $A, B \subset X$, A is acl-independent from B over $A \cap B$. Equivalently,

$$\text{acl-dim}(A \cup B) = \text{acl-dim}(A) + \text{acl-dim}(B) - \text{acl-dim}(A \cap B).$$

In [29], Pillay gives a direct proof that strongly minimal complex tori (for example, generic complex tori) of dimension greater than 1 are locally modular. On the other hand, it is not very difficult to see that a projective curve is not locally modular.

In [31], Pillay points out that every strongly minimal set in $\mathcal{A}$ that is not locally modular is essentially algebraic. This follows from the deep results of Hrushovski and Zilber [21] on Zariski geometries, and it is worth the digression to give a brief description of this abstract setting. Recall that a

[9] In the absence of ω_1-compactness one must reformulate this condition to hold uniformly in the parameters.

topological space is *noetherian* if it satisfies the descending chain condition on closed sets. In a noetherian topological space, the *noetherian dimension* of a closed set, denoted by Ndim, is the maximal length of a chain of irreducible closed subsets. A *Zariski geometry* on a set X is a noetherian topology on X^n for each $n \geq 1$, such that the following conditions hold:

$Z0$. *If $f: X^n \to X^m$ is given by $f(x) = (f_1(x), \ldots, f_m(x))$, where each $f_i: X^n \to X$ is either a coordinate projection or a constant map, then f is continuous. Moreover, the diagonals $x_i = x_j$ are closed in X^n.*

$Z1$. *Suppose $\pi: X^n \to X^k$ is the projection map onto the first k coordinates, and $C \subset X^n$ is a closed set. There is a proper closed subset $F \subset \mathrm{cl}(\pi C)$ (where* cl *denotes topological closure), such that $\mathrm{cl}(\pi C) \setminus F \subset \pi C$.*

$Z2$. *Suppose $C \subset X^n \times X$ is closed, and denote by $C_a = \{x \in X : (a, x) \in C\}$ the fibre of C above $a \in X^n$. Then there exists $m \geq 0$, such that for all $a \in X^n$, either C_a or $X \setminus C_a$ is of size at most m.*

$Z3$. *Suppose $F \subset X^n$ is an irreducible closed set and Δ_{ij} is the diagonal $x_i = x_j$ in X^n. Then every irreducible component of $F \cap \Delta_{ij}$ has noetherian dimension at least* $\mathrm{Ndim}(F) - 1$.

For example, if K is an algebraically closed field and C is a smooth algebraic curve over K, then C equipped with the usual Zariski topology on its cartesian powers is a Zariski geometry. Indeed, the main purpose of [21] is to isolate topological and geometric conditions that characterise this example.

Suppose X is a Zariski geometry. Then we can view X as a structure by taking the closed subsets of X^n as the basic relations. Equipped with this structure, X admits quantifier elimination and is strongly minimal (this follows from $Z1$ and $Z2$). One of the main theorems in [21] is that if X is in addition not locally modular, then there is an algebraically closed field K interpretable in X. Moreover, there is a finite-to-one surjective map $f: X \to \mathbb{P}(K)$, such that for each $n \geq 1$, $f^n: X^n \to \mathbb{P}(K)^n$ is continuous and maps constructible sets to constructible sets (where the topology on $\mathbb{P}(K)^n$ is taken to be the usual Zariski topology). This says that X is rather close to being a smooth algebraic curve over K.

How does this theory apply to the structure $\mathcal{A}$? Suppose X is a strongly minimal set in $\mathcal{A}$ that is given as a nonempty Zariski open subset of a compact complex variety $\overline{X}$. For each $n \geq 1$, we have a noetherian topology on X^n coming from the relatively Zariski closed subsets (intersections of Zariski closed sets in $\overline{X}^n$ with X^n). Notice that X equipped with these subsets of its powers is exactly the induced structure on X from $\mathcal{A}$. Now assume, moreover, that X is smooth. That is, the analytic structure on X inherited from $\overline{X}$ is smooth. It is pointed out in [20] that X together with this Zariski topology on each of its cartesian powers, is a Zariski geometry. It follows that if X is not locally modular, then there is an algebraically closed field interpretable in X. By elimination of imaginaries, this field is definable in $\mathcal{A}$. However, the only infinite definable field in $\mathcal{A}$, up to definable isomorphism, is the complex field $(\mathbb{C},+,\times)$ living on the sort $\mathbb{P}(\mathbb{C})$.[10] The results on Zariski geometries mentioned above, thus imply that if X is not locally modular then there is a definable, finite-to-one, surjection from X to $\mathbb{P}(\mathbb{C})$. It follows that there is a finite-to-one meromorphic surjection from $\overline{X}$ to $\mathbb{P}(\mathbb{C})$. In particular, $\dim \overline{X} = 1$, and by the Riemann Existence Theorem, $\overline{X}$ is a projective curve. To summarise, if X is not locally modular then it is a Zariski open subset of a projective curve.

It was assumed in the previous paragraph that X was given as a Zariski open subset of a compact complex variety, and that it was smooth. These conditions are somewhat superfluous. By quantifier elimination, after possibly removing finitely many points, every strongly minimal set X in $\mathcal{A}$ is a Zariski open subset of a compact complex variety. Also, the non-smooth locus of a compact complex variety is a proper analytic subset[11]—and hence definable. By strong minimality, after possibly removing another finite set of points, X is smooth. Hence, up to finitely many points, every strongly minimal set in $\mathcal{A}$ is a Zariski geometry, and the results described above apply. That is, the following dichotomy holds:[12]

THEOREM 3.1. *Every strongly minimal set in $\mathcal{A}$ is either locally modular or a projective curve up to finitely many points.*

§**4. Definable groups.** Recall that a *definable group* is a definable set G with a definable map from $G \times G$ to G that satisfies the conditions of a group operation. Definable groups are an intrinsic part of pure model theory, in the sense that they arise naturally when studying the structural properties of a stable theory. *Binding groups*, which are infinitely definable groups of automorphisms that play a role rather similar to that of Galois groups, arise when two definable sets (or types) are related (i.e., nonorthogonal) but the interaction requires additional parameters.[13] Definable groups also appear when one tries to classify strongly minimal sets according to the behaviour of the algebraic closure relation. For example, among strongly minimal sets there is a further dividing line coming from the notion of *triviality*—when the algebraic closure of the union of sets is the union of their algebraic closures. For Zariski geometries, nontriviality is witnessed by the interpretability of an inifinite definable group, in the way that not being locally modular is witnessed by the interpretability of an infinite definable field.

[10] An argument for this is sketched in [31] (the discussion following Remark 3.10), and uses the classification of locally compact fields as well as the Riemann Existence Theorem.

[11] For example, see 2.14 in [7].

[12] See the appendix for a recent proof of this theorem that does not appeal to the theory of Zariski geometries.

[13] See Poizat [37] Section 2.5 for a discussion of binding groups in stable theories.

Recall that a definable group is *definably connected* if it has no definable subgroups of finite index, and *definably simple* if it has no proper infinite normal definable subgroups. For example, any strongly minimal group is definably connected and definably simple. In a theory of finite Morley rank, an understanding of the definable groups can often be reduced to an understanding of those groups that are definably connected and definably simple.

