

INVARIANT HYPERSURFACES

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ABSTRACT. The following theorem, which includes as very special cases results of Jouanolou and Hrushovski on algebraic D -varieties on the one hand, and of Cantat on rational dynamics on the other, is established: Working over a field of characteristic zero, suppose $\phi_1, \phi_2 : Z \rightarrow X$ are dominant rational maps from a (possibly nonreduced) irreducible scheme Z of finite-type to an algebraic variety X , with the property that there are infinitely many hypersurfaces on X whose scheme-theoretic inverse images under ϕ_1 and ϕ_2 agree. Then there is a nonconstant rational function g on X such that $g\phi_1 = g\phi_2$. In the case when Z is also reduced the scheme-theoretic inverse image can be replaced by the proper transform. A partial result is obtained in positive characteristic. Applications include an extension of the Jouanolou-Hrushovski theorem to generalised algebraic \mathcal{D} -varieties and of Cantat's theorem to self-correspondences.

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1. INTRODUCTION

Fix an algebraically closed field K of characteristic zero. The following is the main result of this paper:

Theorem 1.1. *Suppose X is an algebraic variety, Z is an irreducible algebraic scheme, and $\phi_1, \phi_2 : Z \rightarrow X$ are rational maps whose restrictions to Z_{red} are dominant, all over K . Then the following are equivalent:*

- (1) *There exist nonempty Zariski open subsets $V \subseteq Z$ and $U \subseteq X$ such that the restrictions $\phi_1^V, \phi_2^V : V \rightarrow U$ are dominant regular morphisms, and there exist infinitely many hypersurfaces H on U satisfying*

$$(\phi_1^V)^{-1}(H) = (\phi_2^V)^{-1}(H).$$

- (2) *There exists $g \in K(X) \setminus K$ such that $g\phi_1 = g\phi_2$.*

If Z is reduced then these are also equivalent to:

- (3) *There exist infinitely many hypersurfaces H on X satisfying $\phi_1^*H = \phi_2^*H$.*

Here, and throughout this paper, we only consider *algebraic schemes*, i.e. separated schemes of finite type. By an *algebraic variety* we mean an integral (so reduced and irreducible) algebraic scheme, and by a *hypersurface* we mean a Zariski closed subset of pure codimension one. If $\phi : Z \rightarrow X$ is a morphism of schemes and $H \subseteq X$ is a Zariski closed subset, then we use $\phi^{-1}(H)$ to denote the *scheme-theoretic inverse image* of H . If $\phi : Z \rightarrow X$ is a dominant rational map of algebraic varieties then ϕ^*H denotes the *proper transform* of H , i.e. the Zariski closure of the set-theoretic inverse image of a sufficiently small Zariski dense and open subset of H .

As motivation, let us consider two well-known special cases of the theorem.

The first is from differential-algebraic geometry. By an *algebraic D -variety* we mean an affine algebraic variety X over K equipped with a regular section to the tangent bundle, $s : X \rightarrow TX$. A closed subvariety $H \subseteq X$ is a *D -subvariety* if $s|_H : H \rightarrow TH$. Note that s corresponds to a K -linear derivation δ on the coordinate ring $K[X]$, and that a D -subvariety corresponds to a δ -ideal of $K[X]$. Note also that the derivation δ extends uniquely to a derivation on the fraction field $K(X)$. The following is a consequence of unpublished work of Hrushovski [5] in the mid-nineties on model-theoretic implications of a theorem of Jouanolou [6] on foliations from the seventies; see [4, Theorem 4.2] for a published account.

Corollary 1.2 (Jouanolou-Hrushovski). *Suppose (X, s) is an algebraic D -variety with infinitely many D -subvarieties of pure codimension one. Then there exists $g \in K(X) \setminus K$ such that $\delta(g) = 0$.*

Proof. If $X = \text{Spec}(R)$, apply Theorem 1.1 to

- $Z = \text{Spec}(R[\epsilon]/\epsilon^2)$,
- $\phi_1 : Z \rightarrow X$ the morphism induced by the K -algebra homomorphism from R to $R[\epsilon]/\epsilon^2$ given by $r \mapsto r + \delta(r)\epsilon$, and
- $\phi_2 : Z \rightarrow X$ the morphism induced by the natural inclusion of R in $R[\epsilon]/\epsilon^2$.

See §5 for details. □

In fact, we obtain the Jouanolou-Hrushovski result for the general setting of “algebraic D -varieties”, where the derivation is replaced by any system \mathcal{D} of *generalised operators* in the sense of Moosa-Scanlon [11]. This is Theorem 5.7 below.

We also recover from Theorem 1.1 a result in rational dynamics. By a *rational dynamical system* we mean an algebraic variety X over K equipped with a dominant rational self-map $\phi : X \rightarrow X$. A Zariski closed subset $H \subseteq X$ is *totally invariant* if $\phi^*H = H$. The following is the algebraic case of Theorem B in [3].

Corollary 1.3 (Cantat [3]). *Suppose (X, ϕ) is a rational dynamical system with infinitely many totally invariant hypersurfaces. Then there exists $g \in K(X) \setminus K$ such that $g\phi = g$.*

Proof. Apply Theorem 1.1 to

- $Z = X$,
- $\phi_1 = \phi$, and
- $\phi_2 = \text{id}_X$.

See §6 for details. □

Again, we actually get more: we can replace the dominant rational self-map ϕ in the above corollary with an arbitrary self-correspondence. This is Corollary 6.2 below. In fact, it is, we think, useful to view the data of Theorem 1.1, namely the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\phi_1} & X \\ \phi_2 \downarrow & & \\ X & & \end{array}$$

as a generalised notion of self-correspondence on X , a self-correspondence that need not be reduced and need not be finite-to-finite.

Our theorem thus unifies these two well-known results, yielding at the same time natural generalisations in both cases.

A word about the proof of Theorem 1.1. Our approach is algebraic, thus differing significantly from the methods of Jouanolou-Hrushovski, and Cantat in the special cases. We first reduce to a situation where everything is defined over a finitely generated subfield and the hypersurfaces have principal vanishing ideals. In that setting our result appears as Theorem 3.1 below, whose proof is where the main technical work of the paper is done. When Z is reduced we follow to some extent the approach of [2, Theorem 1.2] which is related to Cantat's theorem but obtained independently. A separate argument (appearing in Section 6) is required to replace the scheme-theoretic inverse image with the proper transform in the reduced case. When Z is non-reduced we concoct a derivation and rely on a refinement of [1, Proposition 6.10] that is itself a refinement (but independent) of Jouanolou-Hrushovski. Besides this use of [1], which is substantial, our proof of Theorem 1.1 is largely self-contained.

Once Theorem 1.1 is proved, we look closer in Section 7 at the case when Z is reduced, and are lead to study the birational geometry of algebraic varieties equipped with a set of hypersurfaces. More precisely, we consider the category whose objects are normal varieties X equipped with a set of prime divisors S , and where a morphism $(X, S) \rightarrow (Y, T)$ is a dominant rational map $X \rightarrow Y$ with generic fibre irreducible, and such that, up to a finite set, S is obtained as the proper transform of elements of T . For this category, in the case when we are working over a field of finite transcendence degree, we give a geometric proof that every object (X, S) admits a terminal morphism; one that factors through every other morphism

originating at (X, S) . See Theorem 7.2 below for a precise statement. Combining this theorem – which may be of independent interest – with Cantat’s theorem, we obtain a more conceptual alternative proof of Theorem 1.1 in the special case when Z is reduced and ϕ_1, ϕ_2 have irreducible generic fibres.

In a final section we discuss what goes through in positive characteristic. Because of our reliance on the characteristic zero differential-algebraic geometry of [1] when Z is non-reduced, we restrict our attention to the reduced case. But even that – namely, the equivalence of conditions (2) and (3) of Theorem 1.1 when Z is reduced – does not hold as stated in positive characteristic. We do expect it to hold if we ask the generic fibres of ϕ_1 and ϕ_2 to be *geometrically reduced* – something that is automatically satisfied in characteristic zero. But we are only able to prove the equivalence if we add to (3) the additional constraint that infinitely many of the invariant hypersurfaces H are defined over (the separable closure of) a fixed finitely generated subfield. This is Theorem 8.1 below.

Our current methods have a couple of drawbacks that we present here as suggesting possibilities for future work. The first is regarding effective uniform bounds. Tracing through the proofs it is possible to compute explicitly a bound N such that the “infinitely many” in (1) and (3) of Theorem 1.1 can be replaced by “more than N ”. But N will depend not only on natural geometric invariants associated to the data, but also on the rank of the divisor class group of X (over a minimal field of definition). So these bounds are worse, less uniform, than those that arise in the special cases dealt with by the work of Jouanolou-Hrushovski and Cantat. It would be useful to find an effective bound N that remains constant as Z, X, ϕ_1, ϕ_2 vary in an algebraic family.

A second deficiency is that we are not able to work in the complex analytic setting. The methods of Jouanolou-Hrushovski and Cantat, in contrast, extend to compact complex manifolds and meromorphic maps. In particular, Cantat’s results in [3] include as a special case Krasnov’s theorem [8] that a compact complex manifold without nonconstant meromorphic functions has only finitely many hypersurfaces. A generalisation of Theorem 1.1 that includes complex analytic spaces would therefore be of significant interest.

Throughout this paper all rings are assumed to be commutative, unitary, and all fields are of characteristic zero except in the final Section 8.

2. SOME DIFFERENTIAL ALGEBRA PRELIMINARIES

By a *derivation* we mean a linear map $\delta : R \rightarrow S$, where $R \subseteq S$ is an extension of integral domains of characteristic zero, such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$. If $A \subseteq R$ is a subring, then we say that δ is *A-linear* to mean that $\delta(a) = 0$ for all $a \in A$ – which we note is equivalent to δ being a morphism of A -modules. By the *constants* of the derivation $\delta : R \rightarrow S$ we mean the subring $R^\delta := \{a \in R : \delta a = 0\}$. If $R = S$ then we call (R, δ) a *differential ring*. A differential ring whose underlying ring is a field is called a *differential field*.

Here is a basic fact about derivations that we record now for later use, and that is deduced by a straightforward computation using the Leibniz rule.

Fact 2.1. *Suppose $\delta : R \rightarrow S$ is a derivation, P is a polynomial in $R[x_1, \dots, x_n]$, and $a = (a_1, \dots, a_n) \in R^n$. Then*

$$\delta P(a) = \sum_{j=1}^n \frac{\delta P}{\delta x_j}(a) \delta(a_j) + P^\delta(a)$$

where $P^\delta \in S[x_1, \dots, x_n]$ is obtained by applying δ to the coefficients of P .

The following two lemmas are also very elementary and well-known.

Lemma 2.2. *Suppose K/k is a function field extension, δ is a k -linear derivation on K , and $R \subseteq K$ is a finitely generated k -subalgebra. Then there exists a finitely generated k -algebra extension R' of R in K such that δ restricts to a differential ring structure on R' .*

Proof. We may as well assume that $R = k[a]$ for some $a = (a_1, \dots, a_n)$ such that $K = k(a)$. Because of Fact 2.1, it suffices to show that after possibly extending a to a longer finite tuple $a' = (a_1, \dots, a_m)$ from K , and setting $R' := k[a']$, that $\delta a_i \in R'$ for all $i \leq m$. If we write $\delta(a_i) = \frac{P_i(a)}{Q_i(a)}$ for $i \leq n$, then it is not hard to see, using Fact 2.1 again, as well as the quotient rule for derivations, that $a' := (a_1, \dots, a_n, \frac{1}{Q_1(a)}, \dots, \frac{1}{Q_n(a)})$ works. \square

Lemma 2.3. *If $(L, \delta) \supseteq (K, \delta)$ is a differential field extension then*

$$(K^\delta)^{\text{alg}} \cap L = K^{\text{alg}} \cap L^\delta.$$

In particular, taking $L = K$, we have that K^δ is relatively algebraically closed in K .

