Contributions to the model theory of algebraic differential equations

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The bulk of this thesis is based on 4 co-authored papers and one singly authored paper in preparation. I have only included results from these papers to which I have made a significant contribution. Moreover, the presentation here, its exposition and the writing itself, are entirely my own work.

Chapter 3 is based on

- Christine Eagles and Léo Jimenez. Internality of autonomous algebraic differential equations, 2025. arxiv:2409.01863.

Chapter 4 is based on

- Christine Eagles and Léo Jimenez. Splitting differential equations using Galois theory. *Trans. Amer. Math. Soc.*, 378(12):8783–8820, 2025.

Chapter 5 is based on

- Yutong Duan, Christine Eagles, and Léo Jimenez. Algebraic independence of solutions to multiple Lotka-Volterra systems. 2025. preprint, arXiv:2507.17090.

Chapter 6 is based on

- Christine Eagles and Léo Jimenez. Domination, fibrations, and splitting. *The Journal of Symbolic Logic*, page 1–23, 2025. Published online at https://doi.org/10.1017/jsl.2025.10156.

The main results of Chapter 7 are my own solo-authored work, and are being prepared for publication. The exceptions are Theorem 7.1.4, Example 7.1.5 and Theorem 7.3.10, which appear in “Domination, fibrations, and splitting" cited above.

Abstract

This thesis deals with semiminimal analyses of finite rank types, primarily in the stable theory of differentially closed fields of characteristic zero (DCF_0). The two main themes considered in this thesis are determining when a type is minimal or semiminimal, and understanding what invariants of finite rank types are captured by a semiminimal analysis.

In DCF_0, a central concern of this thesis is determining when a type is almost internal to the field of constants. Partially generalising a result of Rosenlicht, algebraic criteria are provided in two different contexts: rational vector fields on affine n-space, and pullbacks under the logarithmic derivative of certain types which are internal to the constants. The criteria in the former case answers a question posed by Freitag, Jaoui, Marker and Nagloo about when the Poizat equations are internal to the constants. In both cases, the theory of binding groups in stable theories plays a significant role.

Results of Duan and Nagloo are improved upon to completely classify when the generic types of Lotka-Volterra systems are minimal. In the minimal case, a characterization of the possible relations that may exist between solutions of distinct Lotka-Volterra systems is given.

In the general setting of a totally transcendental theory, it is shown that the multiplicity with which a minimal type arises in a semiminimal analysis of a finite rank type is invariant, i.e., it is independent of the semiminimal analysis. A conjecture is proposed for the possible ways for two semiminimal analyses of the same finite rank type to differ. Along the way, the connection between semiminimal analyses and domination decompositions, is clarified.

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My parents have been incredibly and unconditionally supportive, even though they do not quite understand what I do. I want to thank my sister Chantelle and her family for providing me some home away from home. My cat Margot has been a much needed source of distraction, and I thank her for attending every Zoom meeting. I would like to thank my friends who kept in touch, even though I lived in Waterloo, Abetare and Karen. Thank you for continuing to support me and listening to me complain about the never ending writing process. I started my academic math journey with Abetare and am so glad she was the first peer with whom I discussed math. Particularly special is that Karen is becoming a doctor at the same time, although she will be much more useful on a plane.

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Last but not least, I can not adequately express my gratitude for my partner Jérémy Champagne. He has provided me with support, laughter, love and a cat. These past years with you have been wonderful and I know that we will have many more together.

Dedication

To Lilou, the kitty, and Sam, the bunny.

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Chapter 1

Introduction

This thesis is concerned with applications of model theory to the theory of differentially closed fields of characteristic zero (DCF_0). These are fields equipped with a derivation, i.e. an additive operator which satisfies the Leibniz rule, in which every consistent finite system of differential polynomial equations has a solution. That is they are the existentially closed differential fields. The theory of differentially closed fields was shown to be axiomatizable by Robinson [67] and a simpler axiomatization, as well as a proof of the theory being totally transcendental, was given by Blum [7].

Types in DCF_0 correspond to differential varieties. Differential varieties may be infinite dimensional, but those of finite dimension are of particular interest and correspond to types of finite (Morley or U-) rank. Finite rank stability theory, in particular the study of finite rank types in DCF_0, has a rich past with major applications to geometry. One of the most-notable applications is Hrushovki's work on the Mordell-Lang conjecture [34]. Recently, finite rank stability theory has found exciting applications to algebraic differential equations and transcendence problems in arithmetic geometry (see for example [12], [22], [26], [27]).

The main focus of this thesis is to study how finite rank types are built from minimal types (types of U-rank 1). In algebraic geometry, minimality corresponds to one-dimensionality, whereas in DCF_0, there are interesting minimal types of dimension larger than 1. A deep and important theorem about minimal types in DCF_0 is the Zilber dichotomy which says that all minimal types either come from algebraic curves in the field of constants or have a sparse geometry.

In any stable theory, a key step in going from minimal to finite rank types is *internality*. Roughly speaking, a definable set X is *internal* to a definable set Y if, after extending the base parameter set, X is the definable image of a definable subset of a finite cartesian power of Y. The dependence on new parameters is an important feature of internality, and is controlled by the *binding group*. If we loosen the notion of internality to allow infinite-to-finite definable correspondences, we obtain what is often called *almost internality*. These notions make sense for, and indeed are best suited to, complete types. We say that a complete type is *semiminimal* if it is non-algebraic (so has infinitely many realizations) and is almost internal to a minimal type. It is semiminimal types that are the actual building blocks of finite rank types: every finite rank type admits a *semiminimal analysis*, which is a sequence of definable functions whose fibers are semiminimal, and which terminates in a semiminimal type after finitely many steps.

The goal of this thesis is to better understand semiminimal analyses, mostly in DCF_0, but also in stable theories more generally. In particular, we focus on two related themes:

1. Characterizing when a type is itself minimal or semiminimal,
2. Understanding what invariants of a finite rank type are captured by a semiminimal analysis.

We focus on finite rank types in DCF_0 because they correspond to certain algebraic-geometric objects of interest. For example, when working over subfields of the constants (the so-called autonomous case) finite rank types precisely capture the birational geometry of algebraic vector fields. In the non-autonomous situation, it is not algebraic vector fields that arise, but rather the natural twists of these by the differential structure on the field of definition, called *algebraic D-varieties*. Behind most applications of the model theory of differential fields is this correspondence between the structure of finite rank types and the birational geometry of D-varieties.

We delay further introductory remarks to the beginning of each chapter, where more precise articulations of the main results can be found. For now, we provide an informal summary of the results in the thesis.

Chapter 2 is a brief review of the necessary background of the theory DCF_0 and some general stability theory. Of particular interest is the definition of generic types for both a D-variety and a system of algebraic differential equations, and that every finite rank type is interdefinable with the generic type of a D-variety. For general stability theory, we focus on orthogonality, internality, and analyzability for finite rank types. While working in the general stability-theory setting, we keep an eye on specializing to DCF_0.

In Chapters 3 and 4, we are interested in almost internality to the field of constants in DCF_0. That is types which, up to finite-to-finite correspondences, come from pure algebraic geometry after allowing additional parameters. For example, the generic type of a rational vector field of dimension n is almost internal to the constants if and only if it admits n algebraically independent rational first integrals. The overall theme of these chapters is to obtain an algebraic criterion for (almost) internality to the constants in the spirit of Rosenlicht [68]. Rosenlicht characterized almost internality to the field of constants of the generic type of a rational vector field on the affine line.

In Chapter 3 we provide an algebraic criterion for when the generic type, in DCF_0, of a rational vector field on affine n-space is almost internal to the field of constants. This is Theorem 3.2.4. We apply our criterion to answer a question of Freitag, Jaoui, Marker [21], characterising when certain order 2 algebraic equations, called Poizat equations, are almost internal to the constants (Proposition 3.3.1). We also apply the criterion to Lotka-Volterra systems, which are classically interesting systems of algebraic differential equations that we will further explore in Chapter 5. The contents of this chapter are based on joint work with Léo Jimenez appearing in [19].

In Chapter 4, we focus on generalising a result of Jin and Moosa [45] on an algebraic criterion for almost internality to the field of constants in certain non-autonomous settings. First, we look at a fibration $f : p \to q$ where q and the fibers of f are internal to the field of constants. In this case, the types have binding groups, which are permutations of the set of realizations fixing the constants pointwise induced by automorphisms of the universe. Under certain conditions on the binding groups of q and the fibers, we show that the type p is almost internal to the constants if and only if $f : p \to q$ almost splits, which roughly means that p is interalgebraic with the Morley product $q \otimes s$ for some type s which is almost internal to the constants (Theorem 4.2.6). We then restrict our attention to when f is given by the logarithmic derivative $x \mapsto \frac{\delta(x)}{x}$. This case is studied in [45], but under the assumption that q is minimal. Removing the minimality assumption, but imposing conditions on the binding group, we obtain an algebraic condition equivalent to the type p being almost internal to the constants (Corollary 4.3.4). This chapter consists of joint work with Léo Jimenez, appearing in [20].

The Lotka-Volterra system is the vector field on $\mathbb{A}^2$ defined by a system of differential equations of the form

$$LV_{a,b,c,d} = \begin{cases} x' = x(ay+b) \\ y' = y(cx+d) \end{cases}$$

for some non-zero constants a, b, c, d. These are important equations arising in the study of predator-prey populations. Duan and Nagloo [17] show that the generic type of $LV_{a,b,c,d}$ in DCF_0 is minimal (even *strongly minimal*) when b and d are $\mathbb{Q}$-linearly independent, and the type is not minimal if $b = d$. In Chapter 5, using a method of Jaoui [39], we show that the generic type is minimal whenever $b \neq d$ (Proposition 5.2.3). We are then able to deduce transcendence results about solutions to such equations: we show that given finitely many distinct Lotka-Volterra systems, satisfying some mild conditions on the parameters, there are no algebraic relations over the constants among any set of solutions from the various systems. This is Corollary 5.3.3. Chapter 5 is based on joint work with Yutong Duan and Léo Jimenez, appearing in [16].

In Chapter 6, we leave the context of DCF_0 and work in a general totally transcendental theory eliminating imaginaries. We study fibrations $f : p \to q$ whose fibers are semiminimal. This is a first step in understanding the structure of semiminimal analyses. We make use of the notion of domination in stable theories. In particular we show that if some (equivalently any) fiber of f is almost internal to a minimal type r, then p is domination equivalent to $r^{(m)} \otimes q$ for some m between 0 and $U(p) - U(q)$ (Theorem 6.3.1). We focus on the power m, examining conditions that force m to be either 0 or $U(p) - U(q)$. This chapter is based on joint work with Léo Jimenez, appearing in [18].

Finally, Chapter 7 examines some uniqueness properties of semiminimal analyses in the general setting. The goal of the chapter is to show that the multiplicity with which a minimal type arises in a semiminimal analysis for a given type p, does not depend on the choice of analysis (Theorem 7.4.2). Along the way we show that this notion of multiplicity of r in p extends the existing notion of local r-weight, introduced by Shelah and further developed by Hrushovski [30] (Proposition 7.5.7). Lastly, while it is well-known that a type need not have a unique semiminimal analysis, we examine the possible ways that analyses may differ. We give a conjectural description of the possible deviations (Conjecture 7.6.5) and prove a preliminary case of the conjecture (Proposition 7.6.6). While some of the results appear in joint work with Léo Jimenez [18], the main theorems here are solo work by the author and are currently in preparation for publication.

Chapter 2

Preliminaries

2.1 Differentially closed fields

2.1.1 ACF_0

We begin by recalling the basic model theory of algebraically closed fields om characteristic zero (ACF_0). Working in the language of rings $L = \{0, 1, +, -, \times\}$, we consider the theory T of integral domains of characteristic zero in the language L. To better understand this theory, we study its *existentially closed* models. Being an existentially closed model M of T means that any finite system of polynomial equations and inequations with coefficients from M that has a solution in some integral domain extending M, has a solution in M. The existentially closed models of T are precisely algebraically closed fields of characteristic zero, whose theory we denote by ACF_0. The theory ACF_0 is a strongly minimal theory which eliminates imaginaries and admits quantifier elimination.

We fix $\mathcal{U} \models \mathrm{ACF}_0$, a *sufficiently saturated* model meaning that any consistent set of formulas over a small subfield of $\mathcal{U}$, meaning of cardinality strictly less than $|\mathcal{U}|$, has a solution in $\mathcal{U}$. All parameter sets will we consider will be small subsets of $\mathcal{U}$, unless explicitly stated otherwise. For any $A \subseteq \mathcal{U}$, $\mathrm{dcl}(A)$ is the field generated by A and $\mathrm{acl}(A)$ is the field-theoretic algebraic closure of the field generated by A. In particular, $\mathrm{acl}(\emptyset) = \mathbb{Q}^{\mathrm{alg}}$.

A *Zariski closed set over* F in $\mathcal{U}^n$ is the zero set of finitely many polynomials in n variables whose coefficients are from F. The Zariski closed sets over F form the closed sets of a noetherian topology called the Zariski topology over F. There is also an absolute Zariski topology on $\mathcal{U}^n$ where a subset is Zariski closed if it is Zariski closed over some field (i.e. over $\mathcal{U}$). A Zariski closed set is *irreducible over a field* F if it cannot be written as the union of any two proper Zariski closed sets over F. A Zariski closed set is *absolutely irreducible* if it cannot be written as the union of any two proper Zariski closed sets. As a consequence of quantifier elimination, the definable sets are finite boolean combinations of Zariski closed sets. We use the convention that definable means definable with parameters, unless stated otherwise.

We denote *affine* n*-space* over F by $\mathbb{A}^n_F$ which we view as a functor taking a field L extending F to the set L^n. By an *affine variety over* F, we mean a subfunctor $V \subseteq \mathbb{A}^n_F$ for some n, defined by a set of polynomials $\Sigma \subseteq F[x_1, ..., x_n]$. That is, for any $L \supseteq F$, $V(L)$ is the set of $a \in L^n$ such that $f(a) = 0$ for all $f \in \Sigma$. Since we are working in a universal domain $\mathcal{U}$, we will often identify affine varieties with their $\mathcal{U}$-points, so with Zariski closed subsets of $\mathcal{U}^n$.

Recall that a complete type in n variables over a set A, denoted $p \in S_n(A)$, is a maximal consistent collection of L_A formulas in n-variables. Often we will write $p \in S(A)$ when the number of variables is clear from context. By $\mathrm{tp}(a/A)$, we mean the collection of all L_A-formulas that $a \in \mathcal{U}^n$ satisfies. In ACF_0, there is a bijective correspondence between types $p \in S_n(F)$ and irreducible subvarieties of $\mathbb{A}^n$ over F. One direction of the correspondence is given by mapping a type p to its *Zariski locus*, denoted by $\mathrm{loc}(p)$. This locus is equal to $\mathrm{loc}(a/F)$, for any $a \models p$, which is the smallest Zariski closed set over F containing a. The reverse direction of the correspondence is given by mapping V, an irreducible variety over F, to the *generic type* of V over F, that is $\mathrm{tp}(a/F)$ for some $a \in V(\mathcal{U}^n)$ such that a lies in no proper Zariski closed set over F.

We have a well-behaved notion of independence. Let A and B be subsets of $\mathcal{U}$ and F be a sub-field. We say that A is *independent* from B over F, denoted $A \underset{F}{\overset{\mathrm{ACF}_0}{\perp}} B$ if $\mathrm{trdeg}(F(A)/F) = \mathrm{trdeg}(F(A,B)/F(B))$. This agrees with Shelah's non-forking independence. A *non-forking extension* of a type $p \in S(F)$ to $L \supseteq F$ is $\mathrm{tp}(a/L)$ for some $a \models p$ such that $a \underset{F}{\overset{\mathrm{ACF}_0}{\perp}} L$. We say a type over F is *stationary* if it has a unique non-forking extension to any field extension of F. A type $p \in S(F)$ is stationary if and only if p is the generic type of an absolutely irreducible variety over F.

2.1.2 DCF_0

We provide a short introduction to the model theory of differential fields. The contents of this section can be found in [54, Chapter 2] and [57], for example.

We begin with a brief review of differential rings. All rings will be unital and all fields will be of characteristic zero. A *derivation* on a ring R is an additive function $\delta : R \to R$ satisfying the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$. We call the pair (R, δ) a differential ring. When R is a field, we say that (R, δ) is a *differential field*. We say that (L, δ') is a *differential field extension* of (F, δ) if $F \subseteq L$ and $\delta'|_F = \delta$. The *constants* of (R, δ) is the set $\{a \in R \mid \delta(a) = 0\}$. We denote the nth derivative of some a in R by $\delta^n(a)$.

Let (F, δ) be a differential field, and $f \in F[x_1, ..., x_k]$. By f^δ, we mean the polynomial obtained by applying δ to the coefficients. So if $\delta = 0$ on F, then $f^\delta = 0$.

A key consequence of the Leibniz rule is the following easily verified identity.

Fact 2.1.1. *Let (F, δ) be a differential field, $f \in F[x_1, ..., x_k]$ and $a = (a_1, ..., a_k) \in F^k$. Then, $\delta(f(a)) = f^\delta(a) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(a)\delta(a_i)$.*

A differential polynomial over (F, δ) in variables $x_1, ..., x_k$ is an ordinary polynomial in the (infinite collection of) variables $\{x_i^{(j)} \mid j < \omega\}$, where we interpret $x_i^{(j)}$ as $\delta^j(x_i)$. That is, we have the ring of differential polynomials $F\{x_1, ..., x_k\} = F[x_i^{(j)} \mid i = 1, ..., k, j < \omega]$ equipped with the derivation extending δ on F and given by $\delta(x_i^{(j)}) = x_i^{(j+1)}$. The *order* of a differential polynomial is the highest power of $\delta^i(x)$ appearing in the polynomial.

We work in the language of rings together with a function symbol for the derivation, $L_\delta = \{0, 1, +, -, \times, \delta\}$, and consider T_δ, the L_δ theory of differential integral domains of characteristic zero. The existentially closed models of T_δ are those differential integral domains M with the property that any finite system of differential polynomial equations and inequations with coefficients in M and with a solution in some differential integral domain extending M, already has a solution in M. This is axiomatised as follows: (M, δ) is an existentially close model of T_δ if and only if $M \models ACF_0$ and for any $f, g \in M\{x\}$ with $g \neq 0$ and $\mathrm{ord}(f) > \mathrm{ord}(g)$, there is $a \in M$ such that $f(a) = 0 \neq g(a)$. The theory of existentially closed models of T_δ is called the theory of differentially closed fields and is denoted by DCF_0. The axiomatisation we have given is due to Blum [7], though the first axiomatisation was due to Robinson, influenced by even earlier work by Seidenberg. Much later, a geometric axiomatisation was given by Pierce and Pillay in [62].

The theory DCF_0 is a tame theory: it is ω-stable, admits quantifier elimination, and eliminates imaginaries. We fix $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$, a sufficiently saturated model that will serve as a universal domain for differential-algebraic geometry. So, implicitly, every differential field we consider, unless stated otherwise, is assumed to be a small differential subfield of $(\mathcal{U}, \delta)$, and all types (partial or complete) are over small sets of parameters, and hence are realized in $(\mathcal{U}, \delta)$.

Similarly to ACF_0, we have algebraic descriptions of model-theoretic definable and algebraic closures. For $A \subseteq \mathcal{U}$, $\mathrm{dcl}(A) = \langle A \rangle$ is the differential field generated by A, and $\mathrm{acl}(A) = \langle A \rangle^{\mathrm{alg}}$ is the field-theoretic algebraic closure of the differential field generated by A. In particular, $\mathrm{acl}(\emptyset) = \mathbb{Q}^{\mathrm{alg}}$.

We have a refinement of the Zariski topology given by the Kolchin topology. A *Kolchin closed* subset of $\mathcal{U}^n$ over a differential field F is the zero set of finitely many differential polynomials whose coefficients are from F. The Kolchin closed sets over F form the closed sets of the noetherian Kolchin topology over (F, δ). A Kolchin closed set is *irreducible over* (F, δ) if it cannot be written as the union of two proper Kolchin closed sets over F. We again have an absolute Kolchin topology on $\mathcal{U}^n$ where a subset is Kolchin closed if it is Kolchin closed over some differential subfield (i.e. over $\mathcal{U}$) A Kolchin closed set is *absolutely irreducible* if it cannot be written as the union of two proper Kolchin closed subsets. As a consequence of quantifier elimination, the definable sets are finite boolean combinations of Kolchin closed sets. We use the convention that definable means definable with parameters.

An important definable set in $(\mathcal{U}, \delta)$ is the field of constants, denoted by $\mathcal{C}$, which is defined by $\delta(x) = 0$. This is an algebraically closed definable subfield of $\mathcal{U}$ that is "pure" in the sense that every definable set in $(\mathcal{U}, \delta)$ that lives in some cartesian product of $\mathcal{C}$ is definable in the pure field structure $(\mathcal{C}, 0, 1, +, -, \times)$. We will sometimes consider types over $\mathcal{C}$, which break the convention of parameter sets being small. However, one consequence of ω-stability is the *stable embeddedness* of $\mathcal{C}$ in $(\mathcal{U}, \delta)$, meaning that any complete type over $\mathcal{C}$ is isolated by its restriction to some finitely generated subfield of $\mathcal{C}$. See for example [14, Appendix] for more details on stable embeddedness.

Similarly to ACF_0, we have a bijective correspondence between types in $S_n(F)$ and irreducible Kolchin closed sets over (F, δ). This bijection maps p to the *Kolchin locus* of p, denoted by $\mathrm{loc}_\delta(p)$. This locus is given by $\mathrm{loc}_\delta(a/F)$ for any $a \models p$, which is the smallest Kolchin closed set over F containing a. For the reverse direction, if V is an irreducible Kolchin closed set over F, then V maps to $\mathrm{tp}(a/F)$ for some *generic point* $a \in V$. By a generic point, we mean that $a \in V$ and a lies in no proper Kolchin closed set over F.

There is also a well-behaved notion of independence which agrees with Shelah's notion of non-forking independence. We say that A is *independent* from B over F, denoted $A \mathop{\smile\!\!\!\!\!\!|}_F B$, if $\langle A \rangle \mathop{\smile\!\!\!\!\!\!|}_F^{\mathrm{ACF}_0} \langle B \rangle$, i.e., $\mathrm{trdeg}(F\langle A \rangle / F) = \mathrm{trdeg}(F\langle A, B \rangle / F\langle B \rangle)$. Here by $F\langle A \rangle$ we mean the differential field generated by A over F If $(L, \delta) \supseteq (F, \delta)$, then a non-forking extension of $p \in S(F)$ is $\mathrm{tp}(a/L)$ such that $a \mathop{\smile\!\!\!\!\!\!|}_F L$. Under this notion of independence, a type over (F, δ) being stationary is equivalent to the type being the generic type of an absolutely irreducible Kolchin closed set over F. When F is an algebraically closed differential field, any type $p \in S(F)$ is stationary.

2.1.3 Finite dimensional types

In this thesis we are interested in finite dimensional types in the theory DCF_0. Fix (F, δ) a small differential subfield of $(\mathcal{U}, \delta)$. A type $\mathrm{tp}(a/F)$ is *finite dimensional* if $\mathrm{trdeg}(F\langle a\rangle/F)$ is finite for some (equivalently any) $a \models p$. Recall that $F\langle a\rangle = F(a, \delta(a), \delta^2(a), ...)$. This transcendence degree is finite if and only if $F\langle a\rangle$ is finitely generated over F as a field. In the finite dimensional case, we say that the *dimension* of $p \in S(F)$, denoted $\dim(p)$, is $\mathrm{trdeg}(F\langle a\rangle/F)$. Forking for finite dimensional types is capture by a change in dimension; if $\mathrm{tp}(a/F)$ is finite dimensional and $(L, \delta) \supseteq (F, \delta)$, $a \underset{F}{\downarrow} L$ if and only if $\dim(a/L) = \dim(a/F)$.

Every finite dimensional type arises, up to interdefinability, from a D-variety, a notion we now explain. First we briefly recall algebraic vector fields.

Recall that given an affine variety $X \subseteq \mathbb{A}^k$ where $I(X)$ is the ideal of polynomials vanishing on X, the *tangent bundle of* X is the subvariety $TX \subseteq \mathbb{A}^{2k}$ defined by $\begin{cases} f(x_1, ..., x_k) = 0 \\ \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} y_i = 0 \end{cases}$ for all $f \in I(X)$. An *algebraic vector field* is a pair (X, s) where X is a variety defined over a field F and $s : X \to TX$ is a regular section map to the projection map $\pi : TX \to X$. If the section $s : X \to TX$ is only assumed to be a rational map rather than regular, we call it a *rational vector field.*

If X is defined over a differential field (F, δ), then we can twist the above construction by the derivative δ.

Definition 2.1.2. Let $X \subseteq \mathbb{A}^k$ be an affine variety over a differential field (F, δ). The *prolongation* of X over (F, δ) is the subvariety $\tau X \subseteq \mathbb{A}^{2k}$ over F defined by

$\begin{cases} f(x_1, ..., x_k) = 0 \\ f^{\delta}(x_1, ..., x_k) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} y_i = 0 \end{cases}$ for every $f \in I(X)$. A (algebraic) *D-variety* over (F, δ) is the pair (X, s) where $X \subseteq \mathbb{A}^k$ is an affine variety and $s : X \to \tau X$ is a regular section to the co-ordinate projection $\pi : \tau X \to X$. If $s : X \to \tau X$ is only a rational map, as opposed to a regular map, we say that (X, s) is a *rational D-variety.*

The *autonomous* case corresponds to when $\delta = 0$ on F. Note that in the autonomous case, prolongations are tangent bundles and D-varieties are vector fields.

A D-variety can be seen as encoding a system of algebraic differential equations; namely the equations $\delta(x_i) = s_i(x_1, ..., x_k)$ where $s = (\mathrm{id}_X, s_1, ..., s_k)$.

Let (X, s) be a D-variety over (F, δ). A *D-subvariety* of (X, s) over (L, δ), a differential field extension of (F, δ), is an algebraic subvariety $Y \subseteq X$ over L such that $s|_Y : Y \to \tau Y$. A D-variety (X, s) is *irreducible* if X cannot be written as the union of two proper subvarieties Y and W such that $(W, s|_W)$ and $(Y, s|_Y)$ are D-subvarieties.

The following fact expresses the δ-twisted version of the well-known fact that algebraic vector fields on affine varieties correspond to derivations on the coordinate ring.

Fact 2.1.3. *Suppose $X \subseteq \mathbb{A}^k$ is an affine variety over (F, δ). Let $s = (\mathrm{id}_X, s_1, ..., s_k)$ be a rational D-variety structure on X over F. Then there is a unique derivation δ_s on $F(X)$ extending δ on F such that $\delta_s(x_i + I(X)) = s_i$ for all $i = 1, .., k$.*

Moreover, every derivation on $F(X)$ extending δ on F is of the form δ_s for some rational D-variety (X, s).

In particular, when $\delta = 0$ on F we obtain a correspondence between finitely generated extensions of F with a derivation that is zero on F, and rational vector fields. Similarly, rational D-varieties correspond bijectively to derivations on function field extensions of f equipped with a derivation extending δ.

Definition 2.1.4. Suppose (X, s) is a rational D-variety over (F, δ). Let $x = (x_1, ..., x_k)$ and let $s(x) = (\mathrm{id}_X, s_1(x), ...s_k(x))$. A *$D$-point* of (X, s) is some point $a = (a_1, ..., a_k) \in X(\mathcal{U})$ such that for all $i = 1, ..., k$, $\delta(a_i) = s_i(a)$.

The set of all D-points of (X, s) is the definable set

$$(X, s)^\sharp := \{a \in X(\mathcal{U}) \mid \delta(a_i) = s_i(a) \text{ for all } i = 1, ..., k.\}$$

For example, $\mathcal{C} = (\mathbb{A}, 0)^\sharp$ where $0(x) = (x, 0)$ is the trivial derivation on the affine line. Notice that the set of D-points is a Kolchin closed subset of X.

Definition 2.1.5. Let (X, s) be an irreducible D-variety over (F, δ). The *generic type* of (X, s) over F is the type that says $x \in (X, s)^\sharp$ and that X is the Zariski locus of x over F.

Fact 2.1.6. *If (X, s) is an irreducible D-variety over (F, δ), then its generic type over F is finite dimensional of dimension $\dim(X)$. Moreover, every finite dimensional complete type over F is interdefinable with the generic type of an irreducible D-variety over (F, δ).*

Here we say that two types $p, q \in S(F)$ are *interdefinable* if for some (equiv. any) $a \models p$, there is $b \models q$ such that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Fb)$, i.e., $F\langle a\rangle = F\langle b\rangle$.

2.2 Some geometric stability theory

We now work in a general stability-theoretic setting, namely that of a fixed totally transcendental theory T which eliminates imaginaries. (We can drop this last requirement by working in the many sorted theory T^{eq}.) The only particular example of T that we deal with in this thesis is DCF_0, but there are of course many other examples of interest. A good reference for general stability theory is [63]. Fix a sufficiently saturated model $\mathcal{U} \models T$ which we treat as a universal domain.

2.2.1 Stationary types and fibrations

Let a be a tuple, $A \subseteq B$ be parameter sets. We say that C is *independent* from B over A, denoted by $C \underset{A}{\downarrow} B$, if $\mathrm{tp}(C/B)$ does not fork over A (see Section 1.1 of [63]). In DCF_0, this notion of independence agrees with the definition given in terms of transcendence degree, that is $C \underset{A}{\downarrow} B$ if and only if $\mathrm{trdeg}(\langle AC\rangle/\langle A\rangle) = \mathrm{trdeg}(\langle ABC\rangle/\langle AB\rangle)$.

For a tuple $c_1, ..., c_n$, we will write $c_1, ..., c_n \underset{A}{\downarrow} B$ to mean the set $\{c_1, ..., c_n\}$ is independent from B over A. We say that a tuple $c_1, ..., c_n$ is A-independent if for all $i = 1, ..., n$, $c_i \underset{A}{\downarrow} c_1, ..., c_{i-1}, c_{i+1}, ..., c_n$. Equivalently, a tuple is A-independent if and only if for every $i = 1, ..., n$, $c_i \underset{A}{\downarrow} c_1, ..., c_{i-1}$.

We define a partial ordering on complete types in the variables $x = (x_1, ..., x_n)$ by $q < p$ if and only if q is a forking extension of p. That is, $A \subseteq B$, $p \in S(\mathcal{U})$, $q \in S(B)$, $p \subseteq q$ and for some (equiv. any) $b \models q$, $b \underset{A}{\not\downarrow} B$. U-rank is the foundation rank of this partial ordering. We often write $U(a/A)$ instead of $U(\mathrm{tp}(a/A))$. Note that a type is *algebraic*, meaning that its realizations lie in the acl of the parameter set, if and only if $U(p) = 0$. Similarly, if $b \in \mathrm{acl}(aA)$, then $U(b/A) \leq U(a/A)$ with equality if and only if $\mathrm{acl}(aA) = \mathrm{acl}(bA)$. In ACF_0, $U(a/F) = \mathrm{trdeg}(F(a)/F)$. In DCF_0, if p is finite-dimensional, then $U(p) \leq \dim(p)$. But, one does not always have equality (this is part of the concern if Chapter 5). In the finite-dimensional case, both U -rank and dimension witness forking, so that one often has a choice of which one to work with.

Recall that a type $p \in S(A)$ is *stationary* if for every $B \supseteq A$, p has a unique non-forking extension to B. We denote this unique extension by $p|_B$. Every type over an acl-closed set is stationary. By the *strong type* of a over A we mean the stationary type

$\mathrm{stp}(a/A) := \mathrm{tp}(a/\mathrm{acl}(A))$. Stationarity is preserved by taking non-forking extensions and by automorphisms. We include the following well-known argument.

Remark 2.2.1. *If* $\mathrm{tp}(a/A)$ *is stationary and* $b \in \mathrm{dcl}(Aa)$*, then* $\mathrm{tp}(b/A)$ *is stationary.*

Proof. Let $B \supseteq A$ and fix $b_1, b_2 \models \mathrm{tp}(b/A)$ where $b_i \underset{A}{\downarrow} B$ for $i = 1, 2$. Let $\sigma_i \in \mathrm{Aut}_A(\mathcal{U})$ such that $\sigma_i(b) = b_i$ for $i = 1, 2$. Since $b \in \mathrm{dcl}(Aa)$, there is some A-definable function such that $b = f(a)$. Hence, $b_i = f(\sigma_i(a))$ for $i = 1, 2$ as f is A-definable. Set $a_i := \sigma_i(a)$. Choose $a_i' \models \mathrm{tp}(a_i/Ab_i)$ such that $a_i' \underset{b_iA}{\downarrow} B$. Notice that $a_i' \models \mathrm{tp}(a/A)$. Since $b_i \underset{A}{\downarrow} B$, we have that $a_i' \underset{A}{\downarrow} B$. By stationarity of $\mathrm{tp}(a/A)$, $\mathrm{tp}(a_1'/B) = \mathrm{tp}(a_2'/B)$. Hence, there is some $\tau \in \mathrm{Aut}_B(\mathcal{U})$ such that $\tau(a_1') = a_2'$.

We have that $\tau(b_1) = \tau(f(a_1')) = f(\tau(a_1')) = f(a_2') = b_2$. Hence, $\mathrm{tp}(b_1/B) = \mathrm{tp}(b_2/B)$. So there is a unique non-forking extension of $\mathrm{tp}(b/A)$ to any $B \supseteq A$. □

For stationary types $p, q \in S(A)$, the *Morley product* of p and q is $p \otimes q := \mathrm{tp}(ab/A)$ where $a \models p$ and $b \models q$ and $a \underset{A}{\downarrow} b$. We denote the Morley product of a stationary type with itself n-times as $p^{(n)}$. A *Morley sequence* in p is an A-independent sequence of realizations.

The *canonical base* of a stationary type $p \in S(A)$, denoted $\mathrm{Cb}(p)$, is the smallest definably closed subset of $\mathrm{dcl}(A)$ over which p does not fork and p restricted to $\mathrm{Cb}(p)$ is stationary (see [63, Remark 2.26] for why this exists). We write $\mathrm{Cb}(\mathrm{tp}(a/A))$ as $\mathrm{Cb}(a/A)$. In ACF_0 the canonical base of a type $p \in S(F)$ is the minimal field of definition for $\mathrm{loc}(p)$. Similarly in DCF_0, the canonical base of a finite dimensional type $p \in S(F)$ is the minimal differential field of definition for $\mathrm{loc}_\delta(p)$.

Note that in both theories, the canonical base for a type is unique and is finitely generated (as a field or differential field). That is, taking such a minimal tuple of generators, $\mathrm{Cb}(p) = \mathrm{dcl}(c)$ and for some finite tuple c, we will sometimes identify the canonical base with this finite tuple. Of course, the finite tuple itself is only unique up to interdefinability.

Fact 2.2.2. *[63, Remark 2.26] Let* $p \in S(A)$ *be stationary.*

1. $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(A)$.
2. *For any* $B \subseteq A$*,* p *does not fork over* B *if and only if* $\mathrm{Cb}(p) \subseteq \mathrm{acl}(B)$.
3. $\mathrm{Cb}(\mathrm{tp}(a/\mathrm{acl}(A))) \subseteq \mathrm{dcl}(Aa)$.

Fact 2.2.3. *[63, Lemma 1.3.19] Let $p = \mathrm{tp}(a/A)$ be stationary and non-algebraic. For any Morley sequence $a = a_0, a_1, ...$ in p, there is an $n < \omega$ such that $\mathrm{Cb}(a/A) \subseteq \mathrm{dcl}(a_0, ..., a_n)$.*

It is convenient to introduce some notation and terminology for maps between complete types:

Definition 2.2.4. Let $p, q \in S(A)$ be stationary types.

1. A *definable function* $f : p \to q$ is a relatively A-definable map $f : p(\mathcal{U}) \to q(\mathcal{U})$. That is f is an A-definable function and for some (equivalently any) $a \models p$, $f(a) \models q$. Note that as a function from $p(\mathcal{U})$ to $q(\mathcal{U})$, f is necessarily surjective.

2. A definable function $f : p \to q$ is *finite-to-one* if for some (equivalently any) $a \models p$, $a \in \mathrm{acl}(Af(a))$.

3. We say that p and q are *interdefinable* (over A) if there are definable functions $f : p \to q$ and $g : q \to p$ such that $f \circ g$ and $g \circ f$ are the identity. Equivalently, for all (equivalently some) $a \models p$, there is $b \models q$ such that $\mathrm{dcl}(Aa) = \mathrm{dcl}(Ab)$.

4. We say that p and q are *interalgebraic* (over A) if there is $r \in S(A)$ and finite-to-one definable functions $f : r \to p$ and $g : r \to q$. Equivalently, for all (equivalently some) $a \models p$, there is $b \models q$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Ab)$.

5. A *fiber* of a definable function $f : p \to q$ is a type of the form $\mathrm{tp}(a/Af(a))$ for some $a \models p$. Note that all fibers of f are A-conjugate.

It is often useful to assume that the fibers of a function are stationary:

Definition 2.2.5. Let $f : p \to q$ be a definable function of stationary types $p, q \in S(A)$. We say that $f : p \to q$ is a *fibration* if some (equivalently any) fiber of f is stationary.

Every definable function can be factored into a fibration and a finite-to-one function using the following well-known fact, which can be viewed as a model-theoretic abstraction of Stein factorization in complex geometry.

Lemma 2.2.6. *Let $p, q \in S(A)$ be stationary and $f : p \to q$ be a definable function. There is a fibration $f' : p \to q'$ over A and a finite-to-one definable function $\pi : q' \to q$ such that $f = \pi \circ f'$.*

Proof. Fix $a \models p$ and let $e := \mathrm{Cb}(\mathrm{stp}(a/Af(a)))$. Notice that here we do need to take the strong type as the fiber $\mathrm{tp}(a/Af(a))$ need not be stationary. By Fact 2.2.2, $e \in \mathrm{dcl}(Af(a)a) = \mathrm{dcl}(Aa)$. On the other hand, since e is the canonical base of a stationary type over $\mathrm{acl}(Af(a))$, $e \in \mathrm{acl}(Af(a))$ by Fact 2.2.2. Let $q' := \mathrm{tp}(ef(a)/A)$. By Remark 2.2.1, q' is stationary. Let $f' : p \to q'$ be given by $f'(a) = (e, f(a))$. Notice that $(e, f(a)) \in \mathrm{dcl}(Aa)$, so $f' : p \to q'$ is a definable function. For every $a \models p$, the fiber $\mathrm{tp}(a/Af'(a)) = \mathrm{tp}(a/Af(a)e)$. By definition of e, we have that $\mathrm{tp}(a/e)$ is stationary and $a \underset{e}{\perp} Af(a)$. Hence, $\mathrm{tp}(a/Af(a)e)$ is stationary, showing $f' : p \to q'$ is a fibration.

Let $\pi : q' \to q$ be given by the projection map $\pi(e, f(a)) = f(a)$. Clearly this is definable and since $e \in \mathrm{acl}(Af(a))$, we have that $\pi : q' \to q$ is finite-to-one.

By stationarity of p, q', and q, to show $f = \pi \circ f'$, it suffices to check that $f(a) = \pi(f'(a))$ for a single $a \models p$. Fix $a \models p$, then $\pi(f'(a)) = \pi(e, f(a)) = f(a)$. Hence, $f = \pi \circ f'$. $\square$

Minimal types

A stationary type is *minimal* if it is of U-rank 1. Equivalently $p \in S(A)$ is minimal if and only if for every $B \supseteq A$, p has a unique non-algebraic extension to B. Hence, every non-algebraic extension of a minimal type is again minimal. In the theory of algebraically closed fields of characteristic zero, minimal types correspond to the algebraic curves. However, in DCF_0 there are many minimal types of dimension greater than 1. For example, the focus of Chapter 5 is showing that certain 2-dimensional types in DCF_0 are minimal.

We can further classify minimal types based on the behavior of the model-theoretic algebraic closure on their sets of realizations.

Definition 2.2.7. Let $p \in S(A)$ a minimal type. The type p is *locally modular* if for every $a \models p$ and $B, C \subset p(\mathcal{U})$, we have that $B \underset{\mathrm{acl}(ABa) \cap \mathrm{acl}(ACa)}{\perp} C$.

The type p is *trivial* if every pairwise A-independent set of realizations is A-independent.

Trivial minimal types are locally modular.

Proposition 2.2.8. *Let $p \in S(A)$ be stationary. Then p is trivial if and only if for any $B \supseteq A$ and non-forking extension $p' \in S(B)$, whenever $a, b, c \models p'$ are pairwise B-independent, then a, b, c is A-independent.*

Moreover, when p is minimal, then p is trivial if and only if

$$\mathrm{acl}(AC) \cap p(\mathcal{U}) = \bigcup_{c \in C} \mathrm{acl}(Ac) \cap p(\mathcal{U}) \text{ for any set } C.$$

In the Moreover case, we say that the pregeometry $(p(\mathcal{U}), \mathrm{acl}_A)$ is disintegrated.

Proof. For the forward direction, suppose there exists $B \supset A$ and $a, b, c \models p' \in S(B)$, the non-forking extension of p to B, which are pairwise B-independent, but not B-independent. Without loss of generality, $a \not\downarrow_B bc$.

Let $e = \mathrm{Cb}(abc/B)$. Then $abc \downarrow_e B$, and by Fact 2.2.3, $e \in \mathrm{dcl}(I)$ for any Morley sequence $I = (a_i, b_i, c_i)_{i<\alpha}$ in $\mathrm{tp}(abc/B)$. Note that I is a set of pairwise B-independent realizations of p' since for all i, $a_i, b_i, c_i \models p'$ are pairwise B-independent.

If $abc \not\downarrow_A B$, then by [63, Lemma 1.2.29.], I is not A-independent. But I is a sequence of pairwise A-independent realizations of p: for any a_i, b_j, $a_i \downarrow_B b_j$ and $a_i \downarrow_A B$. Thus $a_i \downarrow_A b_j$. Similarly, we have A-independence for the pairs (a_i, a_j), (b_i, b_j), and (c_i, c_j) with $i \neq j$, (a_i, c_j) and (b_i, c_j). Hence I is sequence of pairwise A-independent realization of p but I is not A-independent. Thus p is not trivial, a contradiction.

If $abc \downarrow_A B$, then $e = \mathrm{Cb}(abc/A)$. Hence, by Fact 2.2.3, $e \in \mathrm{dcl}(I)$ for any Morley sequence in $\mathrm{tp}(abc/A)$, $I = (a_i, b_i, c_i)_{i<\alpha}$. Let $I \downarrow_A a, b, c$. Note that I is A-independent. Hence, $I' = I \cup \{a, b, c\}$ is pairwise A-independent.

Since $abc \downarrow_e B$, we have that $a \downarrow_{bce} B$. If $a \downarrow_A Ibc$, since $e \in \mathrm{dcl}(I)$, $a \downarrow_A ebc$ which implies that $a \downarrow_{eA} bc$. Thus $a \downarrow_e Bbc$, and so $a \downarrow_B bc$, a contradiction. So $a \not\downarrow_A Ibc$, showing that p is not trivial, as I' is pairwise A-independent, but not A-independent.

For the reverse direction, let I be any pairwise A-independent set of realizations of p. Choose k to be minimal such that any k-many elements of I are not A-independent. Notice that $k \geq 3$ since I consists of pairwise A-independent realizations. Then a_1, a_2, a_3 are pairwise $a_4, ..., a_k A$-independent realizations of $p|_{a_4,...,a_k}$ (if $k = 3$, take $a_4 = A$). So, $a_1 \downarrow_{a_4,...,a_k A} a_2, a_3$. Hence, $a_1 \downarrow_A a_2, ..., a_k$, contradicting that $a_1, ..., a_k$ is not A-independent. We have shown that I is A-independent and so p is trivial.

For the moreover statement, suppose that p is minimal. For the forward direction suppose that p is trivial. Clearly $\bigcup_{c \in C} \mathrm{acl}(Ac) \cap p(\mathcal{U}) \subseteq \mathrm{acl}(AC) \cap p(\mathcal{U})$. To show the reverse inclusion, let $C \subseteq p(U)$ and $c \in \mathrm{acl}(AC) \cap p(\mathcal{U})$. Thus $c \in \mathrm{acl}(Ac_1, ..., c_n)$ for some finite subset of C. Let n be minimal possible among all tuples. If $c_i \not\downarrow_A c_j$ for $i \neq j$, by minimality of p, $c_i \in \operatorname{acl}(Ac_j)$. Then $c \in \operatorname{acl}(Ac_1, ..., c_{i-1}, c_{i+1}, ..., c_n)$, contradicting minimality of n. Thus $c_1, ..., c_n$ is a pairwise A-independent tuple of realizations of p. Since $c, c_1, ..., c_n$ is not independent, by triviality, $c \not\downarrow_A c_i$ for some $i = 1, ..., n$. By minimality of p, $c \in \operatorname{acl}(Ac_i)$. Thus $\operatorname{acl}(AC) \cap p(\mathcal{U}) \subseteq \bigcup_{c \in C} \operatorname{acl}(Ac) \cap p(\mathcal{U})$.

For the reverse direction, let C be a pairwise A-independent set of realizations of p. Let $C = (c_i)_{i<\alpha}$. Without loss of generality, suppose that $c_0 \not\downarrow_A (c_i)_{0<i<\alpha}$. By minimality of p, $c_0 \in \operatorname{acl}(A(c_i)_{0<i<\alpha})$. Hence, $c_0 \in \operatorname{acl}(Ac_i)$ for some $i \neq 0$. But, $c_0 \downarrow_A c_i$, a contradiction. Thus, C is A-independent. $\square$

In particular, if a type is trivial, then any of its non-forking extensions are trivial (see for example [2, Part D Chapter XVI 2.2]).

2.2.2 Orthogonality

In this subsection, we recall the notion of orthogonality between two stationary types, a stationary type and a definable set, and a stationary type and a set of parameters. In all cases, orthogonality is meant to capture a lack of "interesting" relations between the two objects.

Definition 2.2.9. Let $p, q \in S(A)$. We say that p is *weakly-orthogonal* to q, denoted by $p \perp^w q$, if for every $a \models p$ and $b \models q$, a is independent from b over A.

Let $p \in S(A)$ and $q \in S(B)$ be stationary types. We say that p is *orthogonal* to q, denoted $p \perp q$, if for every $C \supseteq A \cup B$, the non-forking extension $p|_C$ is weakly orthogonal to $q|_C$.

Note that weak-orthogonality and orthogonality are symmetric notions. In some cases, weak-orthogonality and orthogonality coincide.

Fact 2.2.10. *Let $r, s \in S(A)$ be minimal and trivial types. Then r is non-orthogonal to s if and only if r is not weakly-orthogonal to s.*

For a proof see [2, Lemma 16.2.11].

Orthogonality on minimal types is particularly well-behaved. See Lemma 5.3 in [23] for a proof of the following well-known transitivity fact.

Fact 2.2.11. *Let $p \in S(A)$, $q \in S(B)$ be stationary types of finite U-rank. Let $r \in S(C)$ be minimal. If p is non-orthogonal to r and r is non-orthogonal to q, then p is non-orthogonal to q.*

Every non-algebraic stationary type of finite U-rank has some relation to a minimal type (for example see [11, Corollary 6.3.8.]).

Fact 2.2.12. *Every non-algebraic complete type of finite U-rank is non-orthogonal to a minimal type, potentially over additional parameters.*

Next we discuss what it means for a complete type to be orthogonal to a definable set. This often goes by the term "foreign", see for example [63], but we can reuse the term "orthogonal" without confusion:

Definition 2.2.13. Let $p \in S(A)$ be stationary and X be an A-definable set. We say that p is *weakly orthogonal to X*, denoted $p \perp^w X$, if for every $a \models p$ and every finite tuple $\bar{c} \subset X$, a is independent from $\bar{c}$ over A.

We say p is *orthogonal to X* (or X-orthogonal), denoted $p \perp X$, if for every $B \supseteq A$, $p|_B$ is weakly orthogonal to X.

Remark 2.2.14. *If* $\mathrm{tp}(a/A)$ *is stationary and orthogonal to an A-definable set X, then so is* $\mathrm{tp}(b/A)$ *for any* $b \in \mathrm{acl}(Aa)$.

In the theory DCF_0 we are often concerned with when a complete type is orthogonal to the constants. This is about whether the type has any definable relation with pure algebraic geometry. The following is a useful criterion (see [44, Lemma 2.22]).

Fact 2.2.15. *Suppose $T = \mathrm{DCF}_0$ and (F, δ) is an algebraically closed differential field. Let $p \in S(F)$. Then p is weakly $\mathcal{C}$-orthogonal if and only if $F\langle a\rangle \cap \mathcal{C} = F \cap \mathcal{C}$, for some (equivalently any) $a \models p$.*

Finally, still in the context of DCF_0, let us record the very important following fact about minimal types.