Complex algebraic groups, being definable in $(\mathbb{C}, +, \times)$, are the first examples of groups definable in $\mathcal{A}$. Indeed, every definable group from the sort $\mathbb{P}(\mathbb{C})$ is definably isomorphic to a complex algebraic group.[14] However, there are also non-algebraic groups definable in $\mathcal{A}$. For example, a complex torus is equipped with a holomorphic (and hence definable) group structure, and I have mentioned that generic complex tori of dimension greater than 1 are non-algebraic. Indeed, they are strongly minimal and locally modular. As it turns out, in the definably connected and definably simple case, all non-algebraic groups are strongly minimal and locally modular:

THEOREM 4.1. *A definably connected and definably simple group in $\mathcal{A}$ is either strongly minimal and locally modular or is definably isomorphic to a complex algebriac group.*

I will describe how the above theorem follows from the dichotomy for strongly minimal sets given by Theorem 3.1, and techniques from the model theory of groups of finite Morley rank. Suppose G is a definably connected and definably simple group in $\mathcal{A}$. Using Zilber's Indecomposability Theorem, one obtains a strongly minimal set $X \subset G$, such that G is the image of some cartesian power of X under a definable surjective map, $f : X^n \to G$.[15] Suppose X is not locally modular. By Theorem 3.1, up to finitely many points, X is a projective curve. In particular it is definably isomorphic to a subset of some cartesian power of $\mathbb{P}(\mathbb{C})$. Hence G is interpretable in the sort $\mathbb{P}(\mathbb{C})$ and so definably isomorphic to an algebraic group. Now suppose that X is locally modular. It follows that G, being the image of some cartesian power of X under a definable map, is 1-*based*. 1-basedness is a generalisation of the phenomenon of local modularity from strongly minimal sets to arbitrary definable sets in a stable theory. I will not define 1-basedness here, but instead refer the reader to Section 4.4 of [30]. Hrushovski and Pillay [19] gave a rather complete description of the structure of 1-based groups. For example any definably connected 1-based group is abelian. Applying this to G, which is both definably connected and definably simple, we obtain that G contains no infinite proper definable subgroups. Moreover, every definable subset of a 1-based group is a (finite) boolean combination of cosets of definable subgroups. It follows that every definable subset of G is either finite or cofinite. That is,

[14] This follows from the Weil-vdDries-Hrushovski Theorem, see [37].
[15] See Chapter 2 of [37].

G is strongly minimal and locally modular (local modularity and 1-basedness agree on strongly minimal sets).

In [35], Pillay and Scanlon show that every strongly minimal locally modular group is definably isomorphic to a complex torus. Hence, strongly minimal complex tori of dimension greater than 1 are the only examples of non-algebraic definably connected and definably simple groups. The main result in [35] is much stronger; it classifies all definable groups in $\mathcal{A}$. It is convenient to state this result in the more geometric (though equivalent) category of "meromorphic groups" (Definition 2.2 from [35]).

Definition 4.2. A *meromorphic group* is a complex Lie group G for which there exists a finite open cover $\{W_i\}$, and isomorphisms $\phi_i : W_i \to U_i$, where U_i is a Zariski open subset of some compact complex variety X_i, and such that the following hold:

1 The transition maps are meromorphic. That is, for each $i \neq j$,
$$\phi_i\phi_j^{-1} : \phi_j(W_i \cap W_j) \to \phi_i(W_i \cap W_j)$$
extends to a meromorphic map from X_i to X_j.

2 The group operation is meromorphic. That is, for all i, j, k,
$$\{(x, y) \in U_i \times U_j : \phi_i^{-1}(x)\phi_j^{-1}(y) \in W_k\}$$
is Zariski open in $X_i \times X_j$ and the map $U_i \times U_j \to U_k$ induced by the group operation extends to a meromorphic map $X_i \times X_j \to X_k$.

A *meromorphic subgroup* of G is a closed subgroup H of G, such that for each i, $\phi_i(H \cap W_i)$ is the intersection of a Zariski closed subset of X_i with U_i. A *morphism* (respectively *isomorphism*) of meromorphic groups is a holomorphic (respectively biholomorphic) homomorphism which when restricted to the charts is meromorphic.

Every complex algebraic group is meromorphic. Indeed, the definition of a meromorphic group is designed to extend the notion of a complex algebraic group to the category of compact complex varieties. The relationship between meromorphic groups and groups definable in $\mathcal{A}$ is analogous to the relationship between complex algebraic groups and groups definable in the complex field. By elimination of imaginaries in $\mathcal{A}$, every meromorphic group can be identified with a definable group in $\mathcal{A}$. Conversely, the Weil-vdDries-Hrushovski Theorem for the algebraic case extends to this context: every definably connected group in $\mathcal{A}$ has the structure of a connected meromorphic group that is unique up to isomorphism. This equivalence means that we can move freely from definable groups to meromorphic groups and back. For example, one can see that quotient objects exist in the category of meromorphic groups by applying elimination of imaginaries to groups definable in $\mathcal{A}$.

Here is the classification of meromorphic groups alluded to above:

THEOREM 4.3 (Pillay, Scanlon [35]). *Suppose G is a connected meromorphic group. Then G has a unique normal connected meromorphic subgroup L, such that L is isomorphic to a linear complex algebraic group and G/L is isomorphic to a complex torus.*

Fujiki [9] had proven this result in certain special cases, all of which assumed that G had a "nice" global compactification. This turned out to be a useful notion, and I give the definition here. Let G be a meromorphic group and $\mu\colon G \times G \to G$ the group operation. A *Fujiki-compactification* of G is a compact complex analytic space, G^*, which contains G as a dense Zariski open subset, and a meromorphic map $\mu^*\colon G^* \times G^* \to G^*$ that is holomorphic on $(G \times G^*) \cup (G^* \times G)$ and that agrees with μ on $G \times G$. Essentailly this says that the group operation on G extends meromorphically to the compactification G^*, while the action of G on itself (in both the right and left senses) extends *holomorphically* to an action of G on all of G^*. For example, a complex torus, being already compact, has itself as a Fujiki-compactification. Also, every complex algebraic group has a Fujiki-compactification.[16] Fujiki proved Theorem 4.3 in the case when G is commutative and has a Fujiki-compactification. He also showed that a meromorphic group has a Fujiki-compactification of Kähler-type if and only if it satisfies the conclusion of Theorem 4.3.

Pillay and Scanlon first find Fujiki-compactifications for commutative meromorphic groups that are either strongly minimal or arise as extensions of a 1-dimensional complex algebraic group by a simple complex torus. Once such a compactification has been found, Theorem 4.3 for these cases follows from Fujiki's results. The general case is then deduced using, among other things, techniques from the model theory of groups of finite Morley rank. Notice that as a consequence of Theorem 4.3, and the results of Fujiki mentioned above, we obtain:

COROLLARY 4.4 (Pillay, Scanlon [35]). *Every connected meromorphic group has a Fujiki compactification of Kähler-type.*

At the beginning of this section, I mentioned that there was a further division among strongly minimal sets, triviality and non-triviality, that in the case of Zariski geometries was connected with the interpretability of an infinite group. Using the classification of meromorphic groups, Pillay and Scanlon obtain the following trichotomy for strongly minimal compact complex manifolds (this was already observed by Scanlon in [39]):

THEOREM 4.5. *Let X be a non-trivial strongly minimal compact complex manifold. Then X is either a projective curve or a complex torus.*

[16]This is pointed out in Remark 2.3 of [9], and uses the fact that Theroem 4.3 is true for algebraic groups.

The full trichotomy for strongly minimal compact complex manifolds is witnessed in $\mathcal{A}$. Indeed, smooth projective curves are strongly minimal compact complex manifolds that are not locally modular, and we have seen that generic complex tori of dimension greater than 1 yield examples of strongly minimal compact complex manifolds that are locally modular but non-trivial. In [22], Kowalski and Pillay point out that certain K3 surfaces are trivial.

I will end this section with a brief discussion of some results obtained by Kowalski and Pillay [22] on the structure of a Zariski closed subset of a commutative connected meromorphic group. Suppose G is a connected meromorphic group with a fixed Fujiki-compactification G^*. Then G is naturally equipped with a Zariski topology induced from G^*; the Zariski closed subsets of G being the intersections of G with Zariski closed subsets of the compact complex variety G^*. If $X \subset G$ is a Zariski closed set, then the *stabiliser of* X is the meromorphic subgroup of G given by $\mathrm{stab}(X) = \{g \in G : g + X = X\}$.

