Model Theory of Higher Derivations

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Abstract

We generalise results for differential rings to rings R with additive maps $\partial_1, \ldots, \partial_n : R \to R$ satisfying a certain generalisation of the Leibniz rule, namely $(\mathrm{id}, \partial_1, \ldots, \partial_n)$ is a truncated Hasse derivation. We show that the theory of integral domains of characteristic 0 with such maps admits a model companion, and that this model companion admits quantifier elimination and is stable but not ω-stable for $n \geq 2$.

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1 Introduction

The model theory of integral differential rings of chraracteristic 0 and its model companion, DCF_0, are frequently studied. We wish to extend several results to a more general setting.

We work in the language $L_n := (+, \times, -, 0, 1, \partial_1, \dots, \partial_n)$ extending the language of rings with n unary function symbols $\partial_1, \dots, \partial_n$, and we consider the theory T_n of integral domains of characteristic 0 together with the axioms

$$\forall a \forall b (\partial_m(a+b) = \partial_m a + \partial_m b)$$
$$\forall a \forall b (\partial_m(ab) = a(\partial_m b) + \sum_{j=1}^{m-1} (\partial_j a)(\partial_{m-j} b) + (\partial_m a) b)$$

for all $m = 1, \dots, n$.

This generalises the theory of integral differential rings in characteristic 0, which is the special case where $n = 1$. In fact, we see that ∂_1 is always a derivation, regardless of n.

On the other hand, this theory is a special case of the $\mathcal{D}$-rings in characteristic 0 discussed by Moosa and Scanlon [1]. That paper is highly technical, and works in much greater generality. Our goal is to present this special case in a more concrete and explicit way. In addition, we will address quantifier elimination and stability of the model companion, something that is not considered in [1] as it does not hold in that generality.

The same construction except in positive characteristic and positing an additional iterativity condition, stating that $\partial_a \partial_b = \binom{a+b}{a} \partial_{a+b}$, has been studied by Ziegler [3]. In characteristic 0, iterativity would imply that $\partial_m = \frac{1}{m!} \partial_1^m$ for all $m = 1, \dots, n$, and hence the higher derivations would be completely determined by the derivation ∂_1.

Indeed, for any differential ring (R, ∂) where R is an integral domain of characteristic 0, we have that $(R, \partial, \dots, \frac{1}{n!}\partial^n) \models T_n$. However, we do not assume iterativity, and thus there are other examples.

For example, consider the ring $\mathbb{Q}[t]$ with $\partial_1, \partial_2 : \mathbb{Q}[t] \to \mathbb{Q}[t]$ given by $\partial_1 f = \frac{df}{dt}$ and $\partial_2 f = \frac{df}{dt} + \frac{1}{2}\frac{d^2 f}{dt^2}$. It is easily checked that $(\mathbb{Q}[t], \partial_1, \partial_2) \models T_2$.

Indeed, we will see that we can define $\partial_1, \dots, \partial_n$ by having them take t to any arbitrarily chosen values in $\mathbb{Q}[t]$ to give a model of T_n. This essentially follows from Proposition 4.5 below. In this case, we chose $\partial_1 t = \partial_2 t = 1$. If we took $\partial_1 t = 1$ (so that $\partial_1 f = \frac{df}{dt}$ for all $f \in \mathbb{Q}[t]$) and $\partial_2 t = p(t)$ for an arbitrary $p \in \mathbb{Q}[t]$, then we'd get $\partial_2 f = p\frac{df}{dt} + \frac{1}{2}\frac{d^2 f}{dt^2}$.

In this paper, we show that T_n admits a model companion, D_nCF_0, and that D_nCF_0 admits quantifier elimination, generalising the corresponding results for differential rings. Moreover, we show D_nCF_0 is $\mathfrak{c}$-stable, where $\mathfrak{c}$ is the cardinality of the continuum, but for $n \geq 2$, unlike DCF_0, D_nCF_0 is not ω-stable.

Throughout this paper, all rings are assumed to be commutative and unital, and 0 is assumed to be a natural number. Moreover, irreducible varieties are assumed to be nonempty.

2 ∇-rings

While we are primarily interested in models of T_n, it is useful to work in a more flexible 2-sorted setting:

Definition 2.1. Let $\mathcal{S}_n$ denote the collection of triples (R, S, ∇) where R and S are rings and $\nabla = (\partial_0, \dots, \partial_n)$ is an $(n+1)$-tuple of additive maps $\partial_m : R \to S$ such that:

- $\partial_0 1_R = 1_S$;
- for all $m = 0, \dots, n$, $\partial_m(ab) = \sum_{j=0}^{m} (\partial_j a)(\partial_{m-j} b)$ for all $a, b \in R$.

Note that this forces $\partial_0 : R \to S$ to be a ring homomorphism, giving S the structure of an R-algebra. If $(R, S, \nabla) \in \mathcal{S}_n$, we say that ∇ is an **order n S-valued derivation on R**.

If $(R, S, \nabla) \in \mathcal{S}_n$, $R = S$ and $\partial_0 = \mathrm{id}_R$, then we write (R, ∇) instead of (R, S, ∇) and we say that (R, ∇) is a **∇-ring** and that ∇ is an **order n derivation on** R. If moreover R is a field, we call (R, ∇) a **∇-field**.

So the models of T_n are precisely the ∇-rings (R, ∇) such that R is an integral domain of characteristic 0. (In this case, we ignore $\partial_0 = \mathrm{id}_R$).

Proposition 2.3. Let $(R, S, \nabla) \in \mathcal{S}_n$. Then we have $\partial_m 0 = 0$ for all $m = 0, \dots, n$ and $\partial_m 1 = 0$ for all $m = 1, \dots, n$.

Proof. It's clear that $\partial_m 0 = \partial_m(0 + 0) = \partial_m 0 + \partial_m 0$, so $\partial_m 0 = 0$ for all $m = 0, \dots, n$. Fix $m = 1, \dots, n$ and suppose that $\partial_j 1 = 0$ for all $1 \leq j < m$. Then

$$\partial_m 1 = \partial_m(1 \cdot 1) = \sum_{j=0}^{m} (\partial_j 1)(\partial_{m-j} 1) = (\partial_0 1)(\partial_m 1) + (\partial_m 1)(\partial_0 1) = 2(\partial_m 1)$$

so $\partial_m 1 = 0$. $\square$

For a fixed ring S and $n \in \mathbb{N}$, we'll denote by $\eta_{S,n} : S^{n+1} \to S[\varepsilon]/(\varepsilon^{n+1})$ the natural S-module isomorphism given by $\eta_{S,n}(a_0, \dots, a_n) = \sum_{m=0}^{n} a_m \varepsilon^m$. We will write η_S when n is clear from context. Given $(R, S, \nabla) \in \mathcal{S}_n$, we let $e := \eta_{S,n} \circ \nabla : R \to S[\varepsilon]/(\varepsilon^{n+1})$.

Proposition 2.4. Let R, S be rings. Then

1. If $(R, S, \nabla) \in \mathcal{S}_n$, then $e = \eta_S \circ \nabla$ is a ring homomorphism.
2. If $e : R \to S[\varepsilon]/(\varepsilon^{n+1})$ is a ring homomorphism, then $(R, S, \nabla) \in \mathcal{S}_n$ for $\nabla = \eta_S^{-1} \circ e$.

Proof. Additivity is clear since in one direction, ∇ and η_S are additive, and in the other, η_S^{-1} and e are additive. By Proposition 2.3, if $(R, S, \nabla) \in \mathcal{S}_n$, then $e(1) = \partial_0 1 + (\partial_1 1)\varepsilon + \dots + (\partial_n 1)\varepsilon^n = 1$. And if $e(1) = 1$, then $\partial_0 1 = 1$. Moreover, for all $a, b \in R$,

$$\begin{aligned}
e(a)e(b) &= (\sum_{j=0}^{n} (\partial_j a)\varepsilon^j)(\sum_{k=0}^{n} (\partial_k b)\varepsilon^k) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} (\partial_j a)(\partial_k b)\varepsilon^{j+k} \\
&= \sum_{m=0}^{n} \sum_{j=0}^{m} (\partial_j a)(\partial_{m-j} b)\varepsilon^m
\end{aligned}$$

and $e(ab) = \sum_{m=0}^{n} \partial_m(ab)\varepsilon^m$. Therefore, $e(ab) = e(a)e(b)$ if and only if $\partial_m(ab) = \sum_{j=0}^{m}(\partial_j a)(\partial_{m-j} b)$ for all $m = 0, \dots, n$. $\square$

Definition 2.5. Let (R, ∇) and (S, ∇') be ∇-rings, where $\nabla = (\partial_0, \dots, \partial_n)$ and $\nabla' = (\partial'_0, \dots, \partial'_n)$. A **$\nabla$-ring homomorphism** is a ring homomorphism $\varphi : R \to S$ such that $\varphi \circ \partial_m = \partial'_m \circ \varphi$ for all $m = 0, \dots, n$. A bijective ∇-ring homomorphism is called a **∇-ring isomorphism**.