Fact 2.2.16 (Zilber dichotomy in DCF_0)**.** *Suppose that r is a minimal type in* DCF_0. *Then either r is locally modular or r is non-orthogonal to the constants. In fact, the dichotomy refined into a trichotomy where exactly one of the following is true:*

1. *r is trivial,*

2. *r is locally modular and non-trivial (in which case r is non-orthogonal to the generic type of a minimal one-based group), or*

3. *r is non-orthogonal to the field of constants.*

2.2.3 Internality

A central notion in this thesis is that of internality, which is a kind of opposite notion to orthogonality.

Definition 2.2.17. Suppose that $p \in S(A)$ and X is an A-definable set. We say that p is *internal to X (or X-internal)* if there exists some $B \supseteq A$ such that $p(\mathcal{U}) \subseteq \mathrm{dcl}(BX)$.

In fact, for every $a \models p$ we can choose some finite subset $X_0 \subseteq X$ such that $a \in \mathrm{dcl}(BX_0)$.

Example 2.2.18. Suppose $T = \mathrm{DCF}_0$, $\mathcal{C}$ is the field of constants, and $p(x) \in S(F)$ extends a system of linear (possibly non-homogeneous) differential equations in $x = (x_1, ..., x_n)$ over a differential field F. Then p is $\mathcal{C}$-internal. Indeed, if V is the finite-dimensional $\mathcal{C}$-vector subspace of $\mathcal{U}^n$ defined by the corresponding homogeneous linear system, then $p(\mathcal{U}) \subseteq b+V$ for some tuple $b \in \mathcal{U}^m$. Now, fix basis elements $b_1, ..., b_\ell$ for V, so that every element of V is a $\mathcal{C}$-linear combination of $(b_1, ..., b_\ell)$. Hence, $p(\mathcal{U}) \subseteq \mathrm{dcl}(Fb, b_1, ...b_m\mathcal{C})$.

While much of this thesis is about $\mathcal{C}$-internality in DCF_0, we also sometimes work (especially in chapters 6 and 7) in a general setting.

In addition to internality to definable sets we consider internality to partial type-definable sets, and even to families of partial type-definable sets. We will also sometimes allow the parameters for these partial type-definable sets to vary. Let us set up some notation for dealing with the general situation.

Suppose $\mathcal{Q}$ is a set of partial types over various parameter sets. Given any parameter set B, let us denote by $\mathcal{Q}_B(\mathcal{U})$ the set of all tuples a such that $\mathrm{tp}(a/B)$ extends a member of $\mathcal{Q}$.

Definition 2.2.19. Suppose $p \in S(A)$ and $\mathcal{Q}$ is a family of partial types (over arbitrary parameters). We say that p is *internal to* $\mathcal{Q}$ (or $\mathcal{Q}$-*internal*) if there is some $B \supseteq A$ such that $p(\mathcal{U}) \subseteq \mathrm{dcl}(B\mathcal{Q}_B(\mathcal{U}))$. If we replace dcl by acl, then we get the notion of *almost $\mathcal{Q}$-internality.*

Often, we assume that $\mathcal{Q}$ is *A-invariant* in the sense that it is invariant under the natural action of $\mathrm{Aut}_A(\mathcal{U})$.

Note that if $\mathcal{Q}$ is a single formula ϕ with parameters from A and $X = \phi(\mathcal{U})$, then $\mathcal{Q}$-internality agrees with X-internality as in Definition 2.2.17.

Remark 2.2.20. *We say that a definable set X is* strongly minimal *if X is infinite and every definable subset of X is either finite or co-finite. If X is strongly minimal and r is the generic type of X, then almost X-internality agrees with almost r-internality on stationary types.*

Proof. Let $p \in S(A)$ be stationary. Notice that $r(\mathcal{U}) \subseteq X$, hence if p is almost r-internal, it is almost X-internal. Suppose there is some $B \supseteq A$ such that $p(\mathcal{U}) \subseteq \operatorname{acl}(BX)$. Fix $a \models p|_B$ such that $a \in \operatorname{acl}(Bc_1, ..., c_n)$ for some $c_1, ..., c_n$. As X is strongly minimal, we can replace $c_1, ...c_n$ with an acl-basis over B. If $c_i \not\models r$, then there is some formula $\phi \in r$ such that $c_i \in X \cap \neg\phi(\mathcal{U})$. This set is not finite, since $c_i \notin \operatorname{acl}(A)$ and it is not co-finite, otherwise, r is algebraic. Hence $c_1, ..., c_n \models r$. $\square$

The following characterisation of internality, which is sometimes taken as the definition, is useful in practice:

Fact 2.2.21. *[63, Lemma 7.4.2] Suppose $p \in S(A)$ is stationary. Then p is $\mathcal{Q}$-internal if and only if for any $a \models p$, there exists $B \supseteq A$ with $a \mathop{\smile\!\!\!\!\!\downarrow}_A B$ such that $a \in \operatorname{dcl}(B\mathcal{Q}_B(\mathcal{U}))$. Moreover, almost $\mathcal{Q}$-internality is equivalent to the above statement with* acl *in place of* dcl.

In Example 2.2.18, we saw that a vector space basis for the set of solutions to a system of linear differential equations can be chosen to be the extra parameters necessary to witness internality. This generalises as follows:

Fact 2.2.22. *[63, Lemma 7.4.2] Let $p \in S(A)$ be stationary and $\mathcal{Q}$ be an A-invariant family of (partial) types. If p is $\mathcal{Q}$-internal, then there is some finite sequence of realizations $a_1, ..., a_\ell \models p$ such that, setting $B = Aa_1, ..., a_\ell$, we have that $p(\mathcal{U}) \subseteq \operatorname{dcl}(B\mathcal{Q}_B(\mathcal{U}))$.*

Such a sequence of realizations of p is called a *fundamental system* for p (with respect to $\mathcal{Q}$-internality) and we call p *fundamental* if some (equivalently any) single realization is a fundamental system.

Here are some well known preservation properties of internality:

Lemma 2.2.23. *Let $p \in S(A)$ be stationary and $\mathcal{Q}$ be an A-invariant family of (partial) types.*

1. *If p is (almost) $\mathcal{Q}$-internal, then any extension of p is (almost) $\mathcal{Q}$-internal.*

2. *If p has a non-forking extension that is (almost) $\mathcal{Q}$-internal, then p is (almost) $\mathcal{Q}$-internal.*

3. *Let $p = \operatorname{tp}(a/A)$ be $\mathcal{Q}$-internal and $b \in \operatorname{dcl}(Aa)$. Then $\operatorname{stp}(b/A)$ is $\mathcal{Q}$-internal. Similarly if $\operatorname{tp}(a/A)$ is almost $\mathcal{Q}$-internal and $b \in \operatorname{acl}(Aa)$, then $\operatorname{stp}(b/A)$ is almost $\mathcal{Q}$-internal.*

4. *If $\operatorname{tp}(a/A)$ and $\operatorname{tp}(b/A)$ are stationary types, which are (almost) internal to an A-invariant family of types $\mathcal{Q}$, then $\operatorname{tp}(ab/A)$ is (almost) $\mathcal{Q}$-internal.*

Proof. We will prove the statements for almost-internality, but the proofs work as well for internality.

1. Fix $A' \supseteq A$ and $a \models p$. Let $p' := \operatorname{tp}(a/A')$. First, note that $\mathcal{Q}$ is also A'-invariant. By Fact 2.2.22, there exists $C \supseteq A$ such that for all $a \models p$, there are $b_1, ..., b_n \models \mathcal{Q}_B$ such that $a \in \operatorname{acl}(Bb_1, ..., b_n)$. So $p(\mathcal{U}) \subseteq \operatorname{acl}(B\mathcal{Q}_B(\mathcal{U}))$. Note that for any $b \models \mathcal{Q}_B$, there is some partial type $q \in \mathcal{Q}$ over B such that $b \models q$. Thus, q is a partial type over BA', hence $b \models \mathcal{Q}_{BA'}$. So, $\mathcal{Q}_B(\mathcal{U}) \subseteq \mathcal{Q}_{BA'}(\mathcal{U})$. Hence, $p(\mathcal{U}) \subseteq \operatorname{acl}(BA'\mathcal{Q}_{BA'}(\mathcal{U}))$. On the other hand, since p' is an extension of p, $p'(\mathcal{U}) \subseteq p(\mathcal{U})$. Hence, $p'(\mathcal{U}) \subseteq \operatorname{acl}(BA'\mathcal{Q}_{BA'}(\mathcal{U}))$, and thus for all $a \models p'$, there exists $b_1, ..., b_\ell \models \mathcal{Q}_{BA'}$ such that $a \in \operatorname{acl}(BA'b_1, ..., b_\ell)$.

2. Fix $A' \supseteq A$ and let $p' \in S(A')$ be a non-forking extension of p. There is some $a \models p'$ and $B \supset A'$ with $a \underset{A'}{\downarrow} B$ such that $a \in \operatorname{acl}(B\mathcal{Q}_B(\mathcal{U}))$. Since p' is the non-forking extension of p, we have that $a \underset{A}{\downarrow} A'$, and so $a \underset{A}{\downarrow} B$ with $a \in \operatorname{acl}(B\mathcal{Q}_B(\mathcal{U}))$. By Fact 2.2.21, p is almost $\mathcal{Q}$-internal.

3. Let $B \supseteq A$, $a \models p|_C$ and $b_1, ..., b_n \models \mathcal{Q}_B$ be such that $a \in \operatorname{acl}(Bb_1, ..., b_n)$. For all $i = 1, ..., n$, since b_i realizes a partial type in $\mathcal{Q}$ over B, it realizes a partial type in $\mathcal{Q}$ over $\operatorname{acl}(B)$, meaning that $b_i \models \mathcal{Q}_{\operatorname{acl}(B)}$. Since $b \in \operatorname{acl}(Aa)$, we have that $b \in \operatorname{acl}(Bb_1, ..., b_n) = \operatorname{acl}(\operatorname{acl}(B)b_1, ..., b_n)$ and $b \underset{\operatorname{acl}(A)}{\downarrow} \operatorname{acl}(B)$, witnessing almost $\mathcal{Q}$-internality of $\operatorname{stp}(b/A)$.

4. For a proof of this see [63, Lemma 8.1.2]. □

Even in DCF_0, almost internality is strictly weaker than internality as the equation $\delta(x) = \frac{1}{2x}$ shows [65, Lemma 3.1].

While not every almost $\mathcal{Q}$-internal type is $\mathcal{Q}$-internal, every almost $\mathcal{Q}$-internal type is related to a $\mathcal{Q}$-internal type:

Lemma 2.2.24. *Let $p \in S(A)$ be stationary and $\mathcal{Q}$ be an A-invariant family of (partial) types. If p is almost $\mathcal{Q}$-internal, then there is some finite-to-one definable function $\alpha : p \to q$ with $q \in S(A)$ is $\mathcal{Q}$-internal.*

Proof. Fix some $a \models p$. By [44, Lemma 2.7], there is some tuple b such that $\mathrm{tp}(b/A)$ is $\mathcal{Q}$-internal and $\mathrm{acl}(Aa) = \mathrm{acl}(Ab)$. Let e be the canonical parameter for $\{b_0, ..., b_n\}$, the finite set of Aa-conjugates of b. Then $e \in \mathrm{dcl}(Aa)$ and $a \in \mathrm{acl}(Ae)$ since $b \in \mathrm{acl}(Ae)$. By Lemma 2.2.1, $\mathrm{tp}(e/A)$ is stationary. Since $e \in \mathrm{dcl}(Ab_0, ..., b_n)$ and $\mathrm{tp}(b_i/A)$ is $\mathcal{Q}$-internal, by Fact 2.2.23, $\mathrm{tp}(e/A)$ is $\mathcal{Q}$-internal. Let $q := \mathrm{tp}(e/A)$ and $f : p \to q$ be the A-definable map given by $a \mapsto e$. This is finite-to-one since $a \in \mathrm{acl}(Ae)$. $\square$

One particularly important case for us is when $\mathcal{Q}$ is the set of A-conjugates of a minimal type r (not necessarily over A) it turns out that in this case almost $\mathcal{Q}$-internality and almost r-internality agree:

Fact 2.2.25. *[23, Lemma 5.4] Suppose $p \in S(A)$ and $r \in S(B)$ are stationary with r minimal and $B \supseteq A$. Let $\mathcal{Q}$ be the set of all A-conjugates of r. Then p is almost $\mathcal{Q}$-internal if and only if p is almost r-internal.*

The case of (almost) internality to a minimal type is of particular interest, and we explore this further. First, we have the following useful characterization of almost internality to a minimal type.

Lemma 2.2.26. *Suppose $p \in S(A)$ is stationary and $r \in S(B)$ is minimal. Then p is almost r-internal if and only if there exists $C \supseteq A \cup B$, $a \models p|_C$ and $c_1, ..., c_m$, C-independent realizations of $r|_C$, such that $a \in \mathrm{acl}(Cc_1, ..., c_m)$.*

Proof. The right-to-left direction is immediate from Fact 2.2.21. For the left-to-right direction, almost r-internality tells us, by Fact 2.2.21, that there exists $C \supseteq A \cup B$ $a \models p|_C$ and $c_1, ..., c_m \models r$ such that $a \in \mathrm{acl}(Cc_1, ...c_m)$. Choose C and m such that m is minimal possible among all C.

Suppose that $c_i \mathop{\not\mathrel{\perp\!\!\!\perp}}_B C$, for some for some $i = 1, ..., m$. Then, by minimality of r, $c_i \in \mathrm{acl}(C)$. Hence, $a \in \mathrm{acl}(Cc_1, ..., c_{i-1}, c_{i+1}, ..., c_m)$, contradicting minimality of m. Thus, $c_1, ..., c_m \models r|_C$.

If $c_1, ..., c_m$ is not C-independent, then for some $i = 1, ..., m$, $c_i \mathop{\not\mathrel{\perp\!\!\!\perp}}_C c_1, ..., c_{i-1}$. By minimality of r, $c_i \in \mathrm{acl}(Cc_1, ..., c_{i-1})$. Thus, $a \in \mathrm{acl}(Cc_1, ..., c_{i-1}, c_{i+1}, ..., c_m)$, contradicting minimality of m. We have shown that $(c_1, ..., c_m) \models (r|_C)^{(m)}$. $\square$

Corollary 2.2.27. *If p is almost internal to a minimal type r (over possibly different parameter sets), then p is non-orthogonal to r.*

Proof. Suppose that $p \in S(A)$ and $r \in S(B)$. By the above lemma, there is $C \supseteq A \cup B$, $a \models p|_C$, and $c_1, ..., c_m$ C-independent realizations of r such that $a \in \mathrm{acl}(Cc_1, ..., c_m)$. Choose $0 \leq \ell < m$ minimal such that $a \mathop{\smile\!\!\!\!|}_{C} c_1, ..., c_\ell$. So, $a \mathop{\not\smile\!\!\!\!|}_{Cc_1,...,c_\ell} c_{\ell+1}$ and $a \mathop{\smile\!\!\!\!|}_{A} Cc_1, ..., c_\ell$ and $c_{\ell+1} \mathop{\smile\!\!\!\!|}_{B} c_1, ..., c_\ell$, witnessing that $p \not\perp r$. $\square$

Remark 2.2.28. *It may seem at first glance that the above corollary should not require r to be minimal. The subtlety here is that the parameter set for r is not necessarily the same as that for p. So, for example, there exists $q \in S(A)$ non-algebraic with a non-algebraic forking extension $p \in S(B)$ such that $p \perp q$. As $p(\mathcal{U}) \subseteq q(\mathcal{U})$, p is q-internal.*

Restricting entirely to minimal types, non-orthogonality and almost internality agree:

Lemma 2.2.29. *If $r \in S(A)$ and $s \in S(B)$ are minimal types, then r is almost s-internal if and only if s is non-orthogonal to r.*

Proof. The left-to-right-direction is Corollary 2.2.27 above. For the right-to-left direction, if $r \not\perp s$, by definition there is some $C \supset A \cup B$, $a \models r|_C$ and $b \models s|_C$ such that $a \mathop{\not\smile\!\!\!\!|}_{C} b$. By minimality, $b \in \mathrm{acl}(Ca)$. By Fact 2.2.21, s is almost r-internal. $\square$

While non-orthogonality does not imply almost internality in general, we do have the following:

Fact 2.2.30. *[63, Corollary 7.4.6] Suppose $p \in S(A)$ and $q \in S(B)$ are stationary types and $\mathcal{Q}$ is the set of A-conjugates of q. If p is non-orthogonal to q, then there is $p' \in S(A)$ non-algebraic and $\mathcal{Q}$-internal with a definable function $p \to p'$. In particular, if q is minimal, then we can take p' to be q-internal.*

Note that the "in particular" clause of the above fact follows from the main clause by applying Lemma 2.2.24 and Fact 2.2.25.

The following are useful transitivity statements:

Lemma 2.2.31. *Suppose that $p \in S(A), q \in S(B), r \in S(C)$ are stationary types with r minimal.*

1. *If p is non-orthogonal to q and q is almost r-internal, then p is non-orthogonal to r.*

2. *Suppose now that q is also minimal. If p is almost q-internal and q is non-orthogonal to r, then p is almost r-internal.*

Proof. 1. By taking non-forking extensions we may assume that $C \supseteq A \cup B$. Fix $D \supset A \cup C$ such that for any $b \models q$, $b \in \operatorname{acl}(Dc_1, ..., c_n)$ for some $c_1, ..., c_n \models r$.

Let $D' \supseteq A \cup B$, $a \models p|_{D'}$ and $b \models q|_{D'}$ be such that $a \underset{D'}{\not\perp} b$. Let $a'b' \models \operatorname{tp}(ab/D')$ be such that $a'b' \underset{D'}{\perp} D$. Then $a' \models p|_{D'D}$, $a' \underset{D'}{\not\perp} b'$ and $b' \in \operatorname{acl}(Dc_1, ..., c_n)$ for some $c_1, ..., c_n \models r$. Thus, $a' \underset{D'}{\not\perp} b'D$ and since $a' \underset{D'}{\perp} D$, we have that $a' \underset{DD'}{\not\perp} b'$. Since $b' \in \operatorname{acl}(Dc_1, ..., c_n)$, we have that $a' \underset{DD'}{\not\perp} c_1, ..., c_n$. Since r is minimal, by Lemma 2.2.26, we may assume that $c_1, ..., c_n$ is a DD'-independent set. Then for some $1 \leq k < n$, $a' \underset{DD'c_1,...,c_{k-1}}{\perp} c_k$ and $a' \underset{DD'c_1,...,c_k}{\not\perp} c_{k+1}$. Since $a' \models p|_{DD'c_1,...,c_k}$ and $c_{k+1} \models r|_{DD'c_1,...,c_k}$, $p \not\perp r$.

2. By taking non-forking extensions, we may assume that p, q and r are all over the same parameters, say A. Let $C \supseteq A$, be such that $q|_C \not\perp^w r|_C$. Since p is almost q-internal, by Lemma 2.2.23, $p|_C$ is almost q-internal. Let $D \supseteq C$ and $a \models p|_C$ be such that $a \underset{C}{\perp} D$ and $a \in \operatorname{acl}(Dc_1, ..., c_n)$ for some $c_1, ..., c_n \models q$. By Lemma 2.2.26, we can choose $c_1, ..., c_n \models (q|_D)^{(n)}$.

Let $c \models q|_C$ and $d \models r|_C$ be such that $c \underset{C}{\not\perp} d$ and let $c'd' \models \operatorname{tp}(cd/C)$ be such that $c'd' \underset{C}{\perp} D$. Then, $c' \underset{D}{\not\perp} d'$, and so, by minimality of q and r, we have that $\operatorname{acl}(Dc') = \operatorname{acl}(Dd')$. Since for all $i = 1, ..., n$, $\operatorname{tp}(c_i/D) = \operatorname{tp}(c'/D)$, there is some $\sigma_i \in \operatorname{Aut}_D(\mathcal{U})$ such that $\sigma_i(c') = c_i$. Hence, there is some $d_i := \sigma_i(d') \models r|_D$ such that $\operatorname{acl}(Dc_i) = \operatorname{acl}(Dd_i)$. So, $a \in \operatorname{acl}(Dd_1, ..., d_n)$ witnessing that p is almost r-internal. □

Finally, we will often be concerned with a fibration $p \to q$ whose fibers are each (almost) internal to a minimal type. In principle the minimal type will vary with the fiber, but in fact we can choose a minimal type that holds for all the fibers:

Lemma 2.2.32. *Let $f : p \to q$ be a definable fibration of stationary types $p, q \in S(A)$ such that each fiber is almost internal to some minimal type (potentially over additional parameters including A). The following are equivalent:*

1. *some (equivalently any) fiber of f is nonorthogonal to some stationary type over A, and*

2. *there is a minimal type r such that every fiber of f is almost internal to r.*

Proof. We first show that the family of all fibers $\{\mathrm{tp}(a/Af(a)) \mid a \models p\}$ is an A-invariant family of A-conjugates. Note that the fibers need not be minimal. Fix $a \models p$ and $\sigma \in \mathrm{Aut}_A(\mathcal{U})$. Then $\sigma(a) \models p$ and

$$\begin{aligned}\mathrm{tp}\left(a/Af(a)\right)^{\sigma} &= \mathrm{tp}\left(\sigma(a)/A\sigma(f(a))\right)\\ &= \mathrm{tp}\left(\sigma(a)/Af(\sigma(a))\right).\end{aligned}$$

Hence, $\{\mathrm{tp}(a/Af(a)) \mid a \models p\}$ is A-invariant. For A-conjugacy, note that for any $a, b \models p$, there is $\sigma \in \mathrm{Aut}_A(\mathcal{U})$ such that $\sigma(a) = b$. Since $f : p \to q$ is A-definable, $\mathrm{tp}\left(a/Af(a)\right)^{\sigma} = \mathrm{tp}\left(b/Af(b)\right)$.

Let $B \supseteq A$ and $r \in S(B)$ be a minimal type such that $\mathrm{tp}(a/Af(a))$ is almost r-internal. Fix $C \supset \mathrm{acl}(Af(a))$ and $a' \models \mathrm{tp}(a/Af(a))$ such that $a' \mathop{\smile\!\!\!\!\!\!|}_{f(a)A} C$ and $a' \in \mathrm{acl}(Cb_1, ..., b_n)$ for some $b_i \models r$. Fix $c \models p$ and let $\sigma \in \mathrm{Aut}_A(\mathcal{U})$ be such that $\sigma(a) = c$. Then $\sigma(a') \models \mathrm{tp}(c/Af(c))$ with $\sigma(a') \mathop{\smile\!\!\!\!\!\!|}_{Af(c)} \sigma(C)$ and $\sigma(a') \in \mathrm{acl}(\sigma(C)\sigma(b_1), ..., \sigma(b_n)))$, where $\sigma(b_1), ..., \sigma(b_n) \models r^{\sigma}$, the type obtained by applying σ to r. By definition, the fiber $\mathrm{tp}(c/Af(c))$ is almost internal to r^{σ}, an A-conjugate of r. By definition, $\mathrm{tp}(c/Af(c))$ is almost internal in the family of all A-conjugates of the minimal type r.

For (2) $\implies$ (1), suppose that every fiber is orthogonal to every type over A. Then, by [63, Lemma 1.4.3.3], whenever $f(a), f(b) \models q$ with $f(a) \mathop{\smile\!\!\!\!\!\!|}_{A} f(b)$, we have that $\mathrm{tp}(a/Af(a)) \perp \mathrm{tp}(b/Af(b))$. Since $\mathrm{tp}(a/Af(a))$ is non-algebraic and almost internal to r, then $\mathrm{tp}(a/Af(a))$ is non-orthogonal to r, by Corollary 2.2.27. If $\mathrm{tp}(b/Af(b))$ is also almost r-internal, it is also non-orthogonal to r by Corollary 2.2.27. By Fact 2.2.11, $\mathrm{tp}(a/Af(a)) \not\perp \mathrm{tp}(b/Af(b))$, a contradiction. Hence, not every fiber will be almost internal to the same minimal type.

For (1) $\implies$ (2), fix $a \models p$ and suppose that $\mathrm{tp}(a/Af(a))$ is non-orthogonal to some type over A and is almost internal to the minimal type r. By [63, Lemma 1.4.3.3], there is some $f(b) \models \mathrm{tp}(f(a)/A)$ such that $f(a) \mathop{\smile\!\!\!\!\!\!|}_{A} f(b)$ and $\mathrm{tp}(a/Af(a)) \not\perp \mathrm{tp}(b/Af(b))$. By automorphism, whenever $f(b), f(c) \models q|_{f(a)}$, and $f(b) \mathop{\smile\!\!\!\!\!\!|}_{A} f(c)$, $\mathrm{tp}(b/Af(b)) \not\perp \mathrm{tp}(c/Af(c))$. Moreover, since $\mathrm{tp}(b/Af(c))$ and $\mathrm{tp}(c/Af(c))$ are A-conjugates, we have that if $\mathrm{tp}(b/Af(b))$

is almost internal to the minimal type s, then $\operatorname{tp}(c/Af(c))$ is almost internal to some A-conjugate of s, say s^σ. Since $\operatorname{tp}(c/Af(c))$ is almost s^σ-internal and non-algebraic, $\operatorname{tp}(c/Af(c)) \not\perp s^\sigma$ by Corollary 2.2.27. By Lemma 2.2.31, $\operatorname{tp}(b/Af(b)) \not\perp s^\sigma$. As $\operatorname{tp}(b/Af(b))$ is almost internal to s, $s \not\perp s^\sigma$ again by Lemma 2.2.31. By Lemma 2.2.31, $\operatorname{tp}(c/Af(c))$ is almost s-internal. Hence, whenever $f(b), f(c) \models q$ are independent over A, the fibers $\operatorname{tp}(b/Af(b))$ and $\operatorname{tp}(c/Af(c))$ are almost internal to the same minimal type.

Fix $c \models p$ and let r^σ be an A-conjugate of r such that $\operatorname{tp}(c/Af(c))$ is almost r^σ-internal. Choose $f(b) \models q|_{f(a)f(c)}$. In particular, $f(a) \underset{A}{\perp\!\!\!\perp} f(b)$ and $f(b) \underset{A}{\perp\!\!\!\perp} f(c)$. By the above argument $\operatorname{tp}(b/Af(b))$ is almost internal to r and r^σ. Since $\operatorname{tp}(b/Af(b))$ is non-algebraic and almost internal to the minimal type r, by Fact 2.2.27, $\operatorname{tp}(b/Af(b)) \not\perp r$. Since $\operatorname{tp}(b/Af(b))$ is also almost r^σ-internal, by Lemma 2.2.31, $r \not\perp r^\sigma$. As $\operatorname{tp}(c/Af(c))$ is almost r^σ-internal, by Lemma 2.2.31, $\operatorname{tp}(c/Af(c))$ is almost r-internal. Hence, every fiber is almost internal to the minimal type r. $\square$

Let us specialize back to the case of DCF_0 and internality to the field of constants, $\mathcal{C}$.

Proposition 2.2.33. *Suppose that $T = \mathrm{DCF}_0$, and $p \in S(F)$ is a stationary type over a differential field. Then p is $\mathcal{C}$-internal if and only if there is a differential field extension $L \supseteq F$ such that for all $a \models p$, there is a tuple c from $\mathcal{C}$ with $L\langle a\rangle = L(c)$.*

Proof. Suppose that p is $\mathcal{C}$-internal. By definition there is $K \supseteq F$ and $a \models p|_K$ such that $a \in K(d_1, ..., d_n)$ for some $d_1, ..., d_n \in \mathcal{C}$. Choose K and n so that n is minimal such. By Lemma 2.2.26, we may assume that $d_1, ..., d_n$ is K-independent. We may also assume that $d_i \in K\langle a\rangle^{\mathrm{alg}}$ for all $i = 1, ..., n$, otherwise $a \underset{F}{\perp\!\!\!\perp} Kd_i$, and hence we can take $a \in K(d_i)(d_1, ..., d_{i-1}, d_{i+1}, ..., d_n)$, contradicting minimality of n. Thus $K\langle a\rangle \subseteq K(d_1, ..., d_n)$ and $K(d_1, ..., d_n) \subseteq K\langle a\rangle^{\mathrm{alg}}$. By elimination of imaginaries on the induced structure on $\mathcal{C}$, there is a canonical parameter in $\mathcal{C}$ for the set of all Ka-conjugates of $d_1, ..., d_n$. Let c be the finite tuple interdefinable with the canonical parameter. Then $c \in K\langle a\rangle$ and $a \in K(c)$.

We have found K which satisfies the lemma for the particular choice of a. However, we need to find an L which works for all $a \models p$. Let L be an $|F|^+$ saturated model. Fix $a' \models p$. Let $K' \models \operatorname{tp}(K/F)$ such that $K' \underset{F}{\perp\!\!\!\perp} La'$. Then $\operatorname{tp}(aK/F) = \operatorname{tp}(a'K'/F)$ and $a' \underset{L}{\perp\!\!\!\perp} LK$. Hence, $\operatorname{tp}(a'/LK')$ is definable over some F' in L with the same cardinality as F. We may assume that F' contains F. By saturation of L, there is some $C \subseteq L$ such that $C \models \operatorname{tp}(K'/F')$. Thus $\operatorname{tp}(a'C/F) = \operatorname{tp}(a'K'/F) = \operatorname{tp}(aK/F)$. Let $\sigma \in \operatorname{Aut}_F(\mathcal{U})$ be such that $\sigma(aK) = a'C$ and let $c' = \sigma(c)$. Since $K\langle a\rangle = K(c)$, applying σ yields that $C\langle a'\rangle = C(c')$, so that $L\langle a'\rangle = L(c')$, as desired.

The reverse direction is immediate since there is some $(L, \delta) \supseteq (F, \delta)$, $a \models p|_L$ and $c \subseteq \mathcal{C}$ such that $a \in K(c) = \mathrm{dcl}(Kc)$. Hence, p is $\mathcal{C}$-internal. $\square$

An immediate consequence of this is that for stationary (almost) $\mathcal{C}$-internal types, U-rank and dimension agree.

2.2.4 Analyzability

An interesting phenomenon in geometric stability theory is when a type admits a finite sequence of fibrations where the fibers at each stage are internal to a fixed definable set X. This is the property of being analyzable in X. Here is a formal definition working with a family of partial types in place of a single definable set. It is drawn from Jin's thesis [44]. A good reference for analyzability is Chapter 7 of [63].

Definition 2.2.34. Let $p \in S(A)$ be stationary and $\mathcal{Q}$ be an A-invariant family of (partial) types. We say p is *analyzable* in $\mathcal{Q}$ (or $\mathcal{Q}$-*analyzable*) if there is a realization $a \models p$ and a sequence of tuples $(a_n, ..., a_0)$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Aa_n)$, $a_0 \in \mathrm{acl}(A)$, and for all $i = 0, ..., n-1$, $a_i \in \mathrm{dcl}(Aa_{i+1})$, and $\mathrm{stp}(a_{i+1}/Aa_i)$ is almost $\mathcal{Q}$-internal. Such a sequence is called a $\mathcal{Q}$-*analysis for* p. We call n the *length* of the analysis or say that p is $\mathcal{Q}$-analyzable in n-steps.

Note that being analyzable in $\mathcal{Q}$ is preserved by interdefinability over A. Moreover, it is well known (see [44, Proposition 3.2], for example) that we can replace almost $\mathcal{Q}$-internal with $\mathcal{Q}$-internal in the above definition of being $\mathcal{Q}$-analyzable. Note that 0-step analyzability in $\mathcal{Q}$ is equivalent to algebraicity and 1-step analyzability in $\mathcal{Q}$ is equivalent to almost $\mathcal{Q}$-internality.

Example 2.2.35. Let $T = \mathrm{DCF}_0$ and F be an algebraically closed differential field. Let p be the generic type of the equation $\delta(\frac{\delta(x)}{x}) = 0$. Then p is not almost $\mathcal{C}$-internal, but it is $\mathcal{C}$-analyzable in 2 steps [13, Fact 4.2].

Note that p is called the pullback of the constants under the logarithmic derivative. These pullbacks will be studied in Section 3.1.

Similarly to internality, analyzability behaves well with arbitrary extensions.

Fact 2.2.36. *[63, Lemma 8.1.2] Let $p \in S(A)$ be stationary and $\mathcal{Q}$ be an A-invariant family of types. If p is $\mathcal{Q}$-analyzable, then any stationary extension of p is $\mathcal{Q}$-analyzable.*

We can express analyzability in terms of fibrations.

Lemma 2.2.37. *Let $A = \mathrm{acl}(A)$ and $\mathcal{Q}$ be an A-invariant family of partial types. Let $p \in S(A)$ be stationary and $\mathcal{Q}$-analyzable in n-steps. Then there exists stationary types $p_0, ..., p_n \in S(A)$ and a sequence of definable fibrations $(f_i : p_i \to p_{i-1})_{i=1,...,n}$ such that*

1. *p_n is interalgebraic with p,*
2. *p_0 is algebraic, and*
3. *for every $i = 1, ..., n$, the fibers of $f_i : p_i \to p_{i-1}$ are almost $\mathcal{Q}$-internal.*

We call such a sequence a $\mathcal{Q}$-analysis of p by fibrations in n-steps.

Proof. All we need to prove is that there exists an analysis of p in $\mathcal{Q}$, $(a_n, ..., a_0)$ such that the induced functions $f_i : \mathrm{tp}(a_i/A) \to \mathrm{tp}(a_{i-1}/A)$ are fibrations - that is that they have stationary fibers.

Let $(b_n, ..., b_0)$ be an arbitrary n-step analysis. Set $a_n := b_n$. Let $g_n : \mathrm{tp}(a_n/A) \to \mathrm{tp}(b_{n-1}/A)$. By Lemma 2.2.6, there is a fibration $f_n : \mathrm{tp}(a_n/A) \to \mathrm{tp}(a_{n-1}/A)$ and a finite-to-one definable function $h_{n-1} : \mathrm{tp}(a_{n-1}/A) \to \mathrm{tp}(b_{n-1}/A)$ such that $g_n = h_{n-1} \circ f_n$. As the fibers of f_n are extensions of the fibers of g_n, they are still almost $\mathcal{Q}$-internal. Now consider $g_{n-1} : \mathrm{tp}(b_{n-1}/A) \to \mathrm{tp}(b_{n-2}/A)$ and apply Lemma 2.2.6 to $h_{n-1} \circ g_{n-1} : \mathrm{tp}(a_{n-1}/A) \to \mathrm{tp}(a_{n-2}/A)$ to obtain a fibration $f_{n-1} : \mathrm{tp}(a_{n-1}/A) \to \mathrm{tp}(a_{n-2}/A)$ and a finite-to-one definable function $h_{n-2} : \mathrm{tp}(a_{n-2}/A) \to \mathrm{tp}(b_{n-2}/A)$ such that $h_{n-2} \circ f_{n-1} = h_{n-1} \circ g_{n-1}$.

Iterating this construction we get an analysis $(a_n, ..., a_0)$ with the desired property. $\square$

2.2.5 Binding groups

When a type p is internal to a definable set X, the group of elementary permutations of p fixing X pointwise turns out to be a definable group action that in some sense measures the parameters needed to witness internality.

Definition 2.2.38. Given a definable set X, we denote the group of automorphisms of $\mathcal{U}$ that fix A and X pointwise by $\mathrm{Aut}_A(\mathcal{U}/X)$. Let $p \in S(A)$ be a stationary type which is X-internal. The *binding group of p over X* is the group of permutations

$$\mathrm{Aut}_A(p/X) = \{\sigma|_{p(\mathcal{U})} \mid \sigma \in \mathrm{Aut}_A(\mathcal{U}/X)\}.$$

A very important theorem in geometric stability theory is that these (a priori abstract) permutation groups are definable, see [63, Theorem 7.4.8] for a proof.

Fact 2.2.39. *Suppose that X is an A-definable set and $p \in S(A)$ is stationary and X-internal. Then there is an A-definable group G acting relatively A-definable on $p(\mathcal{U})$ such that G together with this action is isomorphic to $\mathrm{Aut}_A(p/X)$ acting on $p(\mathcal{U})$.*

We will often identify the binding group with this definable group labeling it also as $\mathrm{Aut}_A(p/X)$.

The definable binding group is functorial in the following sense: if $f : p \to q$ is a definable function between stationary types over A that are internal to the same A-definable set X, then there is an induced A-definable surjective homomorphism $\widetilde{f} : \mathrm{Aut}_A(p/X) \to \mathrm{Aut}_A(q/X)$. This homomorphism can be defined as follows: suppose $\sigma \in \mathrm{Aut}_A(p/X)$ and lift it to an automorphism $\widehat{\sigma} \in \mathrm{Aut}_A(\mathcal{U}/X)$. Define $\widetilde{f}(\sigma) := \widehat{\sigma}|_{q(\mathcal{U})}$. See, for example, [45, Lemma 3.1] for details on this. In particular, if p and q are interdefinable, then $\mathrm{Aut}_A(p/X)$ and $\mathrm{Aut}_A(q/X)$ are A-definably isomorphic.

We record some well-known important properties of binding groups.

Fact 2.2.40. *Let $p \in S(A)$ be stationary and almost internal to X, an A-definable set.*

1. *The action of $\mathrm{Aut}_A(p/X)$ on p is determined by its action on a fundamental system of solutions.*
2. *$\mathrm{Aut}_A(p/X)$ acts transitively on $p(\mathcal{U})$ if and only if p is weakly orthogonal to X.*
3. *$\mathrm{Aut}_A(p/X)$ acts freely and transitively on $p(\mathcal{U})$ if and only if p is weakly orthogonal to X and fundamental.*
4. *If $B \supseteq A$, then $\mathrm{Aut}_B(p|_B/X)$ is a B-definable subgroup of $\mathrm{Aut}_A(p/X)$.*

Binding groups in DCF_0

We record some facts about binding groups relative to the field of constants $\mathcal{C}$ in the theory DCF_0.

If $p \in S(F)$ is stationary and $\mathcal{C}$-internal, the binding group of p is definably isomorphic, potentially over some field extension of F, to the $\mathcal{C}$-points of an algebraic group. In the case when the binding group is commutative, no additional parameters are needed to witness this definable isomorphism (see [41, Lemma 2.2]).

Fact 2.2.41. *[41, Lemma 2.6] Let F be an algebraically closed differential field and $p \in S(F)$ be $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal. Then $\mathrm{Aut}_F(p/\mathcal{C})$ is connected.*

Recall that a type is *isolated* if its set of realizations is a definable set instead of merely type definable. In this case, the type is isolated by the formula defining the set of realizations. Note that the generic type of a definable set need not be isolated, for example the generic type of the constants is non-isolated as it is $\emptyset$-definable set with infinitely many points in $\mathrm{acl}(\emptyset)$ and every definable subset of the constants is either finite or co-finite.

Remark 2.2.42. *Let F be an algebraically closed differential field. If $p \in S(F)$ is $\mathcal{C}$-internal type and $\mathcal{C}$-orthogonal, then p is isolated. Indeed if p is weakly $\mathcal{C}$-orthogonal, by Fact 2.2.40, the binding group $\mathrm{Aut}_F(p/\mathcal{C})$ acts transitively on $p(\mathcal{U})$. Hence, $p(\mathcal{U})$ is the orbit of any $a \in p(\mathcal{U})$ under the action of the definable group $\mathrm{Aut}_F(p/C)$. Thus $p(\mathcal{U})$ is a definable set, as opposed to type-definable set, hence p is isolated (by the formula defining $p(\mathcal{U})$).*

We have described the surjective definable morphism on binding groups induced by a definable function between types internal to the same definable set. In the context of DCF_0, we can also describe the kernel of this map (for example see [41, Lemma 2.7]):

Fact 2.2.43. *Let F be an algebraically closed differential field and $f : p \to q$ be a fibration of $\mathcal{C}$-internal stationary types $p, q \in S(F)$.*

Fix $n \geq 1$ and $\overline{a} := (a_1, \ldots, a_n) \models p^{(n)}$ such that $\overline{a}$ is a fundamental system for p and $f(\overline{a})$ is a fundamental system for q. Let $r := \mathrm{tp}(\overline{a}/Ff(\overline{a}))$. Then the kernel of $\widetilde{f} : \mathrm{Aut}_F(p/\mathcal{C}) \to \mathrm{Aut}_F(q/\mathcal{C})$ is definably isomorphic to $\mathrm{Aut}_{Ff(\overline{a})}(r/\mathcal{C})$.

In Section 4.2, we need to understand the subgroup of a binding group that fixes realizations of an auxiliary type. We end these preliminaries by introducing some notation and proving some lemmas about the situation for later use.

Definition 2.2.44. Suppose F is an algebraically closed differential field and $p \in S(F)$ is stationary $\mathcal{C}$-internal type. Suppose that $q \in S(F)$ is another stationary type. We denote $\mathrm{Aut}_F(p/\mathcal{C}, q)$ to be the subgroup of elements $\sigma \in \mathrm{Aut}_F(p/\mathcal{C})$ which lift to an automorphism $\widehat{\sigma} \in \mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$ that also fixes $q(\mathcal{U})$ pointwise.

Lemma 2.2.45. *Let $p \in S(F)$ be $\mathcal{C}$-internal and $q \in S(F)$ be stationary. The group $\mathrm{Aut}_F(p/\mathcal{C}, q)$ is an F-definable normal subgroup of $\mathrm{Aut}_F(p/\mathcal{C})$.*

Proof. For the purposes of this proof, let us be careful about distinguishing between $\mathrm{Aut}_F(p/\mathcal{C})$ and its definable avatar, that is an F-definable group G with an F-definable action on $p(\mathcal{U})$ for which there is an abstract isomorphism $\gamma : \mathrm{Aut}_F(p/\mathcal{C}) \to G$ that preserves the action.

Working with internality to the F-invariant family of partial types $\{\delta(x) = 0, q(x)\}$, we also have a definable avatar of $\mathrm{Aut}_F(p/\mathcal{C}, q)$, namely an F-definable group H acting F-definably on $p(\mathcal{U})$ with an abstract isomorphism $\eta : H \to \mathrm{Aut}_F(p/\mathcal{C}, q)$ that preserves the action.

Composing we obtain $\iota := \gamma \circ \eta : H \to G$. It is not hard to see that if we fix $(a_1, ..., a_n)$, a tuple of realizations of p that is a fundamental system for the $\mathcal{C}$-internality of p, then ι is definable over $F \cup \{a_1, ..., a_m\}$. This is because the action of both $\mathrm{Aut}_F(p/\mathcal{C})$ and $\mathrm{Aut}_F(p/\mathcal{C}, q)$ are both determined by how they acts on the fundamental system $a_1, ..., a_n$.

To show that $\iota = \gamma \circ \eta$ is an F-definable isomorphism, we show that for every $\tau \in \mathrm{Aut}_F(\mathcal{U})$ and $g \in H$, $\tau(\widetilde{\iota}(g)) = \widetilde{\iota}(\tau(g))$.

Fix $\tau \in \mathrm{Aut}_F(\mathcal{U})$ and $g \in H$. Let $\sigma = \eta(g)$. Since every automorphism in the binding group is determined by its action on the fundamental system, and the actions of the bindings groups are definably isomorphic to the actions of the corresponding definable group, $\iota(g)$ and g have the same action on $p(\mathcal{U})$. Let $\mu_G : G \times p(\mathcal{U}) \to (\mathcal{U})$ and $\mu_H : H \times p(\mathcal{U}) \to (\mathcal{U})$ be the F-definable actions given by Fact 2.2.39. For every $\tau \in \mathrm{Aut}_F(\mathcal{U})$, $g \in H$ and $a \models p$,

$$\begin{aligned}
\mu_G(\iota(\tau(g)), a) &= \mu_H(\tau(g), a) \\
&= \tau(\mu_H(g, \tau^{-1}(a))) \\
&= \tau(\mu_G(\iota(g), \tau^{-1}(a))) \\
&= \mu_G(\tau(\iota(g)), a)
\end{aligned}$$

Thus $\iota(\tau(g)) = \tau(\iota(g))$. Hence, $\iota : H \to G$ is an F-definable map and $\mathrm{Aut}_F(p/\mathcal{C}, q)$ is F-definably isomorphic to $K := \gamma^{-1}(\iota(H))$, an F-definable subgroup of $\mathrm{Aut}_F(p/\mathcal{C})$.

That K is a normal subgroup is a common argument: fix $\sigma \in K$ and $\tau \in \mathrm{Aut}_F(p/\mathcal{C})$. Let $\widetilde{\tau}$ be any extension of τ to $\mathrm{Aut}_F(\mathcal{U})$. Then $\widetilde{\tau}|_p \circ \sigma \circ \widetilde{\tau}^{-1}|_p = \widetilde{\tau}(\sigma)$. Since K is F-definable and $\widetilde{\tau}$ fixes F-pointwise, $\widetilde{\tau}(\sigma) \in H$. □

Lemma 2.2.46. *(Definable Goursat's lemma) Let G_1 and G_2 be F-definable groups, and H be an F-definable subgroup of $G_1 \times G_2$. Consider the natural projections $\pi_i : G_1 \times G_2 \to G_i$ for $i = 1, 2$, and assume that the maps $\pi_i|_H$ are surjective. Let $N_1 = \ker(\pi_2|_H)$ and $N_2 = \ker(\pi_1|_H)$. Then $\pi_1(N_1)$ and $\pi_2(N_2)$ are F-definable normal subgroups of G_1 and G_2, respectively, and $G_1/\pi_1(N_1)$ and $G_2/\pi_2(N_2)$ are F-definably isomorphic.*

Proof. Define $\phi : G_1/\pi_1(N_1) \to G_2/\pi_2(N_2)$ as the map defined by $g_1\pi_1(N_1) \mapsto g_2\pi_2(N_2)$ where $(g_1, g_2) \in H$. This map is an A-definable group morphism. $\square$

We say two definable groups G_1 and G_2 are *definably isogenous* if there are finite normal definable subgroups $H_1 < G_1$ and $H_2 < G_2$, such that the quotients G_1/H_1 and G_2/H_2 are definably isomorphic.

Corollary 2.2.47. *Let $p, q \in S(F)$ be $\mathcal{C}$-internal. Then $\mathrm{Aut}_F(p/\mathcal{C})/\mathrm{Aut}_F(p/\mathcal{C}, q)$ and $\mathrm{Aut}_F(q/C)/\mathrm{Aut}_F(q/\mathcal{C}, p)$ are F-definably isomorphic.*

Moreover, if p and q are interalgebraic over F, then $\mathrm{Aut}_F(p/\mathcal{C})$ and $\mathrm{Aut}_F(q/\mathcal{C})$ are F-definably isogenous.

Proof. We will apply the definable version of Goursat's lemma with H being the group of all $(\sigma_1, \sigma_2) \in \mathrm{Aut}_F(p/\mathcal{C}) \times \mathrm{Aut}_F(q/\mathcal{C})$ such that there exists a common extension of σ_1 and σ_2 to $\mathrm{Aut}_F(\mathcal{U})$. Notice that if σ is a common extension for σ_1 and σ_2, then $\sigma|_{\mathcal{C}} = \mathrm{id}$. To apply the lemma, we need H to be A-definable. First we show H is type-definable. Fix $\overline{a} = (a_1, ..., a_n)$ a fundamental system for p with respect to $\mathcal{C}$ and $\overline{b} = (b_1, ..., b_m)$ a fundamental system for q with respect to $\mathcal{C}$.

Then $(\sigma_1, \sigma_2) \in H$ if and only if there is $\sigma \in \mathrm{Aut}_F(\mathcal{U})$ such that $\sigma|_p = \sigma_1$ and $\sigma|_p = \sigma_2$ if and only if $\mathrm{tp}(\overline{a}, \overline{b}/A\mathcal{C}) = \mathrm{tp}(\sigma(\overline{a}), \sigma(\overline{b})/A\mathcal{C}) = \mathrm{tp}(\sigma_1(\overline{a}), \sigma_2(\overline{b})/A\mathcal{C})$. By stable embeddedness of $\mathcal{C}$ (and [14, Appendix Lemma 1]), $\mathrm{tp}(\overline{a}, \overline{b}/\mathrm{dcl}(F\overline{a}\overline{b}) \cap \mathrm{dcl}(F\mathcal{C})) \vdash \mathrm{tp}(\overline{a}, \overline{b}/F\mathcal{C})$. Let S be the set of all F-definable functions whose domain includes $\mathrm{tp}(\overline{a}, \overline{b}/F)$ and whose image lies in $\mathrm{dcl}(F\mathcal{C})$. Then $\mathrm{tp}(\overline{a}, \overline{b}/\mathrm{dcl}(F\overline{a}\overline{b}) \cap \mathrm{dcl}(F\mathcal{C})) = \mathrm{tp}(\sigma_1(\overline{a}), \sigma_2(\overline{b})/\mathrm{dcl}(F\sigma_1(\overline{a})\sigma_2(\overline{b}))) \cap \mathrm{dcl}(F\mathcal{C}))$ if and only if for every $f \in S$, $f(\overline{a}, \overline{b}) = f(\sigma_1(\overline{a}), \sigma_2(\overline{b}))$. Hence, $(\sigma_1, \sigma_2) \in H$ if and only if $\mathrm{tp}(\overline{a}\overline{b}/F\mathcal{C}) = \mathrm{tp}(\sigma_1(\overline{a}), \sigma_2(\overline{b})/F\mathcal{C})$ if and only if for every $f \in S$, $f(\overline{a}, \overline{b}) = f(\sigma_1(\overline{a}), \sigma_2(\overline{b}))$. Thus, H is $\overline{a}, \overline{b}F$-type definable. By ω-stability of DCF_0, it must then be an $F\overline{a}\overline{b}$-definable subgroup (see [54, Theorem 7.5.3] for example).