THEOREM 4.6 (Kowalski, Pillay [22]). *Suppose G is a commutative and connected meromorphic group, and $X \subset G$ is an irreducible Zariski closed set with finite stabiliser. Then X is contained in a translate of an algebraic subgroup.*

Since we can always quotient out by $\mathrm{stab}(X)$, the above theorem implies that every irreducible Zariski closed subset of G is algebraic modulo its stabiliser. This allows one to lift certain results about algebraic groups to meromorphic groups. For example, the following Mordell-Lang type theorem for cyclic subgroups of commutative meromorphic groups follows from the algebraic case:

COROLLARY 4.7 (Kowalski, Pillay [22]). *Suppose G is a commutative connected meromorphic group, X is an irreducible Zariski closed subset of G, and $\Gamma \subset G$ is a cyclic subgroup. If $X \cap \Gamma$ is Zariski dense in X, then X is a translate of a meromorphic subgroup of G.*

Indeed, X is a translate of a meromorphic subgroup if and only if it is a translate of its stabiliser. Equivalently, X', the image of X in $G/\mathrm{stab}(X)$, is a single point. Applying Theorem 4.6 to $X' \subset G/\mathrm{stab}(X)$, we obtain that some translate of X' is contained in an algebraic subgroup. By the truth of Corollary 4.7 for commutative complex algebraic groups, X' must be a translate of $\mathrm{stab}(X')$. Since $\mathrm{stab}(X')$ is trivial, X' is a singleton, and X is a translate of $\mathrm{stab}(X)$ as desired.

The proof of Theorem 4.6 is connected to the "socle argument" which appears in Hrushovski's proof of the Mordell-Lang conjecture for function fields [18]. Suppose G is any commuatative group of finite Morley rank. A definable subgroup $H \subset G$ is *almost strongly minimal* if there is a strongly minimal set $X \subset G$ such that , after passing to a saturated elementary extension of G, $H \subset \mathrm{acl}(F \cup X)$, where F is a finite set of parameters. The *socle of*

G was introduced by Hrushovski in [18], and can be described as the sum of all definably connected almost strongly minimal subgroups of G.[17] Zilber's Indecomposability Theorem implies that $\mathrm{socle}(G)$ is itself a definably connected subgroup of G, and is in fact a finite sum of definably connected almost strongly minimal subgroups. For example, using the dichotomy for strongly minimal sets in $\mathcal{A}$ from Theorem 3.1, one can show that if G is a connected meromorphic group, then $\mathrm{socle}(G) = A + T$, where A is the maximal connected algebraic subgroup of G and T is a sum of strongly minimal locally modular complex tori.

In [18], Hrushovski shows that for commutative groups of finite Morley rank with certain "rigidity" conditions, every definable subset with finite stabiliser is contained in a translate of the socle, up to sets of smaller Morley rank. The rigidity hypothesis need not be satisfied by a commutative meromorphic group. Nevertheless, using the classification of meromorphic groups, together with some arguments from complex analytic geometry, Kowalski and Pillay are able to show that if G is a commutative connected meromorphic group, and $X \subset G$ is an irreducible Zariski closed set with finite stabiliser, then X is contained in some translate of $\mathrm{socle}(G)$. Moreover, following [18], they are then able to conclude that X must be contained in a translate of A, where $\mathrm{socle}(G) = A + T$ as above.[18] Theorem 4.6 follows.[19]

§5. Douady spaces and saturation. One obstacle to applying model-theoretic techniques directly to $\mathcal{A}$ is that $\mathcal{A}$ is not saturated. Since ω_1-saturation and ω_1-compactness are equivalent for countable languages, the reduct of $\mathcal{A}$ to any countable sublanguage of $\mathcal{L}$ is ω_1-saturated. Of course, in reducing the language we may lose some of the structure on $\mathcal{A}$ that we are interested in. It may be the case, nevertheless, that for certain sorts of $\mathcal{A}$ the full structure (allowing parameters) is induced by some countable sublanguage of $\mathcal{L}$. In other words, there may be sorts of $\mathcal{A}$ in which saturation fails for only syntactic reasons. In my thesis [25], I discuss such sorts and give a characterisation of them in terms of their Douady spaces. In this section I will first give a rather detailed exposition of the ideas from complex analytic geometry that are involved, and then discuss their implication on the model theory of $\mathcal{A}$.

Suppose that $f : X \to S$ and $g : T \to S$ are surjective holomorphic maps on compact complex varieties. Then $(T \times_S X)_{\mathrm{red}}$, the set-theoretic fibred

[17] This was not the original definition in [18], however the above characterisation follows from arguments appearing there. See [22] for a more detailed discussion of the socle.

[18] This follows from the fact that A and T are fully orthogonal, T is 1-based, and X has finite stabiliser.

[19] Theorem 4.6 can also be obtained, more directly, from Theorem A.2 of the appendix—see Pillay [32] for details.

product of X and T over S, will be denoted by X_T:

$$X_T = (T \times_S X)_{\mathrm{red}} = \{(t, x) \mid f(x) = g(t)\} \subset T \times X.$$

The first coordinate projection map restricted to X_T will be denoted by $f_T \colon X_T \to T$. The fibres of f_T can be identified (by the second projection map) with analytic sets in X. Hence, $X_T \subset T \times X$ yields a family of analytic sets in X parametrised by T. Under this identification, $(X_T)_t = f^{-1}(g(t))$ for all $t \in T$. So $f_T \colon X_T \to T$ is just the lifting by g of the original family given by $f : X \to S$ to a new set of parameters. The following diagram is illustrative:

$$\begin{tikzcd} X \arrow[d, "f"] & X_T \subset T \times X \arrow[d, "f_T"] \\ S & T \arrow[l, "g"] \end{tikzcd}$$

This process is referred to as *base change*. Note that if $T = S$ and g is the identity map, then X_S is just the graph of f. Also, if $T = \{s\}$ is a point in S and $g \colon \{s\} \hookrightarrow S$ is the identity embedding of this point, then X_s is just the fibre of f above s, which agrees with earlier notation.

One complication is that X_T need no longer be irreducible. However, if the general fibres of $f : X \to S$ are irreducible (that is, f is a *fibre space*), then there is a unique irreducible component of X_T that projects onto T. I will call this compact complex variety the *strict pull back* of X in X_T and denote it by $X_{(T)}$. The general fibres of $X_T \to T$ and $X_{(T)} \to T$ are the same.

Definable families of analytic sets, though natural from the model-theoretic point of view, are not sufficiently well-behaved. The correct geometric notion of a family of analytic sets, involves a flatness condition. Let $f : X \to S$ be a holomorphic map between complex analytic spaces, and suppose $\mathcal{F}$ is a coherent analytic sheaf on X. Then $\mathcal{F}$ is said to be *f-flat* if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module. The map f itself is *flat* if the structure sheaf $\mathcal{O}_X$ is f-flat.[20] Essentially, flatness of a holomorphic surjection between compact complex varieties means that the family of analytic sets it defines varies "nicely" from the geometric perspective. For example, the fibres of a flat map are pure dimensional and of constant dimension.

Fortunately, if we allow bimeromorphic changes, every definable family can be made flat. Suppose X and S are compact complex varieties, and $G \subset S \times X$ is an irreducible Zariski closed set such that $G \to S$ is a not necessarily flat fibre space. Hironaka's Flattening Theorem [17] says that after changing $G \to S$ bimeromorphically, we can obtain a *flat* family of analytic

[20] I suggest Section 6 of [38] as a reference to the theory of coherent analytic sheaves, and Section 2 of [26] for more details on the concept of flatness in complex analytic geometry.

sets in X. Indeed, there exists a modification $g\colon T \to S$ such that the strict pull back $G_{(T)} \to T$ is now flat. Again a diagram is illustrative:

$$\begin{tikzcd} S \times X \supset G \arrow[d] & G_{(T)} \subset T \times_S (S \times X) = T \times X \arrow[l, "g \times id"'] \arrow[d] \\ S & T \arrow[l, "g"'] \end{tikzcd}$$

Since g is a modification, $G_{(T)} \to T$ is a bimeromorphic copy of $G \to S$, that now defines a flat family of analytic sets in X parametrised by T.