3 Prolongations

The goal of this section is to define the prolongation of polynomials and varieties over a ∇-field, by analogy with the case of differential rings. This will be needed in the next section where we show that T_n admits a model companion, by showing the existentially closed models of T_n can be axiomatised. Prolongations will feature in one of these axioms, the Geometric Axiom, which will be introduced at the end of this section.

Let R, S be rings, $g : R \to S$ a function with $g(0) = 0$, $x = (x_1, \dots, x_l)$ be variables and $f \in R[x]$. We let $f^g \in S[x]$ denote the polynomial obtained by applying g to the coefficients of f. Note that if $e : R \to S$ is a ring homomorphism, then the map $f \mapsto f^e : R[x] \to S[x]$ is also a ring homomorphism.

Definition 3.1. Let $(R, S, \nabla) \in \mathcal{S}_n$. Fix variables $x^{\partial_m} = (x_1^{\partial_m}, \dots, x_l^{\partial_m})$ for all $m = 0, \dots, n$, and let $x := x^{\partial_0}$. Let $S' := S[x^{\partial_0}, \dots, x^{\partial_n}]$. We define the **prolongation map** in l variables to be the map $\tau : R[x] \to (S')^{n+1}$ given by $\tau f = \eta_{S'}^{-1}(f^e(\sum_{m=0}^{n} x^{\partial_m}\varepsilon^m))$, and we let $\tau = (\tau_0, \dots, \tau_n)$.

That is, we apply e to the coefficients of f and we replace each variable x_i by $\sum_{m=0}^{n} x_i^{\partial_m}\varepsilon^m$. We then write this in $S'[\varepsilon]/(\varepsilon^{n+1})$ as an S'-linear combination of $1, \varepsilon, \dots, \varepsilon^n$ and take τf to be the tuple of coefficients in S'. Explicitly, in the case when $l = 1$, write $f = \sum_{i=0}^{d} a_i x_1^i$. Then we expand

$$\sum_{i=0}^{d}((\partial_0 a_i) + (\partial_1 a_i)\varepsilon + \dots + (\partial_n a_i)\varepsilon^n)(x_1^{\partial_0} + x_1^{\partial_1}\varepsilon + \dots + x_1^{\partial_n}\varepsilon^n)^i$$

as a polynomial in ε. The coefficients of $\varepsilon^0, \ldots, \varepsilon^n$ in the resulting polynomial are then $\tau_0, \ldots, \tau_n$.

As an example, we'll compute τ for a ∇-ring when $n = 2$ and $l = 2$.

We'll let $x^{\partial_0} = x = (x_1, x_2)$, $x^{\partial_1} = (y_1, y_2)$ and $x^{\partial_2} = (z_1, z_2)$. Then writing $f = \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} x_1^i x_2^j$, we have

$$\begin{aligned}&\tau_0 f + (\tau_1 f)\varepsilon + (\tau_2 f)\varepsilon^2 \\ &= \sum_{i=0}^{d} \sum_{j=0}^{d} (a_{i,j} + (\partial_1 a_{i,j})\varepsilon + (\partial_2 a_{i,j})\varepsilon^2)(x_1 + y_1\varepsilon + z_1\varepsilon^2)^i (x_2 + y_2\varepsilon + z_2\varepsilon^2)^j.\end{aligned}$$

Matching terms which are order 0 in ε, we have

$$\tau_0 f = \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} x_1^i x_2^j = f.$$

It's also clear that this will always hold for ∇-rings (and $\tau_0 f = f^{\partial_0}$ in general). Matching terms which are order 1 in ε, we have

$$\begin{aligned}\tau_1 f = &\sum_{i=0}^{d} \sum_{j=0}^{d} (\partial_1 a_{i,j}) x_1^i x_2^j \\ &+ \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} \binom{i}{1} x_1^{i-1} y_1 x_2^j + \sum_{i=0}^{d} \sum_{j=0}^{d} a_{i,j} x_1^i \binom{j}{1} x_2^{j-1} y_2 \\ &= f^{\partial_1} + \frac{\partial f}{\partial x_1} y_1 + \frac{\partial f}{\partial x_2} y_2.\end{aligned}$$

Matching terms which are order 2 in ε, we have

$$\tau_2 f = \sum_{i=0}^{d} \sum_{j=0}^{d} (\partial_2 a_{i,j}) x_1^i x_2^j + \sum_{i=0}^{d} \sum_{j=0}^{d} (\partial_1 a_{i,j}) \binom{i}{1} x_1^{i-1} y_1 x_2^j$$

$$
\begin{aligned}
&+\sum_{i=0}^{d}\sum_{j=0}^{d}(\partial_1 a_{i,j})x_1^i\binom{j}{1}x_2^{j-1}y_2+\sum_{i=0}^{d}\sum_{j=0}^{d}a_{i,j}\binom{i}{1}x_1^{i-1}z_1x_2^j\\
&+\sum_{i=0}^{d}\sum_{j=0}^{d}a_{i,j}x_1^i\binom{j}{1}x_2^{j-1}z_2+\sum_{i=0}^{d}\sum_{j=0}^{d}a_{i,j}\binom{i}{2}x_1^{i-2}y_1^2x_2^j\\
&+\sum_{i=0}^{d}\sum_{j=0}^{d}a_{i,j}x_1^i\binom{j}{2}x_2^{j-2}y_2^2+\sum_{i=0}^{d}\sum_{j=0}^{d}a_{i,j}\binom{i}{1}x_1^{i-1}y_1\binom{j}{1}x_2^{j-1}y_2\\
=\ &f^{\partial_2}+\frac{\partial f^{\partial_1}}{\partial x_1}y_1+\frac{\partial f^{\partial_2}}{\partial x_2}y_2+\frac{\partial f}{\partial x_1}z_1\\
&+\frac{\partial f}{\partial x_2}z_2+\frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}y_1^2+\frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}y_2^2+\frac{\partial^2 f}{\partial x_1\partial x_2}y_1y_2.
\end{aligned}
$$

As visible from this example, the explicit expression for τ can be computed, but can quickly become unwieldy.

Let $(R,S,\nabla)\in\mathcal{S}_n$ and $a\in R^l$. Then we let $\partial_m a := (\partial_m a_1,\dots,\partial_m a_l)$ and $\nabla a := (\partial_0 a,\dots,\partial_n a)$. Moreover, if $x=(x_1,\dots,x_l)$ are variables and $f=(f_1,\dots,f_r)\in(R[x])^r$, then we let $\tau_m f := (\tau_m f_1,\dots,\tau_m f_r)$ and $\tau f := (\tau_0 f,\dots,\tau_n f)$.

Similarly, if R,S are rings, $e: R\to S$ is a ring homomorphism and $a\in R^l$, we let $e(a) := (e(a_1),\dots,e(a_l))$.

Lemma 3.2.

1. Let $(R,S,\nabla)\in\mathcal{S}_n$ and $x=(x_1,\dots,x_l)$ be variables. Then we have $\tau x = (x^{\partial_0},\dots,x^{\partial_n})$ and $\tau a = \nabla a$ for all $a\in R$.

2. Let S be a ring, $x=(x_1,\dots,x_l)$ be variables and $f\in S[x,\varepsilon]/(\varepsilon^{n+1})$. Then for all $a\in S^l$, we have $\eta_S^{-1}(f(a)) = (\eta_{S[x]}^{-1}(f))(a)$. In particular, for all $b=(b^{\partial_0},\dots,b^{\partial_n}) = (b_1^{\partial_0},\dots,b_l^{\partial_0},\dots,b_1^{\partial_n},\dots,b_l^{\partial_n})\in S^{(n+1)l}$, we have that $(\tau f)(b) = \eta_S^{-1}(f^e(\sum_{m=0}^n b^{\partial_m}\varepsilon^m))$.

3. Let R,S be rings, $x=(x_1,\dots,x_l)$ be variables, $e: R\to S$ be a ring homomorphism, $f\in R[x]$ and $a\in R^l$. Then $e(f(a)) = f^e(e(a))$.

Proof of 1. Fix $j=1,\dots,l$. Let $f_j\in R[x]$ with $f_j(x)=x_j$. Then we have $f_j^e = e(1)x_j = x_j$, so $f_j^e(\sum_{m=0}^n x^{\partial_m}\varepsilon^m) = \sum_{m=0}^n x_j^{\partial_m}\varepsilon^m$. Then $\tau_m x_j = x_j^{\partial_m}$ for all $m=0,\dots,n$ and $j=1,\dots,l$, so $\tau x = (x^{\partial_0},\dots,x^{\partial_n})$.

Let $a \in R$ and let $g \in R[x]$ with $g(x) = a$. Then $g^e(\sum_{m=0}^{n} x^{\partial_m} \varepsilon^m) = e(a)$, so $\tau a = \nabla a$.