Proof. If $a \in L$ is algebraic over K^δ then we apply Fact 2.1 with $P(x)$ a minimal polynomial of a over K^δ and conclude that $\frac{dP}{dx}(a)\delta(a) = 0$. But as $\deg P \geq 1$, we must have that $\frac{dP}{dx} \neq 0$ and of degree strictly less than $\deg P$, so that $\delta(a) = 0$.

Conversely, if $a \in L^\delta$ is algebraic over K then we apply Fact 2.1 with $P(x)$ the minimal monic polynomial of a over K and conclude that $P^\delta(a) = 0$. But as P is monic P^δ will be of strictly smaller degree unless it is identically zero. So it must be identically zero, implying that all of the coefficients of P are in K^δ , and hence that a is algebraic over K^δ . \square

The following is maybe less widely known, but is a consequence of an argument appearing in [5]. We give a proof here for the sake of completeness.

Lemma 2.4. *Suppose F/k is a function field extension, K/F is a field extension, and $\delta : F \rightarrow K$ is a k -linear derivation. Then δ extends to a differential field structure on K such that K^δ is algebraic over F^δ .*

Proof. Suppose $t \in K$ is transcendental over F . For each $\gamma \in k$, consider the derivation $\delta_\gamma : F(t) \rightarrow K$ induced by $\delta_\gamma(t) = \gamma t$. We claim that for some $\gamma \in k$, $F(t)^{\delta_\gamma} = F^\delta$. Fix a sufficiently saturated differentially closed field (\mathcal{U}, D) with constant field \mathcal{C} . Each extension $\delta_\gamma : F(t) \rightarrow K$ embeds into (\mathcal{U}, D) . Moreover, as these extensions all agree on F , we may as well assume that $F \subseteq \mathcal{U}$ and that these embeddings are over F . So we get elements $t_\gamma \in \mathcal{U}$ and subfields $K_\gamma \subseteq \mathcal{U}$, such that $(F(t), K, \delta_\gamma)$ is isomorphic to $(F(t_\gamma), K_\gamma, D \upharpoonright_{F(t_\gamma)})$. In particular, $D(t_\gamma) = \gamma t_\gamma$. Now, if $F(t)^{\delta_\gamma} \neq F^\delta$, and we let $g \in F(t)^{\delta_\gamma} \setminus F$, then the image of g in (\mathcal{U}, D) is an element $h \in F(t_\gamma)^D \setminus F$. It follows by Steinitz exchange that $t_\gamma \in F(h)^{\text{alg}}$. Writing the function field F as $k(a)$, we have that $F(h)^{\text{alg}} \subseteq \mathcal{C}(a)^{\text{alg}}$. That is, t_γ

is a solution to the equation $Dx = \gamma x$ in $\mathcal{C}(a)^{\text{alg}}$. But the set of $\gamma \in \mathcal{C}$ such that $Dx = \gamma x$ has a solution in fixed finite transcendence degree extension of \mathcal{C} – such as $\mathcal{C}(a)^{\text{alg}}$ is – forms a finite rank additive subgroup of \mathcal{C} . This is an old result of Kolchin [7], but see also [4, Fact 4.3]. Hence there must exist $\gamma \in k$ for which $Dx = \gamma x$ has no solution in $\mathcal{C}(a)^{\text{alg}}$. For such a γ , $F(t)^{\delta\gamma} = F^\delta$, as desired.

So, iterating this process, if we let E be a transcendence basis for K over F , then we can find an extension of δ , which we will also call δ , to $F(E)$ with no new constants. Since K is algebraic over $F(E)$ there is by Fact 2.1 a unique further extension of δ to K . By Lemma 2.3, K^δ is algebraic over $F(E)^\delta = F^\delta$. \square

As discussed in the introduction, a special case of our main theorem is the finite-dimensional case of the Jouanolou-Hrushovski theorem on D -subvarieties of codimension one. An algebraic proof of this finiteness theorem in the context of several derivations was given in [1]. We will rely on the following refinement of that result in the case of a single derivation.

Proposition 2.5. *Suppose k is a finitely generated field, A is a finitely generated k -algebra, and $\delta : A \rightarrow A$ is a k -linear derivation. Suppose that there exists an infinite sequence $(r_j : j < \omega)$ in $\text{Frac}(A)$ such that $\delta(r_j)/r_j \in A$ for all $j < \omega$, and such that $(r_j : j < \omega)$ is multiplicatively independent modulo k^{alg} . Then there exists $g \in \text{Frac}(A)^\delta \setminus k^{\text{alg}}$. In fact, if G is the multiplicative group generated by $(r_j : j < \omega)$, then $G \cap \text{Frac}(A)^\delta$ is nontrivial.*

Remark 2.6. By the sequence being *multiplicatively independent modulo k^{alg}* we mean that its image in $\text{Frac}(A)^\times / (\text{Frac}(A)^\times \cap k^{\text{alg}})$ is multiplicatively independent. In other words, no nontrivial product of integer powers of the r_j 's is in k^{alg} .

Proof. This is quite close to [1, Proposition 6.10], but among the differences are that we are working over a finitely generated field rather than an uncountable algebraically closed field, and that the r_j are coming from $\text{Frac}(A)$ rather than from A itself. We have therefore something to do.

First, let us observe that we get the ‘‘in fact’’ clause for free. Indeed, letting $F := \text{Frac}(A)^\delta$, consider the finitely generated F -algebra $A' = FA$. Then F is again a finitely generated field, δ is an F -linear differential structure on A' , and $(r_j : j < \omega)$ in $\text{Frac}(A') = \text{Frac}(A)$ satisfies $\delta(r_j)/r_j \in A'$ for all $j < \omega$. We can apply the main statement of the theorem – which we are assuming we have proved – to this context over F . Since $\text{Frac}(A')^\delta = F$, we must have that $(r_j : j < \omega)$ is not multiplicatively independent modulo F^{alg} . But note that F is relatively algebraically closed in $\text{Frac}(A)$ by Lemma 2.3. So $(r_j : j < \omega)$ is not multiplicatively independent modulo F . That is, some nontrivial product of integer powers of the r_j is in F . We have shown that $G \cap F$ is nontrivial, as desired.

So it suffices to prove that $\text{Frac}(A)^\delta \not\subseteq k^{\text{alg}}$.

Next, observe that we can always replace A by a finitely generated localisation. Indeed, this does not change the fraction field, and, using Fact 2.1, one can compute that any such localisation is a differential subring of $(\text{Frac}(A), \delta)$. So we may assume that A is integrally closed. Moreover, as k is a finitely generated field, some finitely generated localisation of A is a unique factorisation domain – this is by [1, Lemma 6.11] though one expects it to have appeared elsewhere and earlier. So we may also assume that A is a UFD.

Consider $k' := \text{Frac}(A) \cap k^{\text{alg}}$. By integral closedness, $k' \subseteq A$. Moreover, δ is k' -linear. The hypotheses of the theorem hold for (A, k') , and if the conclusion

were true for (A, k') then it would be true of (A, k) . That is, it suffices to prove the theorem for k' in place of k . So we may also assume $\text{Frac}(A) \cap k^{\text{alg}} = k$.

Next, we move the r_j into A itself. For each $j < \omega$, write $r_j = \frac{c_j}{d_j}$ where $c_j, d_j \in A$ are coprime. Since $\frac{\delta(r_j)}{r_j} = \frac{\delta(c_j)}{c_j} + \frac{\delta(d_j)}{d_j}$, it follows that $\frac{\delta(c_j)d_j}{c_j} \in A$. Coprimality of c_j and d_j in A then implies $\frac{\delta(c_j)}{c_j} \in A$. A symmetric argument shows that $\frac{\delta(d_j)}{d_j} \in A$. Note that the multiplicative group generated by $\{c_j, d_j : j < \omega\}$ contains that generated by $\{r_j : j < \omega\}$, so the former must also have infinite rank modulo its intersection with k^{alg} . We can therefore find in A a sequence $(a_j : j < \omega)$ that is multiplicatively independent modulo k^{alg} and such that $\frac{\delta(a_j)}{a_j} \in A$ for all $j < \omega$.

Let K be an uncountable algebraically closed field extending k . It follows that $A_K := A \otimes_k K$ is an integrally closed domain extending A , finitely generated over K , and with the property that $\text{Frac}(A) \cap K = k$ in $\text{Frac}(A_K)$. Hence, no nontrivial product of integer powers of the a_j is in K either. Moreover δ extends to a K -linear derivation on $\text{Frac}(A_K)$ and $\frac{\delta(a_j)}{a_j} \in A_K$ for all $j < \omega$. Proposition 6.10 of [1] now applies and we obtain an element $g_K \in \text{Frac}(A_K)^\delta \setminus K$.

At this point we can use a specialisation argument, or, as we prefer to do, the model-completeness of the first order theory of algebraically closed fields, to see that $\text{Frac}(A_{k^{\text{alg}}})^\delta \setminus k^{\text{alg}} \neq \emptyset$ where $A_{k^{\text{alg}}} := A \otimes_k k^{\text{alg}}$. Indeed, $g_K : (X, s) \rightarrow (\mathbb{A}^1, 0)$ is a nonconstant rational map over K on the D -variety (X, s) associated to (A_K, δ) . This is a first order expressible property over k of the parameters in K over which g is defined. As k^{alg} is an elementary substructure of K , we thus obtain nonconstant $g' : (X, s) \rightarrow (\mathbb{A}^1, 0)$ over k^{alg} . That is, $g' \in \text{Frac}(A_{k^{\text{alg}}})^\delta \setminus k^{\text{alg}}$. Now, $\text{Frac}(A_{k^{\text{alg}}})^\delta$ is algebraic over $\text{Frac}(A)^\delta$, so the canonical parameter for the finite set of Galois conjugates of g' is a tuple from $\text{Frac}(A)^\delta$ not all of whose co-ordinates can be in k^{alg} since g isn't. Hence $\text{Frac}(A)^\delta \setminus k^{\text{alg}} \neq \emptyset$. \square

3. THE PRINCIPAL ALGEBRAIC STATEMENT

The key step in our proof of Theorem 1.1 will be the following statement in commutative algebra. It is given here in slightly greater generality than necessary; we will only apply it in the case that the nilradical of S is prime, and the reader is invited to make this assumption, and thereby remove a few of the technicalities, if he or she desires.

Theorem 3.1. *Suppose*

- k is a finitely generated field of characteristic zero,
- R is a finitely generated k -algebra that is an integral domain and such that k is relatively algebraically closed in $\text{Frac}(R)$,
- S is a finitely generated k -algebra, such that k is relatively algebraically closed in $\text{Frac}(S/P)$ for every minimal prime ideal P of S , and
- $f_1, f_2 : R \rightarrow S$ are k -algebra homomorphisms that take nonzero elements of R to regular elements of S .

Suppose there exists a sequence of nonzero elements $(a_j : j < \omega)$ in R that is multiplicatively independent modulo k , and such that $f_1(a_j)S = f_2(a_j)S$ for all $j < \omega$. Then there exists $g \in \text{Frac}(R) \setminus k$ such that $f_1(g) = f_2(g)$.

In fact, if we let F be the subfield of $\text{Frac}(R)$ on which f_1 and f_2 agree, and we let G be the subgroup of $\text{Frac}(R)^\times$ generated by $(a_j : j < \omega)$, then $G \cap F$ is nontrivial.

Remark 3.2. (a) The assumptions on f_1 and f_2 imply that they extend uniquely to embeddings of $\text{Frac}(R)$ into $\text{Frac}(S)$, where by $\text{Frac}(S)$ we mean the localisation of S at the set of all regular elements. It is with respect to these extensions that we mean $f_1(g) = f_2(g)$ in the conclusion of the theorem.

(b) Any nontrivial element of G is necessarily transcendental over k ; this follows from the multiplicative independence of $(a_j : j < \omega)$ modulo k together with the fact that k is relatively algebraically closed in $\text{Frac}(R)$.

Proof. Let us first consider the case when S is a reduced ring.