A notion from complex analytic geometry that is particularly relevant to saturation issues in $\mathcal{A}$, is the universal flat family of analytic subsets of a complex analytic space, constructed by Douady in [6]:

FACT 5.1 (Existence of Douady Spaces). Let X be a complex analytic space. Then there exists a possibly nonreduced complex analytic space $\mathcal{D} = \mathcal{D}(X)$ and an analytic subspace $\mathcal{Z} = \mathcal{Z}(X) \subset \mathcal{D} \times X$ such that:

(a) The projection $\mathcal{Z} \to \mathcal{D}$ is a flat and proper surjection.
(b) If S is a complex analytic space and G is an analytic set in $S \times X$ that is flat and proper over S, then there exists a unique holomorphic map $g\colon S \to \mathcal{D}$ such that $G \simeq S \times_{\mathcal{D}} \mathcal{Z}$ canonically.

$\mathcal{D}(X)$ is called the *Douady Space of* X, $\mathcal{Z}(X)$ is called the *universal family of* X, and $g\colon S \to \mathcal{D}(X)$ as in (b) is called the *Douady map associated to* $G \subset S \times X$.

What does this say in the cases that we are interested in? Suppose X and S are compact complex varieties and $G \subset S \times X$ is an irreducible Zariski closed subset such that $G \to S$ is flat. So G is a flat family of analytic sets in X parametrised by S. Then Fact 5.1 says that there exists a unique holomorphic map, the Douady map, $g\colon S \to \mathcal{D}(X)$, such that $G \simeq S \times_{\mathcal{D}(X)} \mathcal{Z}(X)$ in a canonical fashion. Set-theoretically speaking, this means that g extends by identity to a holomorphic map from G to $\mathcal{Z}(X)$,

$$\begin{tikzcd} S \times X \supset G \arrow[r, "g \times id"] \arrow[d] & \mathcal{Z}(X) \subset D(X) \times X \arrow[d] \\ S \arrow[r, "g"] & \mathcal{D}(X) \end{tikzcd}$$

such that $G_s = \mathcal{Z}(X)_{g(s)}$ for all $s \in S$. Essentially, every flat family of analytic sets in X lives in $\mathcal{Z}(X) \to \mathcal{D}(X)$. By Hironaka's Flattening Theorem, the assumption of flatness will not be very restrictive in applications.

I point out a particular case. Suppose $A \subset X$ is any given Zariski closed subset. Take S to be a fixed point $\{s\}$, and G to be the product $\{s\} \times A$ viewed as a Zariski closed subset of $\{s\} \times X$. G is a one-member family of analytic sets in X, and is trivially flat over $\{s\}$. Hence there is a Douady map $g\colon \{s\} \to \mathcal{D}(X)$ such that the sheaf-theoretic fibre of $\mathcal{Z}(X)$ over $g(s)$, $\{g(s)\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$, is A. Since A is reduced, so is $\{g(s)\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$. In other words, $\mathcal{Z}(X)_{g(s)} = A$. As A was arbitrary, we have that *every* Zariski closed set in X occurs (uniquely) as a reduced fibre of $\mathcal{Z}(X) \to \mathcal{D}(X)$.

It is already becoming clear that when dealing with Douady spaces, nonreduced spaces naturally enter the picture. I have pointed out that the reduced fibres of the Douady space are in bijective correspondence to the Zariski closed subsets of X. The nonreduced fibres, correspond to subspaces of X equipped with possibly nonreduced structure sheaves. I will avoid most of these nonreduced fibres by only considering a certain subspace of the Douady space (following Fujiki in [8]):

Definition 5.2. Suppose X is a compact complex variety. Let $D(X)$ be the subspace of $\mathcal{D}(X)$ that is obtained by taking the union of all the irreducible components, D_α, of $\mathcal{D}(X)_{\mathrm{red}}$ such that for some $d \in D_\alpha$, the (sheaf-theoretic) fibre $\{d\} \times_{\mathcal{D}(X)} \mathcal{Z}(X)$ is reduced and pure dimensional. Let $Z(X)$ be the subspace of $\mathcal{Z}(X)$ obtained by restricting to $D(X)$. We call $D(X)$ the *restricted Douady space of* X, and $Z(X)$ the *restricted universal family of* X.

By passing from $\mathcal{Z}(X) \to \mathcal{D}(X)$ to $Z(X) \to D(X)$, we are focusing on only certain components of the Douady space. Our choice of which components (those that have at least one reduced and pure dimensional fibre) is justified by the following fact:[21] Suppose D_α is an irreducible component of $\mathcal{D}(X)_{\mathrm{red}}$ and Z_α is the restriction of $\mathcal{Z}(X)$ to D_α. Then the following are equivalent

- D_α is a component of the restricted Douady space, $D(X)$,
- Z_α is reduced and pure dimensional,
- There is a dense Zariski open subset $U \subset D_\alpha$ such that for all $u \in U$, the (sheaf-theoretic) fibre of Z_α over u is reduced and pure dimensional.

A consequence is that the collection of pure dimensional Zariski closed subsets of X are in bijective correspondence with a dense Zariski open subset of $D(X)$. Moreover, if S is a compact complex variety and $G \subset S \times X$ is an irreducible Zariski closed subset that is flat and surjective over S, then the Douady map associated to $G \to S$, will map S to $D(X)$.[22] The restricted Douady space and the corresponding restricted universal family are sufficiently universal for our purposes.

A crucial property of Douady spaces is that it has only countably many irreducible components (due to Fujiki [10]). Notice that we have potentially moved out of the structure $\mathcal{A}$. Even when X is a compact complex variety, it is not in general the case that the components of $D(X)$ are compact. Indeed,

[21] See Lemma 1.4 of [8] and Lemma 3 of [11].

[22] This is because a general (sheaf-theoretic) fibre of $G \to S$ will be reduced and pure dimensional.

the question of when the components of the Douady space are compact is very much related to how "saturated" a sort of $\mathcal{A}$ is.

Before making this notion precise, consider the sort $\mathbb{P}(\mathbb{C})$, which is a typical example of the phenomenon described at the beginning of this section. The full structure induced by $\mathcal{A}$ on $\mathbb{P}(\mathbb{C})$ comes from the complex field $(\mathbb{C}, +, \times)$. In particular, there is a finite language, $\mathcal{L}_0 \subset \mathcal{L}$, such that every $\mathcal{L}$-definable set in a cartesian power of $\mathbb{P}(\mathbb{C})$ is $\mathcal{L}_0$-definable (with parameters). Thus $\mathbb{P}(\mathbb{C})$ both retains all of its structure and becomes ω_1-saturated, when it is considered as an $\mathcal{L}_0$-structure. The following definition is from [25]:

Definition 5.3. Let X be a compact complex variety. A *full countable language for X* is a countable sublanguage $\mathcal{L}_0$ of $\mathcal{L}$, such that every $\mathcal{L}$-definable subset of a cartesian power of X is $\mathcal{L}_0$-definable with parameters from $\mathcal{A}$.

One potential defect in the above definition is that a full countable language, for a compact complex variety X, may involve other sorts from $\mathcal{A}$. That is, the parameters involved in the $\mathcal{L}_0$-formulae may come from outside X. However, by uniform definability of types, we can always pull these parameters into X. It follows that X has a full countable language if and only if *there exists a countable language $\mathcal{L}(X)$ and an $\mathcal{L}(X)$-structure on X such that for all $F \subset X^n$, F is $\mathcal{L}$-definable if and only if F is $\mathcal{L}(X)$-definable with parameters from X.* Since $\mathcal{L}$ and $\mathcal{L}(X)$ induce the same definable sets on X, X is still of finite Morley rank as an $\mathcal{L}(X)$-structure, and every $\mathcal{L}(X)$-definable set in X is still a boolean combination of Zariski closed sets. Also, and this is the point of the definition, ω_1-compactness of $\mathcal{A}$ and the countability of $\mathcal{L}(X)$ imply that X is ω_1-saturated as an $\mathcal{L}(X)$-structure. An argument using finite U-rank considerations (due to Pillay), together with the Baire Category Theorem, yields that X is 2^{ω}-saturated.[23] That is, X is saturated as an $\mathcal{L}(X)$-structure (it is $|X|$-saturated and strongly $|X|$-homogeneous). For this reason, I will often say that X is *saturated*, to mean that it has a full countable language.