To show that H is F-definable, we show that H is fixed by every $\tau \in \mathrm{Aut}_F(\mathcal{U})$. Fix $(\sigma_1, \sigma_2) \in H$. Then $\tau(\sigma_1) = \tau|_p \circ \sigma_1 \circ \tau^{-1}|_p$ and $\tau(\sigma_2) = \tau|_q \circ \sigma_2 \circ \tau^{-1}|_q$. Let $\sigma \in \mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$ be a common extension of σ_1 and σ_2. Then the automorphism $\tau \circ \sigma \circ \tau^{-1}$ fixes $F \cup \mathcal{C}$ pointwise and extends $\tau|_p \circ \sigma_1 \circ \tau^{-1}|_p$ and $\tau|_q \circ \sigma_2 \circ \tau^{-1}|_q$. By definition, $(\tau(\sigma_1), \tau(\sigma_2)) \in H$. Thus, $\mathrm{Aut}_F(\mathcal{U})$ fixes H, and so H is F-definable.

Let π_p and π_q be the projection maps $\pi_p : \mathrm{Aut}_F(p/\mathcal{C}) \times \mathrm{Aut}_F(q/\mathcal{C}) \to \mathrm{Aut}_F(p/\mathcal{C})$ and $\pi_q : \mathrm{Aut}_F(p/\mathcal{C}) \times \mathrm{Aut}_F(q/\mathcal{C}) \to \mathrm{Aut}_F(q/\mathcal{C})$. To apply Lemma 2.2.46, we need to show that the projection maps $\pi_p|_H$ and $\pi_q|_H$ are surjective, $\ker(\pi_q|_H) = \mathrm{Aut}_F(p/\mathcal{C})$ and $\ker(\pi_p|_H) = \mathrm{Aut}_F(q/\mathcal{C})$.

For surjectivicity, let $\sigma_1 \in \mathrm{Aut}_F(p/\mathcal{C})$. Then by definition of the binding group, σ_1 extends to some $\sigma \in \mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$. Set $\sigma_2 = \sigma|_q$. Hence, $(\sigma_1, \sigma_2) \in H$. Thus $\pi_p(\sigma_1, \sigma_2) = \sigma_1$. Similarly, $\sigma_q|_H$ is surjective.

For the condition on the kernels, let $\sigma_1 \in \mathrm{Aut}_F(p/\mathcal{C}, q)$. By definition $(\sigma_1, \mathrm{id}|_q) \in H$, hence $(\sigma_1, \mathrm{id}|q) \in \ker(\pi_2|_H)$. Conversely, if $(\sigma_1, \sigma_2) \in \ker(\pi_2|_H)$, then $\sigma_2 = \mathrm{id}|_q$ and there is a common extension of σ_1 and $\mathrm{id}|_q$ to $\sigma \in \mathrm{Aut}_F(\mathcal{U})$. Then $\sigma_1 = \sigma|_p \in \mathrm{Aut}_F(p/\mathcal{C}q)$, and so $\ker(\pi_2|_H) = \mathrm{Aut}_F(p/\mathcal{C}, q)$. Similarly, $\ker(\pi_1|_H) = \mathrm{Aut}_F(q/\mathcal{C}, p)$.

For the moreover statement, by Lemma 2.2.46, the quotients $\mathrm{Aut}_F(p/\mathcal{C})/\mathrm{Aut}_F(p/q, \mathcal{C})$ and $\mathrm{Aut}_F(q/C)/\mathrm{Aut}_F(q/p, \mathcal{C})$ are F-definably isomorphic. Hence, it suffices to show that $\mathrm{Aut}_F(p/q, C)$ and $\mathrm{Aut}_F(q/p, \mathcal{C})$ are finite. Let $c_i \models q$ be such that $a_i \in \mathrm{acl}(Ac_i)$. Then for any $\sigma \in \mathrm{Aut}_F(p/q, C)$, $\sigma(c_i) = c_i$ for all $i = 1, ..., n$. Hence, there are only finitely many possible images for a_i under σ. Since every $\sigma \in \mathrm{Aut}_F(p/q, \mathcal{C})$ is determined by its action on the fundamental system of p, there are only finitely many possible σ. Hence, $\mathrm{Aut}_F(p/q, \mathcal{C})$ is finite. Similarly, $\mathrm{Aut}_F(q/p, \mathcal{C})$ is finite. $\square$

Chapter 3

Internality of vector fields on affine space

In this chapter, we provide an algebraic criterion for almost internality of generic types of rational vector fields $(\mathbb{A}^k, s)$. Recall as shown in Section 2.1.3, to a rational vector field $(\mathbb{A}^k, s)$ over F, we can associate a derivation δ_s on the field of rational functions $F(x_1, ..., x_k)$. Our criterion for almost internality to the constants is in terms of the differential field-arithmetic of $(F(x_1, ..., x_k), \delta_s)$.

The main result, appearing as Theorem 3.2.4 below, is the following:

Theorem 3. *Let $X = (\mathbb{A}^k, s)$ be a rational vector field over F, an algebraically closed field of constants. Let $p \in S(F)$ be the generic type of X over F. The following are equivalent:*

1. *p is almost internal to the constants*
2. *there are $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ algebraically independent over F such that for all $i = 1, ..., k-1$, $\delta_s(g_i) = \lambda g$ for some $\lambda \in F$, and $\delta_s(g)$ is either also λg for some $\lambda \in F$ or 1.*

Moreover, in this case, p is weakly orthogonal to the constants if and only if the λs appearing in (2) can be taken to be $\mathbb{Q}$-linearly independent.

In the case when $k = 1$ we recover the restatement in [45] of an old Theorem of Rosenlicht. Our proof relies on Kolchin's theorem, or rather its model-theoretic statement in [41], which can be used to relate almost internality of p to a logarithmic-differential equation on $\mathbb{G}_m^\ell \times \mathbb{G}_a^n$ where $n = 0$ or 1.

We present two applications of Theorem 3.

1. We show that the generic type of the equation $\delta^2(x) = \delta(x)f(x)$, where f is a rational function, is internal to the constants if and only if f is a constant. When $b \neq d$, this question of Freitag, Jaoui, Marker, and Nagloo [21] and appears as Proposition 3.3.1 below.

2. We show that the generic type of a Lotka-Volterra system $\begin{cases} \delta(x) = x(ay+b) \\ \delta(y) = y(cx+d) \end{cases}$ is not internal to the constants. This gives the first concrete example — other than logarithmic differential equations — of a 2-step analysis that is not almost internal to the constants. It appears as Proposition 3.3.2 below.

The results of this chapter appear in [19] which is joint work with Léo Jimenez. Throughout we fix a sufficiently saturated model $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$ with field of constants $\mathcal{C}$.

3.1 Logarithmic-differential equations

Central to the proof of Theorem 3, is to show that when the generic type of a vector field on affine k-space is internal and weakly $\mathcal{C}$-orthogonal, its binding group must be either $\mathbb{G}_m^k$ or $\mathbb{G}_m^{k-1} \times \mathbb{G}_a$. To show this, we use a result of Kolchin which says that p is related to the generic type of a particular kind of equation known as a logarithmic-differential equation on an algebraic group G. We only need to consider $G = \mathbb{G}_m^\ell \times \mathbb{G}_a^n$. Here $\mathbb{G}_a$ means the additive group $(\mathcal{U}, +)$ and $\mathbb{G}_m$ means the multiplicative group $(\mathcal{U} \setminus \{0\}, \times)$.

Definition 3.1.1. Let $G = \mathbb{G}_m^\ell \times \mathbb{G}_a^n$ and F be an algebraically closed field of constants. By a *logarithmic-differential equation on* G over F we mean a system of the form

$$\begin{cases} \delta(x_i) = \lambda_i x_i,\ 1 \leq i \leq \ell \\ \delta(y_j) = \eta_j,\ 1 \leq j \leq n \end{cases}$$

where $\lambda_1, ..., \lambda_\ell \in F$ and $\eta_1, ..., \eta_n \in F$. We denote this system by $\delta\log_G(x, y) = (\lambda, \eta)$ where $x = (x_1, ..., x_\ell)$, $y = (y_1, ..., y_n)$, $\lambda = (\lambda_1, ..., \lambda_\ell)$, and $\eta = (\eta_1, ..., \eta_n)$.

Let q be the generic type over F of $\delta\log_G(x, y) = (\lambda, \eta)$. As this is a system of linear differential equations, q is $\mathcal{C}$-internal (see Example 2.2.18).

Since q is $\mathcal{C}$-internal, if q is weakly $\mathcal{C}$-orthogonal it must be isolated, by Remark 2.2.42. In fact, q is isolated by $\delta\log_G(x, y) = (\lambda, \eta)$: first note that $\lambda_1, ..., \lambda_\ell, \eta, ..., \eta_n \neq 0$, otherwise the definable projection map from q onto the generic type of $\delta(z) = 0$, contradicts weak orthogonality to the constants. If $\delta\log_G(\bar{a}, \bar{b}) = (\lambda, \eta)$, but $\dim(\bar{a}, \bar{b}/F) = \text{trdeg}(\bar{a}, \bar{b}/F) < \ell + n = \dim(q)$, then there is some non-zero polynomial over F such that $P(\bar{a}, \bar{b}) = 0$. Thus $\delta(P(\bar{a}, \bar{b})) = 0$. If $\ell > 0$, without loss of generality, the first coordinate of $\bar{a}$, a_1, appears in $P(\bar{a}, \bar{b})$ meaning that $\text{trdeg}(a_1/Fa_2, ..., a_\ell, b_1, ..., b_n) = 0$. Hence, the set defined by the logarithmic-differential equation is interalgebraic with the solution set to the logarithmic-differential equation given by the system

$$\begin{cases} \delta(x_1) = 0 \\ \delta(x_i) = \lambda_i x_i,\ 2 \leq i \leq \ell \\ \delta(y_j) = \eta_j,\ 1 \leq j \leq n \end{cases}$$

Indeed the map given by $(a_1, a_2, ..., a_\ell, \bar{b}) \mapsto (P(\bar{a}, \bar{b}), a_2, ..., a_\ell, \bar{b})$ is F-definable and finite-to one. Thus, q is interalgebraic with the generic type of $\delta\log_G(x_1, x_2, ..., x_\ell, y) = (0, \lambda_2, ..., \lambda_\ell, \eta)$, but the later type is non-weakly orthogonal to the constants. Hence, q is non-weakly orthogonal to the constants, a contradiction. A similar argument works when $l = 0$, taking b_1 to appear in $P(\bar{a}, \bar{b})$. So, we have that $\dim(\bar{a}, \bar{b}/F) = \dim(q)$ and so $\delta\log_G(x, y) = (\lambda, \eta)$ isolates q.

Example 3.1.2. Let F be an algebraically closed field of constants.

1. Let q be the generic type over F of $\delta\log_{\mathbb{G}_m}(x) = \lambda$ for some singleton x and $\lambda \in F$. If $\lambda \neq 0$, then q is weakly $\mathcal{C}$-orthogonal and $\text{Aut}_F(q/\mathcal{C}) = \mathbb{G}_m(\mathcal{C})$.

2. Let q be the generic type over F of $\delta\log_{\mathbb{G}_a}(y) = \eta$ for some singleton y and $\eta \in F$. If $\eta \neq 0$, then q is weakly $\mathcal{C}$-orthogonal and $\text{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$.

3. Let $G = \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$ and q be the generic type over F of $\delta\log_G(x, y) = (\lambda, \eta)$. Then $\text{Aut}_F(q/\mathcal{C})$ is a (definable) subgroup of $\mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$.

Proof. These are all standard differential Galois theory arguments. We show (2) and (3) and leave (1) to the reader.

2. Let $a \models q$. Then $\delta(a) = \eta \neq 0$, so $a \notin \mathcal{C}$. If q is non-weakly $\mathcal{C}$-orthogonal, then for some tuple of constants, c, $a \underset{F}{\not\perp} c$. Since $\dim(q) = \text{trdeg}(a/F) = 1 > \text{trdeg}(a/F(c))$, $a \in F(c)^{\text{alg}} \subseteq \mathcal{C}$. This is a contradiction since $\delta(a) \neq 0$, and so $q \perp^w \mathcal{C}$.

For the binding group, we show that $\mathrm{Aut}_F(q/\mathcal{C})$ is a subgroup of $\mathbb{G}_a(\mathcal{C})$. Let $\alpha \in \mathrm{Aut}_F(q/\mathcal{C})$ and $b \models q$. Note that for any $a \models q$, $\delta(a-b) = \delta(a) - \delta(b) = \eta - \eta = 0$, so $a - b \in \mathcal{C}$. In particular, $\alpha(a-b) = a-b$ since $\alpha|_{\mathcal{C}} = \mathrm{id}$. For any $a \models q$, $\alpha(a) = \alpha(a+b-b) = \alpha(a-b)+\alpha(b) = a-b+\alpha(b)$. Since $\alpha(b) \models q$, $\alpha(b) - b \in \mathcal{C}$. Hence, α acts on $q(\mathcal{U})$ by addition of the constant $\alpha(b) - b$.

Since $\eta \neq 0$, $\delta(x) = \eta$ has no solutions in F. Hence $\mathrm{Aut}_F(q/\mathcal{C})$ is a non-trivial subgroup of $\mathbb{G}_a(\mathcal{C})$. The only non-trivial subgroup of $\mathbb{G}_a(\mathcal{C})$ is the group itself, hence $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$.

3. Let r be the generic type of $\delta\log_{\mathbb{G}_m^\ell}(x) = \lambda$ and s the generic type of $\delta\log_{\mathbb{G}_m^n}(y) = \eta$. Let $(\alpha, \beta) \in \mathrm{Aut}_F(q/\mathcal{C})$ where $\alpha \in \mathrm{Aut}_F(r/\mathcal{C})$ and $\beta \in \mathrm{Aut}_F(s/\mathcal{C})$. Then α acts on $r(\mathcal{U})$ by componentwise multiplication of a tuple of constants and β acts on $s(\mathcal{U})$ by addition of a tuple of constants. Hence, $(\alpha, \beta) \in \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$.

Thus $\mathrm{Aut}_F(q/\mathcal{C})$ is definably isomorphic to $\mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$. □

Recall that the binding group of a $\mathcal{C}$-internal type is always definably isomorphic to the $\mathcal{C}$-points of an algebraic group. If the algebraic group is linear, we say that the binding group is linear.

The following model-theoretic restatement of a theorem of Kolchin is from [41] and explains the importance of logarithmic-differential equations. The fact is stated for a general group G, but we only consider $G = \mathbb{G}_m^\ell \times \mathbb{G}_a^n$ for some $\ell, n \in \mathbb{N}$.

Fact 3.1.3. *(Kolchin) Let F be an algebraically closed differential field and $p \in S(F)$ be almost $\mathcal{C}$-internal, fundamental and weakly $\mathcal{C}$-orthogonal. Then there is a connected algebraic group G defined over $F \cap \mathcal{C}$ such that p is interdefinable with the generic type of a logarithmic-differential equation on G over F.*

Corollary 3.1.4. *Let F be an algebraically closed field of constants. Let $p \in S(F)$ be $\mathcal{C}$-internal, weakly $\mathcal{C}$-orthogonal and $\mathrm{Aut}_F(p/\mathcal{C})$ be linear. Then p is interdefinable with the generic type of a logarithmic-differential equation on either $\mathbb{G}_m^k$ or $\mathbb{G}_m^{k-1} \times \mathbb{G}_a$ where $k = U(p)$.*

Proof. It is well known that over constant parameters, if the binding group of a $\mathcal{C}$-internal type p is linear, the binding group $\mathrm{Aut}_F(p/\mathcal{C})$ must be commutative (see for example the proof of Theorem 3.9 in [22]). Hence, $\mathrm{Aut}_F(p/\mathcal{C})$ is definably isomorphic to $G(\mathcal{C})$ where G is a commutative linear algebraic group over F. So, $G(\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$ for some $\ell, n \geq 0$. Moreover, the proof of Theorem 3.9 [22] shows that $n \leq 1$.

Since p is weakly $\mathcal{C}$-orthogonal, $\mathrm{Aut}_F(p/\mathcal{C})$ acts transitively by Fact 2.2.40. Binding groups always act faithfully, so the commutative binding group of p acts freely and transitively on $p(\mathcal{U})$. Hence, p is fundamental (see Fact 2.2.40) and so $k = U(p) = U(\mathrm{Aut}_F(p/\mathcal{C})) = \ell + n$. So $\ell = k$ or $\ell = k - 1$.

Applying Fact 3.1.3, we have that p is, up to interdefinability, the generic type over F of a logarithmic-differential equation on G. □

The following appears as Lemma 3.1 in [19], which shows that if the generic type of a vector field on affine k-space is $\mathcal{C}$-internal, then its binding group must be linear.

Fact 3.1.5. *Let F be an algebraically closed field of constants. Let p be the generic type over F of a rational vector field $(\mathbb{A}^k, s)$. Let q be an F-definable image of p. If q is $\mathcal{C}$-internal, then its binding group is linear.*

We can characterize weak orthogonality to the constants of the generic type for a logarithmic-differential equation on G.

Lemma 3.1.6. *Let F be an algebraically closed field of constants. Let q be the generic type over F of a differential-logarithmic equation on $G = \mathbb{G}_m^\ell$ given by $\delta\log_G(x) = \lambda$ where $\lambda \in F^\ell$. Then q is weakly orthogonal to the constants if and only if λ is $\mathbb{Q}$-linearly independent.*

Moreover, in this case, $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C})$.

Proof. First suppose that λ is not $\mathbb{Q}$-linearly independent. Fix $a = (a_1, ..., a_\ell) \models q$. For all $i = 1, ..., \ell$, $\delta(a_i) = \lambda_i a_i$. Hence, $\lambda_i = \frac{\delta(a_i)}{a_i} = \delta\log_{\mathbb{G}_m}(a_i)$. Then there are $k_1, ..., k_\ell \in \mathbb{Z}$, not all zero, such that $0 = \sum_{i=1}^{\ell} k_i\lambda_i = \sum_{i=1}^{\ell} k_i\delta\log_{\mathbb{G}_m}(a_i)$. A simple computation shows that $\prod_{i=1}^{\ell} a_i^{k_i} \in \mathcal{C}$, but $\prod_{i=1}^{\ell} a_i^{k_i} \notin F$. Hence, $\mathcal{C} \cap F\langle a\rangle \neq \mathcal{C} \cap F$. By Fact 2.2.15, q is not weakly $\mathcal{C}$-orthogonal.

For the reverse direction, suppose that λ is $\mathbb{Q}$-linearly independent and fix a realization $(a_1, ..., a_\ell) \models q$. By genericity of a, $F(a_1, ..., a_\ell)$ is isomorphic over F to the differential field $F(x_1, ..., x_\ell)$ where δ extends to a derivation with $\delta_s(x_i) = \lambda_i x_i$ for all $j = 1, ..., \ell$. Since F is an algebraically closed field of constants, and $\delta_s(x_i) = \lambda_i x_i$ for $\mathbb{Q}$-linearly independent constants $\lambda_1, ..., \lambda_\ell$, we have that $F(x_1, ..., x_\ell) = \mathcal{C} \cap F$ by [51, Proposition 1.20]. Thus $\mathcal{C} \cap F = \mathcal{C} \cap F\langle a_1, ..., a_\ell\rangle$. By Fact 2.2.15, $q \perp^w \mathcal{C}$.

For the moreover statement, since $\mathrm{Aut}_F(q/\mathcal{C})$ is a definable subgroup of $\mathbb{G}_m^\ell(\mathcal{C})$, we show that $\mathbb{G}_m^\ell(\mathcal{C})$ acts on $q(\mathcal{U})$. Let $c = (c_1, ..., c_\ell) \in \mathcal{C}^\ell$. Since $q \perp^w \mathcal{C}$, q is isolated by the formula $\delta\log_{\mathbb{G}_m^\ell}(x) = \lambda$. It suffices to show that for any $b \models q$, $\delta\log_{\mathbb{G}_m^\ell}(c \cdot b) = \lambda$. Let $a = (a_1, ..., a_\ell) \models q$ and $c = (c_1, ..., c_\ell) \in \mathbb{G}_m^\ell(\mathcal{C})$. For all $i = 1, ..., \ell$, by definition $\delta(a_i) = \lambda_i a_i$. Hence, for all $i = 1, ..., \ell$ and any $c_i \in \mathcal{C}$, $\delta(c_i a_i) = c_i \delta(a_i) = \lambda_i c_i a_i$. Thus, $\delta\log_{\mathbb{G}_m^\ell}(c \cdot a) = \lambda$. □

Lemma 3.1.7. *Let F be an algebraically closed field of constants. Let q be the generic type over F of a logarithmic-differential equation on $G = \mathbb{G}_a^n$ given by $\delta\log_G(y) = \eta$, where $\eta \in F^n$. Then q is weakly orthogonal to the constants if and only if $n = 1$ and $\eta_1 \neq 0$.*

Moreover, in this case, $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$ and q is interdefinable with the generic type of $\delta\log_G(y) = 1$.

Proof. Suppose that $q \perp^w \mathcal{C}$. It is easier to show that $\mathrm{Aut}_F(p/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$ and thus $n = 1$ and $\eta_1 \neq 0$. By Example 3.1.2, $\mathrm{Aut}_F(q/\mathcal{C})$ is a definable subgroup of $\mathbb{G}_a^n(\mathcal{C})$.

Fix $b_1, ..., b_m \models q$ a fundamental system of solutions and some $t \in \mathcal{U}$ such that $\delta(t) = 1$. Notice that for all $i = 1, ..., m$ $(1, b_{i,1}, ..., b_{i,n})$ is a solution to the system $\delta(Y) = AY$ where $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$, $\delta(Y) = \begin{pmatrix} \delta(y_0) \\ \vdots \\ \delta(y_n) \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \eta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \eta_n & 0 & \cdots & 0 \end{pmatrix}$. A fundamental system of solutions to $\delta(Y) = AY$ is $\left\{ \begin{pmatrix} 1 \\ \eta_1 t \\ \eta_2 t \\ \vdots \\ \eta_n t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$. Hence, $b_1, ..., b_n \subseteq F(t)$ since $\eta \in F^n$. Finally, $\mathrm{tp}(t/F)$ is clearly fundamental (as it is minimal), $\mathcal{C}$-internal (as $\delta(t) = 1$) and weakly $\mathcal{C}$-orthogonal, hence its binding group is $\mathbb{G}_a(\mathcal{C})$. In particular, $\mathrm{tp}(b_1, ..., b_n/F)$ is fundamental and weakly $\mathcal{C}$-orthogonal with binding group equal to $\mathrm{Aut}_F(q/\mathcal{C})$ (see for example the discussion following Fact 2.3. in [41]). By the Galois correspondence, $\mathrm{Aut}_F(q/\mathcal{C})$ is definably isomorphic to a quotient of $\mathbb{G}_a(\mathcal{C})$ by some normal definable subgroup. Since the only subgroups of $\mathbb{G}_a(\mathcal{C})$ are the trivial group or $\mathbb{G}_a(\mathcal{C})$ itself, $\mathrm{Aut}_F(q/\mathcal{C})$ is either trivial or $\mathbb{G}_a(\mathcal{C})$.

Since $q \perp^w \mathcal{C}$, the binding group acts transitively on q, by Fact 2.2.40. In particular, $\mathrm{Aut}_F(q/\mathcal{C})$ cannot be trivial as $q(\mathcal{U}) \not\subset \mathcal{C}$, otherwise q would not be weakly $\mathcal{C}$-orthogonal. Hence, $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$. Suppose $n > 1$ and derive a contradiction. Fix $(a_1, ..., a_n) \models q$.

For every $i = 1, ..., n$ and $c \in \mathcal{C}$, $\delta(a_i + c) = \eta_i$. By weak-orthogonality, q is isolated by $\delta\log_{\mathbb{G}_a^n}(y) = \eta$. Hence, $(c + a_1, d + a_2, ..., d + a_n) \models q$ for some constants $c \neq d$. But, for all $e \in \mathcal{C}$, it is impossible for $e + a_1 = c + a_1$ and $e + a_2 = d + a_2$. Hence, there is no $\alpha \in \mathbb{G}_a(\mathcal{C})$ such that $\alpha(a_1, ..., a_n) = (c + a_1, d + a_2, ..., d + a_n)$, contradicting that $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$. So, if $q \perp^w \mathcal{C}$, then $n = 1$. Moreover, if $\eta_1 = 0$ then q is the generic type of the constants, and hence not weakly $\mathcal{C}$-orthogonal. Thus, we also have that $\eta_1 \neq 0$.

For the reverse direction, when $n = 1$ and $\eta_1 \neq 0$ this is exactly Example 3.1.2, and hence $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$.

For the moreover statement, there is an F-definable bijection between q and the generic type of $\delta(y) = 1$ given by $y \mapsto \frac{y}{\eta_1}$, witnessing interdefinability. $\square$

Recall that the dimension and U-rank of types internal to the constants agree.

Corollary 3.1.8. *Let F be an algebraically closed field of constants. Let q be the generic type over F of a logarithmic-differential equation on $G = \mathbb{G}_m^\ell \times \mathbb{G}_a^n$ given by $\delta\log_G(x, y) = (\lambda, \eta)$, where $\lambda \in F^\ell$ and $\eta \in F^n$. Then q is weakly orthogonal to the constants if and only if λ is $\mathbb{Q}$-linearly independent, and either $n = 0$ or $n = 1$ and $\eta_1 \neq 0$.*

Moreover, in this case, $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$ and q is interdefinable with the generic type of $\delta\log_G(x, y) = (\lambda, 1)$.

Proof. First, suppose that $q \perp^w \mathcal{C}$. If $n = 0$, this is the forward direction of Lemma 3.1.6, so we may assume that $n > 0$. The generic types of $\delta\log_{\mathbb{G}_m^\ell}(x) = \lambda$ and $\delta\log_{\mathbb{G}_a^n}(y) = \eta$ are also weakly $\mathcal{C}$-orthogonal as they are obtained by projecting q. By Lemma 3.1.6, λ is $\mathbb{Q}$-linearly independent. If $n > 0$, then by Lemma 3.1.7, $n = 1$ and $\eta_1 \neq 0$.

For the reverse direction, we first show that $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a^n(\mathcal{C})$. In this case, the binding group of q acts transitively on its set of realizations, hence $q \perp^w \mathcal{C}$ by Fact 2.2.40. If λ is $\mathbb{Q}$-linearly independent and $n = 0$, this is exactly Lemma 3.1.6. So we may assume that λ is $\mathbb{Q}$-linearly independent and $n = 1$ with $\eta \neq 0$. Fix $(a_1, ..., a_\ell, b) \models q$. Let $r = \mathrm{tp}(a_1, ..., a_\ell/F)$ and $s = \mathrm{tp}(b/F)$. In particular, r is the generic type of $\delta\log_{\mathbb{G}_m^\ell}(x) = \lambda$ and s is the generic type of $\delta\log_{\mathbb{G}_a}(y) = \eta$. Hence, r and s are $\mathcal{C}$-internal types. Since λ is $\mathbb{Q}$-linearly independent, $r \perp^w \mathcal{C}$ by Lemma 3.1.6. Since $n = 1$ and $\eta_1 \neq 0$, $s \perp^w \mathcal{C}$ by Lemma 3.1.7. Recall that U-rank and dimension agree on $\mathcal{C}$-internal types, so $\ell + 1 = U(q)$, $1 = U(s)$ and $\ell = U(r)$. In particular, $U(b/F) = U(b/Fa_1, ..., a_\ell)$, hence $(a_1, ..., a_\ell, b) \models r \otimes s$.

Since r and s are both weakly $\mathcal{C}$-orthogonal, $\mathrm{Aut}_F(r/\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C})$ and $\mathrm{Aut}_F(s/\mathcal{C}) = \mathbb{G}_a(\mathcal{C})$ by Lemmas 3.1.6 and 3.1.7, respectively. We have shown that $q = r \otimes s$ and $\mathrm{Aut}_F(q/\mathcal{C}) < \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a(\mathcal{C})$, hence the the group $\mathrm{Aut}_F(q/\mathcal{C})$ projects surjectively onto both $\mathbb{G}_m^\ell(\mathcal{C})$ and $\mathbb{G}_a(\mathcal{C})$. Thus $\mathrm{Aut}_F(q/\mathcal{C}) = \mathbb{G}_m^\ell(\mathcal{C}) \times \mathbb{G}_a(\mathcal{C})$ by the structure of algebraic groups. □

3.2 criterion for internality to the constants

As a warm-up, we first characterize generic types of vector fields on affine space that are both internal and weakly orthogonal to the constants. Most of the ideas appearing in the general case, namely characterizing just internality to the constants, already appear here in a more tractable form. This is Theorem 3.2.2. While our main result, Theorem 3.2.4, recovers Theorem 3.2.2, we prove it directly in order to illustrate the ideas appearing in Theorem 3.2.4. We use the following fact:

Fact 3.2.1. *[48, Kolchin-Ostrowski Theorem] Let (F, δ) be some differential field and let $a_1, ..., a_m, b_1, ..., b_n \in \mathcal{U}\backslash\{0\}$ be an F-algebraically dependent set such that for all $1 \leq i \leq m$, $\delta(a_i) \in F$ and for all $1 \leq j \leq n$, $\frac{\delta(b_j)}{b_j} \in F$. Then either there exists $c_1, ..., c_m \in \mathcal{C}$, not all zero, such that $\sum_{i=1}^{m} c_i a_i \in F$ or there exists $e_1, ..., e_n \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^{n} b_i^{e_i} \in F$.*

Theorem 3.2.2. *Let $X = (\mathbb{A}^k, s)$ be a rational vector field over F, an algebraically closed field of constants. Let $p \in S(F)$ be the generic type of X over F. The following are equivalent:*

1. *p is almost $\mathcal{C}$-internal and weakly orthogonal to the constants*

2. *there are $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$, algebraically independent over F such that for all $i = 1, ..., k-1$, $\delta_s(g_i) = \lambda g_i$ for some $\lambda \in F$ and $\delta_s(g) = \begin{cases} 1, & \text{or} \\ \lambda g & \text{for some } \lambda \in F \end{cases}$ and all the λs appearing are $\mathbb{Q}$-linearly independent.*

Proof. (1) $\implies$ (2) Assume that p is almost $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal. By Lemma 2.2.24, there is some definable finite-to-one map $\alpha : p \to q$ where $q \in S(F)$ is $\mathcal{C}$-internal. By Remark 2.2.14, q is weakly $\mathcal{C}$-orthogonal. Since p is the generic type of $(\mathbb{A}^k, s)$, $\alpha : p \to q$ is definable and q is $\mathcal{C}$-internal, by Fact 3.1.5 the binding group of q is linear. So, Corollary 3.1.4 applies to q.

Since p is almost $\mathcal{C}$-internal, $U(p) = \dim(p) = k$. As $\alpha : p \to q$ is finite-to-one, $U(q) = U(p) = k$. By Corollary 3.1.4, up to interdefinability over F, q is the generic type over F of a logarithmic-differential equation on either $\mathbb{G}_m^{k-1} \times \mathbb{G}_a$ or $\mathbb{G}_m^k$. Let us assume we are in the former, more involved, case of $G = \mathbb{G}_m^{k-1} \times \mathbb{G}_a$ and q is the generic type of $\log_\delta(x, y) = (\lambda, \eta)$ where $x = (x_1, ..., x_{k-1})$, y is a singleton and $\lambda = (\lambda_1, ..., \lambda_{k-1}) \in F^{k-1}$. Since q is the generic type of a logarithmic-differential equation on $\mathbb{G}_m^{k-1} \times \mathbb{G}_a$, by Corollary 3.1.8, λ is $\mathbb{Q}$-linearly independent and we may assume that $\eta = 1$, up to interdefinability.

Fix $a = (a_1, ..., a_k) \models p$. Since $\alpha : p \to q$ is F-definable, there are $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ such that $\alpha(a) = (g_1(a), ..., g_{k-1}(a), g(a))$. By construction, $\delta(g(a)) = 1$ and $\delta(g_i(a)) = \lambda_i g_i$ for all $i = 1, ..., k-1$. By genericity of a, $\delta_s(g) = 1$ and $\delta_s(g_i) = \lambda_i g_i$ for all $i = 1, ..., k-1$.

It remains to show that $g_1, ..., g_{k-1}, g$ are F-algebraically independent. Suppose that $g_1, ..., g_k$ are algebraically dependent over F and derive a contradiction. By algebraic dependence of the g_i and g, $g_1(a), ..., g_{k-1}(a), g(a)$ are algebraically dependent over F. If $g(a) \in F$, then $U(q) < k$. So, $g(a) \notin F$. By the Kolchin-Ostrowski Theorem (3.2.1), there are some $e_1, ..., e_{k-1} \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^{k-1} g_i(a)^{e_i} \in F$. Since $F \subseteq \mathcal{C}$,

$$\begin{aligned} 0 &= \delta(\prod_{i=1}^{k-1} g_i(a)^{e_i}) \\ &= \sum_{i=1}^{k-1} (e_i \lambda_i g_i(a)^{e_i} \prod_{i \neq j} g_j(a)^{e_j}) \\ &= \left(\prod_{i=1}^{k-1} g_i(a)^{e_i} \right) \left(\sum_{i=1}^{k-1} e_i \lambda_i \right). \end{aligned}$$

By genericty of a, $g_i(a)^{e_i} \neq 0$ for all $i = 1, ..., k-1$, hence $\prod_{i=1}^{k} g_i(a)^{e_i} \neq 0$. Thus $\lambda_1, ..., \lambda_{k-1}$ are $\mathbb{Q}$-linearly dependent, a contradiction since $\lambda = (\lambda_1, ..., \lambda_{k-1})$ is $\mathbb{Q}$-linearly independent. Therefore, $g_1, ..., g_{k-1}, g$ are algebraically independent over F.

We have $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ algebraically independent over F such that for all $i = 1, ..., k-1$, $\delta_s(g_i) = \lambda g_i$ for some $\lambda \in F$ and $\delta_s(g) = 1$, and the λs appearing are $\mathbb{Q}$-linearly independent.

A similar proof when $G = \mathbb{G}_m^k$ produces $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ algebraically independent over F such that for all $i = 1, ..., k-1$, $\delta_s(g_i) = \lambda g_i$ for some $\lambda \in F$ and $\delta_s(g) = \lambda g$ for some $\lambda \in F$, and the λs appearing are $\mathbb{Q}$-linearly independent.

(2) $\implies$ (1) Let $g_1, ..., g_{k-1}, g$ satisfy condition (2) of the theorem and suppose that $\delta_s(g) = 1$. Let $\lambda = (\lambda_1, ..., \lambda_{k-1}) \in F^{k-1}$ be $\mathbb{Q}$-linearly independent such that $\delta_s(g_i) = \lambda_i g_i$

for every $i = 1, ..., k-1$. Let $G = \mathbb{G}_m^{k-1} \times \mathbb{G}_a$ and let q be the generic type over F of the differential- logarithmic equation $\delta\log_G(x, y) = (\lambda, 1)$ where $x = (x_1, ..., x_{k-1})$, and y is a singleton. Since q is the generic type of a differential-logarithmic equation and λ is $\mathbb{Q}$-linearly independent, by Corollary 3.1.8, the type q is weakly $\mathcal{C}$-orthogonal. Recall that q is $\mathcal{C}$-internal as it is the generic type over F of a logarithmic-differential equation on $\mathbb{G}_m^{k-1} \times \mathbb{G}_a$. We show that p is interalgebraic with q and thus, p is almost $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal.

Fix $a \models p$. Let $\alpha : p \to q$ be the definable map given by $\alpha(a) = (g_1(a), ..., g_{k-1}(a), g(a))$. Since $q \perp^w \mathcal{C}$, q is isolated by $\delta\log_G(x, y) = (\lambda, 1)$. By genericity of $a \models p$, $g_i(a) \neq 0$ for all $i = 1, ..., k-1$ and $g(a) \neq 0$, $\delta(g_i(a)) = \lambda_i g_i(a)$ for all $i = 1, ..., k-1$ and $\delta_s(g(a)) = 1$. Hence, $\alpha(a) \models q$. In particular, α is a surjective map from p to q. Recall that $U(q) = \dim(q) = k \leq U(p) \leq k$, hence $U(q) = U(p)$. In particular, for any $a \models p$, $U(a/\alpha(a)F) = U(a/F) - U(\alpha(a)/F) = 0$. So, $a \in \operatorname{acl}(F\alpha(a))$ and p is almost $\mathcal{C}$-internal by Lemma 2.2.23, and weakly $\mathcal{C}$-orthogonal by Fact 2.2.14.

If $\delta_s(g) = \lambda_k g$ for some $\lambda_k \in F$, then q is the generic type of $\delta\log_G(x) = \lambda$ where $x = (x_1, ..., x_k)$ and $\lambda = (\lambda_1, ..., \lambda_k) \in F^k$ with $\lambda_i \neq 0$ for all $i = 1, ..., k$. A similar proof as above shows that p is almost $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal, applying Lemma 3.1.6 instead of Corollary 3.1.8. $\square$

We use a similar idea to prove the forward direction of Theorem 3.2.4 removing the assumption of weak orthogonality to the constants. In the case when p is $\mathcal{C}$-internal, we obtain some fibration $\pi : p \to p'$ with p' realized by a tuple of constants and the fibers of π are weakly orthogonal to the constants. Note that if p is already weakly orthogonal to the constants, then p' can be taken to be an algebraic type. In this case, we can directly apply the proof of Theorem 3.2.2. In general, let $\ell = U(a/\pi(a)F)$. We obtain $g_1, ..., g_{\ell-1}, g \in F(x_1, ..., x_k)$ similarly as in Theorem 3.2.2, dealing with the extra parameters $\pi(a)$ for a fixed $a \models p$. For $g_{\ell+1}, ..., g_k \in F(x_1, ..., x_k)$, we use the algebraic independence of $h_1, ..., h_{k-1} \in F(x_1, ..., x_k)$ such that $\pi(a) = (h_1(a), ..., h_n(a))$ for all $a \models p$.

Lemma 3.2.3. *Let F be an algebraically closed field of constants. Let $p \in S(F)$ have finite U-rank. Then there is an F-definable fibration $\pi : p \to q$ such that q is realized by a tuple of constants and the fibers of π are weakly $\mathcal{C}$-orthogonal.*

Proof. Fix $a \models p$. Note that $\mathcal{C}$ is stably embedded, hence $\operatorname{tp}(a/\operatorname{dcl}(Fa) \cap \mathcal{C})$ isolates $\operatorname{tp}(a/\mathcal{C})$ (see for example [14, Appendix Lemma 1]).

Since p has finite U-rank, $\operatorname{dcl}(Fa) = F\langle a \rangle$ is finitely generated as a field extension of F. Since $\operatorname{dcl}(Fa) \cap \mathcal{C} = F\langle a \rangle \cap \mathcal{C}$ is definably closed and a subset of a stably embedded pure algebraically closed field, $\mathrm{dcl}(Fa) \cap \mathcal{C}$ is a field. In particular, it is a subfield of the finitely generated field $\mathrm{dcl}(Fa) = F\langle a \rangle$. Let $b = (b_1, ..., b_n) \in \mathcal{C}$ be such that $\mathrm{dcl}(Fa) \cap \mathcal{C} = F(b)$. Hence, $\mathrm{tp}(a/Fb)$ isolates $\mathrm{tp}(a/\mathcal{C})$.

Let $q = \mathrm{tp}(b/F)$. Since $b \in \mathrm{dcl}(Fa)$, there is a definable map $\pi : p \to q$ mapping a to b. Since $\mathrm{tp}(a/F\pi(a))$ isolates $\mathrm{tp}(a/\mathcal{C})$, $\mathrm{tp}(a/\mathcal{C})$ is the unique extension of $\mathrm{tp}(a/F\pi(a))$ to the constants. In particular, $\mathrm{tp}(a/\mathcal{C})$ is the unique non-forking extension of $\mathrm{tp}(a/F\pi(a))$ to the constants. Hence, for every tuple d from $\mathcal{C}$, $a \underset{F\pi(a)}{\downarrow} d$. So, $\mathrm{tp}(a/F\pi(a)) \perp^w \mathcal{C}$.

Finally, since $\mathcal{C}$ is algebraically closed, $F(\pi(a))^{\mathrm{alg}} \subset \mathcal{C}$. Let $e = \mathrm{Cb}(a/F(\pi(a))^{\mathrm{alg}})$. Then $e \subseteq \mathrm{acl}(F\pi(a)) \subset \mathcal{C}$ and $e \in \mathrm{dcl}(F\pi(a)a) = \mathrm{dcl}(Fa)$. Thus $e \in \mathrm{dcl}(Fa) \cap \mathcal{C} = \mathrm{dcl}(F\pi(a))$. Thus $\mathrm{stp}(a/F\pi(a))|_{F\pi(a)} = \mathrm{tp}(a/F\pi(a))$ is stationary. $\square$

We can now prove the main result of this chapter, Theorem 3.

Theorem 3.2.4. *Let $X = (\mathbb{A}^k, s)$ be a rational vector field over F, an algebraically closed field of constants. Let $p \in S(F)$ be the generic type of X over F. The following are equivalent:*

1. *p is almost $\mathcal{C}$-internal*
2. *there are $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ algebraically independent over F such that for all $i = 1, ..., k-1$, $\delta_s(g_i) = \lambda g$ for some $\lambda \in F$ and* $\delta_s(g) = \begin{cases} 1, \text{ or} \\ \lambda g \text{ for some } \lambda \in F \end{cases}$

Moreover, in this case, p is weakly $\mathcal{C}$-orthogonal if and only the λs appearing in (2) can be taken to be all non-zero and $\mathbb{Q}$-linearly independent.

Proof. (1) $\implies$ (2) Suppose that p is almost $\mathcal{C}$-internal. By Lemma 2.2.24, there is some finite-to-one definable map, $\alpha : p \to q$ where $q \in S(F)$ is $\mathcal{C}$-internal. By Fact 3.1.5, the binding group of q is linear.

Let $\pi : q \to \pi(q)$ be the map given by Lemma 3.2.3. In particular, $\pi(q)$ is realized by a tuple of constants and for every $b \models q$ the fiber $q_{\pi(b)} := \mathrm{tp}(b/F\pi(b))$ is stationary, $\mathcal{C}$-internal, and weakly $\mathcal{C}$-orthogonal.

To apply our results about binding groups, we need to work over an algebraically closed set of parameters in the constants. Fix $b \models q$ and let $\overline{q_{\pi(b)}}$ be the unique extension of $q_{\pi(b)}$ to $F(\pi(b))^{\mathrm{alg}}$. This extension is again almost $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal. We now show that the binding group of this type is linear, so that we can apply Corollary 3.1.8 to $\overline{q_{\pi(b)}}$.

The map $\pi : q \to \pi(q)$ induces a morphism $\widetilde{\pi} : \mathrm{Aut}_F(q/\mathcal{C}) \to \mathrm{Aut}_F(\pi(q)/\mathcal{C})$. Since $\pi(q)$ is realized by a tuple of constants, $\mathrm{Aut}_F(\pi(q)/\mathcal{C}) = I$. Hence, the kernel of $\widetilde{\pi}$ is $\mathrm{Aut}_F(q/\mathcal{C})$. By Fact 2.2.43, the binding group of q is definably isomorphic to the binding group of $\mathrm{tp}(b_1, ..., b_m/F\pi(b_i), ..., \pi(b_m))$ for $b_1, ..., b_m$, a fundamental system of q. We call this binding group H. In particular, H is a definable linear group since $\mathrm{Aut}_F(q/\mathcal{C})$ is a definable linear group.

For every $i = 1, ..., m$, $\mathrm{Aut}_{F(\pi(b_i))}(q_{\pi(b_i)}/\mathcal{C})$ is a definable subgroup of H. Since $\pi(b_1), ..., \pi(b_n) \subset \mathcal{C}$, they are fixed by $\mathrm{Aut}_{F(\pi(b_i))}(q_{\pi(b_i)}/\mathcal{C})$. Recall that subgroups of linear groups are again linear, so for every $i = 1, ..., m$, $\mathrm{Aut}_{F(\pi(b_i))}(q_{\pi(b_i)}/\mathcal{C})$ is linear. Since $b \in \mathrm{dcl}(F, b_1, ..., b_n\mathcal{C})$, $\mathrm{Aut}_{F(\pi(b))}(q_{\pi(b)}/\mathcal{C})$ is linear. Then $\mathrm{Aut}_{F(\pi(b))^{\mathrm{alg}}}(\overline{q_{\pi(b)}}/\mathcal{C})$ is a normal definable subgroup of $\mathrm{Aut}_{F(\pi(b))}(q_{\pi(b)}/\mathcal{C})$ (see [41, Lemma 2.1]), and hence is also linear.

By Corollary 3.1.4, the type $\overline{q_{\pi(b)}}$ is interdefinable over $F(\pi(b))^{\mathrm{alg}}$ with the generic type, s, of a logarithmic-differential equation over $F(\pi(b))^{\mathrm{alg}}$ on either $G = \mathbb{G}_m^{\ell-1} \times \mathbb{G}_a$ or $G = \mathbb{G}_m^{\ell}$ where $\ell = U(\overline{q_{\pi(b)}})$. Suppose we are in the case when $G = \mathbb{G}_m^{\ell-1} \times \mathbb{G}_a$. Then s is the generic type of $\delta\log_G(x, y) = (\lambda, 1)$ where $x = (x_1, ..., x_{\ell-1})$, $\lambda = (\lambda_1, ..., \lambda_{\ell-1}) \in F(\pi(b))^{\mathrm{alg}}$ and y is a singleton. Let $\gamma = (\gamma_1, ..., \gamma_\ell)$ be a definable map from $\overline{q_{\pi(b)}} \to s$ such that for every $i = 1, ...\ell - 1$, γ_i maps $\overline{q_{\pi(b)}}$ to the generic type of $\delta(z_i) = \lambda_i z_i$ and γ_ℓ maps $\overline{q_{\pi(b)}}$ to the generic type of $z'_\ell = 1$. Since $q_{\pi(b)} \vdash \overline{q_{\pi(b)}}$, for all $i = 1, ..., \ell$, γ_i defines a map on $q_{\pi(b)}$.

Fix $b' \models q_{\pi(b)}$ and $i = 1, ..., \ell - 1$. We first show that $\lambda_i \in F \setminus \{0\}$. If $\lambda_i = 0$, then $\gamma_i(b') \in \mathcal{C}$, which contradicts that $q_{\pi(b)}$ is weakly $\mathcal{C}$-orthogonal. Since $\mathrm{tp}(b'/F) = q$ is $\mathcal{C}$-internal and $\gamma_i(b') \in \mathrm{acl}(Fb')^{\mathrm{alg}}$, $\mathrm{tp}(b'/F)$ is almost $\mathcal{C}$-internal by Lemma 2.2.23. If $\lambda_i \notin F$, then $\mathrm{tp}(\lambda_i/F)$ is the generic type of the constants over F, and is minimal. Note that, $\delta(\frac{\delta(\gamma_i(b')))}{\gamma_i(b')}) = \delta(\lambda_i) = 0$ as $\lambda_i \in F(\pi(b))^{\mathrm{alg}} \subset \mathcal{C}$. Since $\lambda_i \notin F$, $\mathrm{tp}(\gamma_i(b')/F)$ is the generic type of $\delta(\frac{\delta(x)}{x}) = 0$ and is not almost $\mathcal{C}$-internal by Example 2.2.35, a contradiction. Hence, $\lambda_i \in F$.

Since $\overline{q_{\pi(b)}}$ is weakly $\mathcal{C}$-orthogonal, its image under $(\gamma_1, ..., \gamma_\ell)$ must also be weakly $\mathcal{C}$-orthogonal. Since $F(\pi(b))^{\mathrm{alg}}$ is an algebraically closed field of constants, and the image of $\overline{q_{\pi(b)}}$ under $(\gamma_1, ..., \gamma_\ell)$ is the generic type on $\mathbb{G}_m^{\ell-1} \times \mathbb{G}_a$, $\lambda_1, ..., \lambda_{\ell-1}$ is $\mathbb{Q}$-linearly independent by Corollary 3.1.8.