Since $f_1(a_j)S = f_2(a_j)S$, for each $j < \omega$ there is a unit u_j in S such that $f_1(a_j) = u_j f_2(a_j)$. Let P_1, \dots, P_ℓ be the minimal primes of S , and denote by $S_\mu := S/P_\mu$ the corresponding integral domain for each $\mu = 1, \dots, \ell$. Let

$$\bar{u}_j := (u_j + P_1, \dots, u_j + P_\ell) \in S_1^\times \times \dots \times S_\ell^\times.$$

Now, as k is relatively algebraically closed in S_μ , each S_μ^\times/k^\times is a finite rank group, and hence so is $S_1^\times/k^\times \times \dots \times S_\ell^\times/k^\times$. It follows that for some $N > 0$ and all $r > 0$,

$$\bar{u}_{(r-1)N+1}^{k_{r,1}} \cdot \bar{u}_{(r-1)N+2}^{k_{r,2}} \cdots \bar{u}_{rN}^{k_{r,N}} = \lambda_r$$

for some $\lambda_r = (\lambda_{r,1}, \dots, \lambda_{r,\ell}) \in (k^\times)^\ell$ and some $k_{r,1}, \dots, k_{r,N} \in \mathbb{Z}$ not all zero. (Note that if you assumed in the theorem that the nilradical of S was prime then here S is already a domain and $\ell = 1$ with $P_1 = (0)$.)

Let $f_{1,\mu}, f_{2,\mu} : R \rightarrow S_\mu$ be the k -algebra homomorphisms induced by f_1 and f_2 for $\mu = 1, \dots, \ell$. By assumption, they are still injective. Consider, for each $r > 0$,

$$h_r := a_{(r-1)N+1}^{k_{r,1}} \cdot a_{(r-1)N+2}^{k_{r,2}} \cdots a_{rN}^{k_{r,N}}$$

in $\text{Frac}(R) = k(X)$. By construction $f_{1,\mu}(h_r) = \lambda_{r,\mu} f_{2,\mu}(h_r)$.

Letting m be greater than $\dim X$, we get that $\{h_1, \dots, h_m\}$ is algebraically dependent over k . Let $\sum c_{i_1, \dots, i_m} h_1^{i_1} \cdots h_m^{i_m} = 0$ be a nontrivial algebraic relation over k with a minimal number of nonzero coefficients. Note that as none of the h_r are zero, there are at least two nonzero coefficients in this relation. Fixing $\mu = 1, \dots, \ell$ and applying $f_{1,\mu}$ to this we get

$$\sum c_{i_1, \dots, i_m} \lambda_{1,\mu}^{i_1} \cdots \lambda_{m,\mu}^{i_m} f_{2,\mu}(h_1)^{i_1} \cdots f_{2,\mu}(h_m)^{i_m} = 0$$

while applying $f_{2,\mu}$ yields

$$\sum c_{i_1, \dots, i_m} f_{2,\mu}(h_1)^{i_1} \cdots f_{2,\mu}(h_m)^{i_m} = 0.$$

Suppose that $\lambda_{1,\mu}^{i_1} \cdots \lambda_{m,\mu}^{i_m} \neq \lambda_{1,\mu}^{j_1} \cdots \lambda_{m,\mu}^{j_m}$ for some distinct nonzero c_{i_1, \dots, i_m} and c_{j_1, \dots, j_m} . Manipulating these two equations and then taking $f_{2,\mu}^{-1}$ we would get a relation among the h_1, \dots, h_m with fewer nonzero coefficients. As this is impossible by minimality, it must be that $\lambda_{1,\mu}^{i_1} \cdots \lambda_{m,\mu}^{i_m} = \lambda_{1,\mu}^{j_1} \cdots \lambda_{m,\mu}^{j_m}$ whenever c_{i_1, \dots, i_m} and c_{j_1, \dots, j_m} are nonzero. But then, fixing $(i_1, \dots, i_m) \neq (j_1, \dots, j_m)$ with c_{i_1, \dots, i_m} and c_{j_1, \dots, j_m} nonzero, we get that

$$f_{1,\mu}(h_1^{i_1-j_1} \cdots h_m^{i_m-j_m}) = f_{2,\mu}(h_1^{i_1-j_1} \cdots h_m^{i_m-j_m}).$$

Setting $g := h_1^{i_1-j_1} \cdots h_m^{i_m-j_m}$ we have that $f_{1,\mu}(g) = f_{2,\mu}(g)$ for all $\mu = 1, \dots, \ell$. Since S is reduced, $P_1 \cap \cdots \cap P_\ell = (0)$, and hence $f_1(g) = f_2(g)$. That is, $g \in G \cap F$. It remains only to verify that $g \neq 1$. Since $i_r - j_r \neq 0$ for some $r = 1, \dots, m$, and the corresponding $k_{r,1}, \dots, k_{r,N}$ are not all zero, g is a nontrivial product of integer powers of the a_i . By multiplicative independence modulo k , $g \neq 1$.

Now we deal with the case when the nilradical N of S is nontrivial. For any ideal $I \leq N$, let $f_\nu^I : R \rightarrow S/I$ be the composition of f_ν with $S \rightarrow S/I$. Notice that since $I \leq N$ the image of a regular element in S remains regular in S/I . So f_1^I, f_2^I extend to embeddings of $\text{Frac}(R)$ in $\text{Frac}(S/I)$. Consider the subfield

$$F_I := \{g \in \text{Frac}(R) : f_1^I(g) = f_2^I(g)\}.$$

Note that $F = F_{(0)}$. Note also that $G/G \cap F_N$ is of finite rank. Indeed, otherwise we would have a subsequence $(b_j : j < \omega)$ of $(a_j : j < \omega)$ that is multiplicatively independent modulo F_N . But the reduced case applied to this subsequence, which is *a fortiori* multiplicatively independent modulo k , would tell us that $\langle b_j : j < \omega \rangle \cap F_N$ is nontrivial, contradicting the multiplicative independence modulo F_N .

Reduction 3.3. It suffices to prove the theorem under the assumption that for any nonzero ideal $I \leq N$, $G/(G \cap F_I)$ is of finite rank.

Proof of Reduction 3.3. Assume we have proven the theorem under this additional condition. Set $\mathcal{I} := \{I \leq N : G/(G \cap F_I) \text{ is of infinite rank}\}$. Suppose, toward a contradiction, that \mathcal{I} is nonempty. By noetherianity there is a maximal element $J \in \mathcal{I}$. Let $(b_j : j < \omega)$ be a subsequence of $(a_i : i < \omega)$ that is multiplicatively independent modulo F_J . Notice that the assumptions of the theorem remain true of f_1^J, f_2^J and $(b_\ell : \ell < \omega)$. Moreover, the condition of the claim is true of f_1^J, f_2^J and $(b_\ell : \ell < \omega)$ by maximal choice of J . Applying the theorem to f_1^J, f_2^J and $(b_\ell : \ell < \omega)$, we get that $\langle b_\ell : \ell < \omega \rangle \cap F_J$ is nontrivial. But this contradicts the multiplicative independence of $(b_\ell : \ell < \omega)$ over F_J .

So \mathcal{I} is empty. In particular, $G/(G \cap F)$ is of finite rank. But then certainly $G \cap F$ is nontrivial, as desired. \square

Reduction 3.4. It suffices to prove the theorem under the additional assumption that there exists a nonzero $x \in N$ such that $x^2 = 0$, $Q := \text{ann}(x)$ is prime, and if $(x) = Q_1 \cap \cdots \cap Q_\ell$ is the primary decomposition of (x) then $Q_1 \cup \cdots \cup Q_\ell$ contains no regular element of S .

Proof of Reduction 3.4. As N is not trivial, let $x \in N$ be a nonzero element with maximal annihilator. Then $x^2 = 0$ and primality of Q follow. For each ideal in the primary decomposition of (x) , if that ideal has a regular element, we can work in the extension of S obtained by inverting that regular element. The extension S' we obtain is still finitely generated. We can now apply the theorem with S' in place of S , noting that x is still nonzero in S' , $x^2 = 0$ remains true, $\text{ann}_{S'}(x) = QS'$ is still prime, and now the primary ideals appearing in the decomposition of xS' all have no regular elements. We therefore obtain nonconstant $g \in \text{Frac}(R)$ such that $f_1(g) = f_2(g)$ in $\text{Frac}(S') = \text{Frac}(S)$, as desired. \square

Let x be as in Reduction 3.4. By Reduction 3.3, we have that $G/(G \cap F_{(x)})$ is of finite rank. There must exist a sequence $(b_j : j < \omega)$ in $G \cap F_{(x)}$ that is multiplicatively independent modulo k . Being in G implies that each b_j is product

of integer powers of some a_i 's. Since $f_1(a_i)S = f_2(a_i)S$ we have that $f_1(a_i)$ is a multiple of $f_2(a_i)$ by a unit in S^\times . The same is therefore true of b_j . That is,

$$(3.1) \quad f_1(b_j) = u_j f_2(b_j) \text{ for some unit } u_j \in S^\times \text{ for all } j < \omega.$$

On the other hand, being in $F_{(x)}$ means that

$$(3.2) \quad f_1(b_j) - f_2(b_j) \in x \operatorname{Frac}(S) \text{ for all } j < \omega.$$

Let $T := k[(b_j : j < \omega)]$ be the k -subalgebra of $\operatorname{Frac}(R)$ generated by these elements, and consider the k -linear map $f_1 - f_2$ restricted to T . Note that it satisfies the following twisted Leibniz rule:

$$(3.3) \quad (f_1 - f_2)(uv) = (f_1 - f_2)(u)f_1(v) + f_2(u)(f_1 - f_2)(v).$$

Moreover,

$$(3.4) \quad (f_1 - f_2)(T) \subseteq x \operatorname{Frac}(S).$$

Indeed, equation (3.2) tells us that $(f_1 - f_2)$ takes the generators of T into $x \operatorname{Frac}(S)$, and this property is clearly linear. So assuming that $(f_1 - f_2)(u), (f_1 - f_2)(v) \in x \operatorname{Frac}(S)$, it suffices to show that $(f_1 - f_2)(uv) \in x \operatorname{Frac}(S)$. This follows immediately from the above Leibniz rule.

Let $\pi_Q : \operatorname{Frac}(S) \rightarrow \operatorname{Frac}(S/Q)$ be induced by the quotient $S \rightarrow S/Q$. Then $\pi_Q \circ f_1, \pi_Q \circ f_2 : \operatorname{Frac}(R) \rightarrow \operatorname{Frac}(S/Q)$ are embeddings of fields. Since $x^2 = 0$, we have $x \in Q$, and so equation (3.4) tells us that $\pi_Q \circ f_1$ and $\pi_Q \circ f_2$ agree on T . We use this embedding to view $T \subseteq \operatorname{Frac}(S/Q)$.

Claim 3.5. There exists a k -linear derivation δ on $\operatorname{Frac}(S/Q)$ satisfying:

- (i) For all $t \in T$ and $s \in \operatorname{Frac}(S)$, $\pi_Q(s) = \delta(t)$ if and only if $f_1(t) - f_2(t) = xs$.
- (ii) The constant field $\operatorname{Frac}(S/Q)^\delta$ is algebraic over $\operatorname{Frac}(T)^\delta$.
- (iii) $\operatorname{Frac}(T)^\delta \subseteq F$

Proof of Claim 3.5. We first find a derivation $\delta : T \rightarrow \operatorname{Frac}(S/Q)$ with property (i). Given $t \in T$, we know by (3.4) that $f_1(t) - f_2(t) = xs$ for some $s \in \operatorname{Frac}(S)$. Now for some other $s' \in \operatorname{Frac}(S)$ we have

$$\begin{aligned} f_1(t) - f_2(t) = xs' &\iff (s - s')x = 0 \\ &\iff s - s' \in Q \operatorname{Frac}(S) \text{ since } Q = \operatorname{ann}(x) \\ &\iff \pi_Q(s) = \pi_Q(s'). \end{aligned}$$

So we can define $\delta(t) := \pi_Q(s)$, and it will have the desired property. That δ is k -linear is clear from the construction. That it is a derivation follows from (3.3). Indeed, given $u, v \in T$, let $s_u, s_v \in \operatorname{Frac}(S)$ be such that $(f_1 - f_2)(u) = xs_u$ and $(f_1 - f_2)(v) = xs_v$. Then (3.3) along with the construction of δ gives us:

$$\begin{aligned} \delta(uv) &= \pi_Q(s_u f_1(v) + f_2(u) s_v) \\ &= \pi_Q(s_u)v + u\pi_Q(s_v) \quad \text{by our identification of } T \subseteq \operatorname{Frac}(S/Q) \\ &= \delta(u)v + u\delta(v) \end{aligned}$$

as desired.