Here is the promised connection with Douady spaces. Let X be a compact complex variety, and suppose that for all $n > 0$, each component C of $D(X^n)$ is compact. Then C is a sort in $\mathcal{A}$. Let Z_C denote the restriction of $Z(X^n)$ to C. Then Z_C is a Zariski closed subset of $C \times X^n$, and hence a basic relation in $\mathcal{A}$. Let $\mathcal{L}_0$ be the sublanguage of $\mathcal{L}$ consisting of all such relations Z_C, as C and n vary. Since $D(X^n)$ has only countably many components for each n, $\mathcal{L}_0$ is countable. I have pointed out that every irreducible Zariski closed subset of X^n occurs as a fibre of $Z(X^n) \to D(X^n)$. Hence, for all n, every irreducible Zariski closed subset of X^n is $\mathcal{L}_0$-definable with parameters (where the parameters come from the components of $D(X^n)$). By quantifier elimination, $\mathcal{L}_0$ is a full countable language for X. In fact, the converse of this is also true:

[23] This would of course follow from the Continuum Hypothesis. Fortunately such an assumption is not necessary, see Proposition 0.24 in [25].

THEOREM 5.4 (Moosa [25]). *A compact complex variety X has a full countable language if and only if for all $n > 0$, every irreducible component of the restricted Douady space of X^n is compact.*

Given a full countable language, one uses quantifier elmination and Hironaka's Flattening Theorem to cover each component of $D(X^n)$ by countably many holomorphic images of compact complex varieties. It follows that every such component is in fact the holomorphic image of a single compact complex variety, and hence is itself compact.

EXAMPLE 5.5 (The class $\mathcal{C}$). Fujiki has shown that the components of the Douady space of a Kähler-type variety are compact, and even of Kähler-type themselves ([8] and [11]). Moreover the class of Kähler-type varieties, $\mathcal{C}$, is closed under taking cartesian products. So by Theorem 5.4 they are saturated. In particular, all complex tori and Moishezon varieties are saturated. Moreover, since every meromorphic group has a Fujiki-compactification that is of Kähler-type (Corollary 4.4), every definable group in $\mathcal{A}$ is saturated (or rather, lives on a saturated sort).

Some of the methods used in complex analytic geometry to study the classification problem for Kähler-type varieties have a very model-theoretic flavour. For example, I show in my thesis [25], that the *relative Moishezon reduction* of a fibre space living in a saturated compact complex variety exists.[24] This was shown by Campana [3] and Fujiki [13] (independently) for fibre spaces in the class $\mathcal{C}$, and is used to carry out what is essentially a $\mathbb{P}(\mathbb{C})$-analysis (in the model theoretic sense) of a Kähler-type variety.

There are a number of open questions about compact complex varieties with full countable languages. For example, is every saturated compact complex variety of Kähler-type? In fact, it is not even known whether saturation is closed under bimeromorphism. I will end this section with an example of a compact complex variety that does *not* have a full countable language.

EXAMPLE 5.6 (Hopf manifolds). Consider the original Hopf surface, H, defined as follows: let W be the analytic space $\mathbb{C}^2 - \{(0,0)\}$ and $g\colon W \to W$ the analytic automorphism of W given by $g(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{2}z_2)$. Then the Hopf manifold $H = W/ < g >$ is a compact complex surface. In [9] Fujiki points out that $D(H \times H)$ has a noncompact component (by analysing the group of analytic automorphisms of H). By Theorem 5.4, this implies that H is not a saturated complex variety. In particular, H is not of Kähler-type.

§6. The universal domain. As $\mathcal{A}$ is not a saturated model of $\mathrm{Th}(\mathcal{A})$, and not every sort of $\mathcal{A}$ has a full countable language, it is often necessary to consider elementary extensions of $\mathcal{A}$. The sorts of such an extension no longer possess

[24]See [25] for a definition and discussion of what these are.

the structure of a complex analytic space. All that is retained from the standard model is a formal "Zariski topology".

In classical algebraic geometry the notion of a universal domain in which all the objects live, and where "generic" points can be found, is (or at least was) familiar. There does not seem to be an analogue of this in complex analytic geometry. Passing to elementary extensions of the standard universe is characteristic of the model-theoretic approach, and in the case of compact complex varieties it is one of the "new" techniques that model theory brings to complex analytic geometry.

Let $\kappa > |\mathcal{L}|$ be a fixed cardinal, and let $\mathcal{A}'$ be a κ-saturated elementary extension of $\mathcal{A}$ of cardinality κ.[25] That is, $\mathcal{A}'$ is a saturated model of $\mathrm{Th}(\mathcal{A})$; all types over sets of size less than κ are realised in $\mathcal{A}'$ (κ-saturation), and all elementary bijections between sets of size less than κ extend to automorphisms of $\mathcal{A}'$ (strong κ-homogeneity). I will treat $\mathcal{A}'$ as a universal domain for $\mathrm{Th}(\mathcal{A})$. All parameters sets are from now on asumed to be of size less than κ. As opposed to the situation in $\mathcal{A}$, one is forced to consider parameters when dealing with definable sets in $\mathcal{A}'$. A $\emptyset$-definable set in $\mathcal{A}'$ is just the interpretation in $\mathcal{A}'$ of a definable set in $\mathcal{A}$. If X and Y are compact complex varieties (hence sorts of $\mathcal{A}$), and $G \subset Y^n \times X^m$ is a definable subset, then I will sometimes use the notation $G(\mathcal{A})$ and $G(\mathcal{A}')$ to distinguish between G and its interpretation in $\mathcal{A}'$—or, stated differently, between the $\mathcal{A}$-points and the $\mathcal{A}'$-points of G. An arbitrary definable set in $\mathcal{A}'$ is then obtained as the fibre of a $\emptyset$-definable set. They are of the form

$$G(\mathcal{A}')_s = \left\{x \in X(\mathcal{A}')^m : (s, x) \in G(\mathcal{A}')\right\}$$

where s is an n-tuple from $Y(\mathcal{A}')$. In particular, $G(\mathcal{A}')_s$ is called *Zariski closed* if G can be chosen to be a Zariski closed set. By quantifier elimination, every definable set in $\mathcal{A}'$ is a finite boolean combination of Zariski closed sets.

There is a more canonical description of a Zariski closed set in $\mathcal{A}'$. Suppose $F = G(\mathcal{A}')_s$ is a Zariski closed set, where G and s are as above. Let $S \subset Y^n$ be the smallest Zariski closed set such that $s \in S(\mathcal{A}')$. That is, s realises the generic type of S over the empty set. Recall that this type is determined by the formulae stating that it is contained in S but not in any proper Zariski closed subset of S. We say that s is a *generic point of* S (or $S(\mathcal{A}')$) *over the empty set*, and that S is the $\emptyset$-*locus of* s. Notice that if P is a $\emptyset$-definable property of points in S, then P holds for a generic point of S if and only if P holds for general $x \in S$. Letting $H = G \cap (S \times X^m)$, the restriction of G to S, we obtain F as a *generic fibre* of the family of Zariski closed sets $H \subset S \times X^m$ under the projection map $H \to S$. Such a description is somewhat more stable in the following sense: if F is also given as a generic fibre of some other family $K \subset S \times X^m$, then K and H have the same general fibres. That is, there is a

[25]By total transcendentality, such models exist.

nonempty Zariski open subset $U \subset S$, such that for every $u \in U$, $K_u = H_u$. It follows that K and H share those irreducible components that project onto S. The description of F as a generic fibre of a Zariski closed subset of $S \times X^m$ over S is therefore unique up to irreducible components whose projections are proper subsets of S (at least for a fixed parameter set).