Claim. *There are integers n_i and F-definable functions $\tilde{\gamma}_i$, for $i \in \{1, \cdots \ell\}$ mapping p to the generic type of $\delta\log_{\mathbb{G}_m}(z) = (n_i\lambda_i)$ if $i < \ell$, and $\delta\log_{\mathbb{G}_a}(z) = (1)$ for $i = \ell$.*

Proof of claim: Fix $i = 1, ..., \ell - 1$. Since γ_i is $F(\pi(b))^{\mathrm{alg}}$-definable, it has finitely many images, $\gamma_{i,1}, ..., \gamma_{i,.n_i}$, under $\mathrm{Aut}_{F\pi(b)}$ which map $q_{\pi(b)}$ to the generic type of $\delta(z) = \lambda_i z$. Let $\widetilde{\gamma}_i := \gamma_{i,1} \times \cdots \times \gamma_{i,n_i}$.

Then for any $b' \models q_{\pi(b)}$,

$$\begin{aligned}
\delta(\widetilde{\gamma}_i(b')) &= \delta(\gamma_{i,1}(b') \times \cdots \times \gamma_{i,n_i}(b')) \\
&= \sum_{j=1}^{n_i} \delta(\gamma_{i,j}(b')) \prod_{j \neq k} \gamma_{i,k}(b') \\
&= \sum_{j=1}^{n_i} \lambda_i \gamma_{i,1}(b') \times \cdots \times \gamma_{i,n_i}(b') \\
&= (\sum_{j=1}^{n_i} \lambda_i) \gamma_{i,1}(b') \times \cdots \times \gamma_{i,n_i}(b') \\
&= n_i \lambda_i \tilde{\gamma}_i(b').
\end{aligned}$$

Hence, $\widetilde{\gamma}_i$ maps $q_{\pi(b)}$ to the generic type of $\delta(z) = n_i \lambda_i z$.

If $\widetilde{\gamma}_i$ is constant, then $\widetilde{\gamma}_i$ is a constant solution to $\delta(z) = n_i \lambda_i z$. Since $n_i \neq 0$ and $\lambda_i \neq 0$, we obtain that $\widetilde{\gamma}_i(b') = 0$ for all $b' \models q_{\pi(b)}$. Thus, for some $j = 1, ..., n_i$, $\gamma_{i,j}(b') = 0$. Then, $0 = \dim(\gamma_{i,j}(b')/F)$, contradicting that $\gamma_{i,j}(b')$ is a generic solution of the one dimensional generic type of $\delta(z) = \lambda_i z$. Hence, $\widetilde{\gamma}_i$ is non-constant.

If $i = \ell$, γ_ℓ maps to the generic type of $z'_\ell = 1$. Let $\gamma_{\ell,1}, ..., \gamma_{\ell,n_\ell}$ be the $F(\pi(b))$ conjugates of γ_ℓ and set $\widetilde{\gamma}_\ell := \gamma_{\ell,1} + \cdots + \gamma_{\ell,n_\ell}$. Then $\widetilde{\gamma}_\ell$ is an $F(\pi(b))$-definable map from $q_{\pi(b)}$ to the generic type of $\delta(z_\ell) = \sum_{i=1}^{n_\ell} 1 = n_\ell$. By Corollary 3.1.8, we may assume that $n_\ell = 1$.

Fix $i = 1, ...\ell$ and $a \models p$ such that $\alpha(a) = b'$. Since $\alpha : p \to q$ is F-definable, the map $\widetilde{\gamma}_i$ is $F(\pi(b))$-definable and $\pi(b) = \pi(b') \in F(a)$, up to interdefinability, we have some $h_1, ..., h_n \in F(x_1, ..., x_k)$ such that $\pi(b) = (h_1(a), ..., h_n(a))$. Replacing $\pi(b)$ in the formula defining $\widetilde{\gamma}_i$ with $(h_1, ..., h_n)$, we obtain an F-definable map $\widetilde{\gamma}_i \in F(x_1, ..., x_k)$ from p to the generic type of either $\delta\log_{\mathbb{G}_m}(z) = (\lambda_i)$ or $\delta\log_{\mathbb{G}_m}(z) = (1)$. ■

Since λ is $\mathbb{Q}$-linearly independent, so is $\widetilde{\lambda} = (n_1\lambda_1, ..., n_{\ell-1}\lambda_{\ell-1})$. We have obtained F-definable maps from $\widetilde{\gamma_1, ..., \gamma_\ell}$ which map p to the generic type of $\delta\log_{\mathbb{G}_m^{\ell-1} \times \mathbb{G}_a}(x_1, ..., x_{\ell-1}, y) = (\widetilde{\lambda}, 1)$ where $\widetilde{\lambda}$ is $\mathbb{Q}$-linearly independent. Since $\tilde{\lambda}$ is $\mathbb{Q}$-linearly independent, by Corollary 3.1.8, the generic type of $\delta\log_{\mathbb{G}_m^{\ell-1} \times \mathbb{G}_a}(x, y) = (\widetilde{\lambda}, 1)$ is weakly $\mathcal{C}$-orthogonal. Note that the the generic type of this equation is $\widetilde{\gamma}(p)$.

Since $\widetilde{\gamma} = (\widetilde{\gamma}_1, ..., \widetilde{\gamma}_\ell)$ is F-definable, there are non-constant $g_1, ..., g_\ell \in F(x_1, ..., x_k)$ such that $\widetilde{g}(a) = (g_1(a), ..., g_\ell(a))$. The same argument as in the proof of Theorem 3.2.2 shows that $\widetilde{\gamma}(\widetilde{g})$ is algebraically independent over F and for all $i = 1, ..., \ell - 1$, $\delta_s(g_i) = \widetilde{\lambda}_i g_i$ and $\delta_s(g_\ell) = 1$. Set $g := g_\ell$.

A similar argument works if $G = \mathbb{G}_m^\ell$ giving $\delta_s(g_\ell) = \lambda_\ell g_\ell$.

Now we construct the remaining $k - \ell$ many g_is. Fix some $a \models p$ such that $\alpha(a) = b$. Since $\pi(b)$ is a tuple of constants, by quantifier elimination, $\pi \circ \alpha(a) = (h_1(a), ..., h_n(a))$ for some $h_1, ..., h_n \in F(x_1, ..., x_k)$ and since p is almost $\mathcal{C}$-internal, $U(p) = \dim(p) = k$. We compute that:

$$\begin{aligned} \ell &= U(a/F\pi(\alpha(a)) \\ &= U(a/F) - U(\pi(\alpha(a))/F) \\ &= k - U(\pi(\alpha(a))). \end{aligned}$$

Since $\pi(\alpha(a))$ is a tuple of constants, its U-rank is equal to its transcendence degree. Thus, a maximal algebraically independent tuple over F from $\pi(\alpha(a))$ has length $k - \ell$. Without loss of generality we may assume that $h_1(a), ..., h_{k-\ell}(a)$ are algebraically independent over F. By genericity of a, $h_1, ..., h_{k-\ell}$ are algebraically independent over F. Since $\widetilde{\gamma}(p)$ is weakly $\mathcal{C}$-orthogonal and $\pi \circ \alpha(p)(\mathcal{U}) \subset \mathcal{C}$, we have that $\widetilde{\gamma}(a) \underset{F}{\downarrow} \pi(\alpha(a))$. Hence, by genericity of a, the rational functions $g_1, ..., g_\ell, h_1, ..., h_{k-\ell}$ are algebraically independent over F. Notice that for all $i = 1, .., k - \ell$, $\delta(h_i(a)) = 0$ since $h_i(a) \in \mathcal{C}$. By genericity of a, $\delta_s(h_i) = 0$. Take $g_{\ell+1}, ..., g_{k-\ell}$ to be $h_1, ..., h_{k-\ell}$.

We have constructed $g_1, ..., g_{k-1}, g \in F(x_1, ..., x_k)$ algebraically independent over F such that $\delta_s(g) = 1$ and for all $i = 1, ..., k - 1$, $\delta_s(g_i) = \lambda_i g_i$ for some $\lambda_i \in F$. Moreover, our proof shows that if $p \perp^w \mathcal{C}$, then all λ_i are non-zero and $\mathbb{Q}$-linearly independent.

A similar argument works if $\delta_s(g) = \lambda_k g$ for some $\lambda_k \in F \setminus \{0\}$.

(2) $\implies$ (1) Let $g_1, ..., g_{k-1}, g$ satisfy condition (2) and suppose that $\delta_s(g) = 1$. Let $\lambda_1, ..., \lambda_{k-1} \in F$ be such that $\delta_s(g_i) = \lambda_i g_i$ for all $1 \leq i \leq k - 1$. Let $\lambda_1, ..., \lambda_\ell$ be a maximal $\mathbb{Q}$-linearly independent set for some $1 \leq \ell \leq k - 1$.

Let q be the generic type of $\delta\log_G(x, y) = ((\lambda_1, ..., \lambda_\ell), 1)$ where $G = \mathbb{G}_m^\ell \times \mathbb{G}_a$, $x = (x_1, ..., x_\ell)$, and y is a singleton. Since $\lambda_1, ..., \lambda_\ell$ are $\mathbb{Q}$-linearly independent, by Corollary 3.1.8, q is weakly $\mathcal{C}$-orthogonal. Since q is $\mathcal{C}$-internal, $U(q) = \dim(q) = \ell + 1$.

Let $h : p \to q$ be the F-definable map given by $h(a) = (g_1(a), ..., g_\ell(a), g(a))$. By genericity of $a \models p$, $g_i(a) \neq 0$ for all $i = 1, ..., \ell$ and $g(a) \neq 0$. Since $q \perp^w \mathcal{C}$ and it is the generic type of $\delta\log_G(x, y) = ((\lambda_1, ..., \lambda_\ell), 1)$, q is isolated by $\delta\log_G(x, y) = ((\lambda_1, ..., \lambda_\ell), 1)$. The usual computation shows that $\delta(g_i(a)) = g_i(a)$ for all $i = 1, ..., \ell$ and $\delta(g(a)) = 1$. Hence, $h : p \to q$ is a surjective F-definable map.

Let $\lambda_j \in \{\lambda_{\ell+1}, ..., \lambda_{k-1}\}$. By assumption on ℓ, there are $\frac{a_1}{b_1}, ..., \frac{a_\ell}{b_\ell} \in \mathbb{Q}$, written in lowest form, such that $\lambda_j = \sum\limits_{i=1}^{\ell} \frac{a_i}{b_i}\lambda_i$. Since $\lambda_i = \frac{\delta_s(g_i)}{g_i}$, the map

$$h_j = \frac{\prod\limits_{i=1}^{\ell} g_i^{(a_i \cdot \prod\limits_{l\neq i} b_l)}}{g_j^{(\prod\limits_{i=1}^{\ell} b_i)}}$$

satisfies $\delta_s(h_j) = 0$, hence $h_j : p \to r$, where r is the generic type of $\mathcal{C}$ over F. By construction, h_j is interalgebraic with g_j over $F(g_1, ..., g_\ell, g)$. By assumption, $g_1, ..., g_k, g$ are algebraically independent over F, hence $g_1, ..., g_\ell, h_{\ell+1}, ..., h_{k-1}, g$ are F-algebraically independent. Since $h_{\ell+1}, ..., h_{k-1}$ are F-algebraically independent, $h_{\ell+1}(a), ..., h_{k-1}(a)$ is an F-algebraically independent tuple of constants. Notice that $h_i(a) \notin F$, by genericity of a. Hence, $(h_{\ell+1}(a), ..., h_{k-1}(a)) \models r^{(k-\ell-1)}$. So $\pi = (h_{\ell+1}, ..., h_{k-1}) : p \to r^{(k-\ell-1)}$ is an F-definable map from p to a tuple of constants. Since q is weakly $\mathcal{C}$-orthogonal, $q \perp^w r^{(k-\ell-1)}$. Recall that $h(a) \models q$. Hence, $U(h(a)/F) + U(\pi(a)/F) = k$. Thus, $(h, \pi) : p \to q \otimes r^{(k-\ell-1)}$ is finite-to-one since $U(p) = U(q \otimes r^{(k-\ell-1)})$. Since q is $\mathcal{C}$-internal and $r^{(k-\ell-1)}$ is $\mathcal{C}$-internal, by Lemma 2.2.23, $q \otimes r^{(k-\ell-1)}$ is almost $\mathcal{C}$-internal. As p is interalgebraic with the product, p is almost $\mathcal{C}$-internal, by Lemma 2.2.23.

The moreover statement is an immediate consequence of combining the result with Theorem 3.2.2. □

3.3 Applications

An application of Theorem 3.2.4 answers Question 7.6 in [21]. Let F be a field of constants and consider the Poizat equations $\delta^2(x) = \delta(x)f(x)$ for some $f \in F(x)$. Freitag, Jaoui, Marker and Nagloo in [21] show that its generic type is almost $\mathcal{C}$-internal when $f(x)$ is a constant. They ask if almost $\mathcal{C}$-internality forces $f(x)$ to be a constant. We prove that this is indeed the case.

Note that the generic type of $\delta^2(x) = \delta(x)f(x)$ is interdefinable with the generic type of the rational vector field $(\mathbb{A}^2, s)$ where $s(x_1, x_2) = (x_1, x_2, x_2, x_2 f(x_1))$.

Key to the application is the use of Laurent series. Recall that for any field F, the field of rational functions $F(x)$ embeds into the field of Laurent series $F((x))$ which consists of all formal sums $\sum\limits_{i=N}^{\infty} a_i x^i$ with $N \in \mathbb{Z}$, $a_i \in F$ and $a_N \neq 0$. Let δ be a derivation on $F(x)$ which extends naturally to $F((x))$.

Proposition 3.3.1. *Let F be an algebraically closed field of constants and let p be the generic type of $(\mathbb{A}^2, s)$ where $s(x_1, x_2) = (x_1, x_2, x_2, x_2 f(x_1))$ for some $f \in F(x)$. Then p is almost $\mathcal{C}$-internal if and only if $f \in F$.*

Proof. Suppose that p is almost $\mathcal{C}$-internal. Since p is non-algebraic, $p \not\perp \mathcal{C}$. By Theorem 3.2.4, there is some $g_1, g_2 \in F(x_1, x_2)$ that are algebraically independent over F such that $\delta_s(g_1) = \lambda g_1$ for some $\lambda \in F$ and $\delta_s(g_2) = 1$ or $\delta_s(g_2) = \lambda g_2$ for some $\lambda \in F$.

Suppose that $\delta_s(g_i) = 0$ for both $i = 1, 2$. Fix $a = (a_1, a_2) \models p$. Then $\delta(g_i(a)) = 0$ for $i = 1, 2$. As g_1, g_2 are algebraically independent over F, it follows that $F(a) \subseteq F(g_1(a), g_2(a))^{\text{alg}} \subseteq \mathcal{C}$ so that $\delta(a_1) = 0$. But, the equations defining s force $\delta(a_1) = a_2$; hence $a_2 = 0$. This contradicts the fact that $\dim(p) = 2$. Therefore, it cannot be the case that both $\delta_s(g_1)$ and $\delta_s(g_2)$ are zero.

Suppose now that $\delta_s(g_2) = 1$. Then

$$
\begin{aligned}
1 &= \frac{\partial g_2}{\partial x_1}(x_1, x_2)\delta_s(x_1) + \frac{\partial g_2}{\partial x_2}(x_1, x_2)\delta_s(x_2) \\
&= \frac{\partial g_2}{\partial x_1}(x_1, x_2)x_2 + \frac{\partial g_2}{\partial x_2}(x_1, x_2)x_2 f(x_1)
\end{aligned}
$$

Write $g_2 = \sum\limits_{i=N}^{\infty} a_i x_2^i$ as the Laurent series in x_2 over $F(x_1)$ with N the least integer such that $a_N \neq 0$. An easy computation shows that if $N = 0$, then $\frac{\partial g_2}{\partial x_1} x_2 + \frac{\partial g_2}{\partial x_2} x_2 f$ has no constant term as a Laurent series in x_2, contradicting the fact that it was found to be equal to 1 above. So, $N \neq 0$. In that case, we get that $1 = \sum\limits_{i=N}^{\infty} \frac{da_i}{dx_1} x_2^{i+1} + \sum\limits_{i=N} i a_i x_2^i f$. The coefficient of x_2^N on the right hand side is $N a_N f$, so $N a_N f = 0$. Since $N \neq 0$ and $a_N \neq 0$, this implies that $f = 0$. In particular, $f \in F$, as desired. Hence, we may assume that $\delta_s(g_2) \neq 1$.

The only remaining possibility is that for either $i = 1, 2$, $\delta_s(g_i) = \lambda g_i$ for some non-zero $\lambda \in F$. Again, we expand $\delta_s(g_i)$ to get

$$\lambda g_i = \frac{\partial g_i}{\partial x_1} x_2 + \frac{\partial g_i}{\partial x_2} x_2 f.$$

Write $g_i = \sum_{j=N}^{\infty} a_j x_2^j$ as the Laurent series in x_2 over $F(x_1)$ with N the least integer such that $a_N \neq 0$. If $N = 0$, an easy computation shows that $0 = \lambda a_N$, contradicting that $\lambda \neq 0$ and $a_N \neq 0$. So, $N \neq 0$. In that case, we get that $\lambda \sum_{j=N}^{\infty} a_j x_2^j = \frac{d}{dx_1} a_j x_2^{j+1} + \sum_{j=N} j a_j x_2^j f$. Comparing coefficients of x_2^N on the left and right hand sides, we get that $N a_N f = \lambda a_N$. Hence, $f = \frac{\lambda}{N} \in F$, as desired.

By Theorem 3.2.4, we have shown that if p is (almost) $\mathcal{C}$-internal, then $f \in F$. The reverse direction is [21, Section 7.2]. $\square$

Another application of the theorem is to show that the generic type of certain Lotka-Volterra systems is not almost $\mathcal{C}$-internal. The Lotka-Volterra system is more closely examined in Chapter 5.

Proposition 3.3.2. *Let F be an algebraically closed field of constants containing non-zero $a, b, c \in F$. Let p be the generic type over F of the system of differential equations* $\begin{cases} \delta(x) = x(ay+b) \\ \delta(y) = y(cx+b). \end{cases}$ *Then p is non-orthogonal to the constants, but not almost $\mathcal{C}$-internal.*

Proof. Note that p is the generic type over F of the vector field $(\mathbb{A}^k, s)$ where $s(x,y) = (x, y, x(ay+b), y(cx+b))$. First we show that p is weakly $\mathcal{C}$-orthogonal and then apply Theorem 3.2.4. We show that the only $g \in F(x_1, x_2)$ such that $\frac{\partial h}{\partial x_1} x(ay+b) + \frac{\partial h}{\partial x_2} y(cx+b) = 0$ are constant functions.

We can write $g \in F(x_1, x_2)$ as the Laurent series $g = \sum_{i=0}^{\infty} a_i x_2^i$ with N the least integer such that $a_N \neq 0$ and $a_i \in F(x_1)$. Inputting g in the expression on the left hand side, the coefficient for x_2^N is $N a_N (c x_1 + b) + \frac{\partial a_N}{\partial x_1} b x_1$.

Comparing coefficients for x_2^N, we get $N a_N (c x_1 + b) + \frac{\partial a_N}{\partial x_1} b x_1 = 0$. If $N \neq 0$, since $a_N \neq 0$, $\frac{\frac{\partial a_N}{\partial x_1}}{a_N} = -N(\frac{1}{x_1} + \frac{c}{b})$ which has no solution in $F(x_1)$. If $N = 0$, then $\frac{\partial a_N}{\partial x_1} b x_1 = 0$. Since $b \neq 0$, $\frac{\partial a_N}{\partial x_1} = 0$, and hence $a_N \in F$. For any $i \geq 1$, comparing the coefficients of x_2^i gives $\frac{\partial a_{i-1}}{\partial x_1} a x_1 + \frac{\partial a_i}{\partial x_1} b x_1 + i a_i (c x_1 + b) = 0$.

When $i = 1$, we have $\frac{\partial a_0}{\partial x_1} = 0$, hence $\frac{\partial a_1}{\partial x_1} b x_1 + a_1 (c x_1 + b) = 0$. If $a_1 \neq 0$, then $\frac{\frac{\partial a_1}{\partial x_1}}{a_1} = \frac{-(c x_1 + b)}{b x_1}$, which has no solution in $F(x_1)$. Hence, $a_1 = 0$. Suppose we have shown that $a_i = 0$. If $a_{i+1} \neq 0$, then we get $\frac{\frac{\partial a_{i+1}}{\partial x_1}}{a_{i+1}} = \frac{-(c x_1 + b)}{b x_1}$ which has no solution in $F(x_1)$.

Hence, $a_{i+1} = 0$ and $g = a_0 \in F$. Thus, the only solutions to $\frac{\partial h}{\partial x_1} x(ay+b) + \frac{\partial h}{\partial x_2} y(cx+b) = 0$ in $F(x_1, x_2)$ are constant functions.

Now suppose, for sake of contradiction, that $p \not\perp^w \mathcal{C}$. By Fact 2.2.15, there is some $(u, v) \models p$ and $d \in \mathcal{C}$ with $d \in \operatorname{dcl}(Fu, v) \setminus F$. Hence, there is some $g' \in F(x_1, x_2) \setminus F$ such that $g(u, v) = d$. Thus, $0 = \delta(g'(u, v)) = \frac{\partial g'}{\partial x_1}(u, v) u(av + b) + \frac{\partial g'}{\partial x_2}(u, v) v(cu + b)$. By genericty of u, v, we have that $0 = \frac{\partial g'}{\partial x_1} x_1(ax_2 + b) + \frac{\partial g'}{\partial x_2} x_2(cx_1 + b)$. But, there is no non-constant solution to the above equation, a contradiction. Hence, $p \perp^w \mathcal{C}$.

Since $p \perp^w \mathcal{C}$, to show that p is not almost $\mathcal{C}$-internal using Theorem 3.2.4, we need only show that if $g_1, g_2 \in F(x_1, x_2)$ are non-constant solutions to

$$\frac{\partial h}{\partial x_1} x(ay + b) + \frac{\partial h}{\partial x_2} y(cx + b) = \begin{cases} 1 \\ \lambda_h h \end{cases}$$

for some $\mathbb{Q}$-linearly independent $\lambda_{g_1}, \lambda_{g_2} \in F$, then g_1 and g_2 are F-algebraically dependent.

Consider the equation $\frac{\partial h}{\partial x_1} x_1(ax_2 + b) + \frac{\partial h}{\partial x_2} x_2(cx_1 + b) = 1$ and compare the coefficients for x_2^N. If $N \neq 0$, then $Na_N(cx_1 + b) + \frac{\partial a_N}{\partial x_1} bx_1 = 0$ which we previously showed has no solution for a_N in $F(x_1)$. If $N = 0$, then we have $\frac{\partial a_N}{\partial x_1} bx_1 = 1$, but $\frac{\partial a_N}{\partial x_1} = \frac{1}{bx_1}$ has no solutions in $F(x_1)$.

Finally, consider the equation $\frac{\partial h}{\partial x_1} x(ay + b) + \frac{\partial h}{\partial x_2} y(cx + b) = \lambda g$ for $\lambda \neq 0$. Comparing coefficients of the x_2^N term, we obtain the formula $\frac{\frac{\partial a_N}{\partial x_1}}{a_N} = \frac{1}{b}(\frac{\lambda - N}{x_1} - cN)$. A rational function can be a logarithmic derivative only if it has no zeroes and only simple poles with integer residues (see for example [29, Example 2.20]), hence $N = 0$ and $\frac{\lambda}{b} \in \mathbb{Z}$. Thus $\frac{\frac{\partial a_0}{\partial x_1}}{a_0} = \frac{\lambda}{bx_1}$, and so $a_0 = d_0 x_1^{\frac{\lambda}{b}}$ for some $d_0 \in F \setminus \{0\}$. Note that for any $\alpha \in F$, $g = \alpha(cx_1 - ax_2)^{\frac{\lambda}{b}}$ is a solution to $\frac{\partial h}{\partial x_1} x(ay+b) + \frac{\partial h}{\partial x_2} y(cx+b) = \lambda g$. Suppose $h \in F(x_1, x_2)$ is another solution. Fix $(u, v) \models p$. Then $\delta(\frac{g(u,v)}{h(u,v)}) = \frac{\delta(g(u,v))h(u,v) - \delta(h(u,v))g(u,v)}{h(u,v)^2} = \frac{\lambda g(u,v)h(u,v) - \lambda g(u,v)h(u,v)}{h(u,v)^2} = 0$. Hence, $\delta(\frac{g(u,v)}{h(u,v)}) \in \mathcal{C} \cap F(u, v)$. Since $p \perp^w \mathcal{C}$, by Fact 2.2.15, $\frac{g(u,v)}{h(u,v)} \in \mathcal{C} \cap F(u, v) = \mathcal{C} \cap F = F$. By genericity of (u, v), $\frac{g}{h} \in F$. Hence, no two solutions of $\frac{\partial h}{\partial x_1} x(ay + b) + \frac{\partial h}{\partial x_2} y(cx + b) = \lambda g$ are F-algebraically independent. We have shown that p is not almost $\mathcal{C}$-internal by Theorem 3.2.4.

Finally, we show that p is non-orthogonal to the constants. Let $g = (cx_1 - ax_2)^{\frac{\lambda}{b}}$. Then for any $(u, v) \models p$, $\operatorname{tp}(g(u, v)/F)$ is interdefinable with the generic type of the logarithmic-differential equation $\delta\log_{\mathbb{G}_m}(z) = \lambda$. Thus, $\operatorname{tp}(g(u, v)/F)$ is almost $\mathcal{C}$-internal. Let $L \supseteq F$, $g(u', v') \models \operatorname{tp}(g(u, v)/F)|_L$ and $c_1, ..., c_n \in \mathcal{C}$ be such that $g(u', v') \in \operatorname{acl}(Lc_1, ..., c_n)$. Since

$g(u',v') \underset{F}{\perp} L$ and $g(u',v') \notin \operatorname{acl}(F)$, $b \underset{L}{\not\perp} c_1, ..., c_n$. We have $g : \operatorname{tp}(u,v/F) \to \operatorname{tp}(g(u,v)/F)$ and the realizations of $\operatorname{tp}(g(u,v)/F)|_L$ are a subset of the realizations for $\operatorname{tp}(g(u,v)/F)$. Thus, there is some $(u',v') \models \operatorname{tp}(u,v/F)$ such that $g(u',v') \in \operatorname{dcl}(Fu'v')$. Since we have that $g(u',v') \underset{F}{\perp} L$, $(u',v') \underset{F}{\perp} L$. We have that $(u',v') \models \operatorname{tp}(u,v/F)|_L$ and $u'v' \underset{L}{\not\perp} c_1, ..., c_n$, witnessing that $p = \operatorname{tp}(u,v) \not\perp \mathcal{C}$. □

This gives us another example of a type where the shortest possible $\mathcal{C}$-analysis is of length two. The first example we saw comes from the logarithmic differential equation, see Example 2.2.35.

Lemma 3.3.3. *The type p in Proposition 3.3.2 is 2-step analyzable in $\mathcal{C}$. In particular, for any $(u,v) \models p$, $((u,v), \frac{u}{a} - \frac{v}{c}, 1)$ is a $\mathcal{C}$-analysis.*

Proof. Fix $(u,v) \models p$. Clearly $\frac{u}{a} - \frac{v}{c} \in \operatorname{dcl}(Fu,v)$. Note that $\delta(\frac{u}{a} - \frac{v}{c}) = \frac{\delta(u)}{a} - \frac{\delta(v)}{c} = \frac{u(ay+b)}{a} - \frac{v(cu+b))}{c} = b(\frac{u}{a} - \frac{v}{c})$. Thus, $q = \operatorname{stp}(\frac{u}{a} - \frac{v}{c}/F)$ is the generic type of a linear differential equation and is $\mathcal{C}$-internal by Example 2.2.18. Finally, $\operatorname{stp}(u,v/F(\frac{u}{a} - \frac{v}{c}))$ is interdefinable with the strong type of the generic type of the equation $\delta(y) = ay^2 + (b + acz)y$ where $z := \frac{u}{a} - \frac{v}{c}$. The map $y \mapsto \frac{1}{y}$ witnesses that $\operatorname{tp}(u,v/Fz)$ is interdefinable with the generic type p' of the linear differential equation $0 = \delta(w) + (a + b + acz)w$. By Example 2.2.18, p' is $\mathcal{C}$-internal. By interalgebracity over $\operatorname{acl}(Fz)$ and Lemma 2.2.23, $\operatorname{stp}(u,v/F(\frac{u}{a} - \frac{v}{c}))$ is almost $\mathcal{C}$-internal. Thus, $((u,v), \frac{u}{a} - \frac{v}{c}, 1)$ is a $\mathcal{C}$-analysis. □

Chapter 4

Splitting two-step analyses

As in Chapter 3, we continue to search for algebraic criteria for (almost) internality to the constants. This time, we focus on algebraic D-varieties (so possibly nonautonomous and possibly not on $\mathbb{A}^k$) that admit a two-step analysis in the constants. So, the general problem we address in this chapter is the following: given a type which is two-step analyzable in the constants, when is it actually almost internal to the constants? More precisely, we are given a fibration $f : p \to q$ such that q is $\mathcal{C}$-internal and the fibers of f are $\mathcal{C}$-internal. How can we tell if p itself is (almost) $\mathcal{C}$-internal?

Jin studied this question in his thesis [44], in the special case when f is the differential logarithm $\delta\log_{\mathbb{G}_m}$ and q is minimal. He conjectured that, in this case, p is almost $\mathcal{C}$-internal if and only if f *almost splits* in the sense that p is interalgebraic over F with $s \otimes q$ for some almost $\mathcal{C}$-internal type s. He proved his conjecture when the binding group of p is either $\mathbb{G}_m(\mathcal{C})$ or $\mathbb{G}_a(\mathcal{C})$. Jin and Moosa [45] later found a counter example when the binding group of q is $\mathrm{PSL}_2(\mathcal{C})$ and proved the conjecture in all other cases — assuming still that q is minimal and $f = \delta\log_{\mathbb{G}_m}$.

We generalize these results here in several ways. We drop the assumption that q is minimal and assume only that the binding group of q is linear and nilpotent. We also weaken the assumption that $f = \delta\log_{\mathbb{G}_m}$ by asking only that the binding group of the fibers are multiplicative tori. Here is our main result that appears as Theorem 4.2.6 below.

Theorem 4A. *Let F be an algebraically closed differential field and suppose $f : p \to q$ is a fibration of stationary types where q is $\mathcal{C}$-internal, p is weakly $\mathcal{C}$-orthogonal and the fibers of f are $\mathcal{C}$-internal. Assume that the binding group of q is linear and nilpotent and that the binding group of the fibers are definably isomorphic to $\mathbb{G}_m^\ell(\mathcal{C})$ for some $\ell \geq 1$.*

Then p is almost $\mathcal{C}$-internal if and only if $f : p \to q$ almost splits.

As an application, we restrict our attention back to the case when $f = \delta\log_{\mathbb{G}_m}$ and give an explicit algebraic criterion for p to be almost $\mathcal{C}$-internal in the spirit of Rosenlicht [68] and the theorem above.

Theorem 4B. *Let F be an algebraically closed differential field and $q \in S_1(F)$ be the generic type over F of differential equation $\delta^n(x) = f(x, \delta(x), ..., \delta^{n-1}(x))$ for some $n \geq 1$ and $f \in F(x_0, ..., x_{n-1})$ a rational function. Assume that q is weakly $\mathcal{C}$-orthogonal and $\mathcal{C}$-internal with a linear nilpotent binding group. Let $p = \delta\log^{-1}_{\mathbb{G}_m}(q)$. Then the following are equivalent,*

1. *p is almost $\mathcal{C}$-internal*
2. *there is a non-zero $h \in F(x_0, ..., x_n)$, a constant $e \in F$ and a non-zero integer k such that $(kx_0 - e)h = h^\delta + \sum_{i=0}^{n-2} \frac{\partial h}{\partial x_i} x_{i+1} + \frac{\partial h}{\partial x_{n-1}} f$.*

This appears as Corollary 4.3.4 below. Here, by $\delta\log^{-1}_{\mathbb{G}_m}(q)$ we mean the (unique) complete type of an element $a \in \mathcal{U}$, over F, such that $\frac{\delta(a)}{a}$ realises q and $a \notin \mathrm{acl}(F, \frac{\delta(a)}{a})$ (see section 2 of [45]). The contents of this chapter are based on joint work with Léo Jimenez appearing in [20]

Again, throughout this chapter, we work in a sufficiently saturated model $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$ with field of constants $\mathcal{C}$.

4.1 Preliminaries on definable algebraic groups

We begin with some preliminaries on algebraic groups which are essential in our proofs. A good reference for algebraic groups is [38]. By a *unipotent group* we mean an algebraic group that is isomorphic to an algebraic subgroup of U_n, the group of $n \times n$ uppertriangular matrices with 1's along the diagonal. Every unipotent group is isomorphic to an algebraic subgroup of $\mathbb{G}_a^k$ for some k. A *torus* is an algebraic group that is isomorphic to $\mathbb{G}_m^k$ for some k and a *diagonalizable group* is an algebraic group that isomorphic to an algebraic subgroup of $\mathbb{G}_m^k$ for some k. If a torus is defined over some algebraically closed differential field F, then this isomorphism is defined over F (see for example [38, Section 34.3]). Every connected diagonalizable group is a torus.

A fact we frequently use is that there is no non-trivial regular function (a function with non-trivial image) from a unipotent group to a torus. Briefly, this is because every element of a torus is semisimple, but a rational function takes a unipotent element to a unipotent element. The only semisimple element of an algebraic group which is also unipotent is the identity. Hence, the image of a unipotent group in a torus is trivial.

Recall the multiplicative *Jordan decomposition*, which is a consequence of the Lie-Kolchin Theorem (see [[38], Chapter 19] for a proof):

Fact 4.1.1. *Let G be a connected linear nilpotent algebraic group. There is a split short exact sequence of algebraic groups*

$$1 \to G_u \to G \to T \to 1$$

such that

1. *T is a torus and $G_u \trianglelefteq G$ is the unique maximal unipotent subgroup of G,*
2. *T and G_u commute, and*
3. *$T \cap G_u = \{1\}$.*

We list some definable versions of well-known facts about algebraic groups. We are interested in definable groups that are definably isomorphic, potentially over additional parameters, to the $\mathcal{C}$-points of algebraic groups (usually linear algebraic groups) over the constants.

Lemma 4.1.2. *Suppose (F, δ) is an algebraically closed differential field and G is a connected nilpotent definable group over F that is definably isomorphic to the $\mathcal{C}$-points of a linear algebraic group.*

1. *There exists a unique maximal F-definable subgroup of G with the property that it is definably isomorphic to the $\mathcal{C}$-points of a unipotent linear algebraic group. We denote this F-definable subgroup by G_u and call it the F-definable unipotent part of G.*
2. *There exists a unique maximal F-definable subgroup of G with the property that it is definably isomorphic to the $\mathcal{C}$-points of an algebraic torus. We denote this F-definable subgroup by T and call it the F-definable torus part of G.*
3. *The group G is F-definably isomorphic to $G_u \times T$.*
4. *Suppose G acts F-definably on a set X and $a \in X$. Then $G_u \cdot a \cap T \cdot a = \{a\}$ where $H \cdot a$ denotes the orbit of a under H.*

Proof. Parts 1,2 and 3 are just definable versions of Fact 4.1.1, but we give some details. We first prove parts (1), (2) and (3). Let $\widehat{G}$ be a connected linear nilpotent algebraic group and $f : G \to \widehat{G}(\mathcal{C})$ be an F-definable isomorphism. By [38, Proposition 19.2], $\widehat{G} = \widehat{G}_u \times \widehat{T}$ where $\widehat{G}_u$ is the maximal unipotent subgroup of $\widehat{G}$ and $\widehat{T}$ is the maximal torus of $\widehat{G}$. Let $G_u = f^{-1}(\widehat{G}_u(\mathcal{C}))$ and $T = f^{-1}(\widehat{T}(\mathcal{C}))$.

First we show that G_u and T are maximal subgroups of G that are definably isomorphic to the $\mathcal{C}$-points of a unipotent algebraic group and a torus, respectively. Let $H \leq G$ be a definable subgroup and let $g : H \to \widehat{H}(\mathcal{C})$ be a definable isomorphism where $\widehat{H}$ a linear unipotent algebraic group. Since $\mathcal{C}$ is a pure field and for any $a \in H$ and $b \in G$, $g(a), f(b) \in \mathcal{C}$, we have that f, g are given by rational functions over F. Hence, $f \circ g^{-1}$ is a morphism of algebraic groups. Thus $f(H)$ is a unipotent algebraic subgroup of $\widehat{G}(\mathcal{C})$. By Fact 4.1.1, $\widehat{H}(\mathcal{C}) \leq \widehat{G}_u(\mathcal{C})$, and so $H \leq G_u$. We have shown that if H is definably isomorphic to the $\mathcal{C}$-points of a unipotent group, then $H \leq G_u$, hence G_u is a maximal group with respect to this property. If H is another maximal subgroup of G which is definably isomorphic to the $\mathcal{C}$-points of $\widehat{H}$, a unipotent algebraic group, then $H \leq f^{-1}(\widehat{H}(\mathcal{C}))$. If $H \neq f^{-1}(\widehat{H}(\mathcal{C}))$, then H is a proper subgroup of $f^{-1}(\widehat{H}(\mathcal{C}))$, which is definably isomorphic to the $\mathcal{C}$-points of a unipotent group. This contradicts maximality of H. Hence, $H = f^{-1}(\widehat{H}(\mathcal{C}))$. By uniqueness of $\widehat{G_u}$, $H = f^{-1}(\widehat{H}(\mathcal{C})) = f^{-1}(\widehat{G}_u(\mathcal{C})) = G_u$.

For F-definability of G_u, note that any F-conjugate of G_u is a maximal definable subgroup of G, which is definably isomorphic to the $\mathcal{C}$-points of a unipotent algebraic group. By uniqueness of G_u, any F-conjugate is then equal to G_u. Hence, G_u is fixed by $\mathrm{Aut}_F(\mathcal{U})$, and so is F-definable. Since $\widehat{G}_u$ is normal in $\widehat{G}$ and f is an isomorphism of groups, we have that G_u is a normal subgroup of G.

Similarly, T is the unique maximal subgroup of G that is definably isomorphic to a torus, and hence is fixed by $\mathrm{Aut}_F(\mathcal{U})$. So T is F-definable. Since $\widehat{T}$ is a normal subgroup of $\widehat{G}$ (see for example [38, Proposition 19.4]), $T \trianglelefteq G$. Finally, by maximality of G_u and T, $G = G_u \times T$.

For part (4), let $\pi_1 : G \to G_u$ and $\pi_2 : G \to T$ be projection maps. Let $H < G_u \times T$ be a definable subgroup. such that $\pi_1|_H$ and $\pi_2|_H$ are surjective. Let $N_1 = \ker(\pi_2|_H)$ and $N_2 = \ker(\pi_1|_H)$. By Lemma 2.2.46, there is a definable isomorphism between $\pi_1(H)/\pi_1(N_1)$ and $\pi_2(H)/\pi_2(N_2)$. Since quotients of unipotent groups are unipotent and $\pi_1(H) < G_u$, we have that $\pi_1(H)/\pi_1(N_1)$ is unipotent. Since $\pi_2(H)/\pi_2(N_2)$ is a quotient of a torus, if there is a non-trivial isomorphism between $\pi_1(H)/\pi_1(N_1)$ and $\pi_2(H)/\pi_2(N_2)$, then they must both be trivial groups. Hence, $\pi_1(H) = \pi_1(N_1)$ and $\pi_2(H) = \pi_2(N_2)$. Thus $H = \pi_1(H) \times \pi_2(H)$.

The action of G on X is definably isomorphic to its action on G/H, where H is the stabilizer of X. Since $H = \pi_1(H) \times \pi_2(H)$, the action on G/H is definably isomorphic to

$G_u/\pi_1(H) \times T/\pi_2(H)$. Thus the action is definably isomorphic to the action of $G = G_u \times T$ on $G_u/\pi_1(H) \times T/\pi_2(H)$. In particular, for any $a \in X$, we have $G_u \cdot a \cap T \cdot a = \{a\}$. $\square$

The following appears as Lemma 2.18 in [20], providing an F-definably split short exact sequence of groups.

Fact 4.1.3. *Suppose (F, δ) is an algebraically closed differential field, and G is a connected definable group over F that is definably isomorphic to the $\mathcal{C}$-points of an algebraic group. Suppose there is a short exact sequence of F-definable homomorphisms*

$$1 \to K \to G \xrightarrow{\pi} G/K \to 1,$$

where K is a definably isomorphic to the $\mathcal{C}$-points of an algebraic subgroup of an algebraic torus, and G/K is definably isomorphic to the $\mathcal{C}$-points of a linear nilpotent algebraic group. Then G is definably isomorphic to the $\mathcal{C}$-points of a linear nilpotent algebraic group.

Moreover, if K is connected and the torus parts of G and G/K are F-definably isomorphic to the $\mathcal{C}$-points of tori, then the short exact sequence F-definably splits.

4.2 Splitting fibrations

In this section, we prove Theorem 4A from the introduction. The hard direction of this equality is that almost $\mathcal{C}$-internality implies almost splitting. First we consider when p is outright $\mathcal{C}$-internal and provide some sufficient conditions for splitting of the short exact sequence of binding groups induced by $f : p \to q$ to force splitting of the fibration $f : p \to q$. Let us first be more precise about what we mean for a fibration to (almost) split.

Definition 4.2.1. Suppose $f : p \to q$ is a fibration of stationary types over A. Then we say that f *splits* if for all (equivalently some) $a \models p$, there is some b which is independent from $f(a)$ over F, and $\mathrm{dcl}(Aa) = \mathrm{acl}(Af(a)b)$. (In particular, p is interdefinable over A with $\mathrm{tp}(b/A) \otimes q$).

We say that f *almost splits* if the above holds with acl in place of dcl.

We are primarily interested in the case when $f : p \to q$ splits and $\mathrm{tp}(b/A)$ is almost $\mathcal{C}$-internal.

Recall from Subsection 2.2.5, when p is internal to the constants, $q \in S(A)$ and $f : p \to q$ is a fibration, f induces a group homomorphism $\widetilde{f} : \mathrm{Aut}_A(p/\mathcal{C}) \to \mathrm{Aut}_A(q/\mathcal{C})$. Hence, we obtain a short exact sequence

$$1 \to K \to \mathrm{Aut}_A(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_A(q/\mathcal{C}) \to 1.$$

It is natural to ask if splitting of the short exact sequence implies that $f : p \to q$ splits. When q is fundamental and weakly $\mathcal{C}$-orthogonal this is indeed the case. The proof of this which appears in [20] uses definable groupoids (see [42]). Here we use the ideas from the application of groupoids without needing the full groupoid machinery.

Lemma 4.2.2. *Suppose (F, δ) is an algebraically closed differential field. Let $p, q \in S(F)$ and $f : p \to q$ be a fibration of stationary $\mathcal{C}$-internal types. Suppose that q is weakly $\mathcal{C}$-orthogonal. If the induced short exact sequence*

$$1 \to K \to \mathrm{Aut}_F(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C}) \to 1$$

F-definably splits, then there is $t \in S(F)$ and a definable function $\pi : p \to s$ such that q is weakly orthogonal to $\pi(p)$. Moreover, if q is fundamental, then $f : p \to q$ splits. In particular, p is interdefinable with $t \otimes q$.

Proof. Suppose that $s : \mathrm{Aut}_F(q/\mathcal{C}) \to \mathrm{Aut}_F(p/\mathcal{C})$ is an F-definable section map witnessing the splitting of the short exact sequence. We denote the fibers of $f : p \to q$ by $p_{f(a)} = \mathrm{tp}(a/Ff(a))$. Define an equivalence relation E on $p(\mathcal{U})$ by $E(a, b)$ if and only if $s(\sigma)|_{p_{f(a)}}(a) = b$ for some $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ where $\sigma(f(a)) = f(b)$. By working with the A-definable avatar of the binding group, and its deniable action, one can see that E is an A-definable equivalence relation. Let $\pi : p \to r$ be the quotient map induced by E. Since $\pi(a) \in \mathrm{dcl}(Fa)$ and p is $\mathcal{C}$-internal, $\pi(p)$ is $\mathcal{C}$-internal by Lemma 2.2.23.

Recall that in Fact 2.2.40, we saw that the $\mathrm{Aut}_F(q/\mathcal{C})$ acts transitivity on $q(\mathcal{U})$ if and only if for all $b \models q$ and tuples $\bar{c} \subset \mathcal{C}$, $b \mathop{\smile\!\!\!\!\!\!\!\;|}_{F} \bar{c}$. We can similarly show that if the subgroup $\mathrm{Aut}_F(q/\mathcal{C}, r)$ acts transitively on $q(\mathcal{U})$, then $q \perp^w r$. First we note that this group does act transitively on the set of realizations of q.

For transitivity, fix $b, b' \models q$. Since $q \perp^w \mathcal{C}$, by Fact 2.2.40 there is some $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\sigma(b) = b'$. Let $\widehat{\sigma}$ be some extension of σ to $\mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$. To show that $\widehat{\sigma}|_q \in \mathrm{Aut}_F(q/\mathcal{C}, r)$, we show that $\widehat{\sigma}$ fixes each E-class of p. Fix $a \models p$ such that $f(a) = b$. As the short exact sequence splits, $\hat{\sigma}|_p = s(\hat{\sigma}|_q) \circ \tau$ for some $\tau \in K = \ker(\widetilde{f})$. Hence, τ lifts to some $\widehat{\tau} \in \mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$ and $\widetilde{f}(\tau) = \widetilde{f}(\widehat{\tau}|_p) = \mathrm{id}$. Hence,

$$\begin{aligned}\widehat{\sigma}|_q(f(\tau(a))) &= \widehat{\sigma}|_q(\widehat{\tau}|_q(f(a))) \text{ as } f \text{ is } F\text{-definable} \\ &= \widehat{\sigma}|_q(\widetilde{f}(\widehat{\tau}|_p)(f(a))) \\ &= \widehat{\sigma}|_q(f(a)) \\ &= f(\widehat{\sigma}(a)) \text{ as } f \text{ is } F\text{-definable.}\end{aligned}$$

Since $s(\widehat{\sigma}|_q)(\tau(a)) = \widehat{\sigma}|_p(a)$ and $\widehat{\sigma}|_q(f(\tau(a))) = f(\hat{\sigma}|_p(a))$, by definition $E(a, \widehat{\sigma}(a))$. Hence, σ fixes the E-class of a, and so $\widehat{\sigma}(\pi(a)) = \pi(a)$ showing that $\widehat{\sigma}|_r = \mathrm{id}$. So $\widehat{\sigma}|_q \in \mathrm{Aut}_A(q/\mathcal{C}, r)$ and $\widehat{\sigma}|_q(b) = b'$. Thus, $\mathrm{Aut}_F(q/\mathcal{C}, r)$ acts transitively on $q(\mathcal{U})$. To show the independence condition, since $\mathrm{Aut}_F(q/\mathcal{C}, r)$ acts transitively on $q(\mathcal{U})$, fix $b \models q$ and $e \models r$. Let $b' \models q$ be such that $b' \underset{F}{\downarrow} e$. Since the binding group $\mathrm{Aut}_F(q/\mathcal{C}, r)$ acts transitively on $q(\mathcal{U})$, there is some $\sigma \in \mathrm{Aut}_F(q/\mathcal{C}, r)$ such that $\sigma(b') = b$. Since e is fixed by σ, we have that $b \underset{A}{\downarrow} e$. In particular, $q \perp^w r$.

For the moreover statement, assume that q is also fundamental. Fix some $a \models p$. Since $q \perp^w r$, $f(a) \underset{F}{\downarrow} \pi(a)$. We conclude that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$. Since $f : p \to q$ and $\pi : p \to s$ are F-definable, $\mathrm{dcl}(Ff(a)\pi(a)) \subseteq \mathrm{dcl}(Fa)$. So we prove the reverse inclusion. Let $\alpha \in \mathrm{Aut}_{Ff(a)\pi(a)}(\mathcal{U})$. Then $\alpha(a) \models \mathrm{tp}(a/F\pi(a))$, meaning that $E(a, \alpha(a))$. Hence, there exists $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\sigma(f(a)) = f(\alpha(a))$ and $s(\sigma)(a) = \alpha(a)$. On the other hand, $\sigma(f(a)) = f(a)$ and since q is fundamental, σ acts freely (see Fact 2.2.40). Thus, $\sigma = \mathrm{id}$. Since s is a morphism of groups, $s(\sigma) = s(\mathrm{id}) = \mathrm{id}$. Thus, the orbit of a under $\mathrm{Aut}_{Ff(a)\pi(a)}(\mathcal{U})$ is a singleton, showing that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$.