Proof of 2. Write $f = \sum_{m=0}^{n} f_m \varepsilon^m$ for $f_0, \ldots, f_n \in S[x]$. Then for all $a = (a_1, \ldots, a_l) \in S^l$, we have

$$\begin{aligned}
\eta_S^{-1}(f(a)) &= \eta_S^{-1}(\sum_{m=0}^{n} f_m(a)\varepsilon^m) \\
&= (f_0(a), \ldots, f_n(a)) \\
&= (f_0, ..., f_n)(a) \\
&= (\eta_{S[x]}^{-1}(\sum_{m=0}^{n} f_m \varepsilon^m))(a) \\
&= (\eta_{S[x]}^{-1}(f))(a)
\end{aligned}$$

so $(\tau f)(b) = (\eta_{S[x^{\partial_0}, \ldots, x^{\partial_n}]}^{-1}(f^e(\sum_{m=0}^{n} x^{\partial_m} \varepsilon^m)))(b) = \eta_S^{-1}(f^e(\sum_{m=0}^{n} b^{\partial_m} \varepsilon^m))$.

Proof of 3. Write $f = \sum_{i_1=0}^{d} \cdots \sum_{i_l=0}^{d} b_{i_1,\ldots,i_l} x_1^{i_1} \cdots x_l^{i_l}$ and $a = (a_1, \ldots, a_l)$. Then

$$\begin{aligned}
e(f(a)) &= e(\sum_{i_1=0}^{d} \cdots \sum_{i_l=0}^{d} b_{i_1,\ldots,i_l} a_1^{i_1} \cdots a_l^{i_l}) \\
&= \sum_{i_1=0}^{d} \cdots \sum_{i_l=0}^{d} e(b_{i_1,\ldots,i_l}) e(a_1)^{i_1} \cdots e(a_l)^{i_l} \\
&= f^e(e(a)).
\end{aligned}$$

□

Corollary 3.3. Let $(R, S, \nabla) \in \mathcal{S}_n$, $x = (x_1, \ldots, x_l)$ be variables. Then $\nabla f(a) = \tau f(\nabla a)$ for all $a \in R^l$ and $f \in R[x]$.

Proof. We have

$$\nabla f(a) = \eta_S^{-1}(e(f(a))) = \eta_S^{-1}(f^e(e(a))) = \eta_S^{-1}(f^e(\sum_{m=0}^{n} (\partial_m a)\varepsilon^m)) = \tau f(\nabla a).$$

$\square$

Proposition 3.4. Let $(R, S, \nabla) \in \mathcal{S}_n$, $x = (x_1, \ldots, x_l)$ be variables and $b = (b^{\partial_0}, \ldots, b^{\partial_n}) = (b_1^{\partial_0}, \ldots, b_l^{\partial_0}, \ldots, b_1^{\partial_n}, \ldots, b_l^{\partial_n}) \in S^{(n+1)l}$ be arbitrary. Then $(R[x], S, \nabla_b) \in \mathcal{S}_n$, where ∇_b is given by $\nabla_b f = \tau f(b)$. Moreover, it is the unique S-valued order n derivation ∇' on $R[x]$ extending ∇ with $\nabla' x = b$.

Proof. By Proposition 2.4, we only need to show the equivalent statement holds for $\eta_S \circ \nabla_b : R[x] \to S[\varepsilon]/(\varepsilon^{n+1})$, i.e. the map $f \mapsto f^e(\sum_{m=0}^n b^{\partial_m}\varepsilon^m)$. That is, we need to show that this map is the unique ring homomorphism extending $e : R \to S[\varepsilon]/(\varepsilon^{n+1})$ such that $e(x) = \sum_{m=0}^n b^{\partial_m}\varepsilon^m$.

But this a composition of the map $f \mapsto f^e : R[x] \to S[x]$, which we noted is a ring homomorphism, and the inclusion map $S[x] \to (S[\varepsilon]/(\varepsilon^{n+1}))[x]$ and the evaluation map $g \mapsto g(\sum_{m=0}^n b^{\partial_m}\varepsilon^m) : (S[\varepsilon]/(\varepsilon^{n+1}))[x] \to S[\varepsilon]/(\varepsilon^{n+1})$, which are clearly ring homomorphisms. Therefore, their composition is also a ring homomorphism. Uniqueness is clear since the value of e is fixed on R and x, which generate $R[x]$. By Lemma 3.2(1), we have that $\nabla_b a = (\tau a)(b) = \nabla a$ for all $a \in R$, so ∇_b extends ∇, and that $\nabla_b x = (\tau x)(b) = b$. $\square$

Definition 3.5. If (k, ∇) is a ∇-field and X is a subvariety of $\mathbb{A}_k^l$, we define the **prolongation** of X to be the subvariety τX of $\mathbb{A}_k^{(n+1)l}$ defined by $\tau_m f$ for $f \in I(X)$ and $m = 0, \ldots, n$.

We define the map $\pi : \mathbb{A}_k^{(n+1)l} \to \mathbb{A}_k^l$ to be the projection on the first l coordinates. Note that π restricts to a map $\pi : \tau X \to X$ since for all $f \in I(X)$, $f = \tau_0 f \in I(\tau X)$. Note also that if $a \in X(K)$ for some field extension $K \supseteq k$, then $\nabla a \in \tau X(K)$ since by Corollary 3.3, we have that $\tau_m f(\nabla a) = \partial_m f(a) = \partial_m 0 = 0$.

Definition 3.6. Let (k, ∇) be a ∇-field. We say (k, ∇) satisfies the **Geometric Axiom (GA)** if for every irreducible subvariety $X \subseteq \mathbb{A}_k^l$ and every irreducible subvariety $Y \subseteq \tau X$ such that $\pi(Y)$ is Zariski dense in X, there exists $a \in X(k)$ such that $\nabla a \in Y(k)$.

The goal of the next section is to show that the existentially closed models of T_n are precisely the algebraically closed ∇-fields of characteristic 0 satisfying GA, and to deduce from this the existence of a model companion for T_n.

4 Model Companion

First, we wish to show that existentially closed models of T_n are algebraically closed fields. We do this by showing that if $(R, \nabla) \models T_n$, then ∇ extends to algebraic extensions of the fraction field of R.

Lemma 4.1. Let R be a ring. Suppose $a_0, \ldots, a_n \in R$ with a_0 invertible. Then $a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n$ is invertible in $R[\varepsilon]/(\varepsilon^{n+1})$.

Proof. Note that $a_1\varepsilon + \cdots + a_n\varepsilon^n = (a_1 + \cdots + a_n\varepsilon^{n-1})\varepsilon$ is nilpotent, and hence lies in the Jacobson radical of $R[\varepsilon]/(\varepsilon^{n+1})$. Thus, $1 + a_0^{-1}(a_1\varepsilon + \cdots + a_n\varepsilon^n)$ is a unit in $R[\varepsilon]/(\varepsilon^{n+1})$, and hence so is $a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n$. $\square$

Recall that if R is an integral domain, S is a ring and $e : R \to S$ is a ring homomorphism such that $e(b)$ is invertible in S for all $0 \neq b \in R$, then e extends uniquely to ring homomorphism $e : \mathrm{Frac}(R) \to S$ given by $e(\frac{a}{b}) = \frac{e(a)}{e(b)}$.

Corollary 4.2. Let R, S be integral domains and $(R, S, \nabla) \in \mathcal{S}_n$, and suppose $\ker(\partial_0) = \{0\}$. Then ∇ extends uniquely to a $\mathrm{Frac}(S)$-valued order n derivation on $\mathrm{Frac}(R)$. Moreover, if $\partial_0 : R \to S$ is the inclusion map (in which case the assumption that $\ker(\partial_0) = \{0\}$ comes for free), then $\partial_0 : \mathrm{Frac}(R) \to \mathrm{Frac}(S)$ is also the inclusion map.

For the first part of the statement, it suffices to show $e : R \to S[\varepsilon]/(\varepsilon^{n+1})$ extends uniquely to a ring homomorphism $e : \mathrm{Frac}(R) \to \mathrm{Frac}(S)[\varepsilon]/(\varepsilon^{n+1})$. Consider e as a homomorphism $R \to \mathrm{Frac}(S)[\varepsilon]/(\varepsilon^{n+1})$. It suffices to show $e(b)$ is invertible for all $b \neq 0$.

But $\ker(\partial_0) = \{0\}$, so $\partial_0 b \neq 0$ is invertible in $\mathrm{Frac}(S)$, and so by Lemma 4.1, $e(b) = \partial_0 b + (\partial_1 b)\varepsilon + \cdots + (\partial_n b)\varepsilon^n$ is invertible in $\mathrm{Frac}(S)[\varepsilon]/(\varepsilon^{n+1})$.

If $\partial_0 : R \to S$ is the inclusion map, then $\partial_0(\frac{a}{b}) = \frac{\partial_0 a}{\partial_0 b} = \frac{a}{b}$, and therefore $\partial_0 : \mathrm{Frac}(R) \to \mathrm{Frac}(S)$ is the inclusion map. $\square$

Lemma 4.3. Let $k \subseteq K$ be fields of characteristic 0 and let $(k, K, \nabla) \in \mathcal{S}_n$ with ∂_0 the inclusion map. Let $\alpha \in K$ be algebraic over k and let $f \in k[t]$ be its minimal polynomial. Then there exist unique $b_1, \ldots, b_n \in K$ such that $f^e(\alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n) = 0$ in $K[\varepsilon]/(\varepsilon^{n+1})$.

Proof. We proceed by induction on n.