Now, there is a unique extension of δ to $\operatorname{Frac}(T)$ using the usual quotient rule: $\delta\left(\frac{u}{v}\right) := \frac{v\delta u - u\delta v}{v^2}$. Since $\operatorname{Frac}(T)$ is finitely generated over k , we can apply Lemma 2.4 and extend δ further to a derivation $\delta : \operatorname{Frac}(S/Q) \rightarrow \operatorname{Frac}(S/Q)$ whose constant field is algebraic over the constants in $\operatorname{Frac}(T)$. That is, it satisfies property (ii).

Finally we show (iii). Suppose $g = \frac{u}{v} \in \text{Frac}(T)$ and $\delta(g) = 0$. This means $v\delta(u) = u\delta(v)$. Letting $s_u, s_v \in \text{Frac}(S)$ be such that $(f_1 - f_2)(u) = xs_u$ and $(f_1 - f_2)(v) = xs_v$, we have by (i) that $\pi_Q(s_u) = \delta(u)$ and $\pi_Q(s_v) = \delta(v)$. So, $\pi_Q(f_1(v)s_u - f_1(u)s_v) = v\delta(u) - u\delta(v) = 0$ so that $f_1(v)s_u - f_1(u)s_v \in Q \text{Frac}(S)$. Hence $0 = (f_1(v)s_u - f_1(u)s_v)x = f_1(v)(f_1 - f_2)(u) - f_1(u)(f_1 - f_2)(v)$. That is, $\frac{(f_1 - f_2)(u)}{f_1(u)} = \frac{(f_1 - f_2)(v)}{f_1(v)}$, which implies that $1 - \frac{f_2(u)}{f_1(u)} = 1 - \frac{f_2(v)}{f_1(v)}$, and so $\frac{f_1(u)}{f_1(v)} = \frac{f_2(u)}{f_2(v)}$. That is, $f_1(g) = f_2(g)$, as desired. \square

Let $M := \{s \in \text{Frac}(S) : sx \in S\}$. This is an S -submodule of $\text{Frac}(S)$ that contains $Q \text{Frac}(S)$.

Claim 3.6. $\pi_Q(M) = S/Q$.

Proof of Claim 3.6. As M contains S , it suffices to show that $\pi_Q(M) \subseteq S/Q$. Suppose $\frac{c}{d} \in M$. So $cx = yd$ for some $y \in S$. Let $(x) = Q_1 \cap \dots \cap Q_\ell$ be the primary decomposition of (x) in S . It follows that for each $i = 1, \dots, \ell$, $yd \in Q_i$. Since d is regular, Reduction 3.4 implies no power of d can be in Q_i . Hence $y \in Q_i$ for all i . So $y = y'x$ for some $y' \in S$. Hence $cx = dy'x$, so that $(c - dy') \in \text{ann}(x) = Q$. It follows that $\pi_Q(\frac{c}{d}) = \pi_Q(y') \in S/Q$. \square

Let A be a finitely generated k -subalgebra of $\text{Frac}(S/Q)$ that contains S/Q and is preserved by the derivation – by Fact 2.2 this is possible. We show that for all $j < \omega$, $\frac{\delta(b_j)}{b_j} \in A$. Choose $s_j \in \text{Frac}(S)$ such that $f_1(b_j) - f_2(b_j) = xs_j$. By (3.1) we also have $f_1(b_j) - f_2(b_j) = f_2(b_j)(u_j - 1)$ where $u_j \in S^\times$. So $\frac{s_j}{f_2(b_j)}x \in S$, and hence $\frac{s_j}{f_2(b_j)} \in M$. Applying π_Q we get by Claims 3.5(i) that $\frac{\delta(b_j)}{b_j} \in \pi_Q(M)$. Now by Claim 3.6, $\frac{\delta(b_j)}{b_j} \in S/Q \subseteq A$.

To recap then, $(b_j : j < \omega)$ is a sequence in $\text{Frac}(A) = \text{Frac}(S/Q)$ that is multiplicatively independent modulo k , and hence modulo k^{alg} since k is relatively algebraically closed in R , and such that $\frac{\delta(b_j)}{b_j} \in A$ for all $j < \omega$. We are thus in the context of Proposition 2.5, and we can conclude that $\langle b_j : j < \omega \rangle \cap \text{Frac}(A)^\delta$ is nontrivial. But 3.5(ii) tells us that $\text{Frac}(A)^\delta$ is algebraic over $\text{Frac}(T)^\delta$, and 3.5(iii) says the latter is in F . It follows that $\langle b_j : j < \omega \rangle \cap F^{\text{alg}}$ is nontrivial.

Suppose $g \in \langle b_j : j < \omega \rangle \cap F^{\text{alg}}$ is nontrivial. We claim, finally, that $g \in F$. Indeed, suppose toward a contradiction that for some $m > 1$,

$$P(X) = X^m + c_{m-1}X^{m-1} + \dots + c_0$$

is the minimal polynomial of g over F . As f_1 and f_2 agree on F we may as well identify F with its image in $\text{Frac}(S)$ so that f_1, f_2 become F -linear. Applying f_ν to $P(g) = 0$ for $\nu = 1, 2$ yields

$$(3.5) \quad f_\nu(g)^m + c_{m-1}f_\nu(g)^{m-1} + \dots + c_0 = 0$$

in $\text{Frac}(S)$. Since $g \in F_{(x)}$ we have that $f_1(g) = f_2(g) + sx$ for some $s \in \text{Frac}(S)$. Substituting this into (3.5) for $\nu = 1$ we get

$$\begin{aligned} 0 &= (f_2(g) + sx)^m + c_{m-1}(f_2(g) + sx)^{m-1} + \dots + c_0 \\ &= (f_2(g)^m + c_{m-1}f_2(g)^{m-1} + \dots + c_0) + \\ &\quad sxmf_2(g)^{m-1} + sx(m-1)c_{m-1}f_2(g)^{m-2} + \dots + sxc_1 \\ &= sxmf_2(g)^{m-1} + sx(m-1)c_{m-1}f_2(g)^{m-2} + \dots + sxc_1 \\ &= sx \cdot f_2(mg^{m-1} + (m-1)c_{m-1}g^{m-2} + \dots + c_1) \end{aligned}$$

where the second equality uses (3.5) for $\nu = 2$, and the fact that $x^2 = 0$. So, if we let $g' := mg^{m-1} + (m-1)c_{m-1}g^{m-2} + \dots + c_1$, then $sf_2(g') = 0$. Note that $g' \neq 0$ by the minimality of the degree m , and hence $f_2(g')$ is regular in S . It follows that $sx = 0$. But this means that $f_1(g) = f_2(g)$, so that $g \in F$, contradicting $m > 1$.

We have proved that $\langle b_j : j < \omega \rangle \cap F$, and hence $G \cap F$, is nontrivial. \square

4. PROOF OF THE MAIN THEOREM

We now deduce the main part of Theorem 1.1 as stated in the introduction from the algebraic statement given in Theorem 3.1. We will deal with the rest of the statement, namely the improvement in the reduced case, in §6 below.

Theorem 4.1. *Suppose X is an algebraic variety, Z is an irreducible algebraic scheme, and $\phi_1, \phi_2 : Z \rightarrow X$ are rational maps whose restrictions to Z_{red} are dominant, all over K . Then the following are equivalent:*

- (1) *There exist nonempty Zariski open subsets $V \subseteq Z$ and $U \subseteq X$ such that the restrictions $\phi_1^V, \phi_2^V : V \rightarrow U$ are dominant regular morphisms, and there exist infinitely many hypersurfaces H on U satisfying*

$$(\phi_1^V)^{-1}(H) = (\phi_2^V)^{-1}(H).$$

- (2) *There exists $g \in K(X) \setminus K$ such that $g\phi_1 = g\phi_2$.*

Proof. That (2) implies (1) is more or less clear: we can choose nonempty Zariski open sets $V \subseteq Z$ and $U \subseteq X$ such that the restrictions $\phi_1^V, \phi_2^V : V \rightarrow U$ are dominant regular morphisms and such that $g : U \rightarrow \mathbb{A}^1$ is a nonconstant morphism to the affine line. We have the commuting diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi_1^V} & U \\ \phi_2^V \downarrow & & \downarrow g \\ U & \xrightarrow{g} & \mathbb{A}^1 \end{array}$$

so that level sets of g over the K -points of \mathbb{A}^1 yield infinitely many hypersurfaces H on U satisfying $(\phi_1^V)^{-1}(H) = (\phi_2^V)^{-1}(H)$.

Assume that (1) holds.

Let $k \subseteq K$ be a finitely generated subfield over which $Z, X, V, U, \phi_1, \phi_2$ are defined. That is, $X = X_k \times_k K$ for some geometrically irreducible algebraic k -variety X_k , and $Z = Z_k \times_k K$ where $(Z_k)_{\text{red}}$ is a geometrically irreducible algebraic k -variety. We have similar descent statements to k for V, U, ϕ_1, ϕ_2 as well.

We first claim that k can be chosen so that there are infinitely many hypersurfaces H on U defined over k satisfying $(\phi_1^V)^{-1}(H) = (\phi_2^V)^{-1}(H)$. Indeed, fix k and suppose there exists such a hypersurface H that is not defined over k^{alg} . Then H is defined over a finitely generated nonalgebraic extension L of k . Now $\text{Aut}(L^{\text{alg}}/k)$ acts naturally on the whole situation, and there are infinitely many $\text{Aut}(L^{\text{alg}}/k)$ -conjugates of H in U . All these conjugates are defined over L^{alg} and satisfy the property that their inverse images under ϕ_1^V and ϕ_2^V agree. So choosing L instead of k , we may as well assume that we have infinitely many such hypersurfaces over k^{alg} to start with. Replacing these with their conjugates under the action of $\text{Gal}(k)$, we may in fact assume they are over k itself.

Suppose therefore that $(H_j : j < \omega)$ is an infinite sequence of hypersurfaces over k on U with $(\phi_1^V)^{-1}(H_j) = (\phi_2^V)^{-1}(H_j)$ for all $j < \omega$, and such that $H_j \not\subseteq \bigcup_{i < j} H_i$.

Replacing V and U by smaller nonempty Zariski open subsets, we may assume $U = \text{Spec}(R_K)$ and $V = \text{Spec}(S_K)$ where R is a finitely generated k -algebra that is an integral domain, S is a finitely generated k -algebra whose nilradical is prime, $R_K := R \otimes_k K$ and $S_K := S \otimes_k K$, and ϕ_1^V, ϕ_2^V are induced by injective k -algebra embeddings $f_1, f_2 : R \rightarrow S$ which when composed with the quotient by the nilradical of S remain injective. Note that as X_k is geometrically irreducible, k is relatively algebraically closed in $\text{Frac}(R)$. Similarly, as $(Z_k)_{\text{red}}$ is geometrically irreducible, k is relatively algebraically closed in $\text{Frac}(S/N)$.