We have shown that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$ with $f(a) \underset{F}{\downarrow} \pi(a)$. By definition, $f : p \to q$ splits. $\square$

The assumption that q is fundamental can always be obtained by replacing q with a Morley product $q^{(n)}$, for sufficiently large n (see for example [41, Fact 2.4]). However, the lemma would then give that $(f, ..., f) : p^{(n)} \to q^{(n)}$ splits instead of $f : p \to q$ splits, so the lemma alone is not enough to obtain splitting of f.

The key use of q being fundamental (and weakly $\mathcal{C}$-orthogonal) is that for every $a, b \models q$, there is a unique $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\sigma(a) = b$. This means that $s(\sigma)$ should map the fiber above a to the fiber above b, where $s : \mathrm{Aut}_F(q/\mathcal{C}) \to \mathrm{Aut}_F(p/\mathcal{C})$ is the F-definable section map witnessing splitting of the short exact sequence. We can replace the assumption of q being fundamental with the following related assumption:

Definition 4.2.3. Let (F, δ) be an algebraically closed differential field and $f : p \to q$ be a fibration of $\mathcal{C}$-internal types $p, q \in S(F)$. Suppose that q is weakly $\mathcal{C}$-orthogonal and the short exact sequence

$$1 \to K \to \mathrm{Aut}_F(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C}) \to 1$$

F-definably splits with a section map s. Let H be a definable subgroup of $\mathrm{Aut}_F(q/\mathcal{C})$. We say that H satisfies $(\bigstar)$ if for any $b \models q$ and $\sigma \in H$ where $\sigma(b) = b$, then $s(\sigma)$ restricted to the fiber of f above b is the identity.

While this definition is quite technical, it is satisfied by some natural conditions from algebraic groups. Note that binding groups for $\mathcal{C}$-internality are always definably isomorphic to the $\mathcal{C}$-points of algebraic groups. We therefore say that a binding group is linear, or unipotent etc., to mean that the corresponding algebraic group is.

Example 4.2.4. Let (F, δ) be an algebraically closed differential field and $f : p \to q$ be a fibration of stationary $\mathcal{C}$-internal types $p, q \in S(F)$. Suppose that q is weakly $\mathcal{C}$-orthogonal and the short exact sequence

$$1 \to K \to \mathrm{Aut}_F(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C}) \to 1$$

F-definably splits with a section map s.

1. If H is a central subgroup of $\mathrm{Aut}_F(q/\mathcal{C})$, then H satisfies $(\bigstar)$.

2. If H is a subgroup of $\mathrm{Aut}_F(q/\mathcal{C})$ which is definably isomorphic to the $\mathcal{C}$-points of a linear unipotent algebraic group and the binding group of the fibers of f are diagonalizable, then H satisfies $(\bigstar)$.

3. If H is a subgroup of $\mathrm{Aut}_F(q/\mathcal{C})$ which is definably isomorphic to the $\mathcal{C}$-points of a linear nilpotent algebraic group and the binding group of the fibers of f are diagonalizable, then H satisfies $(\bigstar)$.

Proof. 1. Since $q \perp^w \mathcal{C}$, its binding group acts transitively by Fact 2.2.40. Fix $b \models q$. Hence, every stabilizer is conjugate to the stabilizer of b. Thus the stabilizers intersected with H are all conjugate. Then, for any σ in the stabilizer of b intersected with H, we have that for every $b' \models q$, there is some $\tau \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\tau \circ \sigma \circ \tau^{-1}$ is in the stabilizer of b'. Since H is central, $b' = \tau \circ \sigma \circ \tau^{-1}(b') = \sigma(b')$. Hence, for any $\sigma \in H$, if $\sigma(b) = b$, then $\sigma = \mathrm{id}$. So, $s(\sigma) = s(\mathrm{id}) = \mathrm{id}|_p$ which is condition $(\bigstar)$.

2. Every morphism from a unipotent group to a diagonalizable group has trivial image. Hence, for any $b \models q$ and $\sigma \in H$ such that $\sigma(b) = b$, $s(\sigma) \in \mathrm{Aut}_{Ff(a)}(\mathrm{tp}(a/Ff(a))/\mathcal{C})$. So $s : G \to \mathrm{Aut}_{Ff(a)}(\mathrm{tp}(a/Ff(a))/\mathcal{C})$, where $G = \{\sigma \in H \mid \sigma(b) = b\}$, is a morphism from a unipotent group to a diagonalizable group. Thus $s(\sigma)$ restricted to the fiber above b is the identity. So H satisfies ($\bigstar$).

3. Since $\mathrm{Aut}_F(q/\mathcal{C})$ is linear nilpotent, by Fact 4.1.1, it is F-definably isomorphic to $G_u \times T$ where G_u is the maximal unipotent subgroup and T is its maximal torus. Since q is $\mathcal{C}$-internal and $q \perp^w \mathcal{C}$, q is isolated by Remark 2.2.42. In particular, $q(\mathcal{U})$ is a definable set, so we may apply Lemma 4.1.2 to get that for all $b \models q$, $G_u \cdot b \cap T \cdot b = \{b\}$.

Recall that tori in connected nilpotent algebraic groups are central, hence by (1), T satisfies ($\bigstar$). Since G_u is definably isomorphic to the $\mathcal{C}$-points of a linear unipotent algebraic group, it satisfies ($\bigstar$) by (2). Fix $b \models q$ and $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\sigma(b) = b$. Since $\mathrm{Aut}_F(q/\mathcal{C})$, is definably nilpotent, $\sigma = \sigma_u \sigma_T$ for some $\sigma_u \in G_u$ and $\sigma_T \in T$. Since $\sigma(b) = b$, $\sigma_T(b) = \sigma_u^{-1}(b)$. Since $G_u \cdot b \cap T \cdot b = \{b\}$, we have that $\sigma_u(b) = \sigma_T(b) = b$. Thus $s(\sigma_u)$ and $s(\sigma_T)$ both restricted to the fiber above b are the identity. Hence, $s(\sigma)$ restricted to the fiber over b is the identity. □

We can then show that when the binding group of q is linear nilpotent and the binding groups of the fibers of p are diagonalizable, splitting of the induced short exact sequence of binding groups forces $f : p \to q$ to split and the fibers of f to be $\mathcal{C}$-algebraic as opposed to simply $\mathcal{C}$-internal. The latter claim is used to prove Theorem 4.3.3.

Lemma 4.2.5. *Let (F, δ) be an algebraically closed differential field and $f : p \to q$ be a fibration of stationary $\mathcal{C}$-internal types $p, q \in S(F)$. Suppose that q is weakly $\mathcal{C}$-orthogonal and $\mathrm{Aut}_F(q/\mathcal{C})$ is linear nilpotent. If*

$$1 \to K \to \mathrm{Aut}_F(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C}) \to 1$$

F-definably splits, then $f : p \to q$ splits. If in addition, K is finite, then the fibers of f are $\mathcal{C}$-algebraic.

Proof. Suppose that q is weakly $\mathcal{C}$-orthogonal, its binding group is linear nilpotent and the short exact sequence F-definably splits. Applying Example 4.2.4, $\mathrm{Aut}_F(q/\mathcal{C})$ satisfies condition ($\bigstar$) meaning that for any $b \models q$, if $\sigma(b) = b$, then $s(\sigma)$ restricted to the fiber above b is the identity. We denote the fibers $p_b := \mathrm{tp}(a/Fb)$.

We use ($\bigstar$) we show that fibers of $f : p \to q$ are $\mathcal{C}$-algebraic, and we use the definable function $\pi : p \to \pi(p)$ from Lemma 4.2.2 to prove that $f : p \to q$ almost splits.

We let E be the equivalence relation on $p(\mathcal{U})$ and $\pi : p \to r$ the F-definable quotient map given by Lemma 4.2.2, or rather its proof. Recall that E is defined by $E(a, a')$ if and only if $s(\sigma)|_{p_{f(a)}}(a) = a'$ for some $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ where $\sigma(f(a)) = f(a')$ and $\pi : p \to r$ is the induced quotient map. By Lemma 4.2.2, $q \perp^w r$ and so $f(a) \underset{F}{\downarrow} \pi(a)$. Since π is F-definable and p is $\mathcal{C}$-internal, $\pi(p)$ is $\mathcal{C}$-internal by Lemma 2.2.23.

Finally, we conclude that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$. Since $f : p \to q$ and $\pi : p \to r$ are F-definable, $\mathrm{dcl}(Ff(a)\pi(a)) \subseteq \mathrm{dcl}(Fa)$. So, we prove the reverse inclusion. Let $\alpha \in \mathrm{Aut}_{Ff(a)\pi(a)}(\mathcal{U})$. Then $\alpha(a) \models p_{f(a)}$ and $\alpha(f(a)) = f(a)$, so $s(\alpha)|_q(a) = \alpha(a)$. This means that $E(a, \alpha(a))$, and hence there exists $\sigma \in \mathrm{Aut}_F(q/\mathcal{C})$ such that $\sigma(f(a)) = f(\alpha(a))$ and $s(\sigma)(a) = \alpha(a)$. On the other hand, $\sigma(f(a)) = f(a)$ and, since $\mathrm{Aut}_F(q/\mathcal{C})$ satisfies Condition ($\bigstar$), $s(\sigma|_{p_{f(a)}}) = \mathrm{id}$. Thus, the orbit of a under $\mathrm{Aut}_{Ff(a)\pi(a)}(\mathcal{U})$ is a singleton showing that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$.

We have shown that $\mathrm{dcl}(Fa) = \mathrm{dcl}(Ff(a)\pi(a))$ and $f(a) \underset{F}{\downarrow} \pi(a)$. By definition, $f : p \to q$ splits.

Now suppose that K is finite. For $\mathcal{C}$-algebraicity of the fibers, we show that for any $a \models p$, $a \in \mathrm{dcl}(Ff(a)\mathcal{C})$. Fix $a \models p$. For any $\sigma \in \mathrm{Aut}_{Ff(a)}(\mathcal{U}/\mathcal{C})$, $\sigma|_p = s(\sigma|_q) \circ \tau$, for some $\tau \in K$. Note that $\sigma|_q$ fixes $f(a)$, so by ($\bigstar$) $s(\sigma|_q)|_{\mathrm{tp}(a/Ff(a))} = \mathrm{id}$. Hence,

$$\begin{aligned}\sigma(a) &= s(\sigma|_q)(\tau(a)) \\ &= s(\sigma|_q)|_{p_{f(\tau(a))}}(\tau(a)) \\ &= \mathrm{id}(\tau(a)).\end{aligned}$$

Since K is finite, the orbit of a under $\mathrm{Aut}_{Ff(a)}(\mathcal{U}/\mathcal{C})$ is finite. Hence, $a \in \mathrm{acl}(Ff(a)\mathcal{C})$. $\square$

Note that in the above proof that K will not in general be the binding group of the fibers of $f : p \to q$, but the binding group of $\mathrm{tp}(\eta/Ff(\eta))$ where there is a fundamental system η for p of length n such that $f(\eta)$ is also a fundamental system of p (see Fact 2.2.43).

We can now prove Theorem 4A.

Theorem 4.2.6. *Let (F, δ) be an algebraically closed differential field and suppose $f : p \to q$ is a fibration of stationary types where q is $\mathcal{C}$-internal, p is weakly $\mathcal{C}$-orthogonal, and the fibers of f are $\mathcal{C}$-internal. Assume the binding group of q is linear nilpotent and that the binding group of the fibers are diagonalizable.*

Then, p is almost $\mathcal{C}$-internal if and only if $f : p \to q$ almost splits.

Proof. Suppose that p is almost $\mathcal{C}$-internal. We first show that we may assume that p is $\mathcal{C}$-internal. Fix $a \models p$. Let $b \in \mathrm{dcl}(Fa)$ be the finite tuple such that $\mathrm{dcl}(Fb) = \mathrm{dcl}(Fa) \cap \mathrm{Int}_F(\mathcal{C})$ (see [58] for existence of such a tuple). Set $p' = \mathrm{tp}(b/F)$. Let $g : p \to p'$ be the definable function given by $a \mapsto b$. Since p is almost $\mathcal{C}$-internal the definable function $g : p \to p'$ is finite-to-one. Since p is weakly $\mathcal{C}$-orthogonal, p' is also weakly $\mathcal{C}$-orthogonal. Since $f(a) \in \mathrm{dcl}(Fa)$ and $\mathrm{tp}(f(a)/F)$ is $\mathcal{C}$-internal, by definition $f(a) \in \mathrm{dcl}(Fb)$. Let $q = \mathrm{tp}(b/F)$ and $h : p' \to q$ be the definable function given by $b \mapsto f(a)$. Note that the fibers of h are $\mathrm{tp}(b/Fh(b)) = \mathrm{tp}(b/Fh(g(a))) = \mathrm{tp}(b/Ff(a))$, which is the F-definable image of a stationary type, and so is stationary by Remark 2.2.1. Then, $f = h \circ g$. If $h : p' \to q$ almost splits, then there is some $c \underset{F}{\downarrow} h(b)$ such that $\mathrm{acl}(Fb) = \mathrm{acl}(Fch(b))$. Since $g : p \to p'$ is finite-to-one, $\mathrm{acl}(Fa) = \mathrm{acl}(Fb)$, $\mathrm{acl}(Ff(a)) = \mathrm{acl}(Fh(b))$. Hence, $\mathrm{acl}(Fa) = \mathrm{acl}(Ff(a)c)$ showing that $f : p \to q$ almost splits.

Since $\mathrm{tp}(b/F)$ is $\mathcal{C}$-internal, the extension $\mathrm{stp}\,(b/Fh(b))$ is $\mathcal{C}$-internal by Lemma 2.2.23. The definable function $g : p \to p'$ induces a function on the fibers $g : \mathrm{tp}(a/Ff(a)) \to \mathrm{tp}(b/Ff(a))$ which in turn induces a definable surjective homomorphism

$$\widetilde{g} : \mathrm{Aut}_{Ff(a)}(\mathrm{tp}(a/Ff(a))/\mathcal{C}) \to \mathrm{Aut}_{Fh(b)}(\mathrm{tp}(b/Fh(b))/\mathcal{C})$$

where $f(a) = h(b)$. Since $\mathrm{Aut}_{Fh(b)}(\mathrm{tp}(b/Fh(b))/\mathcal{C})$ is the image of the diagonalizable group $\mathrm{Aut}_{Ff(a)}(\mathrm{tp}(a/Ff(a))/\mathcal{C})$, it is diagonalizable. Lastly, since q and q' are interalgebraic over F, their binding groups are definably isogeneous. Since quotients and extensions of linear nilpotent algebraic groups are again linear nilpotent, the binding group of q' is linear nilpotent.

We have shown that $h : p' \to q'$ is a fibration where p' is $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal, the binding group of q' is linear nilpotent and the binding group of the fibers of h are diagonalizable. Hence, to show that $f : p \to q$ almost splits, we may assume that p is outright $\mathcal{C}$-internal. For every $a \models p$, we denote the fiber above $f(a)$ as $p_{f(a)} = \mathrm{tp}(a/Ff(a))$.

Since p is $\mathcal{C}$-internal and weakly $\mathcal{C}$-orthogonal, $\mathrm{Aut}_F(p/\mathcal{C})$ is connected, by Fact 2.2.41. The definable function $f : p \to q$ induces an F-definable morphism $\widetilde{f} : \mathrm{Aut}_F(p/\mathcal{C}) \to \mathrm{Aut}_F(q/\mathcal{C})$ and a short exact sequence

$$1 \to K \to \mathrm{Aut}_F(p/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C}) \to 1.$$

By Fact 2.2.43, K is definably isomorphic to $\mathrm{Aut}_{Ff(\bar{a})}(\mathrm{tp}(\bar{a}/Ff(\bar{a}))/\mathcal{C})$, where $\bar{a} = (a_1, ..., a_n)$ is a fundamental system for p and $f(\bar{a}) = (f(a_1), ..., f(a_n))$ is a fundamental system for q. Then, K is definably isomorphic to a subgroup of $\prod_{i=1}^{n} \mathrm{Aut}_{Ff(a_i)}(p_{f(a_i)}/\mathcal{C})$. Since the binding groups of the fibers of $f : p \to q$ are diagonalizable, meaning they are subgroups of $\mathbb{G}_m^{n_i}$ for some n_i, we have that K is definably isomorphic to a subgroup of $\mathbb{G}_m(\mathcal{C})^\ell$ for some $\ell \geq 1$. If K is connected, then K is a torus and we can apply Fact 4.1.3 to obtain F-definable splitting of the short exact sequence. However, K may not be connected. We show that we can replace K with its connected component by replacing p with an interalgebraic type.

We have $\mathrm{Aut}_F(p/\mathcal{C})$ is connected, K is diagonalizable and $\mathrm{Aut}_F(q/\mathcal{C})$ is linear nilpotent. By Fact 4.1.3, $\mathrm{Aut}_F(p/\mathcal{C})$ is linear nilpotent. Since tori are commutative and K is a subgroup of the maximal torus of $\mathrm{Aut}_F(p/\mathcal{C})$, which exists as $\mathrm{Aut}_F(p/\mathcal{C})$ is connected, K is commutative.

Claim 1. *K is F-definably isomorphic to the $\mathcal{C}$-points of a diagonalizable algebraic group $\widehat{K}$ defined over $A \cap \mathcal{C}$.*

Proof of claim: This proof is a slight modification of the proof of [41, Lemma 2.2]. Since K is commutative, $K \subset \mathrm{dcl}(F\mathcal{C})$ (see for example the proof of Lemma 2.2 in [41]). By compactness, K is the image under an F-definable function of an F-definable subset of some cartesian power of $\mathcal{C}$. Since $\mathcal{C}$ is a stably embedded pure algebraically closed field, its induced structure eliminates imaginaries, hence we obtain an F-definable isomorphism between K and D, an $F \cap \mathcal{C}$-definable group in the field structure on $\mathcal{C}$. Since K is diagonalizable, D must also be diagonalizable. Thus D is of the form $\widehat{K}(\mathcal{C})$, where $\widehat{K}$ is a diagonalizable algebraic group. ■

Since $F \cap \mathcal{C}$ is an algebraically closed field and $\widehat{K}$ is diagonalizable, $\widehat{K} = \widehat{R} \times \mathbb{G}_m^\ell(\mathcal{C})$ for some $\ell \geq 0$ and some finite $F \cap \mathcal{C}$-definable group $\widehat{R}$ (see Proposition 8.7 and 8.11 in [8]). Let $g : K \to \widehat{K}$ be an F-definable isomorphism. Then $f^{-1}(\widehat{R} \times \mathbb{G}_m^\ell(\mathcal{C})) = R \times K^\circ$ for some finite group R and torus K°. Thus K is F-definably isomorphic to $R \times K^\circ$ for some F-definable finite group R and torus K°.

The subgroup R induces an equivalence relation E on $p(\mathcal{U})$ such that $E(x, y)$ if and only if there exists some $\sigma \in R$ such that $\sigma(x) = y$. Let $\eta : p \to s$ be the quotient map. It induces an F-definable morphism $\widetilde{\eta} : \mathrm{Aut}_F(p/\mathcal{C}) \to \mathrm{Aut}_F(s/\mathcal{C})$.

Claim 2. $\ker(\widetilde{\eta}) = R$

Proof of claim: The kernel of $\widetilde{\eta}$ is all $\sigma \in \mathrm{Aut}_F(p/\mathcal{C})$ such that for all $a \models p$, there exists some $\tau \in R$ where $\sigma(a) = \tau(a)$. Since $R < K$ and K is a subgroup of the maximal torus of $\mathrm{Aut}_F(p/\mathcal{C})$, if σ is in the maximal unipotent subgroup of $\mathrm{Aut}_F(p/\mathcal{C})$, by Lemma 4.1.2, $\sigma(a) = a = \tau(a)$. Hence, by the proof of Lemma 4.1.2, $\ker(\widetilde{\eta}) < T$. Since $p \perp^w \mathcal{C}$, by Fact 2.2.40, $\mathrm{Aut}_F(p/\mathcal{C})$ acts transitively on $p(\mathcal{U})$. Since T is central, $\ker(\widetilde{\eta})$ is central. Then, the stabilizers of any two realizations of p are equal. If $\tau \in \ker(\widetilde{\eta})$ fixes some $x \models p$, it fixes all $x \models p$. Since the action is faithful, $\tau = \mathrm{id}$. Hence, $\ker(\widetilde{\eta})$ acts freely. Thus, for all $\tau \in \ker(\widetilde{\eta})$, $\tau = \sigma$ in R. ∎

Since R is a definable subgroup of $K = \ker(\widetilde{f})$, there is a definable function $g : s \to q$ given by mapping $\eta(a) \mapsto f(a)$ (recall that $s = \mathrm{tp}(\eta(a)/F)$). By Lemma 2.2.6, there exists a stationary type $q' \in S(F)$, a fibration $g' : s \to q'$ and a finite-to-one definable function $\pi : q' \to q$ such that $g = \pi \circ g'$. If $g' : s \to q'$ splits, then $\eta(a) \in \mathrm{acl}(Fcg'(\eta(a)) = \mathrm{acl}(Fcf(a))$ for some $c \underset{F}{\perp\!\!\!\!\perp} f(a)$. Since $\eta : p \to s$ is finite-to-one, we obtain $a \in \mathrm{acl}(Fcf(a))$, hence $f : p \to q$ almost splits. It suffices to show that $g' : s \to q'$ splits. Since q' and q are interalgebraic over F and the binding group of q is linear nilpotent, by Lemma 2.2.47, we have that the binding group of q' is linear nilpotent as images and quotients of linear nilpotent algebraic groups are again linear nilpotent.

The definable function $g' : s \to q'$ induces a short exact sequence

$$1 \to K^\circ \to \mathrm{Aut}_F(s/\mathcal{C}) \xrightarrow{\widetilde{g'}} \mathrm{Aut}_F(q'/\mathcal{C}) \to 1$$

where K° is a torus, $\mathrm{Aut}_F(q'/\mathcal{C})$ is linear nilpotent and the maximal tori of $\mathrm{Aut}_F(s/\mathcal{C})$ and $\mathrm{Aut}_F(q'/\mathcal{C})$ are F-definably isomorphic to the $\mathcal{C}$-points of a tori, by the same proof of Claim 1. Thus, by Lemma 4.1.3, $\mathrm{Aut}_F(s/\mathcal{C})$ is linear nilpotent and the short exact sequence induced by $g' : s \to q'$ F-definably splits. Since q is weakly $\mathcal{C}$-orthogonal and q' is interalgebraic with q, q' is weakly $\mathcal{C}$-orthogonal. So, we can apply Lemma 4.2.5 to obtain that $g' : s \to q'$ splits. By the previous discussion, this shows that $f : p \to q$ almost splits.

For the reverse direction, if $f : p \to q$ almost splits, then $\mathrm{acl}(Fa) = \mathrm{acl}(Fbf(a))$ for some $b \underset{F}{\perp\!\!\!\!\perp} f(a)$. Then $\mathrm{tp}(b/Ff(a))$ is interalgebraic with the $\mathcal{C}$-internal fiber $\mathrm{tp}(a/Ff(a))$. Since $\mathrm{tp}(b/Ff(a))$ is a non-forking extension of $\mathrm{tp}(b/F)$, by Lemma 2.2.23, $\mathrm{tp}(b/F)$ is almost $\mathcal{C}$-internal. Since $\mathrm{tp}(b/F)$ and $\mathrm{tp}(f(a)/F)$ are almost $\mathcal{C}$-internal, by Lemma 2.2.23 $\mathrm{tp}(bf(a)/F)$ is almost $\mathcal{C}$-internal. By Lemma 2.2.23, $\mathrm{tp}(a/F)$ is almost $\mathcal{C}$-internal. □

4.3 Splitting the logarithmic derivative map

Recall that in Section 3.1, we defined logarithmic-differential maps on $\mathbb{G}_m^k \times \mathbb{G}_a^l$. In the rest of this chapter we restrict to $\mathbb{G}_m$. So $\delta\log_{\mathbb{G}_m} : \mathcal{U} \setminus \{0\} \to \mathcal{U}$ is the definable function $\delta\log_{\mathbb{G}_m}(a) = \frac{\delta(a)}{a}$. It is a homomorphism from the multiplicative group to the additive group and its kernel is $\mathbb{G}_m(\mathcal{C})$.

We prove Theorem 4B, obtaining an algebraic criterion for internality for a type p where $\delta\log_{\mathbb{G}_m}$ is a fibration from p to q. Such types are defined as:

Definition 4.3.1. Let $q \in S_1(F)$. Its *pullback under the logarithmic derivative* $\delta\log_{\mathbb{G}_m}$, denoted by $p := \delta\log_{\mathbb{G}_m}^{-1}(q)$, is the type of a such that $\delta\log_{\mathbb{G}_m}(a) \models q$ and $a \notin \mathrm{acl}(F, \delta\log_{\mathbb{G}_m}(a))$.

It is well-known that p exists and is unique, and that each fiber of $\delta\log_{\mathbb{G}_m} : p \to q$ is a minimal $\mathcal{C}$-internal type (see for example [44, Proposition 5.3]). Suppose $q \in S_1(F)$ is stationary. If q is finite dimensional, its pullback p under the logarithmic derivative is also finite dimensional: for $a \models p$,

$$\begin{aligned}
\dim(a/F) &= \mathrm{trdeg}_F\left(F\langle a\rangle\right) \\
&= \dim_{F\langle\frac{\delta(a)}{a}\rangle}\left(F\langle a\rangle\right) + \dim_F\left(F\langle\frac{\delta(a)}{a}\rangle\right) \\
&= 1 + \dim(q).
\end{aligned}$$

If q is a $\mathcal{C}$-internal type, the pullback under the logarithmic derivative need not be $\mathcal{C}$-internal, but $\delta\log_{\mathbb{G}_m}$ witnesses that $p = \delta\log_{\mathbb{G}_m}^{-1}(q)$ is 2-step analyzable in the constants. For example, the generic type of $\delta(\frac{\delta(x)}{x}) = 0$, considered in Example 2.2.35 is the pullback under the logarithmic derivative of the generic type of the constants, but is not almost $\mathcal{C}$-internal.

In his thesis [44], Jin introduced the notion of product-splitting (which he called splitting):

Definition 4.3.2. Let $p \in S(F)$. We say $\delta\log_{\mathbb{G}_m} : p \to q$ *product-splits* if there is an integer $k \neq 0$ such that for some $a \models p$ there are w_1, w_2 such that $a^k = w_1 w_2$ and:

- $w_1 \in \mathrm{dcl}(F\delta\log_{\mathbb{G}_m}(a))$
- $\delta\log_{\mathbb{G}_m}(w_2) \in F$.

We show that under the same assumptions as Theorem 4A, product-splitting of $\delta\log_{\mathbb{G}_m} : p \to q$ is equivalent to almost splitting, and hence to almost $\mathcal{C}$-internality of p. We then use product splitting to obtain an algebraic criterion for almost $\mathcal{C}$-internality under these conditions. The following relies heavily on the proof of Theorem 4.2.6.

Theorem 4.3.3. *Let (F, δ) be an algebraically closed differential field and let $q(x) \in S_1(F)$ be weakly $\mathcal{C}$-orthogonal and $\mathcal{C}$-internal with a linear nilpotent binding group. Let $p = \delta\log_{\mathbb{G}_m}^{-1}(q)$. Then the following are equivalent:*

1. $\delta\log_{\mathbb{G}_m} : p \to q$ *product splits,*

2. $\delta\log_{\mathbb{G}_m} : p \to q$ *almost splits,*

3. *p is almost $\mathcal{C}$-internal.*

Proof. (1) $\implies$ (2) This is in [44, Remark 5.5], but we give the argument here. Fix $a \models p$, $k \neq 0$ and let w_1, w_2 be such that $a^k = w_1 w_2$ where $w_1 \in \mathrm{dcl}(F, \delta\log_{\mathbb{G}_m}(a))$ and $\delta\log_{\mathbb{G}_m}(w_2) \in F$. In particular, $a \in \mathrm{acl}(Fw_1w_2) \subseteq \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)w_2)$.

Since w_2 is a singleton and $\frac{\delta(w_2)}{w_2} \in F$, $\dim(w_2/F) = 1$. Hence, either $\delta\log_{\mathbb{G}_m}(a) \underset{F}{\downarrow} w_2$ or $w_2 \in \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a))$. If $w_2 \in \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a))$, then $a \in \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)w_2)) = \mathrm{acl}(F, \delta\log_{\mathbb{G}_m}(a))$, a contradiction since the fibers of $\log_\delta : p \to q$ are minimal. Hence, $\delta\log_{\mathbb{G}_m}(a) \underset{F}{\downarrow} w_2$. Since $w_1 \in \mathrm{dcl}(Fa)$, $w_2 = \frac{a^k}{w_1} \in \mathrm{dcl}(Fa)$. Hence, $\delta\log_{\mathbb{G}_m}(a), w_2 \in \mathrm{acl}(Fa)$. We have shown that $\delta\log_{\mathbb{G}_m}(a) \underset{F}{\downarrow} w_2$ and $\mathrm{acl}(Fa) = \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)w_2)$. Thus $\log_\delta : p \to q$ almost splits.

(2) $\implies$ (3) Assume that $\delta\log_{\mathbb{G}_m} : p \to q$ almost splits. Let $a \models p$ and b be such that $b \underset{F}{\downarrow} \delta\log_{\mathbb{G}_m}(a)$ and $\mathrm{acl}(Fa) = \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)b)$. Since $\mathrm{tp}(b/F\delta\log_{\mathbb{G}_m}(a))$ is interalgebraic with the $\mathcal{C}$-internal fiber $\mathrm{tp}(a/F\delta\log_{\mathbb{G}_m}(a))$ and $b \underset{F}{\downarrow} \delta\log_{\mathbb{G}_m}(a)$, by Lemma 2.2.23, $\mathrm{tp}(b/F)$ is almost $\mathcal{C}$-internal.

Since $q = \mathrm{tp}(\delta\log_{\mathbb{G}_m}(a)/F)$ and $\mathrm{tp}(b/F)$ are almost $\mathcal{C}$-internal, by Lemma 2.2.23, $\mathrm{tp}(\delta\log_{\mathbb{G}_m}(a)b/F)$ is almost $\mathcal{C}$-internal. Hence, by interalgebraicity and Lemma 2.2.23, $p = \mathrm{tp}(a/F)$ is almost $\mathcal{C}$-internal.

(3) $\implies$ (1) Suppose that p is almost $\mathcal{C}$-internal.

First, suppose that $p(\mathcal{U}) \not\subseteq \mathrm{acl}(Fq(\mathcal{U})\mathcal{C})$. Then $\delta\log_{\mathbb{G}_m} : p \to q$ product splits by [20, Theorem 4.5]. So we may assume that $p(\mathcal{U}) \subseteq \mathrm{acl}(Fq(\mathcal{U})\mathcal{C})$. We show that the fibers of $\delta\log_{\mathbb{G}_m} : p \to q$ are not $\mathcal{C}$-orthogonal. In that case, we show that $\delta\log_{\mathbb{G}_m} : p \to q$.

Suppose that $p(\mathcal{U}) \subseteq \mathrm{acl}(Fq(\mathcal{U})\mathcal{C})$ and that the fibers $\delta\log_{\mathbb{G}_m} : p \to q$ are weakly $\mathcal{C}$-orthogonal and derive a contradiction. If $p \not\perp^w \mathcal{C}$, then $a \underset{F}{\not\downarrow} c$ for some tuple $c \subseteq \mathcal{C}$. Hence, $a\delta\log_{\mathbb{G}_m}(a) \underset{F}{\not\downarrow} c$. So either $\delta\log_{\mathbb{G}_m}(a) \underset{F}{\not\downarrow} c$, contradicting that $q \perp^w \mathcal{C}$, or $a \underset{F\delta\log_{\mathbb{G}_m}(a)}{\not\downarrow} c$ contradicting that $\mathrm{tp}(a/F\delta\log_{\mathbb{G}_m}(a)) \perp^w \mathcal{C}$. Thus, we may assume that $p \perp^w \mathcal{C}$ and the fibers of $\delta\log_{\mathbb{G}_m} : p \to q$ are weakly $\mathcal{C}$-orthogonal and derive a contradiction.

By Lemma 2.2.24, there exists a finite-to-one definable function $\pi : p \to s$ such that $s \in S(F)$ is $\mathcal{C}$-internal.

Let $s' := \mathrm{tp}(\pi(a)\delta\log_{\mathbb{G}_m}(a)/F)$. Since $s'(\mathcal{U}) \subseteq \mathrm{dcl}(Fp(\mathcal{U}))$ and $p(\mathcal{U}) \subseteq \mathrm{acl}(Fq(\mathcal{U})\mathcal{C})$, $s'(\mathcal{U}) \subseteq \mathrm{acl}(Fq(\mathcal{U})\mathcal{C})$. Since s' is $\mathcal{C}$-internal, by Fact 2.2.23, the proof of Corollary 2.2.47 shows that $\mathrm{Aut}_F(s'/q,\mathcal{C})$ is finite. This is the kernel of $\widetilde{f} : \mathrm{Aut}_F(s'/\mathcal{C}) \to \mathrm{Aut}_F(q/\mathcal{C})$. Note that the fibers of $f : s' \to q'$ are of the form $\mathrm{tp}(\pi(a)/F\delta\log_{\mathbb{G}_m}(a))$, which is a definable image of $\mathrm{tp}(a/F\delta\log_{\mathbb{G}_m}(a))$. By assumption, $\mathrm{tp}(a/F\delta\log_{\mathbb{G}_m}(a))$ is stationary with a diagonalizable binding group. By Lemma 2.2.1, $f : s' \to q$ is a fibration and its fibers also have diagonalizable binding group, as the image of a diagonalizable group is again diagonalizable. We have that q is weakly $\mathcal{C}$-orthogonal with a linear nilpotent binding group. Since s' is $\mathcal{C}$-internal, $f : s' \to q$ almost splits by Theorem 4.2.6, and in particular, the short exact sequence

$$1 \to \mathrm{Aut}_F(s'/q,\mathcal{C}) \to \mathrm{Aut}_F(s'/\mathcal{C}) \xrightarrow{\widetilde{f}} \mathrm{Aut}_F(q/\mathcal{C})$$

F-definably splits (see the proof of Theorem 4.2.6). Since $\mathrm{Aut}_F(s'/q,\mathcal{C})$ is finite, by Lemma 4.2.5, $\pi(a) \in \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)\mathcal{C})$. Since $\pi : p \to s$ is finite-to-one, we have that the fibers of $\delta\log_{\mathbb{G}_m}$ are $\mathcal{C}$-algebraic, contradicting that they are weakly $\mathcal{C}$-orthogonal.

Finally, we may assume that the fibers of $\delta\log_{\mathbb{G}_m} : p \to q$ are not weakly $\mathcal{C}$-orthogonal. In particular, $\mathrm{stp}(a/F\delta\log_{\mathbb{G}_m}(a))$ is non-weakly $\mathcal{C}$-orthogonal, and so there is some $e \in \mathcal{C} \cap F\langle a\rangle^{\mathrm{alg}}$ and $e \notin F\langle \delta\log_{\mathbb{G}_m}(a)\rangle^{\mathrm{alg}}$ by Fact 2.2.15. Since $\delta\log_{\mathbb{G}_m}(a) = \frac{\delta(a)}{a} \in F\langle \delta\log_{\mathbb{G}_m}(a)\rangle$, $\delta(a) \in F(a)$. Hence, $F\langle a, \delta\log_{\mathbb{G}_m}(a)\rangle = F(a)$. Thus $e = \frac{f(a)}{g(a)}$ for some $f = \sum_{i=0}^{n} b_i x_i$ and $g = \sum_{j=0}^{m} c_j x^j$ in $F\langle \delta\log_{\mathbb{G}_m}(a)\rangle[x]$ with $b_n, c_m \neq 0$. We may assume that $n \geq m$, otherwise replace $e \neq 0$ with $1/e$.

Since e is a constant, we get that $0 = \delta(e) = \delta(f(a))g(a) - f(a)\delta(g(a))$, a polynomial expression of a over $F\langle \delta\log_{\mathbb{G}_m}(a)\rangle$. Since $a \notin \mathrm{acl}(F\delta\log_{\mathbb{G}_m}(a)) = F\langle \delta\log_{\mathbb{G}_m}(a)\rangle^{\mathrm{alg}}$, we have that the right hand side of the equations must have all its coefficients be zero. Computing the coefficient for a^{n+m}, we obtain $c_m\left(\delta(b_n) + nb_n\delta\log_{\mathbb{G}_m}(a)\right) = b_n\left(\delta(c_m) + mc_m\delta\log_{\mathbb{G}_m}(a)\right)$. Since $b_n, c_m \neq 0$, rearranging we get, $\delta\log_{\mathbb{G}_m}(a^{n-m}) = \delta\log_{\mathbb{G}_m}(\frac{c_m}{b_n})$. Thus, there is some $\mu \in \mathcal{C}$ such that $a^{n-m} = \mu\frac{c_m}{b_n}$. If $n \neq m$, let $k = n - m > 0$, $w_1 = \frac{c_m}{b_n}$ and $w_2 = \mu$. Then $w_1 \in F\langle \delta\log_{\mathbb{G}_m}(a)\rangle = \mathrm{dcl}(F\delta\log_{\mathbb{G}_m}(a))$ and $\delta\log_{\mathbb{G}_m}(w_2) = 0 \in F$. Then $a^{n-m} = w_1w_2$ showing that $\delta\log_{\mathbb{G}_m} : p \to q$ product splits.

If $n = m$, then $\frac{b_n}{c_m} = \mu \in \mathcal{C}$. Using Euclidean division, we find an $h \in F\langle \delta\log_{\mathbb{G}_m}(a)\rangle[x]$ with $\delta\log_{\mathbb{G}_m}(h) < \delta\log_{\mathbb{G}_m}(f)$ such that $\frac{h(a)}{g(a)} = -\mu$. Since $e \notin F\langle \delta\log_{\mathbb{G}_m}(a)\rangle$ and $\frac{b_n}{c_m} = \mu \in$

$F\langle\delta\log_{\mathbb{G}_m}(a)\rangle$, we have that $e-\mu \notin F\langle\delta\log_{\mathbb{G}_m}(a)\rangle$. Since $e,\mu \in \mathcal{C}$, $e-\mu \in \mathcal{C}$. Repeating the proof with $\frac{h(a)}{g(a)} = e-\mu$, we obtain $w_1 \in \mathrm{dcl}(F\delta\log_{\mathbb{G}_m}(a))$, $\delta\log_{\mathbb{G}_m}(w_2) \in F$ and $k \neq 0$ such that $a^k = w_1 w_2$. Hence, $\delta\log_{\mathbb{G}_m} : p \to q$ product-splits. Notice that we only used that the fibers were weakly orthogonal to the constants and not that $\delta\log_{\mathbb{G}_m}$ almost splits. $\square$

The following is Theorem 4B.

Corollary 4.3.4. *Let (F,δ) be an algebraically closed differential field and $q \in S_1(F)$ be the generic type over F of the differential equation $\delta^n(x) = f(x,\delta(x),...,\delta^{n-1}(x))$ where $f \in F(x_0,...,x_{n-1})$ is a rational function and $n \geq 1$.*

Assume that q is weakly $\mathcal{C}$-orthogonal and $\mathcal{C}$-internal with a linear nilpotent binding group. Let $p = \log_\delta^{-1}(q)$. Then the following are equivalent,

1. *p is almost $\mathcal{C}$-internal*
2. *there is a non-zero $h \in F(x_0,...,x_n)$, a constant $e \in F$ and a non-zero integer k such that $(kx_0 - e)h = h^\delta + \sum_{i=0}^{n-2} \frac{\partial h}{\partial x_i} x_{i+1} + \frac{\partial h}{\partial x_{n-1}} f$.*

Proof. By Theorem 4.3.3, p is almost $\mathcal{C}$-internal if and only $\delta\log_{\mathbb{G}_m} : p \to q$ product splits. Hence, it suffices to show that $\delta\log_{\mathbb{G}_m} : p \to q$ product splits if and only if the differential equation in (2) is satisfied.

Suppose $\delta\log_{\mathbb{G}_m} : p \to q$ product splits. Let $a \models p$, $k \neq 0$ and w_1, w_2 be such that $a^k = w_1 w_2$, $w_1 \in \mathrm{dcl}(F\delta\log_{\mathbb{G}_m}(a))$, and $\delta\log_{\mathbb{G}_m}(w_2) \in F$. Since $a \neq 0$, $w_1, w_2 \neq 0$. Let $b = \delta\log_{\mathbb{G}_m}(a)$ and $e = \delta\log_{\mathbb{G}_m}(w_2)$. Since q is the generic type of $\delta^n(x) = f(x,\delta(x),...,\delta^{-1}(x))$, $\delta^n(b) = f(b,....,\delta^{n-1}(b))$. Let $\bar{b} = (b,\delta(b),...,\delta^{n-1}(b))$. Since $w_1 \in \mathrm{dcl}(F,\delta\log_{\mathbb{G}_m}(a)) = \mathrm{dcl}(F,b)$. Hence, there is a non-zero $h \in F(x_1,...,x_n)$ such that $w_1 = h(\bar{b})$. Then, $a^k = h(\bar{b})w_2$. Applying $\delta\log_{\mathbb{G}_m}$ to both sides, we obtain $\frac{k\delta(a)a^{k-1}}{a^k} = \frac{\delta(h(\bar{b}))w_2 + h(\bar{b})\delta(w_2)}{h(\bar{b})w_2}$.

Simplifying gives

$$\begin{aligned}(kb-e)h(\bar{b}) &= \delta(h(\bar{b}))\\ &= h^\delta(\bar{b}) + \sum_{i=0}^{n-1} \frac{\partial h}{\partial x_i}(\bar{b})\delta(b_i)\\ &= h^\delta(\bar{b}) + \sum_{i=0}^{n-2} \frac{\partial h}{\partial x_i}(\bar{b})b_{i+1} + \frac{\partial h}{\partial x_n}(\bar{b})f(\bar{b}).\end{aligned}$$

Since b is a generic realization of q, $\dim(q) = n-1$ and this equation is order $n-1$, $(kx_0 - e)h = h^\delta + \sum_{i=1}^{n-2} \frac{\partial h}{\partial x_i} x_{i+1} + \frac{\partial h}{\partial x_n} f$.

For the reverse direction, suppose there exists h, e, and k as in the conclusion. Fix $a \models p$ and let $b = \delta\log_{\mathbb{G}_m}(a)$. Let $\bar{b} = (b, \delta(b), ..., \delta^{n-1}(b))$. If $h(\bar{b}) = 0$, then

$$\operatorname{trdeg}(b, \delta(b), ..., \delta^{n-1}(b)/F) < n = \dim(q),$$

contradicting genericity of b. Let $w_1 = h(\bar{b})$ and $w_2 = \frac{a^k}{h(\bar{b})}$. Hence, $a^k = w_1 w_2$. Then $w_1 \in \operatorname{dcl}(F, \bar{b}) = \operatorname{dcl}(F, \delta\log_{\mathbb{G}_m}(a))$. We compute $\delta\log_{\mathbb{G}_m}(w_2)$ to show that $\delta\log_{\mathbb{G}_m}(w_2) \in F$.

$$\begin{aligned}
\delta\log_{\mathbb{G}_m}(w_2) &= \frac{k\delta(a)a^{k-1}}{a^k} - \delta\log_{\mathbb{G}_m}(h(\bar{b})) \\
&= kb - \frac{\delta(h(\bar{b}))}{h(\bar{b})} \\
&= kb - \frac{h^\delta(\bar{b}) + \sum_{i=0}^{n-1} \frac{\partial h}{\partial x_i}(\bar{b})\delta^{i+1}(b)}{h(\bar{b})} \\
&= kb - (kb - e) \\
&= e \in F.
\end{aligned}$$

We have shown that $a^k = w_1 w_2$ for some $k \neq 0$, $w_1 \in \operatorname{dcl}(F\delta\log_{\mathbb{G}_m}(a))$ and $w_2 \in \operatorname{dcl}(F)$. Hence, $\delta\log_{\mathbb{G}_m} : p \to q$ product splits. □

Example 4.3.5. Let (F, δ) be an algebraically closed differential field. Suppose q is the generic type of an order n linear differential equation. Assume that q is weakly $\mathcal{C}$-orthogonal with a linear nilpotent binding group. Then $p = \delta\log_{\mathbb{G}_m}^{-1}(q)$ is not almost $\mathcal{C}$-internal.

Proof. Let q be the generic type of a linear differential equation, $\delta^n(x) = c + c_0 x + \sum_{i=1}^{n-1} c_i \delta^i(x)$ for some $c, c_0, ..., c_{n-1} \in F$. Then, q is $\mathcal{C}$-internal (see Example 2.2.18). Recall that the fibers of $\delta\log_{\mathbb{G}_m} : p \to q$ for any types p and q are almost $\mathcal{C}$-internal. Suppose, for sake of contradiction, that p is almost $\mathcal{C}$-internal. Since $q \perp^w \mathcal{C}$ and q has a linear nilpotent binding group, by Corollary 4.3.4, there is some non-zero $h \in F(x_0, ..., x_{n-1})$, $e \in F$ and non-zero integer k such that

$$(kx_0 - e)h = h^\delta + \sum_{i=0}^{n-1} \frac{\partial h}{\partial x_i} x_{i+1} + \frac{\partial h}{\partial x_n}(c + c_0 x + \sum_{i=1}^{n-1} c_i x_i).$$

Let ν_i be the valuation in $F(x_0, ..., x_{n-1})$ with respect to x_i. That is we view $F(x_0, ..., x_{n-1})$ as $F(x_0, ...x_{i-1}, x_{i+1}, ..., x_{n-1})(x_i)$ and equip it with the valuation where for any $P, Q \in F(x_0, ...x_{i-1}, x_{i+1}, ..., x_{n-1})[x_i]$, we set $\nu_i(P) = -\deg(P)$, and $\nu_i(\frac{P}{Q}) = \nu_i(P) - \nu_i(Q)$. We have, for all $P, Q \in F(x_0, ...x_{i-1}, x_{i+1}, ..., x_{n-1})[x_i]$, $\nu_i(\frac{\partial P}{\partial x_j}) \geq \nu_i(P)$ and $\nu_i(P^\delta) \geq \nu_i(P)$, from which we deduce that $\nu_i(\frac{\partial \frac{P}{Q}}{\partial x_j}) \geq \nu_i(\frac{P}{Q})$ and $\nu_i(\frac{P}{Q}^\delta) \geq \nu_i(\frac{P}{Q})$.

We can write $h = \widetilde{h} x_0^{-\nu_0(h)} + g(x_0, ..., x_{n-1})$ with $\widetilde{h} \in F(x_1, ..., x_{n-1})$ and $\nu_0(g) > \nu_0(h)$. We have:

$$k\widetilde{h} = c_0 \frac{\partial \widetilde{h}}{\partial x_{m-1}}.$$

Consider the valuation ν_{n-1} in $F(x_0, ..., x_{n-1})$ with respect to x_{n_1}. The valuation of the left hand side is $\nu_{n-1}(\widetilde{h})$ but the valuation of the right hand side is strictly greater than $\nu_{n-1}(\widetilde{h})$, unless $\widetilde{h} = 0$. If $\widetilde{h} = 0$, then $h = g$, contradicting that $\nu_0(g) > \nu_0(h)$. Hence, p is not $\mathcal{C}$-internal. $\square$

Chapter 5

Lotka-Volterra systems

In this chapter we focus on Lotka-Volterra systems

$$LV_{a,b,c,d} = \begin{cases} \delta(x) = x(ay + b) \\ \delta(y) = y(cx + d) \end{cases}$$

where a, b, c, d are non-zero constants. Note that these equations define a vector field on $\mathbb{A}^2$, and hence are examples of the objects of study in Chapter 3. But, we are now concerned with different issues than internality to the constants.

The first problem is determining when $LV_{a,b,c,d}$ has a minimal (or even *strongly minimal*, see Section 5.1 below) generic type. Duan and Nagloo [17] showed that this is the case when b and d are linearly independent over $\mathbb{Q}$. It is also well known that the generic type is not minimal when $b = d$. We are able to fill the gap here. The following appears as Proposition 5.2.3 below:

Theorem 5A. *If $b \neq d$, then the generic type of $LV_{a,b,c,d}$ is strongly minimal.*

We then ask the related question of whether the generic type of the Lotka-Volterra system is *totally disintegrated*, meaning that any tuple of distinct realizations form an independent set. Total disintegration means there is no non-trivial algebraic relations, over any field of definition, between realizations of the type. In the strongly minimal case, we use a generalization of a theorem of Brestovski [9] to show that the type is totally disintegrated. In fact, we show a stronger result that, under a mild constraint on the parameters, there are no interesting relations between distinct realizations to generic types of multiple Lotka-Volterra systems. This appears as Corollary 5.3.3 below.