The above description also gives us a canonical way of going from one set of parameters to another. Suppse that t is an $n + k$ tuple from $Y(\mathcal{A}')$ that extends s, and that $T \subset Y^{n+k}$ is the $\emptyset$-locus of t. Then F can be viewed as being definable over t as well, and as such is obtained as a generic fibre of some Zariski closed subset of $T \times X^m$. How do these two descriptions of F compare? Let π be the coordinate projection map $Y(\mathcal{A}')^{m+k} \to Y(\mathcal{A}')^m$ which takes t to s. Transferring back to the standard model, we get a surjective holomorphic map $\pi \colon T \to S$. We can lift $G \to S$ to T by base change:

$$\begin{tikzcd} S \times X^m \supset G \arrow[d] & G_T = (T \times_S G)_{\mathrm{red}} \subset T \times X^m \arrow[d] \\ S & T \arrow[l, "\pi"'] \end{tikzcd}$$

Recall that $G_T = \{(y, x) \in T \times X^m \colon (\pi(y), x) \in G\}$. Note that for any $v \in T$, the fibre of G_T above v is exactly $G_{\pi(v)}$. The fibres of $G_T \to T$ and $G \to S$ are the same. In particular, F is also obtained as a generic fibre of $G_T \to T$. Working with additional parameters in $\mathcal{A}'$ corresponds to base change in the standard model.

Using this description of Zariski closed sets with parameters, it is not hard to deduce that they also form a noetherian topology on the sorts of $\mathcal{A}'$. A consequence of the descending chain condition is that every Zariski closed set in $\mathcal{A}'$ has a unique expression as the union of finitely many irreducible Zariski closed sets. I will use the term *absolutely irreducible* (instead of just irreducible) Zariski closed set in order to distinguish it from the following relative notion: Suppose X is a compact complex variety, A is a set of parameters, and $F \subset X(\mathcal{A}')$ is a Zariski closed set definable over A. Then F is said to be *irreducible over A* if it cannot be written as the union of two proper A-definable Zariski closed sets. Note that F is irreducible over A if and only if for any tuple s from A over which F is defined, there is an irreducible Zariski closed set $G \subset S \times X$ where S is the $\emptyset$-locus of s and $F = G(\mathcal{A}')_s$. On the other hand, F is absolutely irreducible if and only if $G \to S$ can be chosen with general fibres irreducible (that is, $G \to S$ is a *fibre space*). Suppose F is an A-definable Zariski closed set in $\mathcal{A}'$. Then its absolutely irreducible components are defined over $\mathrm{acl}(A)$. Indeed, since there are only finitely many such components, and automorphisms of $\mathcal{A}'$ that fix A pointwise must permute them, each absolutely irreducible component of F has only finitely many A-conjugates. By elimination of imaginaries and saturation, they are definable over $\operatorname{acl}(A)$. Irreducibility and absolute irreducibility agree over algebraically closed sets of parameters.

One also obtains, in this way, a notion of dimension. Suppose F is a Zariski closed set in $\mathcal{A}'$ obtained as a generic fibre of a holomorphic surjection $G \to S$. The *dimension* of F is the dimension of the general fibres of $G \to S$. This definition makes sense because of the definability of dimension in $\mathcal{A}$. Indeed, by the dimension formula (Fact 2.2) the general fibres of each irreducible component of G that projects onto S are of constant dimension, and hence the general fibres of $G \to S$ are also of constant dimension. Moreover, if F is obtained as a generic fibre of another analytic family $H \to T$, then by base change, we see that the general fibres of $G \to S$ and $H \to T$ have the same dimension. The dimension of F is well-defined.

Suppose A is a set of parameters, c is a tuple of elements from $\mathcal{A}'$, and F is an A-definable Zariski closed in $\mathcal{A}'$. I will say that F is the *A-locus* of c, and that *c is a generic point in F over A*, if F is the smallest Zariski closed set over A that contains c. By quantifier elimination, c is generic in F over A if and only if $c \in F$, and $c \notin G$ for any proper A-definable Zariski closed subset of F. By saturation, every A-definable A-irreducible Zariski closed set has a generic point. Moreover, the generic type over A is unique. The *dimension of c over A* (or of $\operatorname{tp}(c/A)$), denoted by $\dim(c/A)$, is then the dimension of the A-locus of c.

The structure of definable sets in $\mathcal{A}'$ is still rather mysterious. For example, does the classification of definable groups in $\mathcal{A}$ given by Theorem 4.3 extend to $\mathcal{A}'$? On the other hand, it is known that Corollary 4.4 is *not* true in $\mathcal{A}'$; there are definable groups in $\mathcal{A}'$ that are not definably isomorphic to groups living inside the $\mathcal{A}'$-points of a Kähler-type variety. Pillay and Scanlon give a counterexample that also shows that Morley rank and U-rank differ in $\mathcal{A}'$. I will briefly describe their construction (taken from Lieberman [23]).

Example 6.1 (Pillay, Scanlon [34]). Fix an elliptic curve $(E, 0, +)$, and let $\sigma\colon E \times E \to E \times E$ be the automorphism taking (a, b) to $(a + b, a + 2b)$. Fix a complex number τ whose imaginary part is positive. Then $\mathbb{Z} \times \mathbb{Z}$ acts on $E \times E \times \mathbb{C}$ by

$$(m, n)((a, b), s) = (\sigma^n(a, b), s + m + n\tau).$$

Let X be the the quotient of $E \times E \times \mathbb{C}$ by this action, equipped with the induced structure of a compact complex manifold. The coordinate projection $E \times E \times \mathbb{C} \to \mathbb{C}$ induces a holomorphic surjection $q\colon X \to S$, where S is the elliptic curve given as the quotient of $\mathbb{C}$ by the lattice generated by 1 and τ. It is not difficult to see that each fibre of q is isomorphic to the product of elliptic curves $E \times E$. In fact, $q\colon X \to S$ is locally trivial. That is, for every $p \in S$, there is a neighbourhood about p, U, such that $X|_U = q^{-1}(U)$ is isomorphic to $E \times E \times U$ over U. Moreover, the local trivialisations induce a fibrewise group structure on $X \to S$. That is, there is a holomorphic map $X \times_S X \to X$ such that for all $p \in S$, the induced map $X_p \times X_p \to X_p$ is the group operation on X_p that is obtained as the pull back of addition on $E \times E$. Hence, if we let $s \in S(\mathcal{A}')$ be generic in S over $\emptyset$, then $G = X(\mathcal{A}')_s$ is a definable group.

Since $\dim X = 3$, and the fibres of $X \to S$ are of Morley rank 2, the Morley rank of X is also 3. On the other hand, Campana's analysis of $q\colon X \to S$ in [4] shows that if τ is chosen to be sufficiently general, then X has no 2-dimensional subvarieties that project onto S. It follows that the generic fibre $G = X(\mathcal{A}')_s$ is strongly minimal. As S is a curve, finite U-rank computations imply that X (or rather its unique generic type) is of U-rank 2. This yields an example where Morley rank and U-rank differ.

Now let Z be a Kähler-type variety, and suppose that G is definably isomorphic to a group $H \subset Z(\mathcal{A}')$. Hence H is a strongly minimal group. Using uniform definability of types, we obtain H as a generic fibre of a family of definable groups $Y \to T$, where Y and T are definable sets living in the sort Z. That is, $H = Y(\mathcal{A}')_t$, for $t \in T(\mathcal{A}')$ generic over $\emptyset$. Since H is definably isomorphic to $G = X(\mathcal{A}')_s$, there exists a nonempty definable set $U \subset S$, which is Zariski dense and open in $\overline{S}$, such that for each $u \in U$, Y_u is definably isomorphic to X_p for some $p \in S$. On the other hand, since Z is of Kähler-type, it has a full countable language $\mathcal{L}(Z)$. By saturation of Z with respect to $\mathcal{L}(Z)$, there exists $q \in U$, in the standard model, such that the $\mathcal{L}(Z)$-type of q is the same as that of t. In particular, Y_q is strongly minimal. But as Y_q is definably isomorphic to some X_p, it is definably isomorphic to the product of elliptic curves $E \times E$. This contradiction shows that G cannot be definably isomorphic to a group that lives on the $\mathcal{A}'$-points of a Kähler-type variety.