Base case: $n = 0$. Note that $e : k \to K$ is ∂_0, which is the inclusion map. Thus, we have $f^e(\alpha) = f(\alpha) = 0$. Uniqueness is immediate.

Induction step: Let $n \geq 1$ and suppose $b_1, \ldots, b_{n-1} \in K$ are the unique elements satisfying $f^e(\alpha + b_1\varepsilon + \cdots + b_{n-1}\varepsilon^{n-1}) = 0$ in $K[\varepsilon]/(\varepsilon^n)$.

Note that since k is a field of characteristic 0 and f is the minimal polynomial of α over k, we have $f'(\alpha) \neq 0$. Write $f(t) = a_l t^l + \cdots + a_0$ for $a_0, \ldots, a_l \in k$. Then in $K[\varepsilon]/(\varepsilon^n)$, we have

$$
\begin{aligned}
& f^e(\alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n) \\
= & \sum_{j=0}^{l} (a_j + (\partial_1 a_j)\varepsilon + \cdots + (\partial_n a_j)\varepsilon^n)(\alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n)^j \\
= & \sum_{j=0}^{l} (a_j + (\partial_1 a_j)\varepsilon + \cdots + (\partial_n a_j)\varepsilon^n)(\alpha + b_1\varepsilon + \cdots + b_{n-1}\varepsilon^{n-1})^j \\
& + \sum_{j=0}^{l} a_j \binom{j}{1} \alpha^{j-1} b_n \varepsilon^n \\
= & \tau_n f(\alpha, b_1, \ldots, b_{n-1}, 0) + f'(\alpha) b_n \varepsilon^n
\end{aligned}
$$

which is 0 if and only if $b_n = -\frac{1}{f'(\alpha)} \tau_n f(\alpha, b_1, \ldots, b_{n-1}, 0)$. $\square$

Corollary 4.4. Let $k \subseteq K$ be fields of characteristic 0 and $(k, K, \nabla) \in \mathcal{S}_n$ with ∂_0 the inclusion map. Let $\alpha \in K$ be algebraic over k. Then ∇ extends uniquely to a K-valued order n derivation on $k(\alpha)$ such that ∂_0 is the inclusion map.

Proof. Let $f \in k[t]$ be the minimal polynomial of α over k. By Lemma 4.3, there exist unique $b_1, \ldots, b_n \in K$ such that $f^e(\alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n) = 0$ in $K[\varepsilon]/(\varepsilon^{n+1})$. By Proposition 3.4, ∇ extends to a K-valued order n derivation on $k[t]$ such that $\nabla t = (\alpha, b_1, \ldots, b_n)$. Then for $e : k[t] \to K[\varepsilon]/(\varepsilon^{n+1})$, we have

$$e(f(t)) = f^e(e(t)) = f^e(\alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n) = 0.$$

Thus, e extends to a ring homomorphism $e : k[t]/(f(t)) \to K[\varepsilon]/(\varepsilon^{n+1})$. But $k[t]/(f(t)) \cong k(\alpha)$, and so in fact e extends to a ring homomorphism $e : k(\alpha) \to K[\varepsilon]/(\varepsilon^{n+1})$.

Moreover, $e(\alpha) = e(t) = \alpha + b_1\varepsilon + \cdots + b_n\varepsilon^n$, so $\partial_0\alpha = \alpha$. Since ∂_0 is a ring homomorphism and is the inclusion map on k and α, which generate $k(\alpha)$, ∂_0 is the inclusion map on $k(\alpha)$.

For uniqueness, suppose $(k(\alpha), K, \nabla') \in \mathcal{S}_n$ such that ∇' extends ∇ and $\partial'_0 : k(\alpha) \to K$ is the inclusion map. Then

$$0 = e(0) = e(f(\alpha)) = f^e(e(\alpha)) = f^e(\alpha + (\partial'_1\alpha)\varepsilon + \cdots + (\partial'_n\alpha)\varepsilon^n)$$

so by the uniqueness of Lemma 4.3, the values of $\partial'_1\alpha, \ldots, \partial'_n\alpha$ are fixed. Hence, the value of $e(\alpha) = \alpha + (\partial'_1\alpha)\varepsilon + \cdots + (\partial'_n\alpha)\varepsilon^n$ is fixed. Thus, the value of e is fixed on k and α, and hence on $k(\alpha)$. $\square$

Proposition 4.5. Let $F \subseteq L \subseteq K$ be fields of characteristic 0 and let $(F, K, \nabla) \in \mathcal{S}_n$ with ∂_0 the inclusion map. Let A be a transcendence basis for L over F, and fix $b_a = (b_a^{\partial_0}, \ldots, b_a^{\partial_n}) \in K^{n+1}$ for each $a \in A$ arbitrarily such that $b_a^{\partial_0} = a$. Then ∇ extends to a unique K-valued order n derivation on L such that $\nabla a = b_a$ for all $a \in A$ and ∂_0 is the inclusion map.

Proof. Note that $L = F(A)^{alg} \cap L$. Thus, for all $\alpha \in L$, $\alpha \in F(a)^{alg}$ for some $a = (a_1, \ldots, a_r) \in A^r$. Let $b^{\partial_m} := (b_{a_1}^{\partial_m}, \ldots, b_{a_r}^{\partial_m})$ for all $m = 0, \ldots, n$ and $b := (b^{\partial_0}, \ldots, b^{\partial_n})$. Let $x = (x_1, \ldots, x_r)$ be variables. By Proposition 3.4, ∇ extends uniquely to a K-valued order n derivation on $F[x]$ with $\nabla x = b$. Since $a_1, \ldots, a_r$ are transcendental over F, $F[x] \cong F[a]$, so ∇ extends uniquely to a K-valued order n derivation ∇_a on $F[a]$ with $\nabla_a a = b$.

Moreover, $\partial_0 a = b^{\partial_0} = a$ by assumption. Since $\partial_0 : F[a] \to K$ is a ring homomorphism and is the inclusion map on F and a, it is the inclusion map on $F[a]$. By Corollary 4.2, ∇_a extends uniquely to a K-valued order n derivation on $F(a)$ with ∂_0 the inclusion map. By Corollary 4.4, it further extends uniquely to a K-valued order n derivation on $F(a, \alpha)$ with ∂_0 the inclusion map.

This demonstrates uniqueness. If $\alpha \in F(a)^{alg} \cap F(b)^{alg}$ for $a \in A^r$ and $b \in A^s$, then by uniqueness, $\nabla_{(a,b)}|_{F(a,\alpha)} = \nabla_a$ and $\nabla_{(a,b)}|_{F(b,\alpha)} = \nabla_b$. Thus, $\nabla_a\alpha = \nabla_{(a,b)}\alpha = \nabla_b\alpha$. Thus, we get a well-defined map $\nabla : L \to K^{n+1}$ where $\nabla\alpha = \nabla_a\alpha$ for some $a \in A^r$ such that $\alpha \in F(a)^{alg}$.

If $\alpha, \beta \in L$, then $\alpha \in F(a)^{alg}$ and $\beta \in F(b)^{alg}$ for some $a \in A^r$ and $b \in A^s$, so $\alpha, \beta, \alpha+\beta, \alpha\beta \in F(a, b)^{alg}$. Therefore, $\partial_m(\alpha + \beta) = \partial_m\alpha + \partial_m\beta$

and $\partial_m(\alpha\beta) = \sum_{j=0}^{m}(\partial_j\alpha)(\partial_{m-j}\beta)$ for all $j = 0, \ldots, n$, and $\partial_0\alpha = \alpha$, so ∂_0 is the inclusion map. $\square$

Corollary 4.6. Let (R, ∇) be an existentially closed model of T_n. Then R is an algebraically closed field.

Proof. Let $0 \neq a \in R$. Since R is an integral domain, by Corollary 4.2, ∇ extends to an order n derivation on $\mathrm{Frac}(R)$. Note that $(\mathrm{Frac}(R), \nabla) \models T_n$ and extends (R, ∇). Moreover, since $a \neq 0$, $(\mathrm{Frac}(R), \nabla) \models \exists b(ab = 1)$, hence by existential closure, so does (R, ∇).

Thus, $R = k$ is a field. Let $f \in k[t]$ be a nonconstant polynomial and let $\alpha \in k^{alg}$ be a root. Then ∇ extends by inclusion to a $k(\alpha)$-valued order n derivation on k, where ∂_0 is the inclusion map, and since k is of characteristic 0, by Corollary 4.4, ∇ extends further to an order n derivation on $k(\alpha)$. Note that $(k(\alpha), \nabla) \models T_n$ and extends (k, ∇). Moreover, since $f(\alpha) = 0$, $(k(\alpha), \nabla) \models \exists a(f(a) = 0)$, hence by existential closure, so does (k, ∇). Thus, $R = k$ is an algebraically closed field. $\square$

Next, we show existentially closed models of T_n have GA. In fact, we can prove a slightly stronger statement.

Proposition 4.7. Let (K, ∇) be an existentially closed model of T_n. Let $X \subseteq \mathbb{A}^l_K$ and $Y \subseteq \tau X$ be irreducible subvarieties such that $\pi(Y)$ is Zariski dense in X, and let $Z \subsetneq Y$ be a proper subvariety. Then there is $a \in X(K)$ such that $\nabla a \in Y(K) \setminus Z(K)$.