Theorem 5B. *Let F be an algebraically closed field of constants. For $i = 1, ..., n$, let $a_i, b_i, c_i, d_i \in F$ be non-zero and p_i be the generic type over F of LV_{a_i,b_i,c_i,d_i}. Suppose that*

1. *$b_i \neq d_i$, and*
2. *for all $j \neq i$, $\{b_i, d_i\} \neq \{b_j, d_j\}$,*

If $(u_1, v_1), ..., (u_m, v_m)$ are distinct realizations of any of the p_is, then $\mathrm{trdeg}(u_1, v_1, ..., u_m, v_m/\mathcal{C}) = 2m$.

Finally, in the non-strongly minimal case when $b = d$, Proposition 5.4.1 below classifies all 1-dimensional D-subvarieties of $LV_{a,b,c,d}$ over any (potentially non-constant) differential field of definition:

Theorem 5C. *The only irreducible 1-dimensional D-subvarieties of $LV_{a,b,c,d}$ over any differential field of definition are those defined by:*

1. *$x = 0$,*
2. *$y = 0$, and*
3. *$cx - ay - z_0 = 0$ where $z_0 \in \mathcal{U}$ satisfies $\delta(z) = bz$.*

This chapter is based on joint work with Yutong Duan and Léo Jimenez work appearing in [16]. We continue to work over a sufficiently saturated model $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$ with field of constants $\mathcal{C}$.

5.1 Preliminaries on strong minimality

Recall that by Zilber's dichotomy in DCF_0, the behaviour of minimal types is well-understood. In general, a stronger condition than a type being minimal is being strongly minimal.

Definition 5.1.1. A definable set D is *strongly minimal* if any definable subset of D is either finite or co-finite. A non-algebraic stationary type is *strongly minimal* if it contains a formula defining a strongly minimal set.

Note that any strongly minimal type is itself minimal. The converse is true in certain cases, for example when U-rank and Morley Rank agree. Recall that a stationary type p is minimal if and only if $U(p) = 1$. Similarily, a stationary type p is strongly minimal if and only if $\mathrm{RM}(p) = 1$ (see [53, Corollary 6.2.10]). Moreover, $U(p) \leq \mathrm{RM}(p) \leq \dim(p)$ (see for example [54, Chapter 2 Lemma 5.4]). Hence, a 1-dimensional and non-algebraic type is both strongly minimal and minimal. For definable sets of dimension 2, Freitag and Moosa [25, Theorem 6.1] expand upon a result of Marker and Pillay to show that the U-rank and Morley rank of the generic type of S agree.

Fact 5.1.2. *Let p be the generic type of a 2-dimensional D-variety over an algebraically closed field. Then p is strongly minimal if and only if p is minimal.*

In this chapter we are interested in (regular) algebraic vector fields of dimension 2, hence minimality and strong minimality of their generic types are equivalent. We use the strong minimality language in this chapter as the literature surrounding the questions considered here use this terminology. A good survey of strongly minimal types of dimension greater than 1 can be found in [15].

We note here that in the case of a 2-dimensional algebraic vector field, minimality agrees with a type satisfying exchange. This allows us to use an algebraic result of Jaoui [39] to conclude when certain generic types of vector fields are strongly minimal. Exchange is a property of type in a general stable theory.

Definition 5.1.3. A type $p = \mathrm{tp}(a/A)$ satisfies *exchange* if for any $c \in \mathrm{acl}(Aa) \setminus \mathrm{acl}(A)$, then $a \in \mathrm{acl}(Ac)$.

Note that every minimal type $\mathrm{tp}(a/A)$ satisfies exchange: if $b \in \mathrm{acl}(Aa) \setminus \mathrm{acl}(A)$, then $b \underset{A}{\not\perp} a$. Hence, $U(a/Ab) < U(a/A) = 1$ showing that $U(a/Ab) = 0$ and $a \in \mathrm{acl}(Ab)$. Not every type that satisfies exchange is minimal (see [26, Section 4.2] for an example). However, in DCF_0, non-minimal types satisfying exchange are of a special form.

Fact 5.1.4. *[41, Theorem 6.8] Let F be an algebraically closed differential field of constants and $p \in S(F)$ be a stationary non-algebraic type. If p satisfies exchange then either p is minimal or there is a finite-to-one definable function $f : p \to q$ where q is $\mathcal{C}$-internal and its binding group is definably isomorphic to the $\mathcal{C}$-points of a simple abelian variety of dimension > 1.*

Let $(\mathbb{A}^k, s)$ be a (regular) vector field in the variables $\bar{x} = (x_1, ..., x_k)$ with its section map given by $s(\bar{x}) = (\bar{x}, s_1(\bar{x}), ..., s_k(\bar{x}))$. By $(a_1, ..., a_k) \in (\mathbb{A}^k, s)^\sharp(\mathcal{C})$ we mean that for all

$i = 1, ..., k$, $\delta(a_i) = 0$ and $\delta(a_i) = s_i(a_1, ..., a_k)$, and hence $s_i(a_1, ..., a_n) = 0$. If $(\mathbb{A}^k, s)$ is an algebraic vector field over F, we say that an algebraic vector field $(Y, s|_Y)$ is a *restriction* of $(\mathbb{A}^k, s)$ to Y if Y is a subvariety of $\mathbb{A}^k$ and $s|_Y : Y \to TY$ is a section map to the projection $\pi : TY \to Y$. We allow our restrictions of vector fields to have the subvariety Y defined over a field extension of F.

Corollary 5.1.5. *Let p be the generic type of an algebraic vector field $X = (\mathbb{A}^2, s)$ over an algebraically closed field of constants, F. If there is a point of $X^\sharp(\mathcal{C})$ that is not contained in any 1-dimensional restriction of X defined over any field of constants, then p is strongly minimal.*

Proof. Suppose there is some $a \in X^\sharp(\mathcal{C})$ that is not contained in any 1-dimensional restriction of X. Since $\dim(p) = 2$, these are exactly the assumptions in the restatement of [39, Theorem B] in [16]. Hence, p satisfies exchange.

By Fact 5.1.4, either p is minimal or there is a finite-to-one definable function $f : p \to q$ where q is $\mathcal{C}$-internal and $\mathrm{Aut}_F(q/\mathcal{C}) = A(\mathcal{C})$ for some A, a simple abelian variety of dimension > 1.

On the other hand, since p is the generic type of $(\mathbb{A}^2, s)$ over constant parameters, by Fact 3.1.5, A must be a linear algebraic group of dimension > 1. However, infinite simple linear algebraic groups are all dimension 1. Hence, the binding group of p must be finite. In particular, every realization of p has a finite $F \cup \mathcal{C}$ orbit, and hence must be algebraic over the constants. Since $\mathcal{C}$ is an algebraically closed field, p is realized by a tuple of constants, say $c_1, c_2 \in \mathcal{C}$. But, $2 = \dim(p) = \mathrm{trdeg}(c_1, c_2/F)$ and so $c_1 \in \mathrm{acl}(Fc_1c_2) \setminus \mathrm{acl}(Fc_2)$ and $c_2 \notin \mathrm{acl}(Fc_1)$. Hence, p does not satisfy exchange, a contradiction. Thus p must be minimal. Since $\dim(p) = 2$, by Fact 5.1.2, p is strongly minimal. $\square$

5.2 Strong minimality and the Lotka-Volterra system

In this section, we prove Theorem 5A. We first provide a brief introduction to the generic types of the vector fields induced by the Lotka-Volterra equations. The classical Lotka-Volterra system of equations is:

$$LV_{a,b,c,d} := \begin{cases} \delta(x) = x(ay + b) \\ \delta(y) = y(cx + d) \end{cases}$$

where a, b, c, d are non-zero elements of a field of an algebraically closed field of constants, F. This system of differential equations allows us to define an algebraic vector field $(\mathbb{A}^2, s_{a,b,c,d})$

over F where $s_{a,b,c,d}(x,y) = (x, y, x(ay+b), y(cx+d))$. This is an algebraic vector field of dimension 2, hence minimality of its generic type is equivalent to strong minimality.

Duan and Nagloo [17] show that when b and d are $\mathbb{Q}$-linearly independent, the generic type over F of $(\mathbb{A}^2, s_{a,b,c,d})$ is strongly minimal. They also show that when $b = d$, the type is not strongly minimal. We improve upon this result to show that the generic type of $(\mathbb{A}^2, s_{a,b,c,d})$ is strongly minimal if and only if $b \neq d$. To use Corollary 5.1.5, we need to understand the 1-dimensional restrictions of $(\mathbb{A}^2, s_{a,b,c,d})$ over any field of constant parameters.

Example 5.2.1. Let $(\mathbb{A}^2, s_{a,b,c,d})$ be the algebraic vector field corresponding to a Lotka-Volterra system of equations over F, an algebraically closed field of constants.

1. Let $X, Y \subseteq \mathbb{A}^2$ be subvarieties given by $x = 0$ and $y = 0$, respectively. Then $(X, s_{a,b,c,d}|_X)$ and $(Y, s_{a,b,c,d}|_Y)$ are 1-dimensional algebraic vector field restricions of $(\mathbb{A}^2, s_{a,b,c,d})$.
2. Let $X \subseteq \mathbb{A}^2$ be given by $cx - ay = 0$. If $b = d$, $(X, s_{a,b,c,b}|_X)$ is a 1-dimensional algebraic vector field restriction of $(\mathbb{A}^2, s_{a,b,c,d})$.

Proof. 1. Let $s := s_{a,b,c,d}$. To show $s|_X : X \to TX$, we need only check that for any $(u, v) \in X$ is a solution to $\delta(x) = 0$, since TX is defined by $\begin{cases} x = 0 \\ \delta(x) = 0. \end{cases}$ Since $u = 0$, $\delta(u) = 0$. Then $\dim(0, v/F) = \operatorname{trdeg}(v, \delta(v), .../F) = 1$ since $\delta(v) = v(c0+d) = dv \in F(v)$. Hence, $(X, s|_X)$ is a 1-dimensional algebraic vector field. Similarly, if Y is the algebraic subvariety of $\mathbb{A}^2$ defined by $y = 0$, then $(Y, s|_Y)$ is a 1-dimensional algebraic vector field.

2. Let $s := s_{a,b,c,d}$. To show that $s|_X : X \to TX$, we show that for any $(u, v) \in X$, (u, v) is a solution to $\delta(cx - ay) = 0$.

By definition, $\delta(cx - ay) = c\delta(x) - a\delta(y)$. Let $(u, v) \in X$. Then $u = \frac{a}{c}v$, hence $\delta(cu - av) = c\delta(u) - a\delta(v) = c\delta(\frac{a}{c}v) - a\delta(v) = a\delta(v) - a\delta(v) = 0$. Thus, $(X, s|_X)$ is an algebraic vector field. Since $u, \delta(u), \delta(v) \in F(v)$, $\dim(u, v/F) = 1$. Hence, $(X, s|_X)$ is a 1-dimensional algebraic vector field. □

In Lemma 3.1 of [16], it is shown that these are the only 1-dimensional algebraic vector field restrictions of $(\mathbb{A}^2, s_{a,b,c,d})$ over any field of constants.

Fact 5.2.2. *[16, Lemma 3.1] If $b \neq d$, the only irreducible 1-dimensional algebraic vector field restrictions of $(\mathbb{A}^2, s_{a,b,c,d})$, over any field of constants, are $(X, s_{a,b,c,d}|_X)$ and*

$(Y, s_{a,b,c,d}|_Y)$ where X is defined by $x = 0$ and Y is defined by $y = 0$. If $b = d$, the only other irreducible algebraic vector field restrictions of $(\mathbb{A}^2, s_{a,b,c,d})$, over any field of constants, is $(Z, s_{a,b,c,d}|_Z)$ where Z is defined by $cx - ay = 0$.

Proposition 5.2.3. *Let F be an algebraically closed field of constants and $a, b, c, d \in F$ are non-zero. Let p be the generic type over F of the algebraic vector field $(\mathbb{A}^2, s_{a,b,c,d})$ where $s(x, y) = (x, y, x(ay + b), y(cx + d))$. If $b \neq d$, then p is strongly minimal.*

Proof. Suppose that $b \neq d$. Let $s := s_{a,b,c,d}$. To show strong minimality using Corollary 5.1.5, we find a point in $(\mathbb{A}^2, s)^\sharp(\mathcal{C})$ that is not contained in any 1-dimensional vector field restriction of $(\mathbb{A}^2, s)$ over the constants. Consider the point $(\frac{-d}{c}, \frac{-b}{a}) \in \mathcal{C}^2$. Then $\frac{-d}{c}(a\frac{-b}{a} + b) = 0 = \delta(\frac{-d}{c})$ and $\frac{-b}{a}(c\frac{-d}{c} + d) = 0 = \delta(\frac{-b}{a})$. Hence, $(\frac{-d}{c}, \frac{-b}{a}) \in (\mathbb{A}^2, s)^\sharp$. By Fact 5.2.2, the only 1-dimensional vector field restrictions of $(\mathbb{A}^2, s)$ over $\mathcal{C}$ are $(X, s|_X)$ with X the zero set of $x = 0$ and $(Y, s|_Y)$ with Y the zero set of $y = 0$. Since $b, d \neq 0$, $(\frac{-d}{c}, \frac{-b}{a}) \notin (X, s|_X)$ and $(\frac{-d}{c}, \frac{-b}{a}) \notin (Y, s|_Y)$. Thus, p is strongly minimal. □

That p is not strongly minimal when $b = d$ is well-known, see for example [17, subsubsection 3.1.1]. Hence, the generic type of $(\mathbb{A}^2, s_{a,b,c,d})$ over an algebraically closed field of constants is strongly minimal if and only if $b \neq d$. A consequence of Hrushovski and Sokolovic's classification of non-trivial strongly minimal sets in DCF_0 [37], is that most strongly minimal types over constant parameters are trivial. See [12, Proposition 5.8] for a proof:

Fact 5.2.4. *Let F be an algebraically closed field of constants and let $p \in S(F)$ be strongly minimal. If the dimension of p is greater than 1, then p is trivial.*

When $b \neq d$, we have that the generic type p over F of the algebraic vector field $(\mathbb{A}^2, s_{a,b,c,d})$ is strongly minimal and dimension 2, hence p is a trivial type. This triviality is useful in proving theorems 5B and 5C. Additionally, p is orthogonal to the constants by Zilber's dichotomy (Fact 2.2.16).

In terms of transcendence degree, p is trivial means that for any m realizations of p, $(u_1, v_1), ..., (u_m, v_m)$, if $\mathrm{trdeg}(u_1, v_1, ..., u_m, v_m/F) < 2m$, then for some $1 \leq i < j \leq m$, $\mathrm{trdeg}(u_i, v_i, u_j, v_j/F) < 4$. We can consider another condition on the realizations of p, which is an intermediary step in proving Theorem 5B.

Definition 5.2.5. A stationary type $p \in S(F)$ is *totally disintegrated* if any n distinct realizations of p form an F-independent set.

In terms of transcendence degree, the generic type p over F of the D-variety $(\mathbb{A}^2, s)$, is totally disintegrated means that for any distinct m realizations of p, $\operatorname{trdeg}(u_1, v_1, ..., u_m, v_m/F) = 2m$.

A key idea behind the proof for Theorem 5B is a result of Brestovski which says that given two algebraically dependent generic solutions to a differential equation of a specific form which defines a strongly minimal set, there is a relation of a specific form between these two solutions.

Fact 5.2.6. *[9, Theorem 2] Let F be an algebraically closed field of constants and let $f, f_1, ..., f_n \in F\{x\}$ be differential polynomials of order ≤ 1 and $c_1, ..., c_n \in F$ be $\mathbb{Q}$-linearly independent. Suppose that the generic type over F of the irreducible equation*

$$\delta(f) = \sum_{i=1}^{n} c_i \frac{\delta(f_i)}{f_i}$$

is strongly minimal. Given u, v generic solutions to the equation, if u, v are dependent over F, then $f(u)^m = f(v)^m$ for some $m \neq 0$

Duan, Jimenez and I give a partial generalization of Brestovski's theorem for relations between solutions to two distinct differential equations of the same form without the assumption that $c_1, ..., c_n$ are $\mathbb{Q}$-linearly independent (although the assumption of $\mathbb{Q}$-linear independence is not necessary in Brestovski's proof). This generalization also allows one to say something about the algebraic relations for realizations of strongly minimal generic types of non-strongly minimal definable sets, which is the case in which we apply the result. The relation obtained is weaker than Brestovski's conclusion, and is an intermediary step in Brestovski's original proof. The proof relies heavily on differential forms.

Fact 5.2.7. *[16, Theorem 2.21] Let F be an algebraically closed field of constants and $c_1, ..., c_n, d_1, ..., d_m \in F$. Let $f, f_1, ..., f_n, g, g_1, ..., g_m \in F\{x\}$ be differential polynomials of order ≤ 1. Consider the differential equations*

$$\delta(f) = \sum_{i=1}^{n} c_i \frac{\delta(f_i)}{f_i} \tag{E_1}$$

$$\delta(g) = \sum_{i=1}^{m} d_i \frac{\delta(g_i)}{g_i}. \tag{E_2}$$

For $i = 1, 2$, let u_i be a generic solution of E_i such that the type $q_i = \operatorname{tp}(u_i/F)$ is strongly minimal and 2-dimensional. For any $v_1 \models q_1$ and $v_2 \models q_2$ which are dependent over F, then $f(v_1) = eg(v_2) + l$ for some constants e and l.

Corollary 5.2.8. *Let F be an algebraically closed field of constants and $a, b, c, d \in F$ be non-zero. If $b \neq d$, the generic type over F of the algebraic vector field induced by a Lotka-Volterra system, $(\mathbb{A}^2, s_{a,b,c,d})$, is totally disintegrated.*

Proof. Let $s := s_{a,b,c,d}$ and p be the generic type over F of $(\mathbb{A}^2, s)$. Since $b \neq d$, by Proposition 5.2.3, p is strongly minimal (equivalently $\mathrm{RM}(p) = 1$). Moreover, p is trivial by Fact 5.2.4. Let $(u_1, v_1), \ldots., (u_m, v_m) \models p$ be a non-F-independent. So, we show that for some $i \neq j$, $(u_i, v_i) = (u_j, v_j)$. Since our sequence of realizations of a minimal type is not F-independent and p is trivial, it cannot be a pairwise F-independent sequence. Without loss of generality let $u_1 v_1 \not\perp\!\!\!\perp_F v_2 v_2$. In particular, by minimality of p, $F\langle u_1, v_1\rangle^{\mathrm{alg}} = F\langle u_2, v_2\rangle^{\mathrm{alg}}$.

For $i = 1, 2$, since $(u_i, v_i) \in (\mathbb{A}^2, s)^\sharp$, we have that

$$\delta(cu_i - av_i) = cbu_i - adv_i = b\frac{\delta(v_i)}{v_i} - d\frac{\delta(u_i)}{u_i}.$$

Additionally we compute

$$\begin{aligned}\delta(cu_i - av_i) - b(cu_i - av_i) &= (d-b)av_i, \text{ and}\\ \delta(cu_i - av_i) - d(cu_i - av_i) &= (d-b)cu_i.\end{aligned}$$

Since $b \neq d$, $F\langle cu_i - av_i\rangle = F\langle u_i, v_i\rangle$. In particular, $cu_i - av_i$ is interdefinable over F with (u_i, v_i) and $cu_1 - av_1 \in F\langle cu_2 - av_2\rangle^{\mathrm{alg}}$. Since $p = \mathrm{tp}(u_i, v_i/F)$ and $\mathrm{RM}(p) = 1$, by iteralgebraicity over F, $\mathrm{RM}(cu_i - av_i/F) = 1$. Hence, $\mathrm{tp}(cu_i - av_i/F)$ is also strongly minimal for $i = 1, 2$.

Let $f := x$, $f_1 := \delta(x) - bx$, $f_2 := \delta(x) - dx \in F\{x\}$. Then, $cu_1 - av_1$ and $cu_2 - av_2$ satisfy the equation

$$\delta(f) = b\frac{\delta(f_1)}{f_1} - d\frac{\delta(f_2)}{f_2}. \tag{E_1}$$

Also note that $2 = \dim(p) = \dim(\mathrm{tp}(cu_1 - av_1/F)) = \dim(\mathrm{tp}(cu_2 - av_2/F))$ and $\mathrm{tp}(cu_1 - av_1/F)$ and $\mathrm{tp}(cu_2 - av_2/F)$ are strongly minimal. Since $cu_1 - av_1 \not\perp\!\!\!\perp_F cu_2 - av_2$, we have that $cu_1 - av_1$ and $cu_2 - av_2$ satisfy the assumptions of Fact 5.2.7 with $(E_1) = (E_2)$. Thus, there exists $e, l \in \mathcal{C}$ such that

$$(cu_1 - av_1) = e(cu_2 - av_2) + l. \tag{1}$$

Note that $e \neq 0$ since $cu_1 - av_1 \notin \mathcal{C}$, otherwise $1 = \dim(cu_1 - av_1/F) = \dim(p) = 2$, as p is finite-dimensional and u_1, v_1 is interalgebraic over F with $cu_1 - av_1$.

Differentiating this equation and using that $\delta(u_i) = u_i(av_i + b)$ and $\delta(v_i) = v_i(cu_i + d)$, since $(u_i, v_i) \in (\mathbb{A}^2, s_{a,b,c,d})^\sharp$ for $i = 1, 2$, we obtain the equality

$$bcu_1 - adv_2 = ebcu_2 - eadv_2 \tag{2}$$

Computing $b(1) - (2)$ and $d(1) - (2)$ we obtain the expressions

$$\begin{aligned} u_1 &= eu_2 - \frac{d}{c(b-d)} l, \\ v_1 &= ev_2 - \frac{b}{a(b-d)} l. \end{aligned}$$

By the above equation, $\delta(u_1) = eu_2(av_2 + b)$ and $\delta(v_1) = ev_2(cu_2 + d)$. Using these together with $\delta(u_1) = u_1(av_1 + b)$ and $\delta(v_1) = v_1(ca_1 + d)$, since $(u_1, v_1) \in (\mathbb{A}^2, s)^\sharp$, we obtain a polynomial expression in u_2 and v_2:

$$ae(e-1)u_2v_2 - \left(e\frac{b}{(b-d)} l\right) u_2 - \left(ae\frac{-d}{c(b-d)} l\right) v_2 + \frac{bd}{c(b-d)^2} l^2 - \frac{bd}{c(b-d)} l = 0.$$

If the coefficients of the polynomial are not all zero, then

$$\dim(u_2, v_2/F(e,l)^{\mathrm{alg}}) = \mathrm{trdeg}(u_2, v_2/F(e,l)^{\mathrm{alg}}) = 1 < 2 = \dim(u_2, v_2/F).$$

Hence, $u_2, v_2 \underset{F}{\not\perp} e, l$, where $e, l \in \mathcal{C}$, contradicting that p is orthogonal to the constants as p is minimal and trivial. Hence, $l = 0$ and $e = 1$ which gives that $u_1 = u_2$ and $v_1 = v_2$.

We have shown that any set of distinct realizations for p is F-independent. Hence, p is totally disintegrated. $\square$

Note that we cannot directly apply Brestovski's original theorem to $cu_1 - av_1$ and $cu_2 - av_2$ instead of Fact 5.2.7 since b and d may be $\mathbb{Q}$-linearly dependent. Additionally, the differential equation (E_1) is not irreducible.

5.3 Algebraic relations between solutions of Lotka-Volterra systems

In this section we show that under a mild condition, the generic types of distinct rational vector fields induced by distinct Lotka-Volterra equations are orthogonal. This result together with total disintegration of the generic types give Theorem 5B. To obtain the mild condition for orthogonality, we first look at what relations exist between realizations of distinct generic types.

Example 5.3.1. Let F be an algebraically closed field of constants and $a, b, c, d \in F$ be non-zero. Let p_1 be the generic type over F of the algebraic vector $(\mathbb{A}^2, s_{a,b,c,d})$.

1. Let p_2 be the generic type over F of the algebraic vector field $(\mathbb{A}^2, s_{1,b,1,d})$. We have definable maps $f : p_1 \to p_2$ given by $(u, v) \mapsto (cu, av)$ and $g : p_2 \to p_1$ given by $(u, v) \mapsto (\frac{u}{c}, \frac{v}{a})$. Hence, p_1 and p_2 are interdefinable over F. In particular, $p_1 \not\perp p_2$.

2. Let p_2 be the generic type over F of the algebraic vector field $(\mathbb{A}^2, s_{a,d,c,b})$. The function $g : p_1 \to p_2$ given by $(u, v) \mapsto (v, u)$ is definable. Hence, p_1 and p_2 are interdefinable over F. In particular, $p_1 \not\perp p_2$.

In both examples, we obtain something stronger than the two types being non-orthogonal, but rather that the types are interdefinable. For two types to be orthogonal we thus need to rule out the possibilities:

1. $b_1 = b_2$ and $d_1 = d_2$,

2. $b_1 = d_2$ and $d_1 = b_2$.

The proof that these are the only possible relations between two generic types of rational vector fields induced by distinct Lotka-Volterra systems is similar to that of Theorem 5.2.8.

Theorem 5.3.2. *Let F be an algebraically closed field of constants and fix non-zero $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in F$. For $i = 1, 2$, let p_i be the generic type over F of $(\mathbb{A}^2, s_i)$ where $s_i(x, y) = (x, y, x(a_i y + b_i), y(c_i x + d_i))$. If*

1. *$b_1 \neq d_1$, $b_2 \neq d_2$, and*

2. *$\{b_1, d_1\} \neq \{b_2, d_2\}$,*

then p_1 is orthogonal to p_2.

Proof. Since $b_1 \neq d_1$ and $b_2 \neq d_2$, by Proposition 5.2.3, p_1 and p_2 are strongly minimal types. As p_1 and p_2 are both 2-dimensional, by Fact 5.2.4, p_1 and p_2 are trivial types, and thus are orthogonal to the constants. Since p_1 and p_2 are trivial types, by Fact 2.2.10, $p_1 \perp p_2$ if and only if $p_1 \perp^w p_2$. Hence, it suffices to show that for any $(u_1, v_1) \models p_1$ and $(u_2, v_2) \models p_2$, $u_1v_1 \underset{F}{\downarrow} u_2v_2$. Suppose that $u_1v_1 \underset{F}{\not\downarrow} u_2v_2$ and derive a contradiction.

For $i = 1, 2$, since $(u_i, v_i) \in (\mathbb{A}^2, s_i)^\sharp$, $\delta(c_iu_i - a_iv_i) = b_i\frac{\delta(v_i)}{v_i} - d_i\frac{\delta(u_i)}{u_i}$. Additionally,

$$\begin{cases} \delta(c_iu_i - a_iv_i) - d_i(c_iu_i - a_iv_i) = (d_i - b_i)c_iu_i, \\ \delta(c_iu_i - a_iv_i) - b_i(c_iu_i - a_iv_i) = (d_i - b_i)a_iv_i \end{cases}.$$

Since $b_i \neq d_i$, $F\langle c_iu_i - a_iv_i\rangle = F\langle u_i, v_i\rangle$. Hence, $\mathrm{tp}(c_iu_i - a_iv_i/F)$ is interdefinable over F with $\mathrm{tp}(u_i, v_i/F)$.

For strong minimality, since p_1 and p_2 are strongly minimal, they have Morley rank 1. By interdefinability, $\mathrm{tp}(c_1u_1 - a_1v_1/F)$ and $\mathrm{tp}(c_2u_2 - a_2v_2/F)$ are also strongly minimal.

Let $f := x$, $f_1 := \delta(x) - b_1x$, $f_2 := \delta(x) - d_1x \in F\{x\}$. Then f, f_1, f_2 are differential polynomials of order ≤ 1, $c_1u_1 - a_1v_1$ is a solution of the equation

$$\delta(f) = b_1\frac{\delta(f_1)}{f_1} - d_1\frac{\delta(f_2)}{f_2} \tag{E_1}$$

with $\mathrm{tp}(c_1u_1 - a_1v_1/F)$ strongly minimal and of dimension 2. For the second equation let $g := x$, $g_1 := \delta(x) - b_2x$, $g_2 := \delta(x) - d_2x \in F\{x\}$. Then g, g_1, g_2 are differential polynomials of order ≤ 1, $c_2u_2 - a_2v_2$ is a solution of the equation

$$\delta(g) = b_2\frac{\delta(g_1)}{g_1} - d_2\frac{\delta(g_2)}{g_2} \tag{E_2}$$

with $\mathrm{tp}(c_2u_2 - a_2v_2/F)$ strongly minimal and of dimension 2. By assumption, $c_1u_1 - a_1v_1$ and $c_2u_2 - a_2v_2$ are dependent over F. Hence, (E_1), (E_2), $c_1u_1 - a_1v_1$ and $c_2u_2 - a_2v_2$ satisfy the assumptions of Fact 5.2.7. Thus, there is some $e, l \in \mathcal{C}$ such that

$$c_1u_1 - a_1v_1 = e(c_2u_2 - a_2v_2) + l. \tag{1}$$

Differentiating (1) and using that $\delta(u_i) = u_i(a_iv_i + b_i)$ and $\delta(v_i) = v_i(c_iu_i + d_i)$ for $i = 1, 2$, we obtain that

$$b_1c_1u_1 - a_1d_1v_2 = eb_2c_2u_2 - ea_2d_2v_2. \tag{2}$$

Using $b_1(1) - (2)$ and $d_1(1) - (2)$ we obtain the expressions

$$\begin{cases} u_1 = e\frac{c_2}{c_1}(\frac{b_2-d_1}{b_1-d_1})u_2 + e\frac{a_2}{c_1}(\frac{d_2-d_1}{b_1-d_1})v_2 - \frac{d_1}{c_1(b_1-d_1)}l \\ v_1 = e\frac{c_2}{a_1}(\frac{b_2-b_1}{b_1-d_1})u_2 + e\frac{a_2}{a_1}(\frac{b_1-d_2}{b_1-d_1})v_2 - \frac{b_1}{a_1(b_1-d_1)}l \end{cases}$$

Let $A = \frac{c_2}{c_1}(\frac{b_2-d_1}{b_1-d_1})$, $B = \frac{a_2}{c_1}(\frac{d_2-d_1}{b_1-d_1})$, $C = \frac{c_2}{a_1}(\frac{b_2-b_1}{b_1-d_1})$ and $D = \frac{a_2}{a_1}(\frac{b_1-d_2}{b_1-d_1})$. Note that since $\{b_1, d_1\} \neq \{b_2, d_2\}$, $A, B, C, D \neq 0$.

Since $(u_i, v_i) \in (\mathbb{A}^2, s_i)$, $\delta(u_i) = u_i(a_iv_i + b_i)$ and $\delta(v_i) = v_i(c_iu_i + d_i)$ for $i = 1, 2$. Using these identities and (2) we obtain a polynomial expression in u_2, v_2.

$$\begin{aligned} 0 = e^2a_1ACu_2^2 + e^2a_1BCv_2^2 + e(ea_1AD + ea_1BC - a_2A - c_2B)u_2v_2 \\ +e\left((b_1 - b_2)A - \frac{l}{b_1 - d_1}\left(\frac{b_1}{a_1}A + \frac{d_1}{c_1}C\right)\right)u_2 \\ +e\left((b_1 - d_2)B - \frac{l}{b_1 - d_1}\left(\frac{b_1}{a_1}B + \frac{d_1}{c_1}D\right)\right)v_2 \\ +l^2\frac{b_1d_1}{a_1c_1(b_1 - d_1)^2} + l\frac{b_1d_1}{c_1(b_1 - d_1)} \end{aligned}$$

If the coefficients in the expression are not all zero, then $\operatorname{trdeg}(u_1, v_1/F(e, l)^{\text{alg}}) < 2$, hence by strong minimality of $\operatorname{tp}(u_2, v_2/F)$, $u_2, v_2 \in F(e, l)^{\text{alg}} \setminus F$. But, $e, l \in \mathcal{C}$ and $p_2 \not\perp \mathcal{C}$, a contradiction. Since $A, C \neq 0$ and $e^2AC = 0$, we get that $e = 0$. Then $c_1u_1 - a_1v_1 = l \in \mathcal{C}$, contradicting that $p_1 \perp \mathcal{C}$. Thus, $u_1, v_1 \underset{F}{\downarrow} u_2, v_2$ showing that $p_1 \perp p_2$. $\square$

The orthogonality gives the following transcendence result over the whole field of constants which is Theorem 5B.

Corollary 5.3.3. *Let F be an algebraically closed field of constants and for $i = 1, ..., m$, let $a_i, b_i, c_i, d_i \in F$ be non-zero. Let p_i be the generic type over F of the algebraic vector field $(\mathbb{A}^2, s_i)$ with $s_i(x, y) = (x, y, x(a_iy + b_i), y(c_ix + d_i))$. Let $(u_1, v_1), ..., (u_m, v_m)$ be distinct realizations of any of the p_is. If for all $i = 1, ..., m$*

1. $b_i \neq d_i$*, and*

2. for all $i \neq j$*,* $\{b_i, d_i\} \neq \{b_j, d_j\}$*,*

then $\operatorname{trdeg}(u_1, v_1, ..., u_m, v_m/\mathcal{C}) = 2m$.

Proof. For every $j = 1, ..., m$, there is some $i = 1, ..., n$ such that $(u_j, v_j) \models p_i$. Note that when $j \neq k$, it is possible for (u_j, v_j) and (u_k, v_k) to realize the same type. After relabeling and changing n, we may assume that for all $i = 1, ..., n$, there is some $j = 1, ..., m$ such that $(u_j, v_j) \models p_i$. For every $i = 1, ..., n$, $b_i \neq d_i$, hence p_i is strongly minimal, by Proposition 5.2.3, and trivial, by Fact 5.2.4. In particular, for every $i = 1, ..., n$, p_i is orthogonal to the constants. We first show that $\operatorname{trdeg}(u_1, v_1, ..., u_m, v_m/F) = 2m$. Notice that for all $i = 1, ..., m$, $\delta(u_i), \delta(v_i) \in F(u_i, v_i)$, hence $F\langle u_i, v_i\rangle = F(u_i, v_i)$.

For every $i = 1, ..., n$, let $(u_{i_1}, v_{i_1}, ..., u_{i_{m_i}}, v_{i_{m_i}})$ be the tuple of all (u_j, v_j) realising p_i. Then $m = m_1 + \cdots + m_n$ and by assumption $m_i > 0$ for all $i = 1, ..., n$. Since $b_i \neq d_i$, by Corollary 5.2.8, p_i is totally disintegrated for every $i = 1, ..., n$. Thus, for all $i = 1, ..., n$, $(u_{i_1}, v_{i_1}, ..., u_{i_{m_i}}, v_{i_{m_i}}) \models p_i^{(m_i)}$. In particular, if $n = 1$ then $\operatorname{trdeg}(u_1, v_1, ..., u_m, v_m/F) = 2m$. So we may assume that $m > 1$.

Suppose $(u_1, v_1, ..., u_m, v_m)$ is not F-independent and derive a contradiction. Let $k \leq m$ be minimal such that there is a subset of $(u_1, v_1), ..., (u_k, v_k)$ which is not F-independent. After relabelling, $u_1, v_1 \not\mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_k, v_k$ and $u_1, v_1 \mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_{k-1}, v_{k-1}$. By total disintegration of the p_i, not all $(u_1, v_1), ..., (u_k, v_k)$ realize the same type. After reordering, we may assume that (u_1, v_1) and (u_k, v_k) realize different types, say p_i and p_j respectively. Since $i \neq j$ $\{b_i, d_i\} \neq \{b_j, d_j\}$, $b_i \neq d_i$, and $b_j \neq d_j$, by Theorem 5.3.2, $p_i \perp p_j$. On the other hand, if $u_k, v_k \not\mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_{k-1}, v_{k-1}$, by minimality of the p_is, $u_k, v_k \in \operatorname{acl}(Fu_2, v_2, ..., u_{k-1}, v_{k-1})$. Thus $u_1, v_1 \not\mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_{k-1}, v_{k-1}$, a contradiction. We have that $u_1, v_1 \mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_{k-1}, v_{k-1}$, $u_k, v_k \mathrel{\underset{F}{\perp}} u_2, v_2, ..., u_{k-1}, v_{k-1}$ and $u_1, v_1 \not\mathrel{\underset{Fu_2,v_2,...,u_{k-1},v_{k-1}}{\perp}} u_k, v_k$. Hence, $p_i \not\perp p_j$, a contradiction to $p_i \perp p_j$. We have shown that $(u_1, v_1), ..., (u_m, v_m)$ is F-independent.

Note that if $p_1^{(m_1)} \otimes \cdots \otimes p_n^{(m_n)} \not\perp \mathcal{C}$, then there is $K \supseteq F_1, ..., F_n$, $(a_{1,1}, ..., a_{1,m_1}, ..., a_{n,1}, ..., a_{n,m_n}) \models (p_1^{(m_1)} \otimes \cdots \otimes p_n^{(m_n)})|_K$ and a tuple $\overline{c} \subset \mathcal{C}$ such that $a_{1,1}, ..., a_{1,m_1}, ..., a_{n,1}, ..., a_{n,m_n} \not\mathrel{\underset{K}{\perp}} \overline{c}$. Thus for some $j \leq n, k \leq m_j$ we have that

$a_{j,k} \underset{K}{\perp} a_{1,1}, ..., a_{1,m_1}, ..., a_{j,k-1}$ and $a_{j,k} \underset{Ka_{1,1},...,a_{1,m_1},...,a_{j,k-1}}{\not\perp} \bar{c}$, contradicting that $p_j \perp \mathcal{C}$. Thus $p_1^{(m_1)} \otimes \cdots \otimes p_n^{(m_n)} \perp \mathcal{C}$. Hence, $\text{trdeg}(u_1, v_1, ..., u_m, v_m/\mathcal{C}) = \text{trdeg}(u_1, v_1, ..., u_m, v_m/F) = 2m$ $\square$

5.4 The non-strongly minimal case

We saw that when $b = d$, the generic type of p of $(\mathbb{A}^2, s_{a,b,c,b})$ is not strongly minimal. Hence, there is some non-trivial algebraic relations between realizations of its generic type. To understand the possible algebraic relations between realizations of the generic type, we classify all 1-dimensional D-subvarieties of $(\mathbb{A}^2, s_{a,b,c,d})$ over any, potentially non-constant, differential field extending F.

Let C be the algebraic variety defined by $cx - ay - z_0 = 0$ for any $z_0 \in \mathcal{U}$ satisfying $\delta(z) = bz$. Then $(C, s_{a,b,c,b}|_C)$ is a 1-dimensional D-subvariety of $(\mathbb{A}^2, s_{a,b,c,d})$ over $F(z_0)$. To see this we show that $s_{a,b,c,d}|_C : C \to \tau C$ is a section map to $\pi : \tau C \to C$. Since $\tau C \subseteq \mathbb{A}^4$ is defined by

$$\begin{cases} cx - ay - z_0 = 0 \\ \delta(cx - ay - z_0) = 0 \end{cases}$$

and for any $(u, v) \in C$, $\delta(cu - av - z_0) = b(cu - av - z_0) = 0$. We have that for any $(u, v) \in C$, $s_{a,b,c,d}(u, v) = (u, v, u(av + b), v(cu + b)) \in \tau C$. Hence, $(C, s_{a,b,c,d}|_C)$ is a D-subvariety of dimension 1.

Proposition 5.4.1. *Let F be an algebraically closed field of constants with non-zero $a, b, c \in F$. Let $(\mathbb{A}^2, s)$ be the algebraic vector field given by $s(x, y) = (x, y, x(ay + b), y(cx + b))$. The only irreducible 1-dimensional D-subvarieties of $(\mathbb{A}^2, s)$, over any differential field extension of F, are the D-subvarieties given by $x = 0$, $y = 0$ and $cx - ay - z_0 = 0$ for all $z_0 \in \mathcal{U}$ satisfying $\delta(z) = bz$.*

Proof. Let p be the generic type of $(\mathbb{A}^2, s)$ over F. Let r be the generic type of the constants over $\emptyset$. By Fact 5.2.2, the only 1-dimensional D-subvarities of $(\mathbb{A}^2, s)$ over F are given by $x = 0$, $y = 0$ and $cx - ay = 0$, all of which are irreducible. Let $L \supseteq F$ and p' be the generic type over L of an irreducible 1-dimensional D-subvariety (C, s) of $(\mathbb{A}^2, s)$ over L. Fix $(u, v) \models p'$. Then $\text{trdeg}(u, v/L) = 1$. If $u, v \underset{F}{\perp} L$, then $\text{trdeg}(u, v/F) = 1$. Hence, (u, v) is in one of the 1-dimensional D-subvarieties of $(\mathbb{A}^2, s)$ over F, (C', s). Then $C \subseteq C'$ and since C' is irreducible, $C = C'$. Thus $(C, s) = (C', s)$.

So we may assume that $u, v \not\mathop{\smile\!\!\!\!\!\!\mid}_F L$, and hence $\mathrm{trdeg}(u, v/F) = 2$. In particular, (u, v) realizes p. Recall that, by Lemma 3.3.3, $((u, v), \frac{u}{a} - \frac{v}{c}, 1)$ is a $\mathcal{C}$-analysis for p. Let $q = \mathrm{tp}(\frac{u}{a} - \frac{v}{c}/F)$, so $U(q) = 1$ and q is almost $\mathcal{C}$-internal.

If $\frac{u}{a} - \frac{v}{c} \mathop{\smile\!\!\!\!\!\!\mid}_F L$, then $\mathrm{trdeg}(\frac{u}{a} - \frac{v}{c}/L) = 1$. Since $L(\frac{u}{a} - \frac{v}{c}) \subseteq L(u, v)$ and $\mathrm{trdeg}(u, v/L) = 1$, we have that $L(u, v)^{\mathrm{alg}} = L(\frac{u}{a} - \frac{v}{c})^{\mathrm{alg}}$. Since $(u, v) \in (\mathbb{A}^2, s_{a,b,c,d})^\sharp$, $\delta(\frac{u}{a} - \frac{v}{c}) = b(\frac{u}{a} - \frac{v}{c})$. We show that $\frac{u}{a} - \frac{v}{c} \in L^{\mathrm{alg}}$, and hence every L-conjugate of $\frac{u}{a} - \frac{v}{c}$ also satisfies the equation $\delta(z) = bz$.

Suppose that $\frac{u}{a} - \frac{v}{c} \notin L^{\mathrm{alg}}$. We show this implies that $p := \mathrm{tp}(u, v/F)$, is almost $\mathcal{C}$-internal, contradicting Proposition 3.3.2. Since $\frac{u}{a} - \frac{v}{c} \notin L^{\mathrm{alg}}$ and $U(\frac{u}{a} - \frac{v}{c}/F) = 1$, we have that $\frac{u}{a} - \frac{v}{c} \mathop{\smile\!\!\!\!\!\!\mid}_F L$.

We have that $\frac{u}{a} - \frac{v}{c} \mathop{\smile\!\!\!\!\!\!\mid}_F L$ and $u, v \not\mathop{\smile\!\!\!\!\!\!\mid}_F L$, hence $u, v \not\mathop{\smile\!\!\!\!\!\!\mid}_{F, \frac{u}{a} - \frac{v}{c}} L$. By definition of a $\mathcal{C}$-analysis, $\mathrm{stp}(u, v/F, \frac{u}{a} - \frac{v}{c})$ is almost internal to $\mathcal{C}$. Note that this type is not algebraic since $U(u, v/F, \frac{u}{a} - \frac{v}{c}) = U(u, v/F) - U(\frac{u}{a} - \frac{v}{c}/F) = 1$. Thus, $\mathrm{tp}(u, v/F, \frac{u}{a} - \frac{v}{c})$ is non-algebraic and almost $\mathcal{C}$-internal.

Let $w := (u, v)$, $q := \mathrm{tp}(\frac{u}{a} - \frac{v}{c}/F)$ and $f : p \to q$ be the definable function given by $u, v \mapsto \frac{u}{a} - \frac{v}{c}$. We first show that there is an e such that $\mathrm{tp}(e/F)$ is almost $\mathcal{C}$-internal, $w \not\mathop{\smile\!\!\!\!\!\!\mid}_F e$ and $f(w) \mathop{\smile\!\!\!\!\!\!\mid}_F e$. Let $C = \mathrm{Cb}(wf(w)/L)$, or rather C is the finite tuple inter-algebraic with the canonical base, and let $w_0 f(w_0), w_1 f(w_1), ..., w_m f(w_m)$ be a Morley sequence in $\mathrm{stp}(wf(w)/L)$ such that $w_0 f(w_0) = wf(w)$ and $C \subseteq \mathrm{acl}(w_0, ...w_m)$, which exists by Fact 2.2.3. Hence, $w \not\mathop{\smile\!\!\!\!\!\!\mid}_{Ff(w)} C$. Since $\mathrm{stp}(w_i/Ff(w_i))$ is almost $\mathcal{C}$-internal, by Lemma 2.2.23 $\mathrm{tp}(w_0, ..., w_m/Ff(w_0), ..., f(w_m))$ is almost $\mathcal{C}$-internal. By Lemma 2.2.23, $\mathrm{tp}(C/Ff(w_0), ..., f(w_m))$ is almost $\mathcal{C}$-internal. Since $f(w) \mathop{\smile\!\!\!\!\!\!\mid}_F L$, a simple forking calculus argument shows that $f(w_0), ..., f(w_m) \mathop{\smile\!\!\!\!\!\!\mid}_F L$ and $\mathrm{tp}(C/Ff(w_0), ..., f(w_m))$ is non-algebraic. By Fact 2.2.2, $C \subseteq \mathrm{acl}(L)$, hence $f(w_0), ..., f(w_m) \mathop{\smile\!\!\!\!\!\!\mid}_F L$, and so $f(w_0), ..., f(w_m) \mathop{\smile\!\!\!\!\!\!\mid}_F C$. By Lemma 2.2.23, $\mathrm{tp}(C/F)$ is almost $\mathcal{C}$-internal and non-algebraic. Let $\mathrm{dcl}(Fe) = \mathrm{Cb}(C/Fw)$. Since $w \not\mathop{\smile\!\!\!\!\!\!\mid}_{Ff(w)} C$ and $w \mathop{\smile\!\!\!\!\!\!\mid}_{Fe} C$, forking calculus yields $w \not\mathop{\smile\!\!\!\!\!\!\mid}_{Ff(w)} e$. By Fact 2.2.3, $e \in \mathrm{acl}(FC_0, ..., C_n)$ for some Morley product in $\mathrm{stp}(C/Fa)$. Since $\mathrm{stp}(C_i/F)$ is almost $\mathcal{C}$-internal, by Lemma 2.2.23 and Lemma 2.2.23 $\mathrm{stp}(e/F)$ is almost $\mathcal{C}$-internal and non-algebraic. On the other hand, by Fact 2.2.2, $e \in \mathrm{acl}(Fw)$.

We have that $e \in \mathrm{acl}(Fw)$ and $w \underset{Ff(w)}{\not\perp} e$. Since $e \in \mathrm{acl}(Fw)$, $U(e/F) \leq U(w/F) = 2$. If $U(e/F) = 2$, then $\mathrm{acl}(Fw) = \mathrm{acl}(Fz)$ showing that $p := \mathrm{tp}(w/F)$ is almost $\mathcal{C}$-internal which contradicts Proposition 3.3.2. Since $e \notin F^{\mathrm{alg}}$, we have that $U(e/F) = 1$ and $\mathrm{tp}(e/F)$ is stationary. Thus $\mathrm{tp}(e/F)$ is minimal. If $f(w) \underset{F}{\not\perp} e$, then $e \in \mathrm{acl}(Ff(w))$, which contradicts that $w \underset{Ff(w)}{\not\perp} e$. Hence $f(w) \underset{F}{\perp} e$. Then $U(f(w), e/F) = 2$, hence $\mathrm{acl}(Fw) = \mathrm{acl}(Ff(w)e)$. By Lemma 2.2.23, $\mathrm{tp}(f(w), e/F)$ is almost $\mathcal{C}$-internal, thus $p := \mathrm{tp}(w/F)$ is almost $\mathcal{C}$-internal by Lemma 2.2.23. This contradicts Proposition 3.3.2.

So it must be the case that $f(w) := \frac{u}{a} - \frac{v}{c} \in L^{\mathrm{alg}}$. Let $z_0 := \frac{u}{a} - \frac{v}{c}$ and let $z_0, ..., z_m$ be the finitely many L-conjugates of $\frac{u}{a} - \frac{v}{c}$. In particular, since $b \in L$, $\delta(z_i) = bz_i$ for all $i = 0, ..., m$.