Now, as k is a finitely generated field, the localisation of R at some nonzero element is a unique factorisation domain – this is by [1, Lemma 6.11]. So we may assume that R is already a UFD. The vanishing ideals $I(H_j)$ are of the form $I_j R_K$ where I_j is a (radical) height one ideal in R , and hence of the form $I_j = a_j R$ for some sequence $(a_j : j < \omega)$ in R . The scheme-theoretic inverse images $(\phi_\nu^V)^{-1}(H_j)$ are by definition given by the ideals $f_\nu(I_j)S_K$, for $\nu = 1, 2$. That $(\phi_1^V)^{-1}(H_j) = (\phi_2^V)^{-1}(H_j)$ therefore implies that $f_1(a_j)S = f_2(a_j)S$ for all $j < \omega$. Moreover, since $H_j \not\subseteq \bigcup_{i < j} H_i$, each a_j has an irreducible factor that does not appear in a_i for $i < j$, and so no nontrivial product of integer powers of the a_j can be a constant in $\text{Frac}(R)$. That is, the hypotheses of Theorem 3.1 are satisfied, and there must exist $g \in \text{Frac}(R) \setminus k$ such that $f_1(g) = f_2(g)$. Note that $g \in \text{Frac}(R_K) = K(X)$ and $g\phi_1 = g\phi_2$. In $K(X) = \text{Frac}(R) \otimes_k K$ the intersection of $\text{Frac}(R)$ and K is k , so we have that $g \notin K$. This proves (2). \square

5. AN APPLICATION TO ALGEBRAIC \mathcal{D} -VARIETIES

In this section we specialise Theorem 4.1 to the differential context to see how we recover the finite-dimensional Jouanolou-Hrushovski theorem. In fact we work rather more generally in a setting that appears in the work of the second author and Thomas Scanlon [11] toward the model theory of fields equipped with a general class of operators. We will thus obtain a Jouanolou-Hrushovski type theorem for these generalised operators.

The setting is as follows. Fix an algebraically closed field K of characteristic zero. Let \mathcal{D} denote the following fixed data:

- a finite dimensional K -algebra B ,
- a maximal ideal \mathfrak{m} of B with $\pi : B \rightarrow K$ the quotient map,
- a K -basis $(\epsilon_0, \dots, \epsilon_\ell)$ for B such that $\pi(\epsilon_0) = 1$ and $\epsilon_1, \dots, \epsilon_\ell \in \mathfrak{m}$.

The following notion first appears, with somewhat different notation, in [10]. It was inspired by Alexandru Buium's approach to differential algebra.

Definition 5.1. By a \mathcal{D} -ring we will mean a pair (R, e) where R is a K -algebra and $e : R \rightarrow R \otimes_K B$ is a K -algebra homomorphism satisfying $\pi^R \circ e = \text{id}_R$. Here $\pi^R = (\text{id}_R \otimes_K \pi) : R \otimes_K B \rightarrow R$ is the R -algebra homomorphism induced by π . We denote by $R^{\mathcal{D}} := \{r \in R : e(r) = r \otimes 1\}$ the subring of \mathcal{D} -constants.

We will be applying Theorem 4.1 to $X = \text{Spec}(R)$ when (R, e) is a \mathcal{D} -ring and R is a finitely generated K -algebra that is an integral domain. We will set $Z = \text{Spec}(R \otimes_K B)$, $\phi_1 : Z \rightarrow X$ the morphism induced by e , and $\phi_2 : Z \rightarrow X$

the morphism induced by $r \mapsto r \otimes 1$. Note that the nonreduced nature of Z here is essential; $Z_{\text{red}} = X$ and ϕ_1, ϕ_2 restricted to Z_{red} are both the identity.

But in order to see what the theorem will say in this context, we need to explore \mathcal{D} -rings a bit further. First, two motivating examples.

Example 5.2 (Differential rings). Let \mathcal{D} be given by the the local K -algebra $K[\epsilon]/(\epsilon^2)$ with maximal ideal $\mathfrak{m} = (\epsilon)$ and K -basis $(1, \epsilon)$. Suppose R is a K -algebra equipped with a K -linear derivation $\delta : R \rightarrow R$. Then we can make R into a \mathcal{D} -ring by letting $e : R \rightarrow R[\epsilon]/\epsilon^2$ be $r \mapsto r + \delta(r)\epsilon$. In fact, every \mathcal{D} -ring is of this form.

Example 5.3 (Difference rings). Let \mathcal{D} be given by the K -algebra $K \times K$ with maximal ideal \mathfrak{m} generated by $(0, 1)$ and K -basis $((0, 1), (1, 0))$. Then the \mathcal{D} -rings are precisely the K -algebras R equipped with an endomorphism $\sigma : R \rightarrow R$, where $e : R \rightarrow R \times R$ is given by $r \mapsto (r, \sigma(r))$.

In fact, as suggested by the examples, the \mathcal{D} -ring formalism is really a way to study rings equipped with certain operators. Note that $(1 \otimes \epsilon_0, \dots, 1 \otimes \epsilon_\ell)$ is an R -basis for $R \otimes_K B$, and $e : R \rightarrow R \otimes_K B$ can be written with respect to this basis so that for all $r \in R$,

$$e(r) = r \otimes \epsilon_0 + \partial_1(r) \otimes \epsilon_1 + \dots + \partial_\ell(r) \otimes \epsilon_\ell$$

where $\partial_i : R \rightarrow R$ are K -linear operators on R . (That the ϵ_0 -coefficient of $e(r)$ is r comes from the fact that $\pi^R \circ e = \text{id}_R$ and $\pi(\epsilon_0) = 1$.) Writing $\partial := (\partial_1, \dots, \partial_\ell)$, we can recover e from ∂ and vice versa. We will refer interchangeably to (R, e) and (R, ∂) as the \mathcal{D} -ring.

The class of operators ∂ that can be fit into this context is rather broad and robust, including various combinations and twists of differential and difference operators, and closed under various operations. See paragraphs 3.3 through 3.7 of [11] for a discussion of examples.

Naturally associated to the operators ∂ on R are certain K -algebra endomorphisms of R . Let $\mathfrak{m} = \mathfrak{m}_0, \dots, \mathfrak{m}_t$ be the distinct maximal ideals of B , and $\pi = \pi_0, \pi_1, \dots, \pi_t$ the corresponding quotient maps $B \rightarrow K$. Let $\sigma_i := \pi_i^R \circ e : R \rightarrow R$ for $i = 0, 1, \dots, t$. Note that $\sigma_0 = \text{id}_R$, and that $\sigma_1, \dots, \sigma_t$ are K -algebra endomorphisms of R that are in fact K -linear combinations of the $\partial_1, \dots, \partial_\ell$. We write $\sigma := (\sigma_1, \dots, \sigma_t)$ and call (R, σ) the *difference ring associated to (R, e)* .

Definition 5.4 (Totally invariant \mathcal{D} -ideals). Suppose (R, e) is a \mathcal{D} -ring. An ideal $I \subseteq R$ is said to be a \mathcal{D} -ideal if $\partial_i(I) \subseteq I$ for all $i = 1, \dots, \ell$, and *totally invariant* if $\sigma_j(I) = I$ for all $j = 1, \dots, t$.

Proposition 5.5. *Suppose (R, e) is a \mathcal{D} -ring with R noetherian. Let I be an ideal of R . Then I is a totally invariant \mathcal{D} -ideal if and only if $e(I)(R \otimes_K B) = I(R \otimes_K B)$.*

Proof. Note that $(1 \otimes \epsilon_0, \dots, 1 \otimes \epsilon_\ell)$ is an R -basis for $R \otimes_K B$ and that

$$I(R \otimes_K B) = \{a_0 \otimes \epsilon_0 + \dots + a_\ell \otimes \epsilon_\ell : a_0, \dots, a_\ell \in I\}.$$

Suppose $e(I)(R \otimes_K B) = I(R \otimes_K B)$. If $a \in I$ then

$$e(a) = a \otimes \epsilon_0 + \partial_1(a) \otimes \epsilon_1 + \dots + \partial_\ell(a) \otimes \epsilon_\ell \in I(R \otimes_K B),$$

and hence $\partial_1(a), \dots, \partial_\ell(a) \in I$. So I is a \mathcal{D} -ideal. For total invariance, fixing $j = 1, \dots, t$ and applying π_j^R to $e(I)(R \otimes_K B) = I(R \otimes_K B)$ we get immediately that $\sigma_j(I) = I$.

Conversely, suppose I is a totally invariant \mathcal{D} -ideal. Then $e(a) \in I(R \otimes_K B)$ for all $a \in I$, since $\partial_1(a), \dots, \partial_\ell(a) \in I$. That is, $e(I)(R \otimes_K B) \subseteq I(R \otimes_K B)$.

So it remains to show that $I(R \otimes_K B) \subseteq e(I)(R \otimes_K B)$ whenever I is a totally invariant \mathcal{D} -ideal.

We first improve the choice of K -basis $(\epsilon_0, \dots, \epsilon_\ell)$. Note that changing the basis, as long as $\pi(\epsilon_0) = 1$ and $\epsilon_2, \dots, \epsilon_\ell \in \mathfrak{m}$ remain true, does not affect e or σ as these are intrinsically defined. While it does change ∂ it does so only by replacing these operators with certain K -linear combinations of them. In particular, the property of being a totally invariant \mathcal{D} -ideal is not affected. We may therefore adjust the basis so that $\pi_i(\epsilon_j) = \delta_{i,j}$ for $i, j = 0, \dots, t$, and $(\epsilon_{t+1}, \dots, \epsilon_\ell)$ forms a K -basis for the Jacobson radical $J := \bigcap_{j=0}^t \mathfrak{m}_j$. Note that one of the consequences of this choice of basis is that $\sigma_j = \partial_j$ for $j = 0, \dots, t$. (Recall that $\sigma_0 = \partial_0 = \text{id}_R$.)

Suppose now that $I = (a_1, \dots, a_m)$. For each $j = 0, \dots, t$, since $\sigma_j(I) = I$, there is $a_{j,k} \in I$ such that $\sigma_j(a_{k,j}) = a_k$ for all $k = 1, \dots, m$. Letting

$$y_k := \sum_{j=0}^t e(a_{j,k})(1 \otimes \epsilon_j)$$

we have that for each $i = 0, \dots, t$,

$$\begin{aligned} \pi_i^R(y_k) &= \sum_{j=0}^t \sigma_i(a_{j,k})(1 \otimes \pi_i(\epsilon_j)) \\ &= a_k \\ &= \pi_i^R(a_k \otimes 1). \end{aligned}$$

Hence $(a_k \otimes 1) - y_k$ is of the form $b_{t+1} \otimes \epsilon_{t+1} + \dots + b_\ell \otimes \epsilon_\ell$ for some $b_{t+1}, \dots, b_\ell \in R$. (Despite the notation, the b_i 's depend also on k .) In fact, since $y_k \in e(I)(R \otimes_K B) \subseteq I(R \otimes_K B)$, we get that $b_{t+1}, \dots, b_\ell \in I$. Writing $b_\mu = \sum_{\nu=1}^m r_{\mu,\nu} a_\nu$, and setting $s_{\nu,k} := \sum_{\mu=t+1}^m (-r_{\mu,\nu} \otimes \epsilon_\mu)$, we have that

$$y_k = (a_1 \otimes 1)s_{1,k} + (a_2 \otimes 1)s_{2,k} + \dots + (a_k \otimes 1)(1 + s_{k,k}) + \dots + (a_m \otimes 1)s_{m,k}$$

for all $k = 1, \dots, m$. This can be written in matrix notation as

$$\mathbf{a}(\mathbf{1} + \mathbf{S}) = \mathbf{y}$$

where $\mathbf{a} = (a_1 \otimes 1, \dots, a_m \otimes 1)$, $\mathbf{S} = (s_{\nu,k}) \in \text{Mat}_m(R \otimes_K B)$, $\mathbf{1} = \text{id}_{\text{Mat}_m(R \otimes_K B)}$, and $\mathbf{y} = (y_1, \dots, y_m)$. But since each $s_{\nu,k} \in R \otimes_K J$, and J is a nilpotent ideal of B , we get that \mathbf{S} is nilpotent, and so $\mathbf{1} + \mathbf{S}$ is invertible. Hence,

$$\mathbf{a} = \mathbf{y}(\mathbf{1} + \mathbf{S})^{-1} \in (e(I)(R \otimes_K B))^m.$$

That is, for each generator a_k of I in R we have $a_k \otimes 1 \in e(I)(R \otimes_K B)$. Therefore $I(R \otimes_K B) \subseteq e(I)(R \otimes_K B)$, as desired. \square

We are ready now to specialise Theorem 4.1.

Definition 5.6. By an *algebraic \mathcal{D} -variety* we mean an affine algebraic variety X over K whose co-ordinate ring $K[X]$ comes equipped with a \mathcal{D} -ring structure $e : K[X] \rightarrow B[X]$. A Zariski closed subset of X is said to be *totally \mathcal{D} -invariant* if its corresponding ideal is a totally invariant \mathcal{D} -ideal.