Proof. Let $K[X] := K[x]/I(X)$ and $K[Y] := K[x^{\partial_0}, \ldots, x^{\partial_n}]/I(Y)$. We know $K[X]$ and $K[Y]$ are integral domains since X and Y are irreducible subvarieties. Thus, let $K(X)$ and $K(Y)$ be their respective fraction fields.

Since Y is irreducible, it is nonempty, so $I(Y) \cap K = \{0\}$. Therefore, $K \subseteq K[Y]$ and ∇ extends by inclusion to a $K[Y]$-valued order n derivation on K.

By Proposition 3.4, ∇ extends further to a $K[Y]$-valued order n derivation on $K[x]$ with $\nabla f = \tau f((x^{\partial_0}, \ldots, x^{\partial_n}) + I(Y)) = \tau f + I(Y)$ for all $f \in K[x]$.

Since for all $f \in I(X)$, $\tau_m f \in I(\tau X) \subseteq I(Y)$ for all $m = 0, \ldots, n$, we have $e(f) = \tau_0 f + (\tau_1 f)\varepsilon + \cdots + (\tau_n f)\varepsilon^n + I(Y) = 0 + I(Y)$, so $I(X) \subseteq \ker(e)$. Therefore, e extends to $K[X] \to (K[Y])[\varepsilon]/(\varepsilon^{n+1})$.

Claim: $I(X) = I(Y) \cap K[x]$.

It's clear $I(X) \subseteq I(Y) \cap K[x]$ since for every $f \in I(X)$, $f \in K[x]$ and $f = \tau_0 f \in I(\tau X) \subseteq I(Y)$.

Let $f \in I(Y) \cap K[x]$. Suppose $f \notin I(X)$. Let $X_0 \subseteq X$ be the subvariety defined by adding f to the polynomials defining X. Then $\pi(Y) \subseteq X_0 \subsetneq X$, which is a contradiction since $\pi(Y)$ is Zariski dense in X. Thus, $f \in I(X)$, so $I(Y) \cap K[x] \subseteq I(X)$, and thus $I(X) = I(Y) \cap K[x]$. This proves the claim.

In particular, $\varphi : K[X] \to K[Y]$ given by $\varphi(f + I(X)) = f + I(Y)$ is a well-defined injective ring homomorphism, so we may identify $K[X]$ as a subring of $K[Y]$. And $\partial_0(f + I(X)) = \tau_0 f + I(Y) = f + I(Y) = \varphi(f + I(X))$, so ∂_0 is the inclusion map under this identification.

Note that by Corollary 4.2, ∇ extends to a $K(Y)$-valued order n derivation on $K(X)$, with ∂_0 the inclusion map, and by Proposition 4.5, ∇ extends further to an order n derivation on $K(Y)$. Let $a := x + I(Y) \in K(Y)$.

Note that $\nabla a = \tau x + I(Y) = (x^{\partial_0}, \ldots, x^{\partial_n}) + I(Y)$ by Lemma 3.2(1). In particular, for all $f \in K[x^{\partial_0}, \ldots, x^{\partial_n}]$, we have $f(\nabla a) = f + I(Y)$.

Thus, for all $f \in I(Y)$, $f(\nabla a) = 0 + I(Y)$, so $\nabla a \in Y(K(Y))$. And since $I(X) \subseteq I(Y)$, we have $\nabla a \in X(K(Y))$. Furthermore, since $Z \subsetneq Y$, fix $f \in I(Z) \setminus I(Y)$. Then $f(\nabla a) = f + I(Y) \neq 0 + I(Y)$, so $\nabla a \notin Z(K(Y))$.

Moreover, $K \subseteq K(Y)$ with $\nabla(b + I(Y)) = \tau b + I(Y) = \nabla b + I(Y)$ for all $b = b + I(Y) \in K$ by Lemma 3.2(1), so this ∇ extends the original ∇.

By existential closure, there is $b \in X(K)$ such that $\nabla b \in Y(K) \setminus Z(K)$.

□

Now, we work on the converse. That is, we wish to show that all algebraically closed ∇-fields of characteristic 0 satisfying GA are existentially closed models of T_n.

Let (R, ∇) be a ∇-ring and $x = (x_1, \ldots, x_l)$. Let $\{\partial_1, \ldots, \partial_n\}^*$ denote the set of all words in the alphabet $\partial_1, \ldots, \partial_n$, and let λ denote the empty word. For each $w \in \{\partial_1, \ldots, \partial_n\}^*$, we introduce a variable $x^w = (x_1^w, \ldots, x_l^w)$ with $x^\lambda = x$. Then we let $R\{x\} := R[x^w : w \in \{\partial_1, \ldots, \partial_n\}^*]$, and call $R\{x\}$ the ring of **n-differential polynomials** or **∇-polynomials**.

Although we won't need it below, we make $R\{x\}$ into a ∇-ring by taking $\partial_m x^w = x^{\partial_m w}$ for all $m = 0, .., n$ and $w \in \{\partial_1, \ldots, \partial_n\}^*$. That this defines a unique order n derivation is left as an exercise.

We can evaluate elements of $R\{x\}$ at l-tuples from R, or from ∇-ring extensions of R. Indeed, for $w \in \{\partial_1, \dots, \partial_n\}^*$ and $a \in R^l$, we write wa for the result of applying the operators $\partial_1, \dots, \partial_n$ to a in the order specified by w. For $f \in R\{x\}$ and $a \in R^l$, we write $f(a)$ for $f(wa : w \in \{\partial_1, \dots, \partial_n\}^*)$. For instance, if $f(x) = x_2^{\partial_2\partial_3} x_3^{\partial_1} - 4x_1$ and $a = (a_1, a_2, a_3)$, then we have $f(a) = (\partial_2\partial_3 a_2)(\partial_1 a_3) - 4a_1$.

Proposition 4.8. Let $(K, \nabla) \subseteq (L, \nabla)$ be an algebraically closed ∇-fields of characteristic 0 where (K, ∇) has GA. Let $x = (x_1, \dots, x_l)$ and suppose $f_1, \dots, f_r \in K\{x\}$ are such that there exists $b \in L^l$ with $f_j(b) = 0$ for all $j = 1, \dots, r$. Then there exists $c \in K^l$ such that $f_j(c) = 0$ for all $j = 1, \dots, r$.

Proof. Let W be the set of all suffixes of words in $\{\partial_1, \dots, \partial_n\}^*$ appearing in $f_1, .., f_r$. For instance, if $r = 2$, $f_1 = x^{\partial_1\partial_2} + 3x^{\partial_3}$ and $f_2 = 2x^{\partial_1\partial_4\partial_5} + 1$, then $W = \{\partial_1\partial_2, \partial_2, \partial_3, \partial_1\partial_4\partial_5, \partial_4\partial_5, \partial_5, \lambda\}$. Note that W is finite since the expressions $f_1, \dots, f_r$ are finite, and there are finitely many of them, so they contain only finitely many variables between them, and each word has only finitely many suffixes. Let $N := |W|$.

Let $g_1, \dots, g_r$ be $f_1, \dots, f_r$ considered as elements of $K[x^w : w \in W]$. That is, $g_1, \dots, g_r$ are the same polynomials as $f_1, \dots, f_r$, but in our notation for evaluation, we treat them as polynomials in Nl variables, rather than ∇-polynomials in l variables.

Let $b_W := (wb : w \in W) \in L^{Nl}$. Then $g_j(b_W) = f_j(b) = 0$ for all $j = 1, \dots, r$. Let $X := \operatorname{loc}(b_W/K) \subseteq \mathbb{A}_K^{Nl}$ and $Y := \operatorname{loc}(\nabla b_W/K) \subseteq \mathbb{A}_K^{(n+1)Nl}$, where for $c \in L^s$, $\operatorname{loc}(c/K)$ denotes the Zariski locus of c over K.

Note that X and Y are irreducible since they are each the Zariski locus of a point. Note that since $b_W = \pi(\nabla b_W) \in \pi(Y)(L) \subseteq X(L)$ and b_W is generic in X, $\pi(Y)$ is Zariski dense in X. By GA, there exists $a \in X(K)$ such that $\nabla a \in Y(K)$. Write $a = (a^w : w \in W)$.

Let the variables of $\mathbb{A}_K^{(n+1)Nl}$ be $x^{\partial_m, w} = (x_1^{\partial_m, w}, \dots, x_l^{\partial_m, w})$ for each $m = 0, \dots, n$ and $w \in W$. Write $\nabla b_W =: (b^{\partial_m, w} : m = 0, \dots, n, w \in W)$, where we have $b^{\partial_m, w} = \partial_m wb$ for $m = 0, \dots, n$ and $w \in W$.

Claim: Suppose $\partial_m w \in W$ for some $m = 0, \dots, n$ and $w \in W$. Then we have $a^{\partial_m w} = \partial_m a^w$.