Clearly $cx - ay - z_0 = 0 \in \mathrm{tp}(u, v/F, z_0)$. Hence, (u, v) is in the D-subvariety over L given by $cx - ay - z_0 = 0$. For any $(u', v') \models \mathrm{tp}(u, v/L)$, there exists $\sigma \in \mathrm{Aut}_L(\mathcal{U})$ such that $(u', v') = \sigma(u, v)$. Hence, $\sigma(z_0) = z_i$ for some $i = 0, ..., m$. Thus $cu' - av' - \sigma(z_0) = 0$. So, $C = C_0 \cup \cdots \cup C_m$ where C_i is the algebraic variety over L defined by $cx - ay - z_i = 0$. Since C is irreducible, $C = C_i$ for some $i = 0, ..., m$, and so (C, s) is given by the curve $cx - ay - z = 0$ for some solution of $\delta(z) = bz$. $\square$

To show that $\frac{u}{a} - \frac{v}{c} \in L^{\mathrm{alg}}$ we are using the idea that given a definable function $f : \mathrm{tp}(a/F) \to \mathrm{tp}(b/F)$, if $f(a) \underset{F}{\not\perp} d$ and $a \underset{F}{\perp} d$ for any d, we can choose d such that $\mathrm{tp}(d/F)$ is minimal. In the following chapter, we see that this form of argument is not unique to the Lotka-Volterra systems, or even the DCF_0 setting.

Chapter 6

Semiminimal fibrations

Recall that a stationary type is *semiminimal* if it is non-algebraic and almost internal to a minimal type. We then say a *fibration is semiminimal* if some (equivalently) any fiber is semiminimal. Such fibrations are ubiquitous: by an observation of Moosa and Pillay [58], any fibration that is indecomposable in the natural sense that it does not factor nontrivially through other fibrations, is semiminimal. The goal of this chapter is to give structure theorems which allow us to understand a semiminimal fibration of finite U-rank types in terms of the image and the fiber. These structure theorems are in terms of domination equivalence, a certain important equivalence relation on stationary types that we recall and discuss in Section 6.1 below.

In this chapter we work not necessarily in DCF_0, but rather in an arbitrary totally transcendental theory T eliminating imaginaries.

Our main result is:

Theorem 6. *Let $f : p \to q$ be a semiminimal fibration of stationary finite rank types. Let r be a minimal type such that some fiber of $f : p \to q$ is almost r-internal. Then p is domination equivalent to $r^{(m)} \otimes q$ for some $0 \leq m \leq U(p) - U(q)$.*

This appears as the main clause in Theorem 6.3.1. We show that even in DCF_0 the power m can be strictly between 0 and $U(p)-U(q)$. However, when the fibration is indecomposable we prove that it must be either 0 or $U(p) - U(q)$. This is Corollary 6.3.3 below. We get further precise information when the fibers are almost internal to a minimal *trivial* type; this is Theorem 6.3.6 below.

The contents of this chapter are based on joint work with Léo Jimenez and appear in [18]. Throughout this chapter, we work in a sufficiently saturated model $\mathcal{U} \models T$.

6.1 Preliminaries on domination

Some good references for domination are [2, Part A Chapter VI], [11, Section 5.6] and [63, Section 1.4.5].

Definition 6.1.1. 1. Let a, b be tuples and C a set. We say that a *dominates* b over C, denoted $a \trianglerighteq_C b$, if for any set d, whenever $a \underset{C}{\downarrow} d$, then $b \underset{C}{\downarrow} d$.

2. If $p, q \in S(C)$, we say that p *dominates* q over C, denoted $p \trianglerighteq_C q$ if there are $a \models p$ and $b \models q$ such that a dominates b over C.

3. If p dominates q over C and q dominates p over C, we say that p is *domination equivalent* to q over C, denoted $p \sqsupseteq_C q$.

4. If $p \in S(A), q \in S(B)$ are two stationary types, over potentially different sets of parameters, we say that p *dominates* q, denoted $p \trianglerighteq q$, if there is some $C \supseteq A \cup B$ such that $p|_C$ dominates $q|_C$.

5. If p dominates q and q dominates p, we say that p and q are *domination equivalent*, denoted $p \sqsupseteq q$.

If $p \in S(A)$ is algebraic, then for any stationary type $q \in S(A)$, $q \trianglerighteq p$ and if $p \trianglerighteq q$, then q is also algebraic. Sometimes in the literature $p \trianglerighteq q$ is referred to as eventual domination with domination defined as $p \trianglerighteq_A q$.

Let $p \in S(A)$ and $q \in S(B)$ be stationary types. Then p dominates q if and only if for some (any) $\aleph_1$-saturated model $M \supset A \cup B$, there are $a \models p|_M$ and $b \models q|_M$ such that $a \trianglerighteq_M b$ (for a proof, see for example [63, Proposition 5.6.4]). Domination is also transitive, hence p dominates q and q dominates r implies that p dominates r. Moreover, domination is a condition on parallelism classes, that is domination is invariant under taking non-forking relations. The following is well-known (for example see [63, Lemma 1.4.3.4]).

Fact 6.1.2. *Let a, b be tuples and C be a set.*

1. *Suppose a dominates b over C, $C_0 \subseteq C$ and $ab \underset{C_0}{\downarrow} C$. Then a dominates b over C_0.*

2. *Suppose a dominates b over C, $C \subseteq C_1$ and $a \underset{C}{\downarrow} C_1$. Then a dominates b over C_1 (and $b \underset{C}{\downarrow} C_1$).*

It is well-known that domination is compatible with Morley products (for example see [11, Remark 5.6.3]).

Fact 6.1.3. *Let $p \in S(A), q \in (B)$ and $r \in S(C)$ be stationary. Domination is preserved by taking Morley products. Precisely, if p dominates q, then $p \otimes r$ dominates $q \otimes r$*

On minimal types, domination is equivalent to non-orthogonality.

Fact 6.1.4. *[63, Lemma 1.4.4.2] Let $r \in S(A)$ and $s \in S(B)$ be minimal types. Then r dominates s if and only if s dominates r if and only if r is non-orthogonal to s.*

Every stationary type of finite U-rank is domination equivalent to a finite Morley product of minimal types (see for Example [11, Theorem 6.3.6]). Such a product is called a *domination decomposition* for p.

Classically, domination decompositions are stated for a-models and regular types instead of minimal types. Since we work in a totally transcendental stable theory, every $\aleph_1$-saturated model is an a-model, and we can state the result with minimal types instead of regular types.

Fact 6.1.5 (Domination Decomposition)**.** *Let $p \in S(A)$ be stationary, non-algebraic and finite U-rank. Then for some (any) $\aleph_1$-saturated model $M \supset A$, there are pairwise orthogonal minimal types $r_1, ..., r_n \in S(M)$ such that $p|_M$ is domination equivalent over M with the product $r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$ for some positive integers $k_1, ..., k_n$.*

In particular, p is domination equivalent to the product $r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$.

Up to re-ordering and domination equivalence, the r_i are unique. In particular, we have the following:

Lemma 6.1.6. *Let $p \in S(A)$ be stationary, non-algebraic and finite U-rank. Let $r \in S(B)$ be minimal. Let p be domination equivalent to the product $r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$ for pairwise orthogonal minimal types and $0 < k_1, ..., k_n$. Then r is non-orthogonal to p if and only if r is non-orthogonal to r_i for some $i = 1, ..., n$.*

Proof. By taking non-forking extensions, we may assume that $r, r_1, ..., r_n \in S(B)$. Suppose $p \not\perp r$ and let $C' \supseteq A \cup B$, be such that $p|_{C'} \not\perp^w r|_{C'}$. Let $C \supseteq A \cup B$, $a \models p|_C$ and $\overline{c} = (\overline{c_1}, ..., \overline{c_n}) \models (r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)})|_C$ be such that $\overline{c} \trianglerighteq_C a$. Let $D = C \cup C'$ and $a'\overline{c}' \equiv_C a\overline{c}$ such that $a'\overline{c}' \mathop{\smile\!\!\!\!\!\!\;|}_C D$. By Fact 6.1.2, $\overline{c}' \trianglerighteq_D a'$.

Since $p|_{C'} \not\perp^w r|_{C'}$, the non-forking extensions $p|_D$ and $r|_D$ are also non-weakly orthogonal. Let $a'' \models p|_D$ and $b'' \models r|_D$ be such that $a'' \underset{D}{\not\downarrow} b''$. Then $\mathrm{tp}(a''/D) = \mathrm{tp}(a'/D) = p|_D$ so there is $\sigma \in \mathrm{Aut}_D(\mathcal{U})$ such that $a' = \sigma(a'')$. Let $b' = \sigma(b'')$. Then $a' \underset{D}{\not\downarrow} b'$ and $\bar{c} \trianglerighteq_D a'$, hence $\overline{c_1}', ..., \overline{c_n}' \underset{D}{\not\downarrow} b'$. By definition, $r \not\perp r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$. By minimality of r, $r \not\perp r_i$ for some $i = 1, ..., n$ (see for example [11, Lemma 5.6.1]).

For the reverse direction, without loss of generality, suppose that $r \not\perp r_1$. By Fact 6.1.4, $r_1 \mathrel{\square} r$. Since $p \trianglerighteq r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)} \trianglerighteq r_1$. Thus $p \trianglerighteq r$. Hence, for some $C \supseteq A \cup B$, $a \models p|_C$ and $b \models r|_C$, for any d, $a \underset{C}{\downarrow} d$ implies $b \underset{C}{\downarrow} d$. Since $b \underset{B}{\downarrow} C$, $b \notin \mathrm{acl}(C)$, and hence $b \underset{C}{\not\downarrow} b$. By domination, $a \underset{C}{\not\downarrow} b$ which proves that $p \not\perp r$. $\square$

A useful consequence of domination decompositions is that non-domination equivalence of two types can be witnessed by a minimal type.

Lemma 6.1.7. *Let $p \in S(A)$ and $q \in S(B)$ be finite U-rank stationary types. Then the following are equivalent:*

1. *p dominates q,*
2. *for some (any) $\aleph_1$-saturated model $M \supset A \cup B$, there exists $a \models p|_M$ and $b \models q|_M$ such that whenever $\mathrm{tp}(e/M)$ is minimal and $a \underset{M}{\downarrow} e$, then $b \underset{M}{\downarrow} e$.*

Proof. If $p \trianglerighteq q$, then by taking non-forking extensions to any $\aleph_1$-saturated model $M \supset A \cup B$, there exists $a \models p|_M$ and $b \models q|_M$ such that for all d, $a \underset{M}{\downarrow} d$ implies $b \underset{M}{\downarrow} d$. Hence, for every minimal type $\mathrm{tp}(e/M)$, if $a \underset{M}{\downarrow} e$, then $b \underset{M}{\downarrow} e$.

For the reverse direction, suppose that $p \not\trianglerighteq q$. Then for some (any) $M \supset A \cup B$, $p|_M \not\trianglerighteq_M q|_M$. In particular, q is not algebraic. Fix $a \models p|_M$ and $b \models q|_M$. By Fact 6.1.5, there are pairwise orthogonal minimal types $r_1, ..., r_n \in S(M)$ and $k_1, ..., k_n > 0$ such that $q \mathrel{\square} r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$. We show that for some $i = 1, ..., n$ and $e \models r_i$, $a \underset{M}{\downarrow} e$ and $b \underset{M}{\not\downarrow} e$.

Fix $(\overline{c_1}, ..., \overline{c_n}) \models r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$ with $\overline{c_i} = (c_{i,1}, ..., c_{i,m_i})$ such that $b \trianglerighteq_M \overline{c_1}, ..., \overline{c_n}$. Clearly, for all $i = 1, ..., n$ and $j = 1, ..., k_i$, $b \underset{M}{\not\downarrow} c_{i,j}$.

If for all $i = 1, ..., n$ and $j = 1, ..., k_i$, $a \not\mathop{\downarrow}_M c_{i,j}$, by minimality $c_{i,j} \in \mathrm{acl}(Ma)$. Hence, if $a \mathop{\downarrow}_M d$, then $d \mathop{\downarrow}_M \overline{c_1}, ..., \overline{c_n}$. By definition, $p|_M \trianglerighteq_M r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)} \trianglerighteq_M q|_M$, contradicting that $p|_M \not\trianglerighteq q|_M$. Thus, for some i, j, $a \mathop{\downarrow}_M c_{i,j}$ and $b \not\mathop{\downarrow}_M c_{i,j}$. Taking $e = c_{i,j}$, $\mathrm{tp}(e/M) = r_i$ is minimal. □

In particular, to witness non-domination, the proof shows that we can pick e to be a realization of any minimal type appearing in the domination decomposition of p with a higher Morley power than in the decomposition of q.

6.1.1 Domination decompositions and parameters

In the previous section, to find a domination decomposition we had to let our minimal types be over additional parameters. We give some sufficient conditions for when p is domination equivalent to a Morley product of minimal types over the base parameters. First, when a type is almost internal to a minimal type, its domination decomposition is very simple and we do not need to refer to $\aleph_1$-saturated models. This will be useful when understanding semiminimal fibrations.

Lemma 6.1.8. *Let $p \in S(A)$ be stationary, non-algebraic and finite U-rank. Let $r \in S(B)$ be minimal. Then p is almost r-internal if and only if p is domination equivalent to $r^{(U(p))}$.*

Proof. We first show that if p is almost r-internal, then there exists some $C \supset A \cup B$, $a \models p|_C$ and $(c_1, ..., c_m) \models r^{(m)}|_C$ such that $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_n)$. This is a useful formulation for almost internality. Suppose that p is almost r-internal. Hence, there exists $C \supseteq A \cup B$, $a \models p|_C$ and $c_1, ..., c_m \models r$ such that $a \in \mathrm{acl}(Cc_1, ..., c_m)$. Choose such a C and $c_1, ..., c_m$ such that m is minimal. By Lemma 2.2.26, we may choose C and $c_1, ..., c_m$ such that $c_1, ..., c_m$ is C-independent (indeed in the proof we assume that m in minimal).

To prove that $c_1, ..., c_m \in \mathrm{acl}(Ca)$, by minimality of r, it suffices to show that for all $i = 1, ..., m$, $c_i \not\mathop{\downarrow}_C a$. Assume that for some $i = 1, ..., m$, $c_i \mathop{\downarrow}_C a$, and derive a contradiction. Without loss of generality, we may assume that $i = 1$. Since $c_1, ..., c_m \not\mathop{\downarrow}_C a$ and $c_1 \mathop{\downarrow}_C a$, $c_2, ..., c_m \not\mathop{\downarrow}_{c_1 C} a$. Since $a \mathop{\downarrow}_C c_1$ and $a \mathop{\downarrow}_A C$, we have that $a \mathop{\downarrow}_A Cc_1$. So setting $C' := Cc_1$, this contradicts minimality of m. Hence, $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_m)$.

Thus $n = U(p) = U(a/C) = U(c_1, ..., c_m/C) = \sum_{i=1}^{m} U(c_i/C) = m$. Since for any d, $a \underset{C}{\perp} d \iff a \underset{C}{\perp} c_1, ..., c_m, p|_C \sqsupseteq_C \text{tp}(c_1, ..., c_m/C)$. Therefore, $p \sqsupseteq r^{(m)}$ with $m = U(p)$.

For the reverse direction, suppose that $p \sqsupseteq r^{(n)}$ and let $C \supseteq A \cup B$, $a \models p|_C$ and $(c_1, ..., c_n) \models r^{(n)}|_C$ be such that $a \trianglerighteq_C c_1, ..., c_n$. For every $i = 1, ..., n$, $c_1, ..., c_n \underset{C}{\not\perp} c_i$, hence $a \underset{C}{\not\perp} c_i$. By minimality of r, $c_1, ..., c_n \in \text{acl}(Ca)$. We compute

$$n = U(a/C) = U(c_1, ..., c_n/C) + U(a/Cc_1, ..., c_n) = n + U(a/Cc_1, ..., c_n).$$

Thus $a \in \text{acl}(Cc_1, ..., c_n)$. As $a \models p|_C$, this witnesses that p is almost r-internal. $\square$

Note that in general, when a stationary type $p \in S(A)$ is almost internal to a minimal type r, p is almost internal to every A-conjugate of r by Fact 2.2.25. When we look at a semiminimal fibration $f : p \to q$, the fibers may all be internal to minimal types over additional parameters which depend on the choice of fiber. Since domination decompositions are unique up to non-orthogonality, it will be useful to assume that all minimal types are non-orthogonal, hence domination equivalent, to a minimal type over the base parameters. When p is itself almost internal to a locally modular type, this is immediate.

Lemma 6.1.9. *Let $p = \text{tp}(a/A)$ be stationary, and suppose that p is non-orthogonal to a minimal, locally modular type t, which is potentially over additional parameters. Then there exists a minimal type $r \in S(A)$ such that p is non-orthogonal to r and t is non-orthogonal to r.*

Proof. The proof of this statement is contained in the proof of [58, Proposition 2.3]. Fix $a \models p$. Since $p \not\perp t$ and t is minimal, by Fact 2.2.30, there exists $c \in \text{dcl}(Aa) \setminus \text{acl}(A)$ such that $\text{stp}(c/A)$ is r-internal. Since p is stationary, $\text{tp}(c/A)$ is stationary and t-internal since $c \underset{A}{\perp} \text{acl}(A)$. Recall that every locally modular minimal type is *one-based*, meaning that for any tuple of realizations of t and $B \supseteq A$, $a \underset{\text{acl}(Aa) \cap \text{acl}(B)}{\perp} B$.

As t is one-based, $\text{tp}(c/A)$ is also one-based (this was shown in [70, Corollary 9]). If $\text{tp}(c/A)$ is minimal, then $r = \text{tp}(c/A)$ is minimal with $t \not\perp r$ by Corollary 2.2.27. Since r is t-internal, p is non-algebraic and $p \not\perp t$, we have that $p \not\perp r$ by Lemma 2.2.31. Otherwise, there exists $e = \text{Cb}(c/Ae)$ such that $\text{tp}(c/Ae) = \text{tp}(c/A) - 1$ (see for example [63, Lemma 7.2.7]). By one-basedness of $\text{tp}(c/A)$, $e \in \text{acl}(Ac)$. Hence,

$$U(e/A) = U(c/A) - U(c/Ae) = U(c/A) - U(c/A) + 1 = 1.$$

Moreover, by Fact 2.2.3, $e \in \mathrm{dcl}(c_1, ..., c_l)$ for some Morley sequence in $\mathrm{stp}(c/Ae)$. Hence, $r = \mathrm{tp}(e/A)$ is stationary and minimal. As $\mathrm{tp}(c/A)$ is t-internal, it is almost internal in the family of all A-conjugates of t. Since $e \in \mathrm{acl}(Ac) \setminus \mathrm{acl}(A)$, by Lemma 2.2.23, $\mathrm{tp}(e/A)$ is almost internal in the family of all A-conjugates of t. By Fact 2.2.25, $\mathrm{tp}(e/A)$ is almost t-internal. By Corollary 2.2.27, $r \not\perp t$. Since p is almost t-internal and non-algebraic, $p \not\perp r$ by Lemma 2.2.31. $\square$

When p is almost internal to a non-locally modular minimal type, the conclusion of the lemma need not hold. For example, consider [33, Proposition 19]. Hrushovski gives a theory of Morley rank 2 which is *almost strongly minimal* meaning that given a saturated model $\mathcal{U}$, there is a formula ϕ (maybe using parameters) such that $\phi(\mathcal{U})$ is strongly minimal, and there is some finite set A such that $\mathcal{U} \subseteq \mathrm{acl}(A, \phi(\mathcal{U}))$. Note that this implies in particular that U-rank and Morley rank coincide. Hrushovski shows that in this theory, there are no strongly minimal, and hence there are no minimal types over $\mathrm{acl}^{eq}(\emptyset)$. Taking any type $p \in S(\mathrm{acl}^{eq}(\emptyset))$, there cannot be any minimal type over $\mathrm{acl}^{eq}(\emptyset)$ which is non-orthogonal to p.

The following condition is enough to give our desired conclusion of Lemma 6.1.9 for any minimal type t:

Condition 6.1.10. *There is a non-locally modular minimal type $t \in S(\emptyset)$ such that any non-locally modular minimal type is non-orthogonal to t.*

This condition is a common assumption in geometric stability theory and is satisfied by many theories of interest. In particular, DCF_0 satisfies Condition 6.1.10 with the generic type of the constants over $\emptyset$ as the unique non-locally modular type. Working under this assumption, there is a domination decomposition for $p \in S(A)$ over the base parameters.

Proposition 6.1.11. *Suppose T satisfies Condition 6.1.10. Let $p \in S(A)$ be stationary, non-algebraic and finite U-rank. Then there are $r_1, ..., r_n \in S(A)$ pairwise orthogonal minimal types such that p is domination equivalent to $r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$ for some $k_1, ..., k_n >$ 0.*

Proof. By Fact 6.1.5, there exists an $\aleph_1$-saturated model $M \supset A$, minimal types $t_1, ..., t_n \in S(M)$ and positive integers $k_1, ..., k_n$ such that $p \,\square\, t_1^{(k_1)} \otimes \cdots \otimes t_n^{(k_n)}$. Fix $i = 1, ..., n$. If t_i is non-locally modular, then by Assumption 6.1.10, $t_i \not\perp t \in S(\emptyset)$. Hence, $t_i \not\perp t|_A$. By Fact 6.1.4, $t_i \,\square\, t|_A$. Set $r_i := t|_A$

Now suppose that t_i is locally modular. Since t_i is minimal, t_i satisfies exchange. By Lemma 6.1.9, $t_i \not\perp r_i$ with a locally modular minimal type $r_i \in S(A)$. Since t_i and r_i are minimal types, $t_i \,\square\, r_i$.

In either case, we have that $t_i \,\square\, r_i$. By applying Fact 6.1.3 k_i-times, we obtain $t_i^{(k_i)} \,\square\, r_i^{(k_i)}$. Again by repeated applications of Fact 6.1.3, $p \,\square\, r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$ for $r_1, ..., r_n \in S(A)$ minimal. $\square$

6.2 Indecomposable fibrations

A natural way that semiminimal fibrations arise is by factoring any given fibration of finite U-rank types until it cannot be factored further. The central notion here is the following slight weakening of the property of exchange for stationary types defined in Definition 5.1.3.

Definition 6.2.1. A stationary type $p = \mathrm{tp}(a/A)$ *admits no proper fibrations* if whenever $b \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$, then $a \in \mathrm{acl}(Ab)$.

This definition looks stronger than there being no fibrations from p to a non-algebraic type q with $U(q) < U(p)$ since it is about all definable functions from p and to a type over A. However, [41, Remark 4.3] show that p admits no proper fibrations if and only if whenever $f : p \to q$ is a fibration over A either q is algebraic or f is finite-to-one. In fact, since p is stationary it admits no proper fibrations if and only if whenever $f : p \to q$ is a fibration over A either q is definable or f is bijective.

Recall that a minimal type $p = \mathrm{tp}(a/A)$ satisfies exchange. Hence, for every $b \in \mathrm{dcl}(Aa) \subseteq \mathrm{acl}(Aa)$ if $b \notin \mathrm{acl}(A)$, then $a \in \mathrm{acl}(Ab)$. So every minimal type admits no proper fibrations. Not every type which admits no proper fibrations is minimal, see Example 2.2 in [58]. However, every type that has no proper fibrations is related to a minimal type.

Fact 6.2.2. *[58, Proposition 2.3] Suppose $p \in S(A)$ is a stationary non-algebraic type of finite U-rank that admits no proper fibrations. Then p is semiminimal. In fact, either*

1. *p is almost internal to some non locally modular minimal type, or*
2. *p is interalgebraic over A with $r^{(\ell)}$ for some locally modular minimal type $r \in S(A)$ and $\ell \geq 1$.*

Remark 6.2.3. *If $p \in S(A)$ is a stationary non-algebraic type of finite U-rank that satisfies exchange then p admits no proper fibrations. Hence, by Fact 6.2.2, p is either almost internal to some non locally modular minimal type or p is interalgebraic over A with $r^{(\ell)}$*

for some minimal type $r \in S(A)$. In the later case, we have that $a \models p$ and $c_1, ..., c_\ell \models r^{(\ell)}$ such that $\operatorname{acl}(Aa) = \operatorname{acl}(Ac_1, ..., c_\ell)$. In particular, $c_1 \in \operatorname{acl}(Aa) \setminus \operatorname{acl}(A)$. Since p satisfies exchange, $a \in \operatorname{acl}(Ac_1)$. Thus, $c_2, ..., c_\ell \in \operatorname{acl}(Ac_1)$, and by minimality of r, $c_1 \not\perp_A c_2, ..., c_\ell$ if $\ell > 1$. So $\ell = 1$ and p is interalgebraic with the minimal type r. Since p is stationary, p is minimal. Thus either p is almost internal to a non locally modular minimal type or p is itself a minimal type.

We can talk about semiminimal fibrations whose fibers admit no proper fibrations.

Definition 6.2.4. A fibration $f : p \to q$ of stationary types $p, q \in S(A)$ is *indecomposable* if for any stationary type $r \in S(A)$, and fibrations $g : p \to r$ and $h : r \to q$ over A with $f = h \circ g$, then either $g : p \to r$ or $h : r \to q$ is finite-to-one.

Lemma 6.2.5. *Let $f : p \to q$ be a fibration of stationary types $p, q \in S(A)$. Then $f : p \to q$ is indecomposable if and only if some (equivalently any) fiber of $f : p \to q$ admits no proper fibrations.*

Proof. Suppose for some $a \models p$, the stationary fiber $\operatorname{tp}(a/Af(a))$ admits a proper fibration. We will show that $f : p \to q$ is not indecomposable. Let $g : \operatorname{tp}(a/Af(a)) \to \operatorname{tp}(g(a)/Af(a))$ be a fibration that is not finite-to-one. In particular, the fiber $\operatorname{tp}(a/Af(a))$ is non-algebraic. Let $s = \operatorname{tp}(f(a), g(a)/A)$. Then $(f, g) : p \to s$ has fibers of the form $\operatorname{tp}(a'/Af(a')g(a'))$, which are A-conjugates of the stationary fiber $\operatorname{tp}(a/Af(a)g(a))$ of g. Thus $(f, g) : p \to s$ is a fibration. Let $\pi : s \to q$ be the projection function given by $f(a)g(a) \mapsto f(a)$. The fibers of $\pi : s \to q$ are of the form $\operatorname{tp}(g(a')/Af(a'))$ which are stationary as they are images of the stationary fibers of f. Hence, $\pi : s \to q$ is a fibration. Now for any $a' \models p$, we have that $\pi((f, g)(a')) = \pi(f(a'), g(a')) = f(a')$. Hence, $\pi \circ (f, g) = f$.

Since $g : \operatorname{tp}(a/Af(a)) \to \operatorname{tp}(g(a)/Af(a))$ is not finite-to-one, $a \notin \operatorname{acl}(Af(a)g(a))$ showing that $(f, g) : p \to s$ is not finite-to-one. Since $\operatorname{tp}(g(a)/Af(a))$ is non-algebraic, the projection map $\pi : s \to q$ is not finite-to-one. Thus $f : p \to q$ is not indecomposable.

For the reverse direction, fix $a \models p$ such that the fiber $\operatorname{tp}(a/Af(a))$ admits no proper fibrations. Let $s \in S(A)$ be stationary, and $g : p \to s$ and $h : s \to q$ be fibrations such that $f = h \circ g$. Since $g(a) \in \operatorname{dcl}(Aa)$ and $\operatorname{tp}(a/Af(a))$ admits no proper fibrations, either $g(a) \in \operatorname{acl}(Af(a)) = \operatorname{acl}(Ah(g(a)))$, in which case $h : s \to q$ is finite-to-one, or $a \in \operatorname{acl}(Af(a)g(a)) = \operatorname{acl}(Ag(a))$, in which case $g : p \to s$ is finite-to-one. Thus, $f : p \to q$ is indecomposable. $\square$

Corollary 6.2.6. *An indecomposable fibration $f : p \to q$ of stationary finite U-rank types with $U(p) > U(q)$ is a semiminimal fibration.*

Proof. Let $f : p \to q$ be an indecomposable fibration of stationary finite U-rank types. By Lemma 6.2.5, some (equivalently any) fiber of $f : p \to q$ admits no proper fibrations. Since $U(p) > U(q)$, the fibers of $f : p \to q$ are all non-algebraic. Since p is finite U-rank, the fibers are also all finite U-rank. By Fact 6.2.2, every fiber is semiminimal. By definition $f : p \to q$ is a semiminimal fibration. $\square$

6.3 A domination decomposition of semiminimal fibrations

Domination behaves well with definable maps. For example, if $p, q \in S(A)$ are stationary types and $f : p \to q$ is a definable map, then for any $a \models p$, since $f(a) \in \mathrm{dcl}(Aa)$ we have that a dominates $f(a)$ over A witnessing p dominates q. We now prove Theorem 6.

Theorem 6.3.1. *Let $f : p \to q$ be a semiminimal fibration of stationary finite rank types $p, q \in S(A)$. Let r be a minimal type such that some fiber of $f : p \to q$ is almost r-internal. Then p is domination equivalent to $r^{(m)} \otimes q$ for some $0 \leq m \leq U(p) - U(q)$.*

Moreover, if $r \in S(A)$ then the following are equivalent:

1. *p is domination equivalent to $r^{(n)} \otimes q$ where $n = U(p) - U(q)$*
2. *$f : p \to q$ is* uniformly almost r-internal*, meaning there are $a \models p$ and $C \supseteq A$ with $a \mathop{\smile\!\!\!\!\!\mid}_A C$ and $c_1, ..., c_m \models r$ such that $a \in \mathrm{acl}(Cf(a), c_1, ..., c_m)$.*

Proof. Fix $a \models p$ and some minimal type $r \in S(B)$ such that $\mathrm{tp}(a/Af(a))$ is almost r-internal. First we show that $p \sqsupseteq\!\!\sqsubseteq r^{(k)} \otimes q$ for some $0 \leq k \leq U(a/Af(a))$. Let $M \supset A \cup f(a) \cup B$ be an $\aleph_1$-saturated model. Since $f : p \to q$ is a definable map, $p \trianglerighteq q$. If $p \sqsupseteq\!\!\sqsubseteq q$, then setting $k = 0$ we have that $p \sqsupseteq\!\!\sqsubseteq r^{(k)} \otimes q$. So we may assume that $q \ntrianglerighteq p$.

Fix $b \models p|_M$. We will inductively construct $(c_1, ..., c_l) \models r^{(l)}|_{Mf(b)}$ such that $c_1, ..., c_l \in \mathrm{acl}(Mb)$. In particular, we show that given such $c_1, ..., c_l$, if $r^{(l)} \otimes q \ntrianglerighteq p$, there exists $c_{l+1} \models r|_{Mf(b)c_1,...,c_l}$ and $c_{l+1} \in \mathrm{acl}(Mb)$.

For $l = 1$, if $q \ntrianglerighteq p$, by Lemma 6.1.7, there exists a minimal type $\mathrm{tp}(e/M)$ such that $e \mathop{\smile\!\!\!\!\!\mid}_M f(b)$ and $e \not\mathop{\smile\!\!\!\!\!\mid}_M b$. Thus $\mathrm{tp}(e/M) \not\perp \mathrm{tp}(b/Mf(b))$. Since $\mathrm{tp}(e/M) \not\perp \mathrm{tp}(b/M) = p|_M$. As $\mathrm{tp}(e/M)$ is minimal, by Fact 2.2.11 $\mathrm{tp}(b/Af(b))$, $\mathrm{tp}(b/Af(b)) \not\perp p \in S(A)$. By Lemma 2.2.32, all fibers of f are almost internal the single minimal type r. Hence, the extension

$\mathrm{tp}(b/Mf(b))$ is almost r-internal and non-algebraic since $b \underset{Af(b)}{\perp} M$. Note, this type is non-algebraic since $f : p \to q$ has non-algebraic fibers. By Lemma 2.2.31, $\mathrm{tp}(e/M) \not\perp r$. Since M is $\aleph_1$-saturated, $\mathrm{tp}(e/M) \not\perp^w r|_M$ as orthogonality is equivalent to weak-orthogonality over an $\aleph_1$-saturated model (see for example [63, Lemma 1.4.3.1]). Hence, by minimality of $\mathrm{tp}(e/M)$ and $r|_M$, there exists $c_1 \models r|_M$ such that $\mathrm{acl}(Me) = \mathrm{acl}(Mc_1)$. Thus $c_1 \underset{M}{\perp} f(b)$ and $c_1 \underset{M}{\not\perp} b$. Hence, $(c_1, f(b)) \models (r \otimes q)|_M$ and $c_1, f(b) \in \mathrm{acl}(Mb)$. Thus, $b \trianglerighteq_M f(b)c_1$ witnessing that $p \trianglerighteq r \otimes q$.

If $r \otimes q \trianglerighteq p$, we set $k = 1$ and then $p \mathbin{\square} r \otimes q$. Otherwise, suppose we have constructed $(c_1, ..., c_l) \models r^{(l)}|_{Mf(b)}$ such that $c_1, ..., c_l f(b) \in \mathrm{acl}(Mb)$ and $r^{(l)} \otimes q \not\trianglerighteq p$. Then, by Lemma 6.1.7, there is a minimal type $\mathrm{tp}(e/M)$ such that $e \underset{M}{\perp} f(b)c_1, ..., c_l$ and $e \underset{M}{\not\perp} b$. Hence, $\mathrm{tp}(e/M) \not\perp \mathrm{tp}(b/Mf(b))$. By the same argument as above, there is some $c_{l+1} \models r|_{Mf(b)}$ such that $\mathrm{acl}(Mc_{l+1}) = \mathrm{acl}(Me)$. Hence, $c_{l+1} \underset{M}{\perp} f(b)c_1, ..., c_l$ and $c_{l+1} \underset{M}{\not\perp} b$. By minimality of r, $c_{l+1} \in \mathrm{acl}(Mb)$. We have shown that $c_1, ..., c_{l+1} \models r^{(l+1)}|_{Mf(b)}$ and $c_1, ..., c_{l+1} f(b) \in \mathrm{acl}(Mb)$.

Since $b \underset{M}{\not\perp} c_{l+1}$ and $p \trianglerighteq r^{(l)} \otimes q$ we have that $bc_1, ..., c_l f(b) \underset{M}{\not\perp} c_{l+1}$. As $c_1, ..., c_l f(b) \underset{M}{\perp} c_{l+1}$, we have that $b \underset{c_1,...,c_l f(b)M}{\not\perp} c_{l+1}$. Hence, $U(b/Mc_1, ..., c_{l+1} f(b)) < U(b/Mc_1, ..., c_l f(b)) < \cdots < U(b/Mf(b)) = U(b/Af(b))$. This process must stop in less than or equal to $U(a/Af(a))$-many steps. Hence, for some $0 \leq k \leq U(a/Af(a))$, we have that $p \mathbin{\square} r^{(k)} \otimes q$.

For the moreover statement, suppose that $p \mathbin{\square} r^{(n)} \otimes q$ where $n = U(p) - U(q)$. Then there exists $C \supseteq A$, $a \models p|_C$ and $c_1, ..., c_n, f(a) \models (r^{(n)} \otimes q)|_C$ be such that $a \trianglerighteq_C c_1, ..., c_n f(a)$. Since for all $i = 1, ..., n$, $c_1, ..., c_n f(a) \underset{C}{\not\perp} c_i$, by domination $a \underset{C}{\not\perp} c_i$. By minimality of r, $c_1, ..., c_n \in \mathrm{acl}(Ca)$. We compute,

$$\begin{aligned}
n &= U(a/Af(a)) \\
&= U(a/Cf(a)) \\
&= U(c_1, ..., c_n/Cf(a)) + U(a/Cc_1, ..., c_n f(a)) \\
&= U(c_1, ..., c_n/C) + U(a/Cc_1, ..., c_n f(a)) \\
&= n + U(a/Cc_1, ..., c_n f(a)).
\end{aligned}$$

Hence, $a \in \mathrm{acl}(Cc_1, ..., c_n f(a))$. As $a \models p|_C$, instead of simply $a \underset{Af(a)}{\perp} C$, by definition $f : p \to q$ is uniformly almost r-internal.

For the reverse direction, suppose that $f : p \to q$ is uniformly almost r-internal. We first show that there is some $C \supseteq A$, $a \models p|_C$, $(c_1, ..., c_n, f(a)) \models (r^{(n)} \otimes q)|_C$ such that $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_m, f(a))$. By uniform almost r-internality, there exists $C \supseteq A \cup B$, $a \models p|_C$ and $c_1, ..., c_m \models r$ such that $a \in \mathrm{acl}(Cf(a)c_1, ..., c_m)$. Choose m to be minimal among all C and M. By Lemma 2.2.26, we may assume that $(c_1, ..., c_m) \models r^{(m)}|_C$. If $f(a) \not\mathop{\smile\!\!\!\!\!\mid}_C c_1, ..., c_m$, then by minimality of r, for some $i = 1, ..., m$ we have that $c_i \in \mathrm{acl}(Cf(a)c_1, ..., c_{i-1})$, contradicting minimality of m. Thus, by stationarity, $(c_1, ..., c_m, f(a)) \models (r^{(m)} \otimes q)|_C$.

Suppose that there is some $i = 1, ..., m$ such that $c_i \notin \mathrm{acl}(Ca)$ and derive a contradiction. Then by minimality of r, $c_i \mathop{\smile\!\!\!\!\!\mid}_A aC$. Since $a \mathop{\smile\!\!\!\!\!\mid}_A C$, $c_1 a \mathop{\smile\!\!\!\!\!\mid}_A C$. Since $c_i \notin \mathrm{acl}(C)$, by minimality of r, $c_i \mathop{\smile\!\!\!\!\!\mid}_C a$. Thus, by transitivity of non-forking, $a \mathop{\smile\!\!\!\!\!\mid}_A Cc_i$. Taking $D := Cc_i$, we have that $a \in \mathrm{acl}(Dc_1, ..., c_{i-1}, c_{i+1}, ..., c_m)$ contradicting minimality of m.

Thus $c_1, ..., c_m, f(a) \in \mathrm{acl}(Ca)$. Hence, $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_m, f(a))$. By interalgebraicity we have that

$$
\begin{aligned}
n &= U(a/Af(a)) \\
&= U(a/Cf(a)) \\
&= U(c_1, ..., c_m/Cf(a)) \\
&= m.
\end{aligned}
$$

Hence, $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_n, f(b))$.

Thus, for every set d, $a \mathop{\smile\!\!\!\!\!\mid}_C d \iff c_1, ..., c_n, f(a) \mathop{\smile\!\!\!\!\!\mid}_C d$. By definition, $a \sqsupseteq_C c_1, ..., c_n f(a)$, witnessing that $p \sqsupseteq r^{(n)} \otimes q$. $\square$

It is not always the case that the power of r appearing in the domination decomposition $p \sqsupseteq r^{(m)} \otimes q$ is 0 or $U(p) - U(q)$.

Proposition 6.3.2. *Let $T = \mathrm{DCF}_0$ and $r \in S(\emptyset)$ be the generic type of the field of constants. For every $0 < m < n$, there is a fibration $f : p \to q$ whose fibers have U-rank n and are almost r-internal such p is domination equivalent to $r^{(m)} \otimes q$.*

Proof. Fix $0 < m < n$. Let F be an algebraically closed field of constants. Let s be the generic type over F of the differential equation $\delta(\frac{\delta(x)}{x}) = 0$ from Example 2.2.35. Recall this type is 2-step analyzable in the constants, but not almost $\mathcal{C}$-internal. Then $\delta\log_{\mathbb{G}_m} : s \to r$ is a fibration where $\delta\log_{\mathbb{G}_m}(x) = \frac{\delta(x)}{x}$. This fibration has minimal almost r-internal fibers.

Since $U(s) - U(r) = 1$ and the fibers of $\delta\log_{\mathbb{G}_m} : s \to r$ are almost r-internal, $s \sqsubset r \otimes r$ or $p \sqsubset r$ by Theorem 6.3.1. If $s \sqsubset r^{(2)}$, then since $U(s) = 2$, by Proposition 6.1.8, s is almost r-internal, a contradiction. Thus $s \sqsubset r$.

Let $p = s^{(n-m)} \otimes r^{(m)}$ and let $f : p \to r^{(n-m)}$ be the definable function defined by by $(x_1, ..., x_{n-m}, y_1, ..., y_m) \mapsto (\delta\log_{\mathbb{G}_m}(x_1), ..., \delta\log_{\mathbb{G}_m}(x_{n-m}))$. Since $U(s) = 2$, $U(p) = 2n - m$, hence the fibers of $f : p \to r^{(n-m)}$, which are of the form $\mathrm{tp}(a_1, ..., a_{m-n}, b_1, ..., b_m / F\delta\log_{\mathbb{G}_m}(a_1), ..., \delta\log_{\mathbb{G}_m}(a_{n-m}))$, are almost r-internal with U-rank n. This type is stationary since $a_1, ..., a_{m-n}, b_1, ..., b_m$ is F-independent, and $\mathrm{tp}(a_i / F\delta\log_{\mathbb{G}_m}(a_i))$ and $\mathrm{tp}(b_j/F)$ are stationary for all $i = 1, ..., n-m$ and $j = 1, ..., m$. Hence, $f : p \to r^{(n-m)}$ is a fibration. Let $q = r^{(m)}$.

Then since $s \sqsubset r$, by repeated applications of Fact 6.1.3, $p \sqsubset s^{(n-m)} \otimes r^{(m)} \sqsubset r^{(n-m)} \otimes q$ and $n - m > 0$. □

In the case of indecomposable fibrations we can show that m will always take on the extreme values of 0 or $U(p) - U(q)$.

Proposition 6.3.3. *Let $f : p \to q$ be an indecomposable fibration of stationary finite rank types $p, q \in S(A)$. Then either*

1. *p is domination equivalent to q, or*
2. *there is some fixed minimal type $r \in S(B)$, such that all fibers of $f : p \to q$ are almost r-internal and p is domination equivalent to $r^{(n)} \otimes q$ where $n = U(p) - U(q)$.*

Moreover, if T satisfies Condition 6.1.10, then in (2) one can choose $r \in S(A)$.

Proof. Fix $a \models p$. Since $f : p \to q$ is indecomposable, by Lemma 6.2.5, the fiber $\mathrm{tp}(a/Af(a))$ admits no proper fibrations. By Fact 6.2.2, there is some minimal type $r \in S(B)$ such that $\mathrm{tp}(a/Af(a))$ is almost r-internal.

Suppose that p is not domination equivalent to q. By Lemma 6.3.1, $p \sqsubset r^{(k)} \otimes q$ for some $0 < k \leq U(p) - U(q)$. By Lemma 6.1.6, $r \not\perp p$. Since $\mathrm{tp}(a/Af(a))$ is almost r-internal and non-algebraic, $\mathrm{tp}(a/Af(a)) \not\perp r$ by Corollary 2.2.27. By Lemma 2.2.31, $\mathrm{tp}(a/Af(a)) \not\perp p \in S(A)$. Hence, by Lemma 2.2.32, the fibers are all almost r-internal. In particular, $q \not\trianglerighteq p$ and r appears with a higher Morley power in the domination decomposition for p than for q. Then, by the proof of Lemma 6.1.7, there is some $\aleph_1$-saturated model $M \supset A$, such that for all $a \models p|_M$ there is $e \models r|_M$ such that $a \underset{M}{\not\perp} e$ and $f(a) \underset{M}{\perp} e$. Fix such an a, M, and e. We will produce a d' such that $\mathrm{tp}(d'/A)$ is almost r-internal and $d' \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(Af(a))$.

By no proper fibrations, this will give that $a \in \operatorname{acl}(Af(a)d')$. Since $\operatorname{tp}(d'/A)$ is almost r-internal, we will be able to choose the parameters witnessing that the fiber $\operatorname{tp}(a/Af(a))$ is almost r-internal.

Let d be a finite sub-tuple of $\operatorname{Cb}(\operatorname{stp}(eM/Aa))$ with maximal possible U-rank over A. Then $d \in \operatorname{acl}(Aa)$. By Fact 2.2.3, $d \in \operatorname{dcl}(e_1M_1, ..., e_nM_n)$ for some finite Morley sequence in $\operatorname{stp}(eM/Aa)$. We induct on n to show that $a \mathop{\downarrow}_A M_1, ..., M_n$.

When $n = 1$, $a \mathop{\downarrow}_A M_1$ since $\operatorname{stp}(M_1/Aa) = \operatorname{stp}(M/Aa)$ and $a \mathop{\downarrow}_A M$.

Suppose we have shown that $a \mathop{\downarrow}_A M_1, ..., M_k$. Suppose that $a \mathop{\not\downarrow}_A M_1, ..., M_{k+1}$. By inductive hypothesis, $a \mathop{\downarrow}_A M_1, ..., M_k$, hence $a \mathop{\not\downarrow}_{AM_1,...,M_{k+1}} M_{k+1}$. Thus $aM_1, ..., M_k \mathop{\not\downarrow}_A M_{k+1}$. Since $\operatorname{stp}(M_{k+1}/Aa) = \operatorname{stp}(M/Aa)$, $a \mathop{\downarrow}_A M_{k+1}$. Thus $M_1, ..., M_k \mathop{\not\downarrow}_{Aa} M_{k+1}$, contradicting that $M_1, ..., M_{k+1}$ is Aa-independent. Hence, $a \mathop{\downarrow}_A M_1, ..., M_n$. Since $d \in \operatorname{acl}(Aa)$, $d \mathop{\downarrow}_A M_1, ..., M_n$.

Since $\operatorname{tp}(e_i/A) = \operatorname{tp}(e/A)$ for all $i = 1, ..., n$, e_i is an A-conjugate of e. Hence, e_i is a realization of $\mathcal{R}$, the family of all A-conjugates of r. We have shown that $d \in \operatorname{acl}(M_1, ..., M_n, e_1, ..., e_n)$ with $d \mathop{\downarrow}_A M_1, ..., M_n$ and $e_1, ..., e_n \models \mathcal{R}$. Hence, $\operatorname{tp}(d/A)$ is almost $\mathcal{R}$-internal. By Fact 2.2.25, $\operatorname{tp}(d/A)$ is almost r-internal.

Let d' be the canonical parameter for $\{d_1, ..., d_k\}$ the finite set of Aa-conjugates of d. Then $d' \in \operatorname{dcl}(Aa)$. Since $d_1, ..., d_k$ are Aa-conjugates of d, $\operatorname{tp}(d_i/A)$ is almost $\mathcal{R}$-internal, and hence almost r-internal by Fact 2.2.25. Thus, by Fact 7.5.4, $\operatorname{tp}(d_1, ..., d_k/A)$ is almost r-internal. Finally, by definition, $d' \in \operatorname{acl}(d_1, ..., d_n)$, and hence, $\operatorname{tp}(d'/A)$ is almost r-internal by Lemma 2.2.23 and Fact 2.2.25. Since $d \in \operatorname{acl}(d'A)$, to conclude that $d' \notin \operatorname{acl}(Af(a))$, it suffices to show that $d \notin \operatorname{acl}(Af(a))$. Suppose for sake of contradiction that $d \in \operatorname{acl}(Af(a))$. Since $\operatorname{Cb}(\operatorname{stp}(eM/Aa)) \subseteq \operatorname{acl}(Ad)$, by Fact 2.2.2, $eM \mathop{\downarrow}_{Af(a)} a \implies e \mathop{\downarrow}_{Mf(a)} a$. Since $e \mathop{\downarrow}_M f(a)$, we get that $e \mathop{\downarrow}_M a$, a contradiction. Thus $d' \notin \operatorname{acl}(Af(a))$. So we have that $d' \in \operatorname{dcl}(Aa) \setminus \operatorname{acl}(Af(a))$. Since $\operatorname{stp}(a/Af(a))$ admits no proper fibrations, $a \in \operatorname{acl}(Ad'f(a))$.

As $\operatorname{tp}(d'/A)$ is almost r-internal, there exists $D \supseteq A \cup B$ with $d' \mathop{\downarrow}_A D$ and $c_1, ..., c_k \models r$ such that $d' \in \operatorname{acl}(Dc_1, ..., c_k)$. Choose $D' \models \operatorname{tp}(D/Ad')$ such that $D' \mathop{\downarrow}_{Ad'} a$. In particular, since $d' \underset{A}{\perp} D'$, we have that $a \underset{A}{\perp} D'$. Let $\sigma \in \mathrm{Aut}_{Ad'}(\mathcal{U})$ such that $\sigma(D) = D'$. For all $i = 1, ..., k$, let $c'_i = \sigma(c_i)$. Then $d' \in \mathrm{acl}(D'c'_1, ..., c'_n)$. Thus $a \in \mathrm{acl}(Ad'f(a)) \subseteq \mathrm{acl}(D'c'_1, ..., c'_k f(a))$ and $a \underset{A}{\perp} D'$. Thus, by definition, $f : p \to f(p)$ is uniformly almost r-internal. Thus, by Theorem 6.3.1, $p \sqsubset r^{(n)} \otimes q$.