Theorem 5.7. *Suppose (X, e) is an algebraic \mathcal{D} -variety over K . If (X, e) has infinitely many totally \mathcal{D} -invariant hypersurfaces then there exists a \mathcal{D} -constant rational function $g \in K(X) \setminus K$.*

Remark 5.8. There is a unique \mathcal{D} -field structure on $K(X)$ extending e on $K[X]$, and it is in this \mathcal{D} -field that the theorem concludes there is a nonconstant \mathcal{D} -constant.

Proof. Write $X = \text{Spec}(R)$ with (R, e) a \mathcal{D} -ring. Let $Z := \text{Spec}(R \otimes_K B)$, $\phi_1 : Z \rightarrow X$ the morphism induced by the K -algebra homomorphism $e : R \rightarrow R \otimes_K B$, and $\phi_2 : Z \rightarrow X$ induced by $r \mapsto r \otimes 1$. Note that $Z_{\text{red}} = X$ and hence Z is irreducible. Moreover ϕ_1, ϕ_2 both restrict to the identity on Z_{red} , and hence are dominant. So Theorem 4.1 applies. By Proposition 5.5, if $H \subseteq X$ is a totally \mathcal{D} -invariant hypersurface with ideal $I = I(H)$, then $e(I)(R \otimes_K B) = I(R \otimes_K B)$. This means that $\phi_1^{-1}(H) = \phi_2^{-1}(H)$. Hence, condition (1) of Theorem 4.1 holds with $U = X$ and $V = Z$. The theorem gives us $g \in K(X) \setminus K$ with $g\phi_1 = g\phi_2$. That is, $e(g) = g \otimes 1$ under the canonical extension of e to $\text{Frac}(R) \rightarrow \text{Frac}(R) \otimes_K B$. We have found a nonconstant \mathcal{D} -constant rational function on X , as desired. \square

When \mathcal{D} is given by $B = K[\epsilon]/\epsilon^2$ as in Example 5.2 we recover the following consequence of a theorem of Jouanolou [6] on solutions to rational foliations: *an algebraic D -variety with infinitely many D -hypersurfaces admits a nonconstant rational first integral*. This statement is the finite-dimensional case of Proposition 2.3 of Hrushovski's unpublished manuscript [5]; or, for a published proof, note that it is precisely the $(m, r) = (1, 0)$ case of [4, Theorem 4.2]. In fact, we get (a new proof of) the $r = 0$ case of [4, Theorem 4.2] for arbitrary $m \geq 1$ by applying Theorem 5.7 to the case when \mathcal{D} is given by $B = K[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^2$.

6. THE REDUCED CASE AND AN APPLICATION TO RATIONAL DYNAMICS

In this section we improve Theorem 4.1 in the case when Z is also a (reduced) algebraic variety, and thereby complete the proof of Theorem 1.1. We also deduce the application to rational dynamics discussed in the introduction.

First, for any function $\phi : Z \rightarrow X$ and any subset $H \subseteq X$, let us denote by $\phi^{-1}[H]$ the *set-theoretic inverse image* of the set H . This is to avoid confusion with the notation $\phi^{-1}(H)$ we are using for the scheme-theoretic inverse image. Now, suppose $\phi : Z \rightarrow X$ is a dominant rational map between algebraic varieties. For a hypersurface $H \subseteq X$ with $H \cap \text{Im}(\phi)$ Zariski dense in H , by the *proper transform* of H , denoted by ϕ^*H , we mean the hypersurface on Z that is the union of those irreducible components of the Zariski closure of $\phi^{-1}[H \cap \text{Im} \phi]$ in Z that project dominantly onto some irreducible component of H .

Theorem 6.1. *Suppose Z and X are algebraic varieties and $\phi_1, \phi_2 : Z \rightarrow X$ are dominant rational maps, over an algebraically closed field K of characteristic zero. Then the following are equivalent:*

- (1) *There exist infinitely many hypersurfaces H on X satisfying $\phi_1^*H = \phi_2^*H$.*
- (2) *There exists $g \in K(X) \setminus K$ such that $g\phi_1 = g\phi_2$.*

Note that when $Z = X$ and $\phi_2 = \text{id}$, this theorem says that if a rational dynamical system (X, ϕ) has infinitely many totally invariant hypersurfaces, then ϕ preserves a nonconstant rational function. That is, we recover the algebraic case of [3, Corollary 3.3]. See also the closely related [2, Theorem 1.2]. But we can do better. By a *rational finite self-correspondence* we will mean an algebraic variety X equipped with a closed irreducible subvariety $\Gamma \subseteq X \times X$ such that the co-ordinate

projections $\pi_1, \pi_2 : \Gamma \rightarrow X$ are dominant and generically finite-to-one.¹ Note that we get a rational dynamical system by taking Γ to be the graph of a dominant rational self-map. A Zariski closed subset $V \subseteq X$ is *totally invariant* if its proper transforms in Γ by the two co-ordinate projections agree. A rational function g on X is *preserved by Γ* if $g\pi_1 = g\pi_2$.

Corollary 6.2. *Suppose (X, Γ) is a rational finite self-correspondence with infinitely many totally invariant hypersurfaces. Then Γ preserves a nonconstant rational function on X .*

Proof. Apply Theorem 6.1 to $Z := \Gamma$, $\phi_1 := \pi_1$, and $\phi_2 := \pi_2$. \square

In fact, Theorem 6.1 is precisely the generalisation of the above corollary to arbitrary self-correspondences – where the co-ordinate projections need not be generically finite-to-one. As such, it can be viewed as a statement in generalised rational dynamics.

In order to deduce Theorem 6.1 from Theorem 4.1 we need to observe that when working over a finitely generated field, and restricting attention to sufficiently small Zariski open sets, the scheme-theoretic inverse image and the proper transform agree on hypersurfaces. This is Proposition 6.5 below, and may very well be known, but we could not find it in the literature. Our proof will rely on the following elementary, and certainly well-known, lemmas in commutative algebra.

Lemma 6.3. *Suppose A is a noetherian integral domain and $B = A[x_1, \dots, x_n]_g$ is the localisation of a polynomial algebra over A . If $I \subseteq A$ is a radical ideal then so is IB . Moreover, if A, B are in addition finitely generated k -algebras for some field k , $\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is the induced morphism of k -varieties, $V := V(I) \subseteq \text{Spec } A$ is the corresponding subvariety, and $g \notin IA[x_1, \dots, x_n]$, then $\phi^{-1}(V) = \phi^*V$.*

Proof. It is straightforward to check that localisation preserves radicality. That taking polynomial extensions also preserve radicality follows from:

- (a) if $P \subset A$ is a prime ideal then so is $PA[x_1, \dots, x_n]$, and
- (b) for prime ideals P_1, \dots, P_ℓ of A ,

$$\left(\bigcap_{i=1}^{\ell} P_i \right) A[x_1, \dots, x_n] = \bigcap_{i=1}^{\ell} (P_i A[x_1, \dots, x_n]).$$

The “moreover” clause follows by first noting that since IB is radical, the scheme-theoretic and set-theoretic inverse images of $V = V(I)$ agree. Moreover, if P is a minimal prime ideal of A containing I then, by (a) and the fact that $g \notin PA[x_1, \dots, x_n]$, PB is a prime ideal. That is, the irreducible components of $\phi^{-1}[V]$ are of the form $\phi^{-1}[W]$ where W is an irreducible component of V . Hence the proper transform agrees with the set-theoretic inverse image of V . \square

Lemma 6.4. *If $A \subseteq B$ is an étale extension of noetherian unique factorisation domains, and I is a height one radical ideal of A , then IB is radical. Moreover, if A, B are in addition finitely generated k -algebras for some field k , $\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is the induced morphism of k -varieties, and $H := V(I)$ is the corresponding hypersurface on $\text{Spec}(A)$, then $\phi^{-1}(H) = \phi^*H$.*

¹The model theorist may recognise the pair (X, Γ) as an *algebraic σ -variety* over the fixed field – it corresponds to a finite rank type in the theory ACFA.

Proof. Let P_1, \dots, P_ℓ be the distinct minimal prime ideals of I in A . Since A is a UFD and each P_i is of height one, we have that $P_i = p_i A$ for some irreducible $p_i \in A$. Since $I = P_1 \cap \dots \cap P_\ell$ and the p_i are mutually non-associate, we get that $I = (p_1 \cdots p_\ell)A$. Let $p_i = u_i q_{i,1}^{n_{i,1}} \cdots q_{i,m_i}^{n_{i,m_i}}$ be the prime factorisation of p_i in the UFD B . Now, each $Q_{i,j} := q_{i,j}B$ is a minimal prime ideal of $P_i B$, and hence by the going down theorem for flat extensions, $Q_{i,j}$ lies over P_i . In particular, the $q_{i,j}$ are non-associate even as i varies. But moreover, as B over A is unramified, $P_i B_{Q_{i,j}} = Q_{i,j} B_{Q_{i,j}}$. That is, $q_{i,j}^{n_{i,j}} B_{Q_{i,j}} = q_{i,j} B_{Q_{i,j}}$. This forces each $n_{i,j} = 1$. So $IB = (\prod_{i=1}^\ell \prod_{j=1}^{m_i} q_{i,j})B$ is radical.

For the “moreover” clause, again we first observe that the set-theoretic and scheme-theoretic inverse images of $H = V(I)$ agree because IB is radical. Now, the irreducible components of $\phi^{-1}[H] = V(IB)$ are the $V(q_{i,j})$. That $Q_{i,j}$ lies over P_i says exactly that $V(q_{i,j})$ projects dominantly onto $V(p_i)$, which is an irreducible component of H . Hence $\phi^{-1}[H] = f^*H$, as desired. \square

Proposition 6.5. *Suppose $\phi : Z \rightarrow X$ is a dominant rational map between algebraic varieties over a finitely generated field k . There exist nonempty Zariski open subsets $V \subseteq Z$ and $U \subseteq X$ such that the restriction $\phi^V : V \rightarrow U$ is a dominant regular morphism, and for all but finitely many hypersurface H on U , $(\phi^V)^{-1}(H) = (\phi^V)^*H$.*

Proof. Replacing Z and X by nonempty Zariski open subsets, it suffices to prove the proposition in the case when $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$ are affine k -varieties and ϕ is a dominant k -morphism induced by an injective k -algebra homomorphism $f : R \rightarrow S$. Now, as we have used before, that k is a finitely generated field implies that the localisation of R (respectively S) at some nonzero element is a unique factorisation domain – this is [1, Lemma 6.11]. So we may assume that R and S are already unique factorisation domains.

Next, by Noether’s normalisation lemma, after replacing S with S_g for some nonzero g , we may assume that the homomorphism f factors through injective k -algebra homomorphisms $R \rightarrow R'$ and $R' \rightarrow S$ where R' is a polynomial algebra over R and S is quasi-finite over R' . Localising both R' and S further, we may in fact take $R' \rightarrow S$ to be étale, though now R' is a finitely generated localisation of a polynomial algebra over R .