Note that $b^{\partial_0, \partial_m w} = \partial_m wb = b^{\partial_m, w}$. Thus, ∇b_W satisfies $x^{\partial_0, \partial_m w} = x^{\partial_m, w}$, and since ∇b_W is generic in Y, so does everything in Y. Since $\nabla a \in Y$, we have $a^{\partial_m w} = \partial_0 a^{\partial_m w} = \partial_m a^w$.

By induction, $a^w = wa^\lambda$ for all $w \in W$. And for all $j = 1, \ldots, r$, we have $f_j(a^\lambda) = g_j(a) = 0$, since $g_j(b_W) = 0$, $a \in X$ and b_W is generic in X. Thus, $a^\lambda \in K^l$ works. $\square$

Note that every atomic formula in L_n in the variables $x = (x_1, \ldots, x_l)$ and parameters from a ring R is of the form $f = g$ for $f, g \in R\{x\}$, where x_j^w represents the string wx_j for $j = 1, \ldots, l$ and $w \in \{\partial_1, \ldots, \partial_n\}^*$. In T_n, this is equivalent to the atomic formula $f - g = 0$, so every atomic formula is equivalent to one of the form $f = 0$ for $f \in R\{x\}$.

Theorem 4.9. The existentially closed models of T_n are precisely the algebraically closed ∇-fields of characteristic 0 satisfying GA.

Proof. Let (K, ∇) be an existentially closed model of T_n. We know K is an algebraically closed field of characteristic 0. Moreover, by applying Proposition 4.7 with $Z = \emptyset \subsetneq Y$ since Y is irreducible, we get GA.

Conversely, let (K, ∇) be an algebraically closed ∇-field of characteristic 0 satisfying GA.

Suppose $(K, \nabla) \subseteq (R, \nabla) \models T_n$. Let $x = (x_1, \ldots, x_l)$ be variables and $\varphi(x)$ be a finite conjunction of atomic and negated atomic formulas for which there exists $a \in R^l$ such that $(R, \nabla) \models \varphi(a)$.

Then ∇ extends to an order n derivation on $L := (\mathrm{Frac}(R))^{alg}$. And $a \in L^l$ with $(L, \nabla) \models \varphi(a)$. Let $f_1, \ldots, f_r, g_1, \ldots, g_s \in K\{x\}$ such that

$$\varphi(x) = f_1 = 0 \wedge \cdots \wedge f_r = 0 \wedge g_1 \neq 0 \wedge \cdots \wedge g_s \neq 0.$$

Let $b_j := \frac{1}{g_j(a)}$ for all $j = 1, \ldots, s$ and $b = (b_1, \ldots, b_s)$. Let $y = (y_1, \ldots, y_s)$. Let $h_j(x, y) := g_j(x)y_j - 1$ for all $j = 1, \ldots, s$. Then (a, b) satisfies $f_j = 0$ for all $j = 1, \ldots, r$ and $h_j = 0$ for all $j = 1, \ldots, s$. By Proposition 4.8, there exists $(c, d) \in K^{l+s}$ such that $f_j(c, d) = 0$ for all $j = 1, \ldots, r$ and $h_j(c, d) = 0$ for all $j = 1, \ldots, s$. In particular, $f_j(c) = 0$ for all $j = 1, \ldots, r$ and $g_j(c) \neq 0$ for all $j = 1, \ldots, s$, so $(K, \nabla) \models \varphi(c)$. Thus, (K, ∇) is an existentially closed model of T_n. $\square$

Corollary 4.10. T_n admits a model companion.

Proof. Since T_n is a universal theory, it suffices that existentially closed models of T_n are axiomatisable.

Being an algebraically closed ∇-field of characteristic 0 is clearly elementary. It remains therefore to verify that GA is first-order axiomatisable. This is somewhat subtle, though no subtler than the axiomatisability of the geometric axiom for DCF_0, and we sketch a proof in the appendix. $\square$

Definition 4.11. An **n-differentially closed field** is an algebraically closed ∇-field of characteristic 0 satisfying GA. The theory of n-differentially closed fields, as axiomatised in the appendix, is denoted D_nCF_0.

5 Quantifier Elimination

In this section, we will show that D_nCF_0 admits Quantifier Elimination. Note that this is not a consequence of the results in [1].

Lemma 5.1. Let U be an algebraically closed field of characteristic 0 and let $K, L \subseteq U$ be subfields which are algebraically disjoint over a common further subfield $F \subseteq K \cap L$. Suppose (K, ∇_K) and (L, ∇_L) are ∇-fields and ∇_K and ∇_L agree on F. Then ∇_K and ∇_L jointly extend to an order n derivation on $(KL)^{alg}$.

Proof. Fix a transcendence basis A for K over F. Since K, L are algebraically disjoint over F, A is also a transcendence basis for $(KL)^{alg}$ over L, and so by Proposition 4.5, ∇_L extends uniquely to an order n derivation ∇ on $(KL)^{alg}$ with $\nabla a = \nabla_K a$ for all $a \in A$. Moreover, $\nabla_K|_F = \nabla_L|_F = \nabla|_F$ extends uniquely to an order n derivation ∇' on K with $\nabla' a = \nabla_K a$ for all $a \in A$, so $\nabla|_K = \nabla_K$. $\square$

Proposition 5.2. Suppose $(K, \nabla), (L, \nabla) \models D_nCF_0$ with a common substructure $(R, \nabla) \models T_n$. Then there are embeddings $f : (K, \nabla) \to (M, \nabla)$ and $g : (L, \nabla) \to (M, \nabla)$ where $(M, \nabla) \models D_nCF_0$ with $f|_R = g|_R$.

Proof. Note that by Corollary 4.2 and Proposition 4.5, ∇ extends uniquely to an order n derivation on $F = (\mathrm{Frac}(R))^{alg}$, so (F, ∇) is also a common substructure, and obviously $f|_F = g|_F$ implies $f|_R = g|_R$. Thus, without loss of generality, we may assume $R = F = F^{alg}$.

Let A be a transcendence basis for K over F and $B := \{x^a : a \in A\}$ be a set of distinct variables. Let $U := L(B)^{alg}$ and $K' := F(B)^{alg}$. There is a ring isomorphism $\rho : K \to K'$ preserving F, since K, K' are algebraically closed fields extending F of the same transcendence degree. B is a transcendence basis for K' over F which is algebraically independent over L in U. Thus, K' and L are algebraically disjoint over F.

Define $\nabla' := (\rho \circ \partial_0 \circ \rho^{-1}, \ldots, \rho \circ \partial_n \circ \rho^{-1})$. It is easily checked that this makes (K', ∇') into a ∇-field isomorphic to (K, ∇) via ρ. By Lemma 5.1, (K', ∇') and (L, ∇) extend to a ∇-field (U, ∇) where $U = L(B)^{alg} = (K'L)^{alg}$. Since D_nCF_0 is a model companion of T_n, every model of T_n embeds into a model of D_nCF_0. Thus, extend (U, ∇) further to $(M, \nabla) \models D_nCF_0$. Let $f := \iota_{K'} \circ \rho$ and $g := \iota_L$, where $\iota_{K'} : K' \to M$ and $\iota_L : L \to M$ are the inclusion maps. Then $f : K \to M, g : L \to M$ are injective ∇-ring homomorphisms satisfying $f|_F = g|_F$ since ρ, $\iota_{K'}$ and ι_L all fix F. □

Corollary 5.3. *D_nCF_0 admits Quantifier Elimination.*

Proof. To show D_nCF_0 admits quantifier elimination, it suffices to check that if $(K, \nabla), (L, \nabla) \models D_nCF_0$ with a common substructure (R, ∇), $\varphi(x)$ is a conjunction of atomic and negated atomic L_n-formulas with parameters from R, where x is a single variable, and there exists $a \in K$ realising $\varphi(x)$, then there also exists $b \in L$ realising $\varphi(x)$.

Let $(K, \nabla), (L, \nabla) \models D_nCF_0$ with a common substructure (R, ∇), and let $\varphi(x)$ be a conjunction of atomic and negated atomic L_n-formulas with parameters from R with $a \in K$ realising $\varphi(x)$. By Proposition 5.2, amalgamate $(K, \nabla), (L, \nabla) \models D_nCF_0$ into $(M, \nabla) \models D_nCF_0$. Then $(M, \nabla) \models T_n$ extends $(L, \nabla) \models D_nCF_0$ and $a \in M$ realises $\varphi(x)$, hence by existential closure, there exists $b \in L$ realising $\varphi(x)$. It follows that D_nCF_0 admits quantifier elimination. □

6 Stability

We may ask whether or not D_nCF_0 is stable or ω-stable. Note that D_1CF_0 corresponds to DCF_0, which is known to be ω-stable, and in particular stable. However, we will show that for $n \geq 2$, D_nCF_0 is stable but not ω-stable. Specifically, it is $\mathfrak{c}$-stable, where $\mathfrak{c}$ is the cardinality of the continuum.

Recall that if $(K, \nabla) \models D_nCF_0$, $A \subseteq K$ and $c \in K^l$, then $\mathrm{tp}(c/A)$ denotes the type of c over A, i.e. the set of L_n-formulas $\varphi(x)$ with parameters from A, where $x = (x_1, ..., x_l)$, such that $(K, \nabla) \models \varphi(c)$.