§7. **Nonstandard algebraicity.** The Riemann Existence Theorem states that every 1-dimensional compact complex variety is algebraic—it can be biholomorphically embedded into some complex projective space. An extension of this to the universal domain $\mathcal{A}'$ was asked for by Pillay in [31] and obtained in my thesis [25]. In order to state this theorem, I need to make precise what it means for a Zariski closed set in $\mathcal{A}'$ to be algebraic. It should, of course, have something to do with being embeddable into projective space over the nonstandard algebraically closed field in $\mathcal{A}'$ that extends $\mathbb{C}$, which I will denote by $\mathbb{C}^{\mathcal{A}'}$.[26] One key point is that we should allow such an embedding to be defined with additional patramaters from $\mathcal{A}'$. Another issue is to make precise what kind of embedding is sought. Here are two possible notions that correspond to Moishezon and projective in the standard model:

DEFINITION 7.1. Suppose X and Y are compact complex varieties. A holomorphic surjection $X \to Y$ is *trivially Moishezon* if for some $n \geq 1$, there is

[26]Note that $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$ is just the interpretation of the sort $\mathbb{P}(\mathbb{C})$ in $\mathcal{A}'$.

a meromorphic map, $g: X \to Y \times \mathbb{P}_n(\mathbb{C})$, which is bimeromorphic with its image and which commutes with the projection map $Y \times \mathbb{P}_n(\mathbb{C}) \to Y$:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \times \mathbb{P}_n(\mathbb{C}) \arrow[dl] \\ & Y & \end{tikzcd}$$

If in addition, g and n can be chosen such that for general $y \in Y$, the fibrewise map $g_y: X_y \to \mathbb{P}_n(\mathbb{C})$ is biholomorphic with its image, then $X \to Y$ is called *trivially projective*. An irreducible Zariski closed set F in $\mathcal{A}'$ is *Moishezon* (respectively *projective*) if it can be obtained as a generic fibre of a trivially Moishezon (respectively trivially projective) surjection between compact complex varieties.

The first thing to notice about these definitions is that for a fibre space to be trivially Moishezon or trivially projective is stable under base change (and strict pull backs). Hence for an absolutely irreducible Zariski closed set in $\mathcal{A}'$ to be Moishezon or projective does not depend on a choice of parameters. This can be expressed as follows. Suppose that F is of the form $G(\mathcal{A}')_s$, where $G \subset S \times X$ is an irreducible Zariski closed set, X and S are compact complex varieties, the projection $G \to S$ is a fibre space, and $s \in S(\mathcal{A}')$ is a generic point of S over the empty set. Then F is Moishezon (respectively projective) if and only if for some compact complex variety T and holomorphic surjection $T \to S$, $G_{(T)} \to T$ (the strict pull back of G in the fibred product $S \times_T G$) is trivially Moishezon (respectively trivially projective). In diagrams:

$$\begin{tikzcd} G \arrow[d] & G_{(T)} \arrow[r, "g"] \arrow[d] & T \times \mathbb{P}_n(\mathbb{C}) \arrow[dl] \\ S & T \arrow[l] & \end{tikzcd}$$

where g satisifies the appropriate conditions of Definition 7.1.

The notion of Moishezon in $\mathcal{A}'$, though weaker than projective, is more stable under model-theoretic manipulations. Suppose F is given as above. Using quantifier elimination together with the dictionary set up in the previous section that allows one to translate from Zariski closed sets and additional parameters in $\mathcal{A}'$, to families of Zariski closed sets and base change in $\mathcal{A}$, it is not hard to see that F is Moishezon if and only if:

- *There is a definable embedding of a nonempty Zariski open subset of F into some cartesian power of the sort $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$.*

Moreover, using saturation in $\mathcal{A}'$, this is equivalent to:

- *There is a tuple of parameters t extending s and a generic point of F over t that is interdefinable with a tuple from $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$ over t.*

Finally, from the definition of internality together with the fact that the induced structure on the sort $\mathbb{P}(\mathbb{C})$ eliminates imaginaries (as it is that of a pure algebraically closed field), we can conclude that F is Moishezon if and only if:

- *The generic type of F over s is internal to* $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$.

It should now be clear in what sense being Moishezon in $\mathcal{A}'$ corresponds to being "generically" algebraic.

Theorem 7.2 (Moosa [25]). *Every irreducible* 1*-dimensional Zariski closed set in $\mathcal{A}'$ is Moishezon.*

The above Theorem says that every fibre space of curves in $\mathcal{A}$, possibly after base change, is trivially Moishezon. Indeed, this follows from a result of Campana together with some observations regarding projective linear spaces. Since these spaces also appear in other aspects of the model theory of compact complex varieties, I will give a rather detailed description of them here.

Fix a compact complex variety S and consider the category of compact complex varieties over S. In complex analytic geometry, the analogue of projective space in this relative category is the notion of a projective linear space over S. Let $\mathcal{F}$ be a coherent analytic sheaf on S. I will sketch a (local) construction of the *projective linear space over S associated to $\mathcal{F}$*, denoted by $\pi\colon \mathbb{P}(\mathcal{F}) \to S$. Let $U \subset S$ be a small open subset for which there exists a resolution of $\mathcal{F}_U$ as follows:

$$\mathcal{O}_U^p \xrightarrow{\alpha} \mathcal{O}_U^q \longrightarrow \mathcal{F}_U \longrightarrow 0$$

As an $\mathcal{O}_U$-linear homomorphism, α can be represented by a $q \times p$ matrix $M = (m_{ij})$, where each $m_{ij} \in \mathcal{O}_U$. Letting X be coordinates for U and $(Y_1 : \cdots : Y_q)$ homogeneous coordinates for $\mathbb{P}_{q-1}$, $\mathbb{P}(\mathcal{F})_U$ is the analytic subset of $U \times \mathbb{P}_{q-1}$ defined by the equations:

$$m_{1i}(X)Y_1 + \cdots + m_{qi}(X)Y_q = 0$$

for $i = 1, \ldots, p$. One checks that $\mathbb{P}(\mathcal{F})_U$ depends only on the coherent sheaf $\mathcal{F}_U$, and not on the particular resolution chosen above. We then patch the $\mathbb{P}(\mathcal{F})_U$ to obtain $\mathbb{P}(\mathcal{F})$. The structure of a fibre space over S is induced on $\mathbb{P}(\mathcal{F})$ by the coordinate projection maps $U \times \mathbb{P}_{q-1} \to U$.

For each $s \in S$, the fibre $\mathbb{P}(\mathcal{F})_s$ is isomorphic to $\mathbb{P}_r$, where $r + 1$ is the rank of $\mathcal{F}$ at s. In the special case when $\mathcal{F}$ is locally free, this rank is constant and $\pi\colon \mathbb{P}(\mathcal{F}) \to S$ is called a *projective bundle over S*. Projective bundles are locally trivial in the following strong sense. There is an open cover $\{U_i\}$ of S and local trivialisations

$$\begin{tikzcd} \mathbb{P}(\mathcal{F})_{U_i} \arrow[rr, "h_i", "\simeq"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{P}_r \arrow[dl] \\ & U_i & \end{tikzcd}$$

such that the induced transition functions

$$\begin{tikzcd} U_i \cap U_j \times \mathbb{P}_r \arrow[rr, "h_i h_j^{-1}", "\simeq"'] \arrow[dr] & & U_i \cap U_j \times \mathbb{P}_r \arrow[dl] \\ & U_i \cap U_j & \end{tikzcd}$$

are of the form $h_i h_j^{-1}(x, p) = (x, g_{ij}(x)(p))$, where $g_{ij} \colon U_i \cap U_j \to PGL(r, \mathbb{C})$ is holomorphic. In other words, a projective bundle over S of rank r is locally a product of the base with projective r-space, where the transition functions fix the base points while permuting the fibres as elements of the projective general linear group. A *trivial* projective bundle over S is one of the form $S \times \mathbb{P}_r(\mathbb{C}) \to S$. Equivalently, $\mathcal{F} = \mathcal{O}_S^{r+1}$.