So we have that ϕ factors as $\text{Spec}(S) \rightarrow \text{Spec}(R') \rightarrow \text{Spec}(R)$. Since R' is of the form $R[x_1, \dots, x_n]_g$, Lemma 6.3 tells us that if $I = I(H)$ is a radical height one ideal in R then IR' is radical. Moreover, since $V(g)$ can only contain finitely many hypersurfaces, for all but finitely many such I , IR' is again of height one. Since S is étale over R' , Lemma 6.4 now applies and we get that $(IR')S = f(I)S$ is radical. The “moreover” clauses in the lemmas tell us that $\phi^{-1}(H) = \phi^*H$. \square

Proof of Theorem 6.1. That (2) implies (1) is again clear: the level sets of g will witness (1). Suppose (1) holds. Exactly as in the beginning of the proof of Theorem 4.1 we can find a finitely generated subfield $k \subseteq K$ over which Z, X, ϕ_1, ϕ_2 are defined and such that there is an infinite set \mathcal{H} of hypersurfaces H on X over k satisfying $\phi_1^*H = \phi_2^*H$. Applying Proposition 6.5 to $(\phi_\nu)_k : Z_k \rightarrow X_k$, there exist nonempty Zariski open subsets $V \subseteq Z_k$ and $U \subseteq X_k$ such that, for $\nu = 1, 2$, the restrictions $(\phi_\nu)_k^V : V \rightarrow U$ are dominant regular morphisms and for all but finitely many $H \in \mathcal{H}$, $((\phi_\nu)_k^V)^{-1}(H_k \cap U) = ((\phi_\nu)_k^V)^*(H_k \cap U)$. Noting that proper transforms commute with extending the base field, and observing that U and V

avoid only finitely many hypersurfaces on X and Z respectively, we have that for all but finitely many $H \in \mathcal{H}$, $((\phi_1)_k^V)^*(H_k \cap U) = ((\phi_2)_k^V)^*(H_k \cap U)$. Hence, $((\phi_1)_k^V)^{-1}(H \cap U) = ((\phi_2)_k^V)^{-1}(H \cap U)$ for all but finitely many $H \in \mathcal{H}$. As scheme-theoretic inverse images also commute with extending the base field, we get that $(\phi_1^{V_K})^{-1}(H \cap U_K) = (\phi_2^{V_K})^{-1}(H \cap U_K)$. This witnesses the truth of condition (1) of Theorem 4.1, and so by that theorem, condition (2) holds. \square

7. NORMAL VARIETIES EQUIPPED WITH PRIME DIVISORS

Theorem 6.1 is really about the birational geometry of algebraic varieties equipped with a set of hypersurfaces. We will show how a direct study of this category leads us to an alternative, more geometric and conceptual, proof of Theorem 6.1 in the case when we assume that the fibres of ϕ_1 and ϕ_2 are irreducible. This section is self-contained and largely independent from the rest of the paper.

Fix a field k of characteristic 0, and let $K := k^{\text{alg}}$. We consider the following category \mathcal{V}_k . The *objects* of \mathcal{V}_k are pairs (X, S) where X is a normal geometrically irreducible algebraic variety over k and S is a set of prime divisors (i.e., irreducible hypersurfaces) on $X_K := X \times_k K$. A *morphism* $(X, S) \rightarrow (Y, T)$ will be a dominant rational map $\phi : X \rightarrow Y$ over k whose generic fibre is geometrically irreducible, and such that the symmetric difference between S and $\{\phi_K^* t : t \in T\}$ is finite.

Remark 7.1. Here $\phi_K : X_K \rightarrow Y_K$ is the base extension of $\phi : X \rightarrow Y$ to K . Because the generic fibre of ϕ_K is irreducible, the proper transform of all but finitely many prime divisors on Y_K is a prime divisor on X_K . Indeed, if t is an irreducible hypersurface on Y_K that has nonempty intersection with the Zariski open subset of points in Y_K over which the fibre of ϕ_K is irreducible, then $\phi_K^* t$ will be irreducible.

Note that in this category the underlying varieties and rational maps are over k but the irreducible hypersurfaces they come with may be over the algebraic closure K . Things would become notationally much clearer if we assumed that k is algebraically closed, but in fact the main theorem will only apply when k is a finitely generated field. We will systematically use the subscript K to indicate base extension from k up to K . One exception, however, will be for fields of rational functions: For X a geometrically irreducible algebraic variety over k we will denote by $K(X)$ the field of rational functions on X_K .

The category of algebraic varieties over k has a terminal object, namely $\text{Spec}(k)$. At first sight one might think that $(\text{Spec}(k), \emptyset)$ is the terminal object in \mathcal{V}_k , but this is not the case. If S is a *finite* set then there is a canonical morphism $(X, S) \rightarrow (\text{Spec}(k), \emptyset)$, but if S is infinite then it is not hard to see that the existence of a morphism $(X, S) \rightarrow (Y, T)$ implies $\dim Y > 0$. We seek to repair this lack of terminal object by asking if the *undercategory* of arrows originating at a given (X, S) in \mathcal{V}_k has a terminal object.

Theorem 7.2. *Suppose k is finitely generated. For every object (X, S) in \mathcal{V}_k there is a morphism $\pi : (X, S) \rightarrow (X', S')$ that is terminal with respect to all morphisms originating from (X, S) . That is, given $\phi : (X, S) \rightarrow (Y, T)$ there is a unique $\psi : (Y, T) \rightarrow (X', S')$ such that $\psi\phi = \pi$.*

Proof. First of all, we can embed X as an open subvariety of a normal proper variety \overline{X} . Let \overline{S} denote the set of Zariski closures of elements of S in \overline{X}_K . The embedding of X in \overline{X} induces an isomorphism $(X, S) \cong (\overline{X}, \overline{S})$ in \mathcal{V}_k . It suffices therefore to prove the theorem for $(\overline{X}, \overline{S})$. That is, we may assume X is proper.

Our assumption of normality means that for any rational function $f \in K(X)$ we can consider the Weil divisor $\text{div}(f)$ on X_K . By the *support* of f we mean the set of prime divisors appearing in $\text{div}(f)$ with nonzero coefficient – so it is the set of “zeros” and “poles” of f . Given a set T of prime divisors on X_K , let us denote by $T^\# \subseteq K(X)$ the set of rational functions whose support is contained in T , and by K_T the relative algebraic closure of $K(T^\#)$ in $K(X)$.

Consider the natural action of $\text{Gal}(k)$ on X_K coming from the fact that X is over k . For any set T of prime divisors on X_K , let \widehat{T} denote the closure of T under this action.

Since $K(X)$ has finite transcendence degree over K , and the $K_{\widehat{T}}$ are relatively algebraically closed in $K(X)$ by definition, there must be a cofinite $S_0 \subseteq S$ such that $K_{\widehat{T}} = K_{\widehat{S}_0}$ for all cofinite $T \subseteq S_0$. Since the identity map is an isomorphism between (X, S) and (X, S_0) , it suffices to prove the theorem for (X, S_0) . That is, we may assume that $K_{\widehat{T}} = K_{\widehat{S}}$ for every cofinite $T \subseteq S$.

There is also a natural action of $\text{Gal}(k)$ on $K(X)$. As \widehat{S} is $\text{Gal}(k)$ -invariant, so is the set of rational functions $\widehat{S}^\#$, and hence also the subfield $K_{\widehat{S}} \subseteq K(X)$. This means that $K_{\widehat{S}}$ is the function field of a K -variety that descends to k , that is, $K_{\widehat{S}} = K(X')$ for some normal geometrically irreducible algebraic variety X' over k , and the embedding $K(X') \subseteq K(X)$ comes from a dominant rational map $\pi : X \rightarrow X'$ over k . As $K_{\widehat{S}}$ is relatively algebraically closed in $K(X)$ the generic fibre of π is also geometrically irreducible.

We claim that only finitely many $s \in S$ map dominantly onto X'_K by π_K . Suppose towards a contradiction that infinitely many elements of S map dominantly onto X'_K . By the Mordell-Weil-Néron-Severi theorem (see [9, Corollary 6.6.2] for details) the divisor class group $\text{Cl}(X)$ is finitely generated (as k is a finitely generated field). Let n be bigger than the rank of $\text{Cl}(X)$. Choose $s_1, \dots, s_n \in S$ that map dominantly onto X'_K and have distinct $\text{Gal}(k)$ -orbits. If we let H_i be the union of the $\text{Gal}(k)$ -conjugates of s_i , then H_i descends to k and is k -irreducible. That is, we have distinct prime divisors d_1, \dots, d_n on X over k such that $H_i = d_i K$. By choice of n there are rational (and so integer) numbers r_1, \dots, r_n not all zero, and $f \in k(X) \setminus k$, such that $\sum r_i d_i = \text{div}(f)$ in $\text{Div}(X)$. Note that, working again over K , the support of $f \in K(X)$ is contained in $\{s_i^\sigma : i = 1, \dots, n, \sigma \in \text{Gal}(k)\}$. In particular, $f \in K_{\widehat{S}} = K(X')$. On the other hand, each s_i^σ maps dominantly onto X'_K as π is over k , and so f cannot be (the pull-back of) a function on X'_K . This contradiction proves that only finitely many $s \in S$ map dominantly onto X'_K .

So, for cofinitely many $s \in S$, we have $s \cap \text{dom}(\pi_K) \neq \emptyset$ and the Zariski closure of $\pi_K(s)$ is a proper irreducible subvariety s' of X'_K . By dimension considerations it must be that s' is a prime divisor on X'_K . By Remark 7.1, $\pi_K^* s' = s$ for all but finitely many of these s . Letting S' be the set of prime divisors s' on X'_K obtained in this way, we have that $\pi : (X, S) \rightarrow (X', S')$ is a morphism in \mathcal{V}_k .

It remains to show π is terminal. Given a morphism $\phi : (X, S) \rightarrow (Y, T)$, we seek to complete the triangle

$$\begin{array}{ccc} (X, S) & \xrightarrow{\phi} & (Y, T) \\ \pi \downarrow & & \\ (X', S') & & \end{array}$$

with a morphism $\psi : (Y, T) \rightarrow (X', S')$.

Since $\phi : (X, S) \rightarrow (Y, T)$ is a morphism, it must be that only finitely many $s \in S$ map dominantly onto Y_K under ϕ_K . Replacing S by a cofinite subset, we may assume that there are no such $s \in S$. It follows from the fact that $\phi : X \rightarrow Y$ is over k that also no elements of \widehat{S} will map dominantly onto Y_K . Now, suppose $f \in \widehat{S}^\#$. Then no member of the support of f maps dominantly onto Y_K . This means that f has no zeros or poles on the generic fibre X_η of ϕ_K . By the properness of X , and hence of X_η over $K(Y)$, we must have that f is constant on the generic fibre. So f is the pull-back of a rational function on Y . That is, $\widehat{S}^\# \subseteq K(Y)$, and hence $K(X') = K_{\widehat{S}} \subseteq K(Y)$. We thus obtain a dominant rational map $\alpha : Y_K \rightarrow X'_K$ with irreducible generic fibre such that $\pi_K = \alpha \phi_K$. Since ϕ is dominant there is a unique such α . Since π and ϕ are over k , an automorphism argument shows that α descends to k , that is, $\alpha = \psi_K$ for some dominant rational map $\psi : Y \rightarrow X'$ with geometrically irreducible generic fibre. And we have

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \pi \downarrow & \searrow \psi & \\ X' & & \end{array}$$

It remains to verify that the symmetric difference between T and $\{\psi_K^* s' : s' \in S'\}$ is finite. But a diagram chase shows that for cofinitely many $s' \in S'$, $\psi_K^* s'$ is the Zariski closure of $\phi_K(\pi_K^* s')$. Hence, for cofinitely many $s' \in S'$, $\psi_K^* s' \in T$. On the other hand, for cofinitely many $t \in T$, t is the proper transform under ψ of the Zariski closure of $\pi_K(\phi_K^* t)$ in X'_K , which is in S' for cofinitely many t . \square

While we think the above theorem may be of independent interest, our immediate motivation is the following alternative proof of a special case of Theorem 6.1.

Corollary 7.3. *Suppose Z and X are algebraic varieties and $\phi_1, \phi_2 : Z \rightarrow X$ are dominant rational maps with generic fibres irreducible, all over an algebraically closed field K of characteristic zero. If there exist infinitely many hypersurfaces H on X satisfying $\phi_1^* H = \phi_2^* H$ then there is $g \in K(X) \setminus K$ with $g\phi_1 = g\phi_2$.*

Proof. The general idea of proof is to use Theorem 7.2 to reduce to the case of a rational dynamical system, and then apply the results of Cantat in that setting (namely Corollary 1.3 of the introduction).