Proposition 6.1. D_nCF_0 is not ω-stable for all $n \geq 2$.

Proof. Consider the field $F := \mathbb{Q}(x_S^{(j)} : S \subseteq \mathbb{N}, j \in \mathbb{N})$, where $x_S^{(j)}$ are variables for all $S \subseteq \mathbb{N}$ and $j \in \mathbb{N}$. Let $\partial_0 : \mathbb{Q} \to F$ be the inclusion map and $\partial_m : \mathbb{Q} \to F$ be the zero map for $m = 1, \ldots, n$. Then $(\mathbb{Q}, F, \nabla) \in \mathcal{S}_n$. By Proposition 4.5, we can extend ∇ to an order n derivation on F in such a way that for all $S \subseteq \mathbb{N}$,

- $\partial_1 x_S^j = x_S^{j+1}$ for all j
- $\partial_2 x_S^j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{else} \end{cases}$
- $\partial_m x_S^j = 0$ for all $m > 2$ and all j

Since D_nCF_0 is a model companion of T_n, $(F, \nabla) \models T_n$ embeds into $(M, \nabla) \models D_nCF_0$.

Let $S, S' \subseteq \mathbb{N}$ with $S \neq S'$. Without loss of generality, suppose $m \in S \backslash S'$. Then $\partial_2 \partial_1^m x_S^{(0)} = 1 \neq 0 = \partial_2 \partial_1^m x_{S'}^{(0)}$, so $\mathrm{tp}(x_S^{(0)}/\mathbb{Q}) \neq \mathrm{tp}(x_{S'}^{(0)}/\mathbb{Q})$.

But then D_nCF_0 is not ω-stable because there are uncountably many complete types over $\mathbb{Q}$. $\square$

Note that for all $(K, \nabla) \models D_nCF_0$ and parameters $A \subseteq K$, it can be shown that $\mathrm{dcl}(A) = \mathbb{Q}(wa : w \in \{\partial_1, \ldots, \partial_n\}^*, a \in A)$ and that $\mathrm{acl}(A) = (\mathrm{dcl}(A))^{alg}$, using quantifier elimination. We leave this to the reader to verify as it isn't strictly necessary for the proof below. However, it may aid in intuition.

Proposition 6.2. D_nCF_0 is $\mathfrak{c}$-stable for all $n \in \mathbb{N}$.

Proof. List the countably many words $\{\partial_1, \ldots, \partial_n\}^*$ as $w_0, w_1, \ldots$ and let $(K, \nabla) \models D_nCF_0$ and $A \subseteq K$ with $|A| = \mathfrak{c}$.

Let $k := \mathrm{dcl}(A) = \mathbb{Q}(wa : w \in \{\partial_1, \ldots, \partial_n\}^*, a \in A)$. Note that $|k| = \mathfrak{c}$ since elements of k are of the form $f(w_1' a_1, \ldots, w_l' a_l)$ for some rational function $f \in \mathbb{Q}(x_1, \ldots, x_l)$, $w'_1, \ldots, w'_l \in \{\partial_1, \ldots, \partial_n\}^*$ and $a_1, \ldots, a_l \in A$, where $|\coprod_{l \in \mathbb{N}} \mathbb{Q}(x_1, \ldots, x_l) \times (\{\partial_1, \ldots, \partial_n\}^*)^l \times A^l| = \mathfrak{c}$.

For each $l \in \mathbb{N}$ and $b \in K$, let $k_l^b := k(w_0 b, \ldots, w_{l-1} b)$. Either $w_l b$ is algebraic over k_l^b, in which case let f_l^b be its minimal polynomial over k_l^b, or it is transcendental over k_l^b, in which case let f_l^b be the zero polynomial. Fix $g_l^b \in k(x^{w_0}, \ldots, x^{w_{l-1}})[x_l]$ such that $g_l^b(w_0 b, \ldots, w_{l-1} b) = f_l^b$.

Claim: Suppose $b, c \in K$ such that $g_l^b = g_l^c$ for all $l \in \mathbb{N}$. Then we have $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$.

By quantifier elimination, it suffices to show b, c agree on all atomic formulas with parameters from A, that is that for all $f \in k\{x\}$, $f(b) = 0$ if and only if $f(c) = 0$. So if $f \in k[x^{w_0}, \ldots, x^{w_l}]$ for some $l \in \mathbb{N}$, then $f(w_0 b, \ldots, w_l b) = 0$ if and only if $f(w_0 c, \ldots, w_l c) = 0$.

Suppose $\mathrm{tp}(b/A) \neq \mathrm{tp}(c/A)$, and without loss of generality, suppose there is $f \in k[x^{w_0}, \ldots, x^{w_l}]$ for some minimal $l \in \mathbb{N}$ with $f(w_0 b, \ldots, w_l b) = 0$ but $f(w_0 c, \ldots, w_l c) \neq 0$.

Let $f^b := f(w_0 b, \ldots, w_{l-1} b) \in k_l^b[x^{w_l}]$. Then $f^b(w_l b) = 0$. Suppose $f^b = 0$. Then f doesn't depend on x^{w_l}, contradicting the minimality of l. Thus, $f^b \neq 0$, so $w_l b$ is algebraic over k_l^b.

Since f_l^b is the minimal polynomial of $w_l b$ over k_l^b, we have that f_l^b divides f^b. So there exists $g \in k(x^{w_0}, \ldots, x^{w_{l-1}})[x^{w_l}]$ such that

$$g_l^b(w_0 b, \ldots, w_{l-1} b) g(w_0 b, \ldots, w_{l-1} b) = f(w_0 b, \ldots, w_{l-1} b).$$

Suppose that $g_l^b(w_0 c, \ldots, w_{l-1} c) g(w_0 c, \ldots, w_{l-1} c) \neq f(w_0 c, \ldots, w_{l-1} c)$. Then by clearing denominators and moving everything to one side, we get that $h(w_0 c, \ldots, w_{l-1} c) \neq 0$ but that $h(w_0 b, \ldots, w_{l-1} b) = 0$ for some $h = \sum_{j=0}^r h_j (x^{w_l})^j \in k[x^{w_0}, \ldots, x^{w_{l-1}}][x^{w_l}]$. Then for some $j = 0, \ldots, r$, $h_j(w_0 c, \ldots, w_{l-1} c) \neq 0$ but $h_j(w_0 b, \ldots, w_{l-1} b) = 0$, contradicting the minimality of l.

So $g_l^b(w_0 c, \ldots, w_{l-1} c) h(w_0 c, \ldots, w_{l-1} c) = f(w_0 c, \ldots, w_{l-1} c)$, where we have that $g_l^b(w_0 c, \ldots, w_{l-1} c) = g_l^c(w_0 c, \ldots, w_{l-1} c) = f_l^c$, by definition.

Then $0 \neq f(w_0 c, \ldots, w_l c) = f_l^c(w_l c) h(w_0 c, \ldots, w_{l-1} c, w_l c) = 0$, where the last equality is because f_l^c is the minimal polynomial of $w_l c$. This is a contradiction, proving that $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$.

So $\mathrm{tp}(b/A)$ is fully specified by an element of $\prod_{l\in\mathbb{N}} k(x^{w_0},\ldots,x^{w_{l-1}})[x_l]$. Since $|k(x^{w_0},\ldots,x^{w_{l-1}})[x_l]| = \mathfrak{c}$ for all $l \in \mathbb{N}$, we have that

$$|\prod_{l\in\mathbb{N}} k(x^{w_0},\ldots,x^{w_{l-1}})[x_l]| = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$$

so D_nCF_0 is $\mathfrak{c}$-stable. □

Appendix

In this appendix, we're going to sketch the proof that GA can be expressed in 1st order, and thus D_nCF_0 can be axiomatised.

The difficulty in showing that GA is axiomatisable arises mainly from expressing Zariski density and irreducibility. Ultimately, these are statements about polynomials. It is possible to quantify over polynomials of bounded degree, but not polynomials of arbitrary degree, so we will need several results giving us bounds on the degrees of polynomials.

Lemma A1. Given fixed $r, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every field k, variables $x = (x_1,\ldots,x_l)$ and a single variable t, if $f_1,..,f_r \in k[x,t]$ are of (total) degree at most d and $I \subseteq (f_1,\ldots,f_r) \cap k[x]$ is a prime ideal of $k[x]$ containing every element of $(f_1,\ldots,f_r)\cap k[x]$ of degree at most N, then we have $I = (f_1,\ldots,f_r) \cap k[x]$.

Sketch of proof. Note that $k[x]/I$ is an integral domain since I is a prime ideal. Let $K := \mathrm{Frac}(k[x]/I)$. Using the Euclidean algorithm in $K[t]$ and clearing denominators, we get that there exist $g_1,\ldots,g_r,p_1,\ldots,p_r \in k[x,t]$ and $a_1,\ldots,a_r \in k[x] \setminus I$ such that for $g := g_1f_1 + \cdots + g_rf_r$, we have that $a_jf_j - p_jg \in I$ for all $j = 1,\ldots,r$. By careful analysis of the Euclidean algorithm, we can obtain a bound N on the degree of g depending only on r, d.