A *projective morphism*, $f \colon X \to S$, is a holomorphic map that factors through an embedding into a projective linear space over S. That is, there is a coherent analytic sheaf $\mathcal{F}$ on S and an embedding of X into $\mathbb{P}(\mathcal{F})$ over S. A *Moishezon morphism* is a holomorphic map, $f \colon X \to S$, that is bimeromorphic over S to a projective morphism.

REMARK 7.3. One open question about saturated compact complex varieties is whether they are closed under preimages of projective morphisms. Indeed, the question of whether they are stable under bimeromorphic equivalence can be reduced to this.

Notice that Definition 7.1 (of trivially projective/Moishezon morphisms), is derived from the above notions by replacing "projective linear space" by "trivial projective bundle". Indeed, from the model-theoretic point of view, trivial projective bundles, and not projective linear spaces, seem to be the more natural relative form of projectivity. As it turns out, this discrepancy does not pose too many difficulties. To begin with, while a projective morphism need not embed into a product of the base with projective space, it does embed bimeromorphically into an object that is at least locally of that form. That is, every projective linear space is, after a modification of the base, bimeromorphic to a projective bundle. This is a consequence of Hironaka's Flattening Theorem applied to coherent analytic sheafs (a coherent analytic sheaf is locally free if and only if it is flat).

Moreover, and this turns out to be quite useful, every projective bundle is bimeromorphic after base change to a trivial projective bundle. Following ideas of Fujiki in [12], and using his results on relative Douady spaces, I give a proof of this fact in Proposition 2.37 of [25]. Putting these together, we get that after base change every Moishezon morphism is trivially Moishezon.

Returning to the proof of 7.2, we show that every fibre space of curves in $\mathcal{A}$, $f \colon X \to Y$, possibly after base change, is trivially Moishezon. By the above remarks, Moishezon is enough. Campana has shown (see Lemma 3.10

in [5]) that if f admits a holomorphic section then it is Moishezon. Since we are allowing for additional parameters, we can always obtain a holomorphic section by base change with the fibre space itself and considering the diagonal map. Theorem 7.2 follows.

I will conclude with an application to fields definable in $\mathcal{A}'$. Recall that every infinite field definable in $\mathcal{A}$ is definably isomorphic to $(\mathbb{C}, +, \times)$ (see the discussion after Remark 3.10 in [31]). The only known argument for extending this to $\mathcal{A}'$ uses Theorem 7.2. Indeed, one uses the result for $\mathcal{A}$ to conclude that an infinite field in $\mathcal{A}'$ is of dimension 1, and hence can be definably embedded into a cartesian power of the sort $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$. Since there is a unique definable field in any pure algebraically closed field, one obtains:[27]

COROLLARY 7.4. *Every infinite field definable in $\mathcal{A}'$ is definably isomorphic to $(\mathbb{C}^{\mathcal{A}'}, +, \times)$.*

Appendix: The dichotomy theorem revisited. During preparation of this article, a new development has emerged in the model theory of compact complex spaces that fits in well with the particular approach taken here. Interpreting a result due independently to Campana [2] and Fujiki [11], Pillay has obtained a direct proof of Theorem 3.1 (the dichotomy for strongly minimal sets) that does not use the theory of Zariski Geometries. I will give a brief sketch of the ideas involved, now assuming greater familiarity with notions from stability theory. To begin with, here is the result referred to above, on which Pillay's observations are based:

FACT A.1 (Campana [2], Fujiki [11]). *Suppose X is a compact complex variety, $\mathcal{D}(X)$ is the Douady space of X, $\mathcal{Z}(X) \subset \mathcal{D}(X) \times X$ is the universal family of X, and $B \subset \mathcal{D}(X)$ is a reduced and irreducible compact analytic subspace such that for general $b \in B$, $\mathcal{Z}(X)_b$ is reduced and irreducible. Let $Z \subset B \times X$ denote the restriction of $\mathcal{Z}(X)$ to B, and $\pi : Z \to X$ the second coordinate projection. Then π is Moishezon.*[28]

Since every Moishezon morphism is trivially Moishezon after base change, Fact A.1 says that the generic fibres of $\pi : Z \to X$ in $\mathcal{A}'$ are Moishezon. (See the the discussion on these issues in Section 7.) Equivalently, the generic type of a generic fibre of $\pi : Z \to X$ in $\mathcal{A}'$ is internal to $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$. Pillay translates this fact into the following model-theoretic statement:

THEOREM A.2 (Pillay [32]). *Suppose b and c are finite tuples from $\mathcal{A}'$ such that $\mathrm{tp}(c/b)$ is stationary, and b is the canonical base of $\mathrm{tp}(c/b)$. Then $\mathrm{tp}(b/c)$ is internal to $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$.*

Indeed, let $X = \mathrm{locus}(c), B = \mathrm{locus}(b)$, and $Z = \mathrm{locus}(b, c)$. Using Hironaka's Flattening Thoerem we may assume that $Z \to B$ is flat. Moreover,

[27] See Corollary 2.41 in [25] for a more detailed proof.

[28] Campana's version uses cycle spaces instead of Douady spaces.

as $b = \mathrm{Cb}(c/b)$, the general fibres of $Z \to B$ are distinct as subspaces of X. It follows that the Douady map associated to $Z \to B$ is a meromorphic embedding. We may therefore assume that $B \subset \mathcal{D}(X)$, and Z is the restriction of $\mathcal{Z}(X)$ to B. Note that $\mathrm{tp}(b/c)$ is the generic type of a generic fibre of $Z \to X$. Theorem A.2 now follows from Fact A.1.

The following corollary implies that every strongly minimal set in $\mathcal{A}$ is either locally modular or a projective curve up to finitely many points—that is, we obtain a new and direct proof of Theorem 3.1.

COROLLARY A.3 (Pillay [32]). *Every stationary U-rank 1 type in $\mathcal{A}'$ is either locally modular or nonorthogonal to the generic type of* $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$.

To see how this follows from Theorem A.2, let $p(x)$ be a stationary U-rank 1 type in $\mathcal{A}'$. For ease of dicussion, we suppress the parameters of p. If $p(x)$ is not locally modular, then there is a tuple of realisations of $p(x)$, c, and a tuple b from $\mathcal{A}'$, such that $b = \mathrm{Cb}(c/b)$ and $b \notin \mathrm{acl}(c)$. By the theorem, $\mathrm{tp}(b/c)$ is internal to $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$. Since $\mathrm{tp}(b/c)$ is not (model-theoretically) algebraic and $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$ is minimal, this implies that $\mathrm{tp}(b/c)$ is nonorthogonal to the generic type of $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$. As c is a tuple of realisations of $p(x)$ and $b = \mathrm{Cb}(c/b)$, it follows that $p(x)$ is nonorthogonal to the generic type of $\mathbb{P}(\mathbb{C}^{\mathcal{A}'})$, as desired.

In [32], using Theorem A.2, Pillay also obtains a generalisation of Theorem 4.6 on subvarieties of meromorphic groups, without reference to the theory of the socle.

The ideas involved in this section and in the proof of Fact A.1 have been useful in obtaining similar results in other algebraic/model-theoretic contexts; namely, for differential and difference fields (see [36] and [33]).[29]

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[29] ***Note added in proof***: The results from my thesis discussed in this survey have now been written-up in two papers: "A nonstandard Riemann existence theorem" (*Transactions of the American Mathematical Society*, vol. 356 (2004), no. 5, pp. 1781–1797) and "On saturation and the model theory of compact Kähler manifolds" (to appear in Crelle's Journal). Moreover, my notes entitled "Jet spaces in complex analytic geometry: an exposition", which describe the techniques used to prove Fact A.1, is now available at `http://arxiv.org/abs/math.LO/0405563`.

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DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
200 UNIVERSITY AVE. WEST
WATERLOO, ONTARIO, N2L 3G1, CANADA
E-mail: rmoosa@math.uwaterloo.ca