First we reduce to the case that $\dim X < \dim Z$. Indeed, suppose $\dim X = \dim Z$. Then, both ϕ_1 and ϕ_2 are birational, and we can consider the birational self-map $\alpha := \phi_2 \phi_1^{-1} : X \rightarrow X$. If H is a hypersurface on X for which $\phi_1^* H = \phi_2^* H$ then $\alpha^* H = H$. We thus have infinitely many totally invariant hypersurfaces on X for the rational dynamical system (X, α) . By Corollary 1.3 there is a nonconstant

$g \in K(X)$ such that $g\alpha = g$. Precomposing with ϕ_1 yields $g\phi_2 = g\phi_1$, as desired. We may therefore assume that $\dim X < \dim Z$.

Suppose \mathcal{H} is a countably infinite set of hypersurfaces on X whose strict transforms with respect to ϕ_1 and ϕ_2 agree. One complication is that the $H \in \mathcal{H}$ are not necessarily irreducible, and to deal with that we argue now that we may assume that no two members of \mathcal{H} share an irreducible component in common. First some notation: for H a hypersurface on X , let S_H denote the (finite) set of its irreducible components. Note that $\phi_1^*H = \phi_2^*H$ if and only if $\phi_1^*(S_H) = \phi_2^*(S_H)$ as sets of prime divisors on Z . Now, enumerate $\mathcal{H} = \{H_0, H_1, \dots\}$ and define a new sequence H'_i recursively by setting $H'_0 := H_0$ and H'_{i+1} to be the union of the prime divisors in the set $S_{H_{i+1}} \setminus (\bigcup_{j=0}^i S_{H'_j})$. Then we get a sequence H'_0, H'_1, \dots whose nonempty members are hypersurfaces on Y that still satisfy $\phi_1^*H'_i = \phi_2^*H'_i$, because $\phi_1^*(S_{H'_i}) = \phi_2^*(S_{H'_i})$. No two nonempty members of this sequence share an irreducible component. Moreover, there are infinitely many nonempty H'_i , as at any finite stage $\bigcup_{j=0}^i S_{H'_j}$ is a finite set of irreducible hypersurfaces. So we may as well assume that distinct members of the original \mathcal{H} share no irreducible components.

We now proceed by induction on the dimension of Z , with $\dim Z = 0$ being vacuous. For each $H \in \mathcal{H}$, let S_H denote the (finite) set of irreducible components of H , and T_H the set of irreducible components of $\phi_1^*H = \phi_2^*H$ in Z . Set $S := \bigcup_{H \in \mathcal{H}} S_H$

and $T := \bigcup_{H \in \mathcal{H}} T_H$. Let k_0 be a finitely generated subfield over which Z, X, ϕ_1, ϕ_2 are defined. As the statement of the corollary is preserved under birational equivalence, we may assume that Z and X are normal. Suppose for now that all the members of \mathcal{H} (and hence S and T) are defined over $K_0 := k_0^{\text{alg}}$. So $(Z, T), (X, S)$ are objects in \mathcal{V}_{k_0} , and $\phi_1, \phi_2 : (Z, T) \rightarrow (X, S)$ are morphisms in \mathcal{V}_{k_0} . By Theorem 7.2 there is a terminal morphism $\pi : (Z, T) \rightarrow (Z', T')$. We get an induced diagram in \mathcal{V}_{k_0} ,

$$\begin{array}{ccc} (Z, T) & \xrightarrow{\phi_1} & (X, S) \\ \phi_2 \downarrow & \searrow \pi & \downarrow \psi_1 \\ (X, S) & \xrightarrow{\psi_2} & (Z', T'). \end{array}$$

We want to apply the induction hypothesis to $\psi_1, \psi_2 : X \rightarrow Z'$. To do so, we first remove any H from \mathcal{H} that maps dominantly onto Z' by ψ_1 . There can only be finitely many elements of S with this property since ψ_1 is a morphism in \mathcal{V}_{k_0} . As S is the set of prime divisors appearing as components of elements of \mathcal{H} , and as no two members of \mathcal{H} share an irreducible component, there are only finitely many H 's to remove. Removing finitely many more, we may assume that for all $H \in \mathcal{H}$ the Zariski closure of $\psi_1(H)$, which we denote by H' , is a hypersurface on Z' , and that $\psi_1^*H' = H$. Chasing the above diagram, we get that $\psi_2^*H' = H$ also. We see that there are infinitely many hypersurfaces H' on Z' such that $\psi_1^*H' = \psi_2^*H'$. By the inductive hypothesis (as $\dim X < \dim Z$) there exists nonconstant $g' \in K(Z')$ such that $g'\psi_1 = g'\psi_2 =: g \in K(X)$. So $g\phi_1 = g\phi_2$, as desired.

We still have to consider that case that there is an $H \in \mathcal{H}$ that is not defined over K_0 . But we have seen how to deal with this before: H then is defined over a finitely generated *nonalgebraic* extension k_1 of k_0 . Let $K_1 := k_1^{\text{alg}}$. There are infinitely many $\text{Aut}(K_1/k_0)$ -conjugates of H and they all satisfy the property that

their strict transforms under ϕ_1 and ϕ_2 agree, because ϕ_1 and ϕ_2 are defined over k_0 . Letting $\tilde{\mathcal{H}}$ be this infinite set, and working in \mathcal{V}_{k_1} with $\tilde{\mathcal{H}}$, rather than in \mathcal{V}_{k_0} with \mathcal{H} , we can carry out the above argument. \square

8. POSITIVE CHARACTERISTIC

We have worked so far exclusively in characteristic zero, mostly because the differential algebraic techniques we employ in dealing with the nonreduced case very much require it. But it is reasonable to ask to what extent our proof of the reduced case can be extended to positive characteristic.

The first thing to observe is that even the special case of Cantat's theorem (Corollary 1.3) is false in positive characteristic: consider the dynamical system $(\mathbb{P}^1, \text{Fr}_p)$ on the projective line over the prime finite field \mathbb{F}_p equipped with the p -power Frobenius morphism; there are no preserved nonconstant rational functions, but the $\text{Gal}(\mathbb{F}_p)$ -orbit of any point in $\mathbb{P}^1(\mathbb{F}_p^{\text{alg}})$ is a totally invariant hypersurface. Our proof breaks down in Proposition 6.5 where we replaced scheme-theoretic inverse images by proper transforms; we used the characteristic zero fact that, after localising, a quasi-finite extension can be made étale. The natural way to deal with this would be to impose some separability condition: we should ask that the dominant rational maps $\phi_1, \phi_2 : Z \rightarrow X$ have generic fibres that are *geometrically reduced*, or what is equivalent, that the function field extensions they induce admit separating transcendence basis. This is of course automatic in characteristic zero, and in positive characteristic rules out the Frobenius example. Indeed, the proof of Proposition 6.5 simply goes through in arbitrary characteristic with this additional assumption.²

However, there is another key point in the proof of Theorem 6.1 where characteristic zero is used. In reducing to the case when infinitely many of the invariant hypersurfaces are defined over the same finitely generated field k , we first get them over k^{alg} and then take the union of the Galois-conjugates. In positive characteristic these hypersurfaces will now only be guaranteed to be over the perfect hull of k , which is not necessarily finitely generated. We do not see how to avoid this problem and are thus left with the following partial result in arbitrary characteristic.

Theorem 8.1. *Fix K an algebraically closed field of arbitrary characteristic. Suppose $\phi_1, \phi_2 : Z \rightarrow X$ are dominant rational maps between algebraic varieties over K with geometrically reduced generic fibres. Then the following are equivalent:*

- (1) *There exists a finitely generated subfield $k \subseteq K$ and infinitely many hypersurfaces H on X defined over k^{sep} satisfying $\phi_1^*H = \phi_2^*H$.*
- (2) *There exists $g \in K(X) \setminus K$ such that $g\phi_1 = g\phi_2$.*

Proof. This is obtained by inspecting the proofs in characteristic zero, together with the preceding remarks. We give only a brief sketch.

For (2) \implies (1), let k be a finitely generated field over which Z, X, ϕ_1, ϕ_2, g are defined. Then the level sets of g over k^{sep} give rise to infinitely many hypersurfaces satisfying $\phi_1^*H = \phi_2^*H$.

²Proposition 6.5 does make use of [1, Lemma 6.11] which is stated for characteristic zero. However, the proof given there goes through in positive characteristic if we replace the use of Mordell-Weil with Lang-Néron.

Suppose (1) holds. We may assume that Z, X, ϕ_1, ϕ_2 are all defined over k as well. Replacing the hypersurfaces by the union of their $\text{Gal}(k)$ -conjugates, we may assume that they are all defined over k itself. As discussed above, because of our assumption of geometrically reduced generic fibres, Proposition 6.5 remains true. Hence, exactly as in the proof of Theorem 6.1, after replacing Z and X with sufficiently small nonempty Zariski open subsets, we may assume that we have an infinite sequence $(H_j : j < \omega)$ of hypersurfaces satisfying $\phi_1^{-1}(H_j) = \phi_2^{-1}(H_j)$. We now follow the proof of Theorem 4.1 keeping in mind that Z is reduced, but that the characteristic need not be zero. Possibly shrinking X and Y further, we may assume $X = \text{Spec}(R_K)$ and $Y = \text{Spec}(S_K)$ where R and S are finitely generated k -algebras, R is a UFD, S is an integral domain, k is relatively algebraically closed in $\text{Frac}(R)$ and $\text{Frac}(S)$, and ϕ_1, ϕ_2 are induced by k -algebra embeddings $f_1, f_2 : R \rightarrow S$. The hypersurfaces $(H_j : j < \omega)$ must have principal vanishing ideals and so we get a sequence $(a_j : j < \omega)$ in R that is multiplicatively independent modulo k and, because the H_j satisfy $\phi_1^{-1}(H_j) = \phi_2^{-1}(H_j)$, the a_j satisfy $f_1(a_j)S = f_2(a_j)S$. The hypothesis of Theorem 3.1 are satisfied, except that we may be in positive characteristic. But the proof of Theorem 3.1 in the case when S is an integral domain – this is the first three paragraphs of that proof – did not use characteristic zero. Hence, there exists $g \in \text{Frac}(R) \setminus k$ such that $f_1(g) = f_2(g)$. This proves (2). \square

Question 8.2. Is the assumption in 8.1(1) of the existence of a common finitely generated field of definition necessary?

It may be worth pointing out that Theorem 7.2 on the category of normal varieties equipped with a set of prime divisors remains true in positive characteristic up to applications of Frobenius transforms – but this does not seem to help in answering Question 8.2 even when the generic fibres of ϕ_1, ϕ_2 are assumed to be irreducible.

REFERENCES

- [1] J. Bell, S. Launois, O. Leon Sanchez, and R. Moosa. Poisson algebras via model theory and differential algebraic geometry. *Journal of the European Math Society*, 19(7):20192049, 2017.
- [2] J. Bell, D. Rogalski, and S. J. Sierra. The Dixmier-Moeglin equivalence for twisted homogeneous coordinate rings. *Israel J. Math.*, 180:461–507, 2010.
- [3] S. Cantat. Invariant hypersurfaces in holomorphic dynamics. *Mathematical Research Letters*, 17(5):833–841, 2010.
- [4] J. Freitag and R. Moosa. Finiteness theorems on hypersurfaces in partial differential-algebraic geometry. *Advances in Mathematics*, 314:726–755, 2017.
- [5] E. Hrushovski. Unpublished and untitled notes on “how to deduce the omega-categoricity of degree one strongly minimal sets in DCF from Jouanolou’s work” dating from the mid-nineties.
- [6] J.P. Jouanolou. Hypersurfaces solutions d’une équation de Pfaff analytique. *Mathematische Annalen*, 232(3):239–245, 1978.
- [7] E.R. Kolchin. Algebraic groups and algebraic dependence. *American Journal of Mathematics*, 90:1151–1164, 1968.
- [8] V. A. Krasnov. Compact complex manifolds without meromorphic functions. *Mat. Zametki*, 17:119–122, 1975.
- [9] Serge Lang. *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983.
- [10] R. Moosa and T. Scanlon. Jet and prolongation spaces. *Journal de l’Institut de Mathématiques de Jussieu*, 9(2):391–430, 2010.

- [11] R. Moosa and T. Scanlon. Model theory of fields with free operators in characteristic zero. *Journal of Mathematical Logic*, 14(2):1450009, 43, 2014.

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