Let $h = h_1f_1 + \cdots + h_rf_r \in (f_1,\ldots,f_r)\cap k[x]$. If $h \notin I$, then we have $a_1\cdots a_rh \in k[x]\setminus I$ since I is a prime ideal. But

$$a_1\cdots a_rh + I = a_1\cdots a_rh_1f_1 + \cdots + a_1\cdots a_rh_rf_r + I$$

$$= (a_2 \cdots a_r h_1 p_1 + \cdots + a_1 \cdots a_{r-1} h_r p_r)g + I$$

so $pg = a_1 \cdots a_r h + b \in R \setminus I$ for some $p \in k[x,t]$ and $b \in I$. Thus, $g \notin I$. Furthermore, $p \neq 0$ and since pg is independent of t, so is g. Therefore, $g \in (f_1, \ldots, f_r) \cap k[x]$ is of degree at most N, so $g \in I$. This is a contradiction. Thus, $h \in I$, so $I = (f_1, \ldots, f_r) \cap k[x]$. $\square$

We can use this lemma to deal with Zariski denseness in GA. We'll also need to make use of the following two facts from Schmidt and van den Dries [2]:

1. Given fixed $l, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every field k, variables $x = (x_1, \ldots, x_l)$ and all $f_1, \ldots, f_r, f \in k[x]$ of degree at most d with $f \in (f_1, \ldots, f_r)$, $f = g_1 f_1 + \ldots + g_r f_r$ for some $g_1, \ldots, g_r \in k[x]$ of degree at most N.
2. Given fixed $l, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every field k, variables $x = (x_1, \ldots, x_l)$ and all $f_1, \ldots, f_r \in k[x]$ of degree at most d, if for all $f, g \in k[x]$ of degree at most N with $fg \in (f_1, \ldots, f_r)$, we have $f \in (f_1, \ldots, f_r)$ or $g \in (f_1, \ldots, f_r)$, then $(f_1, \ldots, f_r)$ is a prime ideal of $k[x]$ or $1 \in (f_1, \ldots, f_r)$.

Fact 1 tells us that it is possible to express in 1st order that a particular polynomial lies in an ideal, given bounds on its degree and the degree of the generators of the ideal. Facts 1 and 2 together tell us that it is possible to express in 1st order that an ideal is prime, given bounds on the degree of the generators of that ideal.

With that, we may now define D_nCF_0 as a set of axioms. These will consist of the axioms of algebraically closed ∇-fields of characteristic 0, plus one additional axiom for every fixed $l, d, r_0, \ldots, r_{ln} \in \mathbb{N}$, stating the following, where N is some bound dependent on $d, r_1, \ldots, r_{ln}$ given by Lemma A1:

Let $y = (y_1, \ldots, y_{ln}) := (x_1^{\partial_1}, \ldots, x_l^{\partial_1}, \ldots, x_1^{\partial_n}, \ldots, x_l^{\partial_n})$. For every $j = 0, \ldots, ln$, fix $f_{j,1}, \ldots, f_{j,r_j} \in k[x, y_1, \ldots, y_j]$ of (total) degree at most d, and let $I_j := (f_{j,1}, \ldots, f_{j,r_j}) \subseteq k[x, y_1, \ldots, y_j]$. Suppose that $f_{i,j} \in I_{i+1}$ for all $i = 0, \ldots, ln-1$ and $j = 1, \ldots, r_i$ and that $\tau_m f_{0,j} \in I_{ln}$ for all $m = 0, \ldots, n$ and $j = 0, \ldots, r_0$. Suppose moreover that I_j is a prime ideal of $k[x, y_1, \ldots, y_j]$ for all $j = 0, \ldots, ln$, and for all $j = 0, \ldots, ln - 1$, every element of $I_{j+1} \cap k[x, y_1, \ldots, y_j]$ of degree at most N lies in I_j. Then there exists $a \in k^l$ such that $f_{ln,j}(\nabla a) = 0$ for all $j = 1, \ldots, r_{ln}$.

We've justified that this can be phrased in first order. It remains to show that D_nCF_0 indeed axiomatises the algebraically closed fields of characteristic 0 satisfying GA.

Proposition A2. $(k, \nabla) \models D_nCF_0$ if and only if (k, ∇) is an algebraically closed field of characteristic 0 satisfying GA.

Proof. Suppose (k, ∇) is an algebraically closed ∇-field of characteristic 0 satisfying GA, and fix $l, d, r_0, \ldots, r_{ln} \in \mathbb{N}$ and $f_{j,1}, \ldots, f_{j,r_j} \in k[x, y_1, \ldots, y_j]$ for all $j = 0, \ldots, ln$ as above.

Let $X \subseteq \mathbb{A}_k^l$ be the subvariety defined by $f_{0,1}, \ldots, f_{0,r_0}$ and $Y \subseteq \mathbb{A}_k^{l(n+1)}$ be the subvariety defined by $f_{ln,1}, \ldots, f_{ln,r_{ln}}$.

Since $I_0 = (f_{0,1}, \ldots, f_{0,r_0})$ and $I_{ln} = (f_{ln,1}, \ldots, f_{ln,r_{ln}})$ are prime ideals, X and Y are irreducible subvarieties with $I(X) = I_0$ and $I(Y) = I_{ln}$.

Note that for all $f \in I(X)$, we have $f = h_1 f_{0,1} + \cdots + h_{r_0} f_{0,r_0}$ for some $h_1, \ldots, h_{r_0} \in k[x]$. It follows from Proposition 3.4, evaluating at (x, y), that $(k[x], k[x, y], \tau) \in \mathcal{S}_n$, so for all $m = 0, \ldots, n$, we have

$$\tau_m f = \sum_{j=0}^{m} (\tau_j h_1)(\tau_{m-j} f_{0,1}) + \cdots + \sum_{j=0}^{m} (\tau_j h_{r_0})(\tau_{m-j} f_{0,r_0}) \in I_{ln}$$

since $\tau_m f_{0,j} \in I_{ln}$ for all $m = 0, \ldots, n$ and $j = 1, \ldots, r_0$. Thus, $I(\tau X) \subseteq I(Y)$, so $Y \subseteq \tau X$.

By Lemma A1, we have $I_j = I_{j+1} \cap k[x, y_1, \ldots, y_j]$ for all $j = 0, \ldots, ln-1$, and thus $I(Y) \cap k[x] = I_{ln} \cap k[x] = I_0 = I(X)$. Thus, $\pi(Y)$ is Zariski dense in X, so by the geometric axiom, there exists $a \in X(k) \subseteq K^l$ such that $\nabla a \in Y(k)$. Then $f_{ln,j}(\nabla a) = 0$ for all $j = 1, \ldots, r_{ln}$. Since (k, ∇) is an algebraically closed ∇-field of characteristic 0, $(k, \nabla) \models D_nCF_0$.

Conversely, let $(k, \nabla) \models D_nCF_0$. We know (k, ∇) is an algebraically closed ∇-field of characteristic 0. We wish to show it has GA.

Let $X \subseteq \mathbb{A}_k^l$ and $Y \subseteq \tau X$ be irreducible subvarieties such that $\pi(Y)$ is Zariski dense in X.

Note that $I_j := I(Y) \cap k[x, y_1, \ldots, y_j]$ is a prime ideal of $k[x, y_1, \ldots, y_j]$ for all $j = 0, \ldots, ln$, and $I(X) = I(Y) \cap k[x] = I_0$. For all $j = 0, \ldots, ln$, fix $f_{j,1}, \ldots, f_{j,r_j} \in k[x, y_1, \ldots, y_j]$ such that $I(Y) \cap k[y_1, \ldots, y_j] = (f_{j,1}, \ldots, f_{j,r_j})$.

We have $f_{i,j} \in I_j \subseteq I_{j+1}$ for all $i = 0, \ldots, ln-1$ and $j = 1, \ldots, r_i$, and $\tau_m f_{0,j} \in I(\tau X) \subseteq I(Y)$ for all $m = 0, \ldots, n$ and $j = 0, \ldots, r_0$. And we have $I_{j+1} \cap k[x, y_1, \ldots, y_j] = I_j$.

Fix $d \in \mathbb{N}$ such that $\deg(f_{i,j}) \leq d$ for all $i = 0, \ldots, ln$ and $j = 1, \ldots, r_i$. By the axiom with $l, d, r_0, \ldots, r_{ln}$, we have that there exists $a \in k^l$ such that $f_{ln,j}(\nabla a) = 0$ for all $j = 1, \ldots, r_{ln}$. But then $\nabla a \in Y(k)$, and therefore $a = \pi(\nabla a) \in \pi(Y)(k) \subseteq \pi(\tau X)(k) = X(k)$, so GA holds. $\square$

References

[1] R. Moosa and T. Scanlon. Model theory of fields with free operators in characteristic zero. *Journal of Mathematical Logic*, 14(2):1450009, 2014.

[2] K. Schmidt and L. van den Dries. Bounds in the theory of polynomial rings over fields. A nonstandard approach. *Inventiones mathematicae*, 76:77–92, 1984.

[3] M. Ziegler. Separably closed fields with Hasse derivations. *The Journal of Symbolic Logic*, 68(1):311–318, 2003.