The moreover claim is Proposition 6.1.11. $\square$

When the fibers of a fibration are almost internal to a trivial type, we obtain a stronger decomposition than domination, provided there is some relation between orthogonal fibers. A key result is that internality to a minimal trivial type does not require additional parameters. The following is well-known, see for example [18, Lemma 2.9].

Fact 6.3.4. *Let $B \supseteq A$, $p \in S(B)$ be stationary and $r \in S(A)$ be trivial and minimal. If p is almost r-internal, then p is $r|_B$-algebraic meaning that for any $a \models p$, there are $c_1, ..., c_n \models r|_B$ such that $a \in \mathrm{acl}(Ac_1, ..., c_m)$.*

Lemma 6.3.5. *Let $f : p \to q$ be a fibration of stationary finite U-rank types $p, q \in S(A)$ such that all the fibers of $f : p \to q$ are almost internal to a fixed minimal and trivial type $r \in S(A)$.*

Then for any $a \models p$, there are $c_1, ..., c_n \models r^{(n)}$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Af(a), c_1, ..., c_n)$ and $f(a) \underset{A}{\perp} c_1, ..., c_n$.

Proof. Fix some $a \models p$. If $U(p) = U(q)$, trivially $\mathrm{acl}(Aa) = \mathrm{acl}(Af(a))$. So we may assume the fibers of $f : p \to q$ are non-algebraic. Since $r \in S(A)$ is trivial and $\mathrm{tp}(a/Af(a))$ is almost r-internal, by Fact 6.3.4, $a \in \mathrm{acl}(Af(a)c_1, ..., c_m)$ for some $c_1, ..., c_m \models r$. Choose m to be minimal, then $c_1, ..., c_m \underset{A}{\perp} f(a)$. If not, then for some $i < m$, $c_{i+1} \underset{Ac_1,...,c_i}{\not\perp} f(a)$. By minimality of r, $c_{i+1} \in \mathrm{acl}(Af(a)c_1, ..., c_i)$. Thus $a \in \mathrm{acl}(Af(a)c_1, ..., c_{i-1}, c_{i+1}, ..., c_m)$ contradicting minimality of m. Note that $m > 0$ since $a \notin \mathrm{acl}(Af(a))$. Suppose for some $i = 1, ..., m$ that $c_i \notin \mathrm{acl}(Aa)$. By re-ordering we may assume that there is an $\ell = 1, ..., m$ such that each $c_1, ..., c_l$ is not algebraic over aA and each $c_{i+1}, ..., c_m$ is algebraic over aA. Let $\bar{c} = (c_1, ..., c_l)$ and set $d = \mathrm{Cb}(\mathrm{stp}(\bar{c})/Aa)$. By Fact 2.2.3, $d \in \mathrm{dcl}(\overline{c_0}, ..., \overline{c_k})$ for some infinite Morley sequence $\bar{c} = \overline{c_0}, ..., \overline{c_k}, ...$ in $\mathrm{stp}(\bar{c}/Aa)$. Since $\mathrm{stp}(\bar{c}/Aa) = \mathrm{stp}(\bar{c}_{k+1}/Aa)$, by Fact 2.2.2, $\bar{c}_{k+1} \underset{\bar{c}_1,...,\bar{c}_k}{\perp} aA$. If $\bar{c}_{k+1} \underset{A}{\perp} \bar{c}_1, ..., \bar{c}_k$, $\bar{c}_{k+1} \underset{A}{\perp} a$. Since $c_{l+1}, ..., c_m \in \mathrm{acl}(Aa)$, $a \underset{Ac_{l+1},...,c_m}{\perp} \bar{c}_{k+1}$. Since $a \in \mathrm{acl}(Af(a)\bar{c}_{k+1}c_{l+1}...c_m)$, hence $a \in \mathrm{acl}(Af(a)c_{l+1}...c_m)$, contradicting minimality of m or that $a \notin \mathrm{acl}(Af(a))$ if $l = m$. Thus $\bar{c}_{k+1} \underset{A}{\not\perp} \bar{c}_1, ..., \bar{c}_k$.

Let $j \geq 0$ be minimal such that $\bar{c}_{j+1} \not\mathrel{\mathop{\downarrow}\limits_{A}} \bar{c}_1, ..., \bar{c}_j$. Hence, there is some minimal s such that $c_{j,s} \not\mathrel{\mathop{\downarrow}\limits_{Ac_{j,0},...,c_{j,s-1}}} \bar{c}_1, ..., \bar{c}_j$. Thus $c_{j,0}, ..., c_{j,s-1} \mathrel{\mathop{\downarrow}\limits_{A}} \bar{c}_1, ..., \bar{c}_j$. By minimality and triviality of r, there is some $i = 1, ..., j$ and t (with $t < s$ if $i = j$) such that $\operatorname{acl}(Ac_{k,s}) = \operatorname{acl}(Ac_{i,t})$. By A-indiscernibility of the sequence of $\bar{c}_i s$, $\operatorname{acl}(Ac_{k,s}) = \operatorname{acl}(Ac_{0,t})$. Again by A-indiscernibility, for all i, the sth coordinates $c_{i,s}$ and $c_{0,s}$ are interalgebraic over A. Since $\bar{c}$ is a Morley sequence over Aa, we obtain that $c_s = c_{0,s} \in \operatorname{acl}(Aa)$, a contradiction. $\square$

Theorem 6.3.6. *Let $f : p \to q$ be a fibration of stationary finite U-rank types $p, q \in S(A)$ such that each fiber is almost internal to some trivial minimal type. Then exactly one of the following is true:*

1. *fibers over independent realizations of q are orthogonal, or*
2. *there is a trivial minimal type $t \in S(A)$ such that p is interalgebraic with $t^{(n)} \otimes q$ where $n = U(p) - U(q)$.*

Proof. Fix $(b_1, b_2) \models q^{(2)}$. Let $a_1 \models p$ be such that $f(a_1) = b_1$ and $a_1 \mathrel{\mathop{\downarrow}\limits_{Af(b_1)}} b_2$. Since $b_1 \mathrel{\mathop{\downarrow}\limits_{A}} b_2$, $a_1 \mathrel{\mathop{\downarrow}\limits_{A}} b_2$. Let $a_2 \models p$ be such that $f(a_1) = b_2$ and $a_2 \mathrel{\mathop{\downarrow}\limits_{Ab_2}} a_1$. Since $a_1 \mathrel{\mathop{\downarrow}\limits_{A}} b_2$, $a_2 \mathrel{\mathop{\downarrow}\limits_{A}} a_1$.

Suppose that $\operatorname{tp}(a_1/Ab_1) \not\perp \operatorname{tp}(a_2/Ab_2)$. By assumption, $\operatorname{stp}(a_1/Ab_1)$ is non-algebraic and almost internal to a trivial minimal type $s_1 \in S(B_1)$, hence by Lemma 2.2.31, $\operatorname{stp}(a_2/Ab_2) \not\perp s_1$. By Lemma 6.1.9, there is a minimal type $r_1 \in S(Ab_1)$ such that $\operatorname{stp}(a_2/Ab_2) \not\perp r_1$ and $r_1 \not\perp s_1$. Note that triviality on minimal types is preserved by non-orthogonality ([2, Chapter XVII, Theorem 2.6]), hence $r_1 \in S(Ab_1)$ is also trivial.

Also by assumption, $\operatorname{stp}(a_2/Ab_2)$ is non-algebraic and almost internal to a trivial minimal type $s_2 \in S(B_2)$. By Lemma 2.2.31, $r_1 \not\perp s_2$. By the same argument above, there is a trivial minimal $r_2 \in S(Ab_2)$ such that $r_1 \not\perp r_2$.

Since $\operatorname{tp}(a_1/Ab_1)|_{b_2}$ is stationary, non-algebraic and almost internal to the trivial minimal type r_1, by Fact 6.3.4, $\operatorname{tp}(a_1/Ab_1)$ is $r_1|_{b_2}$-algebraic. Similarly, $\operatorname{tp}(a_2/Ab_2)$ is $r_2|_{b_1}$-algebraic. Since r_1 and r_2 are trivial, by Fact 2.2.10, $r_1|_{b_2} \not\perp^w r_2|_{b_1}$. Hence, $\operatorname{tp}(a_1/Ab_1)$ being $r_1|_{b_2}$-algebraic and $\operatorname{tp}(a_2/Ab_2)$ being $r_2|_{b_1}$-algebraic gives that $\operatorname{tp}(a_1/Ab_1)|_{b_2} \not\perp^w \operatorname{tp}(a_2/Ab_2)|_{b_1}$.

Therefore, there exists $c_2 \models \operatorname{tp}(a_2/Ab_2)|_{b_1}$ such that $a_1 \not\mathrel{\mathop{\downarrow}\limits_{Ab_1b_2}} c_2$. Since $b_2 = f(c_2)$, we have that $a_1 \not\mathrel{\mathop{\downarrow}\limits_{Ab_1}} c_2$. As $c_2 \mathrel{\mathop{\downarrow}\limits_{Ab_2}} b_1$ and $b_1 \mathrel{\mathop{\downarrow}\limits_{A}} b_2$, $c_2 \mathrel{\mathop{\downarrow}\limits_{A}} b_1$, we have that $c_2 \models p|_{Ab_1}$ and

$a_2 \models \mathrm{tp}(a_1/Ab_1)$ and $c_2 \underset{Ab_1}{\not\perp} a_2$. This witnesses that $p \not\perp \mathrm{tp}(a_1/Ab_1)$. Since $\mathrm{tp}(a_1/Ab_1)$ is almost r_1-internal and non-algebraic, $p \not\perp r_1$ by Lemma 2.2.31. Since p is non-algebraic and r_1 is trivial, by Lemma 6.1.9 there is a minimal type $t \in S(A)$ such that $p \not\perp t$ and $r_1 \not\perp t$. Again by [2, Chapter XVII, Theorem 2.6], t is trivial. Since $\mathrm{tp}(a_1/Ab_1)$ is almost r_1-internal and $r_1 \not\perp t$, by Remark 2.2.31, $\mathrm{tp}(a_1/Ab_1)$ is almost t-internal. Since $t \in S(A)$, for any other $d \models p$, $\mathrm{tp}(d/Af(d))$ is an A-conjugate of $\mathrm{tp}(a_1/Ab_1) = \mathrm{tp}(a_1/Af(a_1))$. Thus $\mathrm{tp}(d/Af(d))$ is almost t-internal. Since $t \in S(A)$ is minimal and trivial and every fiber of $f : p \to q$ is almost t-internal, by Lemma 6.3.5, $\mathrm{acl}(Aa_1f(a_1)) = \mathrm{acl}(Af(a_1)c_1, ..., c_n)$ with $n = U(a_1/Af(a_1))$. $\square$

Chapter 7

Semiminimal analyses

In this final chapter, we continue to work in a sufficiently saturated model $\mathcal{U}$ of a totally transcendental theory eliminating imaginaries.

Minimal types are well-understood in many theories. For example, in DCF_0, the Zilber the dichotomy gives a complete characterization. In this chapter, we study fine structural issues around how finite rank types are built up from minimal types. That is, we study semiminimal analyses. Recall that a type is *semiminimal* if it is non-algebraic and almost internal to a minimal type, and a fibration is *semiminimal* if its fibers are semiminimal.

Definition 7.0.1. Let $p \in S(A)$ be stationary. A *semiminimal analysis* for p is a sequence of stationary types $p_0, ..., p_n \in S(A)$ and fibrations $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ such that

1. p_0 is algebraic,
2. p_n is interalgebraic with p, and
3. for every $i = 1, ..., n$, the $f_i : p_i \to p_{i-1}$ is a semiminimal fibration.

There are several points of tension between this definition and the definition of an analysis of a type (in a family of types) given in Definition 2.2.34. First, Definition 2.2.34 is given on the level of realizations and so any such analysis is a sequence of tuples, whereas in Definition 7.0.1 any semiminimal analysis is a sequence of types (connected by fibrations). Secondly, one may be tempted to think of Definition 7.0.1 as the instance of Definition 2.2.34 where one consider an analysis in the family of all minimal types — but this is not the case as the assumption is that each fiber of each fibration is almost internal to a single minimal type, not to the family of all minimal types. In any case, it is the notion of semiminimal analysis given here that will concern us in this chapter

Every non-algebraic finite U-rank type admits a semiminimal analysis. Such analyses are certainly not unique, even up to a natural notion of equivalence that we discuss in Section 7.2 below. The goal of this chapter is to prove that the minimal types arising in a semiminimal analysis, and the multiplicity with which they arise, are invariants of the type, not depending on the particular semiminimal analysis. What we aim to prove is the following:

Theorem 7. *Let p be a stationary type of finite U-rank and $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\ldots,n}$ be a semiminimal analysis of p. For any minimal type r, let*

$$I_r := \{1 \leq i \leq n \mid \text{ some fiber of } f_i : p_i \to p_{i-1} \text{ is almost } r\text{-internal}\}$$

and let $m_r := \sum_{i \in I_r} U(p_i) - U(p_{i-1})$.

Then m_r is an invariant of p and r, independent of the semiminimal analysis $\mathbf{f}$.

We call m_r, provisionally, the multiplicity *of r in $\mathbf{f}$.*

Note that the parameter set for p and r need not be the same. However, if p is over A and r' is an A-conjugate of r, then $I_r = I_{r'}$ and hence the multiplicity of r in $\mathbf{f}$ will be the same as that of r'. This can be seen by an automorphism argument.

We will give an intrinsic characterization of this multiplicity in Theorem 7.4.2 below and connect it to the concept of r-weight in Section 7.5. This chapter is solo work of the author of the thesis, with the exception of Theorem 7.1.4, Example 7.1.5 and Theorem 7.3.10 which are joint with Léo Jimenez and appear in [18].

7.1 Preliminaries on semiminimal analyses

We first show that every semiminimal analysis induces an analysis in the sense of Definition 2.2.34.

Definition 7.1.1. Let $p \in S(A)$ be a stationary finite U-rank type. Given

$$\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\ldots,n}$$

a semiminimal analysis of p, by a *realization* of $\mathbf{f}$ we mean a sequence $(a_n, \ldots, a_0)$ realizing $(p_n, \ldots, p_0)$ and such that $a_{i-1} = f_i(a_i)$ for all $i = 1, \ldots, n$.

If we let $\mathcal{R}$ denote the set of all minimal types, then it should be clear that any realization of $\mathbf{f}$ is an analysis of p in $\mathcal{R}$, in the sense of Definition 2.2.34. In fact, realizations of $\mathbf{f}$ are a special kind of $\mathcal{R}$-analysis as each $\mathrm{tp}(a_i/Aa_{i-1})$ is not just almost $\mathcal{R}$-internal, but almost internal to a single minimal type r in $\mathcal{R}$ (as well as being stationary and non-algebraic).

Every stationary finite U-rank type admits a semiminimal analysis. In fact, if $p, q \in S(A)$ are stationary finite U-rank types with $U(q) < U(p)$ and $f : p \to q$ is a fibration, then $f : p \to q$ decomposes into a sequence of semiminimal fibrations, which can moreover, be chosen to be indecomposable. Recall from Definition 6.2.4 that a fibration $f : p \to q$ of stationary types is *indecomposable* if whenever f factors into two fibrations, one of these fibrations must be finite-to-one. By Corollary 6.2.6, indecomposable fibrations are semiminimal.

Lemma 7.1.2. *Let $f : p \to q$ be a fibration of stationary types $p, q \in S(A)$ with $U(p) > U(q)$. Then there is a sequence of stationary types $p_0, ..., p_n \in S(A)$ and indecomposable fibrations $(f_i : p_i \to p_{i-1})_{i=1,...,n}$ such that*

1. *$f = f_n \circ \cdots \circ f_1$, $p_n := p$ and $p_0 := q$,*
2. *$U(p_i) > U(p_{i-1})$ for all $i = 1, ..., n$,*

Moreover, if $f : p \to q$ is semiminimal witnessed by a minimal type r, then for each $i = 1, ..., n$, some fiber of $f_i : p_i \to p_{i-1}$ is almost r-internal.

Proof. We induct on $U(p)$. If $U(p) = 0$, then the result holds vacuously. If $U(p) = 1$, then $f : p \to q$ is a definable function whose fibers are p, as q must have U-rank 0. Since p is minimal, it admits no proper fibrations. By Lemma 6.2.5, $f : p \to q$ is an indecomposable fibration. Set $p_1 := p$, $p_0 := q$ and $f_1 := f$. Then $(f_1 : p_1 \to p_0)$ is the desired sequence of types and indecomposable fibrations.

Suppose we have shown that for every stationary type of U-rank k, there is a sequence of stationary types and indecomposable fibrations satisfying conditions 1) and 2) of the conclusion of the lemma. Let $U(p) = k + 1$. If $f : p \to q$ is indecomposable then $(f : p \to q)$, is a sequence of indecomposable fibrations satisfying the lemma. If $f : p \to q$ is not indecomposable, then there exists some stationary type $p' \in S(A)$ and fibrations $f' : p \to p'$ and $g : p' \to q$ which are not finite-to-one. Let p' be of maximal possible U-rank. Suppose that, for sake of contradiction, $f' : p \to p'$ is not indecomposable. Then there are fibrations $h : p \to q'$ and $h' : q' \to p'$ such that $U(p) > U(q') > U(p')$. So, we have a pair of fibrations $h : p \to q'$ and $g \circ h' : q' \to q$ such that $f = (g \circ h') \circ h$ and $U(q') > U(p')$

(to see that $g \circ h'$ is a fibration, every realization of q' is $h(a)$ for some $a \models p$. So, the fibers of $g \circ h'$ are $\mathrm{tp}(h(a)/g(h'(h(a))) = \mathrm{tp}(h(a)/g(f'(a))) = \mathrm{tp}(h(a)/f(a))$ which is the image of the stationary fiber $\mathrm{tp}(a/f(a)))$. This contradicts maximality of $U(p')$. Hence, $f' : p \to p'$ is indecomposable.

Since $U(p) > U(p') > U(q)$, we can apply the inductive hypothesis to $g : p' \to q$ and obtain a sequence of stationary types and indecomposable fibrations $(f_i : p_i \to p_{i-1})_{i=1,\dots,n-1}$ such that $g = f_{n-1} \circ \cdots \circ f_1$, $p_{n-1} := p'$, $p_0 := q$ and $U(p_i) > U(p_{i-1})$ for all $i = 1, \dots, n-1$. Set $p_n := p$ and $f_n : p_n \to p_{n-1}$to be $f' : p \to p'$. Then $f = f_n \circ g = f_n \circ \cdots \circ f_1$ and $U(p_i) > U(p_{i-1})$ for all $i = 1, \dots, n$. Hence, $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\dots,n}$ is a sequence of stationary types and indecomposable fibrations satisfying the conclusion of the lemma.

For the moreover statement, fix $a \models p$ and suppose that $\mathrm{tp}(a/Af(a))$ is almost internal to some minimal type r. Set $a_n := a$ and let $(a_n, \dots, a_0)$ be a realization of $\mathbf{f}$. Fix $i = 0, \dots, n-1$. Since $U(p_i) > U(p_{i-1})$, the fiber $\mathrm{tp}(a_i/Aa_{i-1})$ is non-algebraic and almost internal to a minimal type s_i. By Corollary 2.2.27, $\mathrm{tp}(a_i/Aa_{i-1}) \not\perp s_i$. Since $f(a) = a_0 \in \mathrm{dcl}(Aa_{i-1})$, the extension $\mathrm{tp}(a/Aa_{i-1})$ is almost r-internal, by Lemma 2.2.23 and Fact 2.2.25. Since $a_i \in \mathrm{dcl}(Aa)$, by Lemma 2.2.23 and Fact 2.2.25, $\mathrm{tp}(a_i/Aa_{i-1})$ is almost r-internal. By construction of the f_i, $\mathrm{tp}(a_i/Aa_{i-1})$ is non-algebraic. Thus, by Lemma 2.2.31, $r \not\perp s$. By Lemma 2.2.31, $\mathrm{tp}(a_i/Aa_{i-1})$ is almost r-internal. □

Corollary 7.1.3. *Suppose that $p \in S(A)$ is stationary, non-algebraic and finite U-rank. Then there exists a semiminimal analysis $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\dots,n}$ with each $f_i : p_i \to p_{i-1}$ indecomposable.*

In particular, every stationary, non-algebraic finite U-rank type admits a semiminimal analysis.

Proof. Let $f : p \to q$ be a fibration where q is algebraic, and so $U(p) > U(q)$. By Lemma 7.1.2, there is a sequence of indecomposable maps $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\dots,n}$ such that $p_m = p$, $p_0 = q$ and $U(p_i) > U(p_{i-1})$ for all $i = 1, \dots, m$. By Corollary 6.2.6, f_i is semiminimal, and so $\mathbf{f}$ is a semiminimal analysis for p. □

It is natural to compare semiminimal analyses to domination decompositions. Indeed, this was essentially the (unspoken) concern of the previous chapter. As a consequence of iterating Theorem 6.3.1, we obtain the following result:

Theorem 7.1.4. *Let $p \in S(A)$ be stationary and non-algebraic with a semiminimal analysis $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\dots,n}$. For every $i = 1, \dots, n$, fix some minimal type r_i such that a fiber of $f_i : p_i \to p_{i-1}$ is almost r_i-internal. Then p is domination equivalent to the Morley product $r_1^{(m_1)} \otimes \cdots \otimes r_n^{(m_n)}$ for some $0 \leq m_i \leq n_i$ where $n_i = U(p_i) - U(p_{i-1})$.*

Proof. We induct on n, the length of the semiminimal analysis.

If $n = 1$, $f : p_1 \to p_0$ is a semiminimal analysis. In particular, p_1 is almost internal to a minimal type r_1. Since p_1 is interalgebraic with p, by Lemma 2.2.23 and Fact 2.2.25, p is almost r_1-internal. Hence, by Lemma 6.1.8, $p \mathrel{\sqsupseteq\!\sqsubseteq} r_1^{(U(p))}$ where $U(p) = U(p_1) - U(p_0)$.

Suppose the result holds for all semiminimal analyses of length $n-1$. In particular, by the inductive hypothesis we have that $p_{n-1} \mathrel{\sqsupseteq\!\sqsubseteq} r_{n-1}^{(m_{n-1})} \otimes \cdots \otimes r_1^{(m_1)}$ where $0 \leq m_i \leq U(p_i) - U(p_{i-1})$ for all $i = 1, ..., n-1$. Let r_n be a minimal type such that some fiber of $f_n : p_n \to p_{n-1}$ almost r_n-internal. By Lemma 6.3.1, $p_n \mathrel{\sqsupseteq\!\sqsubseteq} r_n^{(m_n)} \otimes p_{n-1}$ for some $0 \leq m_n \leq U(p_n) - U(p_{n-1})$. Since p_n is interalgebraic with p over A, $p \mathrel{\sqsupseteq\!\sqsubseteq} r_n^{(m_n)} \otimes p_{n-1}$. By Fact 6.1.3 $p \mathrel{\sqsupseteq\!\sqsubseteq} r_n^{(m_n)} \otimes r_{n-1}^{(m_{n-1})} \otimes \cdots \otimes r_1^{(m_1)}$ where $0 \leq m_i \leq U(p_i) - U(p_{i-1})$ for all $i = 1, ..., n$. $\square$

Notice that some of the m_i can be zero, so not every minimal type appearing in a semiminimal analysis need appear in the domination decomposition. Even if the minimal types appearing in a semiminimal analysis and a domination decomposition do agree, they may appear with different multiplicities. Here is an example how Theorem 7.1.4 can sometimes be used to compute domination decompositions.

Example 7.1.5. Let $p \in S(A)$ be a stationary type of U-rank 2. Let $r \in S(A)$ be a minimal type. If p is 2-step r-analyzable, but not almost r-internal, then p is domination equivalent to r.

Proof. By the above theorem, either $p \mathrel{\sqsupseteq\!\sqsubseteq} r$ or $p \mathrel{\sqsupseteq\!\sqsubseteq} r \otimes r = r^{(2)}$. If $p \mathrel{\sqsupseteq\!\sqsubseteq} r^{(2)}$, then by Lemma 6.1.8, p is almost r-internal since $U(p) = 2$. But, we assumed that p is not almost r-internal, a contradiction. Thus $p \mathrel{\sqsupseteq\!\sqsubseteq} r$. $\square$

7.2 Minimal types appearing in a semiminimal analysis

Our first goal is Proposition 7.2.3 below. This is a folklore result for which I could not find a clear reference in the literature. It implies, in particular, part of our desired Theorem 7 above; namely that the positivity of the multiplicity of r in $\mathbf{f}$ is independent of the given semiminimal analysis of p.

While orthogonality is a condition on all non-forking extensions of a type $p \in S(A)$, it is possible for p to be orthogonal to a type r and some forking extension of p to be non-orthogonal to r.

Example 7.2.1. This appears as [40, Example 5.6]. Work in DCF_0. Let F be an algebraically closed field of constants. Let p be the generic type over F of $\begin{cases} \delta(x_1) = x_1^3(x_1 - 1) \\ \delta(x_2) = x_1 x_2. \end{cases}$
Then $p \perp \mathcal{C}$, but if $(a, b) \models p$, then $\mathrm{tp}(ab/Fa)$ is a non-algebraic extension of p that is non-orthogonal to the constants.

This motivates:

Definition 7.2.2. Let $p \in S(A)$ and $r \in S(B)$ be stationary. We say that p is *hereditarily orthogonal* to r if every stationary extension of p is orthogonal to r.

We show below that, up to non-orthogonality, the minimal types with positive multiplicity in any semiminimal analysis are exactly the minimal types to which p is non-hereditarily orthogonal. In particular, a minimal type r has *positive multiplicity* in a semiminimal analysis $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,\dots,n}$ means that for some i, some fiber of $f_i : p_i \to p_{i-1}$ is almost r-internal.

Proposition 7.2.3. *Suppose p is a stationary type of finite U-rank and r is a minimal type. Then the following are equivalent:*

1. *p is not hereditarily orthogonal to r.*
2. *r has positive multiplicity in every semiminimal analysis of p.*
3. *r has positive multiplicity in some semiminimal analysis of p.*

Proof. (1) $\implies$ (2) Suppose that p is not hereditarily orthogonal to r. Then for some $B \supseteq A$ and $a \models p$, $\mathrm{tp}(a/B)$ is non-orthogonal to r. By taking non-forking extensions, if necessary, we may assume that $\mathrm{tp}(a/B) \not\perp^w r|_B$ and $B = \mathrm{acl}(B)$. Thus, $\mathrm{tp}(a/B)$ is a stationary and non-algebraic (as algebraic types are orthogonal to every type).

The proof consists of three main steps:

i) First, we show that r has positive multiplicity in some semiminimal analysis $\mathbf{g}$ of $p' := \mathrm{tp}(a/B)$.

ii) Next, we show the given a semiminimal analysis $\mathbf{f}$ of p, we obtain a semiminimal analysis of p'. In particular, if s is a minimal type of positive multiplicity in $\mathbf{f}$, then s has positive multiplicity in some semiminimal analysis of p' (possibly different from the semiminimal analysis obtained in *i*)).

iii) Finally, we show that r has positive multiplicity in any semiminimal analysis for p' and then conclude using *ii*).

Since $p' \not\perp r$, by Fact 2.2.30 there is $q \in S(B)$ and a definable function $g' : p' \to q$ such that q is non-algebraic and r-internal. By Lemma 2.2.6, there is some $q' \in S(B)$, fibration $g : p' \to q'$ and finite-to-one definable function $\pi : q' \to q$ such that $g' = \pi \circ g$. In particular, q' is almost r-internal and non-algebraic. Since q' is already semiminimal, by Lemma 7.1.2, there is a sequence of types over B and definable semiminimal fibrations $\mathbf{g} = (g_i : p'_i \to p'_{i-1})_{i=1,...,m}$ where p'_1 is interalgebraic with q' and p'_0 is some algebraic type. Thus the fibers of $g_1 : p'_1 \to p'_0$ are p'_1, which is almost r-internal by interalgebraicity with the almost r-internal type q'. Thus the multiplicity of r in $\mathbf{g}$ is greater than or equal to $U(p'_1) > 0$.

On the other hand, fix $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ some semiminimal analysis of p. Let $(a_n, ..., a_0)$ be a realization of $\mathbf{f}$ and $r_1, ..., r_n$ be minimal types such that for all $i = 1, ..., n$ $\mathrm{tp}(a_i/Aa_{i-1})$ is almost r_i-internal. Let $q_n = \mathrm{tp}(b_n/B)$ and let $f_n : q_n \to \mathrm{tp}(b_{n-1}/B)$ be the definable function given by $a_n \mapsto a_{n-1}$. First suppose that $\mathrm{tp}(a_n/Ba_{n-1})$ is non-algebraic. By Lemma 2.2.6, there is $q_{n-1} \in S(A)$, fibration $h_n : q_n \to q_{n-1}$ and a definable finite-to-one function $\pi_n : q_n \to q_{n-1}$ such that $f_n = \pi_n \circ h_n$. Then $\mathrm{stp}(a_n/Bh_n(a_n)) = \mathrm{stp}(a_n/Ba_{n-1})$, which is almost r_1-internal.

If $\mathrm{tp}(a_n/Ba_{n-1})$ is algebraic, then let i be greatest such that $\mathrm{tp}(a_n/Ba_i)$ is non-algebraic. Then, $\mathrm{acl}(Ba_n) = \mathrm{acl}(Ba_{i+1})$. Since $\mathrm{tp}(a_{i+1}/Aa_i)$ is almost r_{i+1}-internal, $\mathrm{tp}(a_n/Ba_i)$ is almost r_i-internal. Hence, we have the definable function $f_{i+1} : q_n \to \mathrm{tp}(a_i/B)$ given by $a_n \mapsto a_i$ with non-algebraic fibers which are all almost r_{i+1}-internal. Hence, the same argument as above gives a stationary type $q_i \in S(B)$, a fibration $h_{i+1} : q_n \to q_i$ with some fiber almost r_{i+1}-internal, and a finite-to-one definable function $\pi_{i+1} : q_i \to \mathrm{tp}(a_i/B)$.

Repeating the argument for all $i = n-1, ..., 1$ with $f_i|_B : q_i \to \mathrm{tp}(a_j/B)$ where j is least such that $\mathrm{acl}(Ba_{j+1}) = \mathrm{acl}(Ba_i)$, we obtain a semiminimal analysis for p', $\mathbf{h} = (h_i : q_i \to q_{i-1})_{i=1,...,l}$, where $l \leq n$ and for every $i = 1, ...\ell$, every fiber is non-algebraic and almost internal r_j for some $j = 1, ..., n$. Hence, we have shown that any semiminimal analysis of p induces a semiminimal analysis of p'. In particular, in $\mathbf{h}$, if for some $i = 1, ..., l$, the fibers of $h_i : q_i \to q_{i-1}$ are non-algebraic and almost s-internal, there is some $j = 1, ..., n$ and $f_j : p_j \to p_{j-1}$ whose fibers are non-algebraic and almost s-internal.

Now we show that for some $i = 1, ..., l$, the fibers of $h_i : q_i \to q_{i-1}$ are almost r-internal, hence there is some $j = 1, ..., n$ such that the fibers of $f_j : p_j \to p_{j-1}$ are almost r-internal which means exactly that r has positive multiplicity in $\mathbf{f}$. Fix $(b_l, ..., b_0)$ a realization for $\mathbf{h}$ such that $\mathrm{acl}(Ba) = \mathrm{acl}(Bb_l)$. Let $b = g(a) \models q'$. In particular, $\mathrm{tp}(b/B)$ is almost

r-internal and non-algebraic. Note that

$$\mathrm{acl}(B) \subseteq \mathrm{acl}(Bb_1) \subseteq \cdots \subseteq \mathrm{acl}(Bb_l) = \mathrm{acl}(Ba).$$

Since $b \in \mathrm{acl}(Ba) \backslash \mathrm{acl}(B)$, there is some $1 \leq j \leq l$ such that $b \in \mathrm{acl}(Bb_j) \backslash \mathrm{acl}(Bb_{j-1})$. Since $\mathrm{tp}(b/B)$ is almost r-internal, by Lemma 2.2.23 and Fact 2.2.25, $\mathrm{stp}(b/Bb_{j-1})$ is also almost r-internal. By assumption this type is non-algebraic and almost internal to the minimal type r, hence $\mathrm{stp}(b/Bb_{j-1}) \not\perp r$ by Corollary 2.2.27. On the other hand, by definition of a realization of a semiminimal analysis $\mathrm{tp}(b_j/Bb_{j-1})$ is almost r_j-internal. Since $b \in \mathrm{acl}(Bb_j)$, by Lemma 2.2.23 and Fact 2.2.25, $\mathrm{stp}(b/Bb_{j-1})$ is almost r_j-internal. Since $\mathrm{stp}(b/Bb_{j-1})$ is non-algebraic, non-orthogonal to r and almost internal to the minimal type r_j, by Lemma 2.2.31, $r \not\perp r_j$. By construction, there is some $k = 1, ...n$ such that a fiber of $f_k : p_k \to p_{k-1}$ is (non-algebraic) and almost r_j internal. By Lemma 2.2.31, the fibers of $f_k : p_k \to p_{k-1}$ are almost r-internal.

We have show that for any semiminimal analysis $\mathbf{f}$ of p, there is some i and some non-algebraic fiber of f_i that is almost r-internal. In particular, this show that r has positive multiplicity in every semiminimal analysis of p showing that (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) is immediate.

(3) $\Longrightarrow$ (1) Suppose there is a semiminimal analysis $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ for p such that for some $i = 1, ..., n$ there is a non-algebraic fiber of $f_i : p_i \to p_{i-1}$ which is almost r-internal. Let $(a_n, ..., a_0)$ be a realization of $\mathbf{f}$ such that $\mathrm{tp}(a_i/Aa_{i-1})$ is almost r-internal. Since $\mathrm{tp}(a_i/Aa_{i-1})$ is non-algebraic and almost internal to the minimal type r, by Corollary 2.2.27, $\mathrm{tp}(a_i/Aa_{i-1}) \not\perp r$. Fix $a \models p$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Aa_n)$. Since $a_i \in \mathrm{acl}(Aa)$, $\mathrm{stp}(a/Aa_i) \not\perp r$. Hence, $\mathrm{stp}(a/Aa_i)$ is an extension of p which is non-orthogonal to r, and thus p is not hereditarily orthogonal to r. $\square$

7.3 Levels and reductions

We want to show that the multiplicity of a minimal type in a semiminimal analysis, not just its positivity, is independent of the semiminimal analysis. We use Buechler's theory of *levels* to do so. Buechler introduced levels in [10], with the notion being further studied in [46] and [61]. While Buechler works over the empty set, we fix a base set of parameters A.

Definition 7.3.1. The *first level* of B over C is defined as

$$\ell_{1,A}(B/C) = \{b \in \mathrm{acl}(AB) \mid \mathrm{stp}(b/AC) \text{ is algebraic or semiminimal}\}.$$

Inductively, we define $\ell_{k+1,A}(B/C) = \ell_{1,A}(B/\ell_{k,A}(B/C))$ and set $\ell_{0,A}(B/C) = \mathrm{acl}(AC)$.

In our application of levels, we will typically take $C = \emptyset$, in which case we write $\ell_{k,A}(B)$.

Note that levels are not definably closed. For example, if $b_1, b_2 \in \ell_{1,A}(B) \setminus \mathrm{acl}(A)$, then the pair (b_1, b_2) need not be in $\ell_{1,A}(B)$. Indeed, if $\mathrm{tp}(b_1/A)$ is r_1-semiminimal, $\mathrm{tp}(b_2/A)$ is r_2-semiminimal, and r_1 is orthogonal to r_2, then $\mathrm{tp}(b_1, b_2/A)$ is not semiminimal. However, the definable closure, and indeed algebraic closure, of the kth level is contained in the $(k+1)$-st level, see [10, Remark 3.1(i)].

Levels behave well with independence and domination.

Fact 7.3.2. *[10, Proposition 3.1] For sets B, C and $k \geq 1$*

1. *$\ell_{k,A}(C) \trianglerighteq_{\ell_{k-1,A}(C)} C$*
2. *B is independent from $\ell_{k,A}(C)$ over $\ell_{k,A}(B)$*

We can then define the number of levels for a type:

Definition 7.3.3. We say that a stationary type $p \in S(A)$ has m *levels* if for some (equivalently any) $a \models p$, m is the least k such that $a \in \mathrm{acl}(\ell_{k,A}(a))$.

We verify some basic properties:

Lemma 7.3.4. *1. $\ell_{k,A}(B/C) = \ell_{k,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC))$ for all $k \geq 0$.*

2. *$\mathrm{acl}(\ell_{k+1,A}(B/C)) = \mathrm{acl}(\ell_{k,\ell_{1,A}(B/C)}(B/C))$ for all $k \geq 0$.*
3. *Let $b \in \mathrm{acl}(Aa)$. Then $\mathrm{acl}(\ell_{k,A}(b)) = \mathrm{acl}(\ell_{k,A}(a)) \cap \mathrm{acl}(Ab)$ and $\mathrm{acl}(\ell_{k+1,A}(b)) = \mathrm{acl}(\ell_{k,\ell_{1,A}(a)}(b))$.*

Proof. 1. We induct on k, the number of levels. By definition,

$$\ell_{0,A}(B/C) = \mathrm{acl}(AC) = \ell_{0,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC)).$$

For $k = 1$,

$$\begin{aligned}\ell_{1,A}(B/C) &= \{b \in \mathrm{acl}(AB) \mid \mathrm{stp}(b/AC) \text{ is algebraic or semiminimal}\}\\ &= \{b \in \mathrm{acl}(\mathrm{acl}(A)\mathrm{acl}(AB)) \mid \mathrm{stp}(b/\mathrm{acl}(AC)) \text{ is algebraic or semiminimal}\}\\ &= \ell_{1,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC)\end{aligned}$$

Suppose we have shown that $\ell_{k,A}(B/C) = \ell_{k,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC))$ for $k \geq 0$, then

$$\begin{aligned}\ell_{k+1,A}(B/C) &= \ell_{1,A}(B/\ell_{k,A}(B/C)) \\ &= \ell_{1,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(\ell_{k,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC)))) \\ &= \ell_{k+1,\mathrm{acl}(A)}(\mathrm{acl}(AB)/\mathrm{acl}(AC))\end{aligned}$$

2. We induct on k, the number of levels. By definition,

$\ell_{0,\ell_{1,A}(B/C)}(B/C) = \mathrm{acl}(\ell_{1,A}(B/C)C) = \mathrm{acl}(\ell_{1,A}(B/C))$. Taking the acl on both sides, we get that $\mathrm{acl}(\ell_{0,\ell_{1,A}(B/C)}(B/C)) = \mathrm{acl}(\ell_{1,A}(B/C))$.

Suppose that we have shown that $\mathrm{acl}(\ell_{k,A}(B/C)) = \mathrm{acl}(\ell_{k-1,\ell_{1,A}(B/C)}(B/C))$. Then

$$\begin{aligned}\ell_{k+1,A}(B/C) &= \ell_{1,A}(B/\ell_{k,A}(B/C)) \\ &= \ell_{1,A}(B/\mathrm{acl}(\ell_{k,A}(B/C))) && \text{by 1.} \\ &= \ell_{1,A}(B/\mathrm{acl}(\ell_{k-1,\ell_{1,A}(B/C)}(B/C))) && \text{by inductive hypothesis.}\end{aligned}$$

By definition, $\ell_{1,A}(B/\mathrm{acl}(\ell_{k-1,\ell_{1,A}(B/C)}(B/C)))$ is

$$\{b \in \mathrm{acl}(AB) \mid \mathrm{stp}(b/\ell_{k-1,\ell_{1,A}(B/C)}(B/C)) \text{ is algebraic or semiminimal}\}.$$

Since $\mathrm{acl}(\ell_{1,A}(B/C)) \subseteq \mathrm{acl}(AB)$ and $A \subseteq \mathrm{acl}(\ell_{1,A}(B/C))$, $\ell_{1,A}(B/\mathrm{acl}(\ell_{k-1,\ell_{1,A}(B/C)}(B/C)))$ is

$$\{b \in \mathrm{acl}(\ell_{1,A}(B/C)B) \mid \mathrm{stp}(b/\ell_{k-1,\ell_{1,A}(B/C)}(B/C)) \text{ is algebraic or semiminimal}\}.$$

So,

$$\begin{aligned}\ell_{k+1,A}(B/C) &= \ell_{1,\ell_{1,A}(B/C)}(B/\ell_{k-1,\ell_{1,A}(B/C)}(B/C)) && \text{by definition} \\ &= \ell_{k,\ell_{1,A}(B/C)}(B/C) && \text{by definition.}\end{aligned}$$

3. Let $b \in \mathrm{acl}(Aa)$. We induct on k. By definition,

$$\mathrm{acl}(\ell_{0,A}(b)) = \mathrm{acl}(A) = \mathrm{acl}(\ell_{0,A}(a)) \cap \mathrm{acl}(Ab).$$

Then, $c \in \ell_{k+1,A}(b) = \ell_{1,A}(b/\ell_{k,A}(b))$ if and only if $c \in \mathrm{acl}(Ab) \subseteq \mathrm{acl}(Aa)$ and $\mathrm{stp}(c/\ell_{k,A}(b))$ is algebraic or semiminimal. Since $\mathrm{acl}(\ell_{k,A}(b)) \subseteq \mathrm{acl}(\ell_{k,A}(a))$, $\mathrm{stp}(c/\ell_{k,A}(a))$ is algebraic or semiminimal, by Lemma 2.2.23 and Fact 2.2.25. Thus, we have that

$c \in \ell_{k+1,A}(a) \cap \operatorname{acl}(Ab)$. For the reverse direction, if $c \in \ell_{1,A}(a/\ell_{k,A}(a)) \cap \operatorname{acl}(Ab)$, then $\operatorname{stp}(c/\ell_{k,A}(a))$ is algebraic or semiminimal. By Fact 7.3.2, $c \underset{\ell_{k,A}(c)}{\perp} \ell_{k,A}(a)$. Hence, either $\operatorname{stp}(c/\ell_{k,A}(c))$ is algebraic or $\operatorname{stp}(c/\ell_{k,A}(c))$ is semiminimal. Since $\ell_{k,A}(c) \subseteq \ell_{k,A}(b)$, $\operatorname{stp}(c/\ell_{k,A}(b))$ is algebraic or semiminimal by Lemma 2.2.23 and Fact 2.2.25. Taking the algebraic closure on both sides gives $\operatorname{acl}(\ell_{k+1,A}(b)) = \operatorname{acl}(\ell_{k+1,A}(a)) \cap \operatorname{acl}(Ab)$.

By 2., $\operatorname{acl}(\ell_{k+1,A}(a)) = \operatorname{acl}(\ell_{k,\ell_{1,A}(a)}(a))$. Hence, $\operatorname{acl}(\ell_{k+1,A}(b)) = \operatorname{acl}(\ell_{k,\ell_{1,A}(a)}(b))$. □

Remark 7.3.5. *If a stationary type* $\operatorname{tp}(a/A)$ *has* m *levels, then the type* $\operatorname{stp}(a/\ell_{1,A}(a))$ *has* $m-1$ *levels. In particular, for every* $1 \leq k \leq m$, $\operatorname{stp}(a/\ell_{k,A}(a))$ *has less than* m *levels.*

Similarly, if $b \in \operatorname{acl}(Aa)$, *then* $\operatorname{tp}(b/A)$ *has less than or equal to* m *levels.*

Proof. This follows easily for 2. of the above Lemma. Indeed, $\operatorname{tp}(a/A)$ has m levels means that m is the least integer such that $\operatorname{acl}(Aa) = \operatorname{acl}(\ell_{m,A}(a))$. By Lemma 7.3.4, $\operatorname{acl}(Aa) = \operatorname{acl}(\ell_{m-1,\ell_{1,A}(a)}(a))$. Since $\operatorname{acl}(A) \subseteq \operatorname{acl}(\ell_{1,A}(a)) \subseteq \operatorname{acl}(Aa)$, $\operatorname{acl}(\ell_{1,A}(a)a) = \operatorname{acl}(\ell_{m-1,\ell_{1,A}(a)}(a))$. If $\operatorname{acl}(Aa) = \operatorname{acl}(\ell_{1,A}(a)a) = \operatorname{acl}(\ell_{m-2,\ell_{1,A}(a)}(a))$, then, by Lemma 7.3.4, $\operatorname{acl}(Aa) = \operatorname{acl}(\ell_{m-1,A}(a))$, a contradiction. Hence, $\operatorname{stp}(a/\ell_{1,A}(a))$ has $m-1$ levels. Repeatedly applying the above argument, we get that $\operatorname{acl}(\ell_{k,A}(a)a) = \operatorname{acl}(\ell_{m-k,\ell_{k,A}(a)}(a))$.

If $b \in \operatorname{acl}(Aa)$, by Lemma 7.3.4, $\operatorname{acl}(\ell_{m,A}(b)) = \operatorname{acl}(\ell_{m,A}(a)) \cap \operatorname{acl}(Ab)$. Since $\operatorname{acl}(Aa) = \operatorname{acl}(\ell_{m,A}(a))$, $\operatorname{acl}(\ell_{m,A}(b)) = \operatorname{acl}(Aa) \cap \operatorname{acl}(Ab) = \operatorname{acl}(Ab)$. □

A key observation for us is that the first level is captured by "reductions" of a type p to minimal types that are non-orthogonal to p. Such reductions are a model-theoretic abstraction of algebraic reductions in complex analytic geometry, and have appeared in various forms before ([44], [58], for example). We introduce a variant here that essentially agrees with earlier versions, but is better suited to our needs.

Definition 7.3.6. Suppose that $p = \operatorname{tp}(a/A)$ is stationary and $r \in S(B)$ is minimal. By an *r-reduction of a over A* we mean a tuple $b \in \operatorname{dcl}(Aa)$ such that:

1. $\operatorname{tp}(b/A)$ is r-internal, and
2. for any $c \in \operatorname{dcl}(Aa)$, if $\operatorname{tp}(c/A)$ is almost r-internal, then $c \in \operatorname{acl}(Ab)$.

If b is an r-reduction of a over A, and $b = f(a)$ for some A-definable function f, then we call $f : p \to q := \operatorname{tp}(b/A)$ an *r-reduction of p*.

The r-reduction of a over A is unique, up to interalgebraicity. The r-reduction of a type is also well-defined, up to interalgebraicity.

Lemma 7.3.7. *Suppose $p \in S(A)$ is stationary and $r \in S(B)$ is minimal.*

1. *If $a \models p$ and b_1 and b_2 are both r-reductions of a over A, then b_1 and b_2 are interalgebraic over A.*

2. *Let $a_1, a_2 \models p$ and b_i is an r-reduction of a_i over A for $i = 1, 2$. Then for $i = 1, 2$, there is b_i' interalgebraic with b_i over A such that $\mathrm{tp}(b_1'/A) = \mathrm{tp}(b_2'/A)$.*

Proof. 1. Fix $a \models p$ and let b_1 and b_2 be r-reductions of a over A. By definition $b_1, b_2 \in \mathrm{dcl}(Aa)$, $\mathrm{tp}(b_1/A)$ is almost r-internal and $\mathrm{tp}(b_2/A)$ is almost r-internal. Since b_1 is an r-reduction of a over A, $b_2 \in \mathrm{acl}(Ab_1)$. Since b_2 is an r-reduction of A, $\mathrm{acl}(Ab_1) = \mathrm{acl}(Ab_2)$.

2. Let b_i be an r-reduction of a_i over A for $i = 1, 2$. Since a_1 and a_2 have the same type over A, there is some $\sigma \in \mathrm{Aut}_A(\mathcal{U})$ such that $\sigma(a_1) = a_2$. Let $b_2' := \sigma(b_1)$. Since $b_1 \in \mathrm{dcl}(Aa_1)$, by applying σ, $b_2' \in \mathrm{dcl}(Aa_2)$. Since $\mathrm{tp}(b_1/A)$ is almost r-internal, $\mathrm{tp}(b_2'/A)$ is almost internal in the family of all A-conjugates of r. By Fact 2.2.25, $\mathrm{tp}(b_2'/A)$ is almost r-internal. Since b_2 is an r-reduction of a_2 over A, $b_2' \in \mathrm{acl}(Ab_2)$. Note that $\sigma^{-1}(b_2) \in \mathrm{dcl}(Aa_1)$ and $\mathrm{tp}(\sigma^{-1}(b_2)/A)$ is almost r-internal. Since b_1 is a r-reduction of a_1 over A, $\sigma^{-1}(b_2) \in \mathrm{dcl}(Ab_1)$. Hence, $b_2 \in \mathrm{dcl}(Ab_2')$. By definition, b_2' is an r-reduction of a_2 over A. Since $\sigma(b_1) = b_2'$, $\mathrm{tp}(b_1/A) = \mathrm{tp}(b_2'/A)$. Set $b_1' := b_1$. □

Lemma 7.3.8. *If $p \in S(A)$ is stationary with finite U-rank and $r \in S(B)$ is minimal, then an r-reduction of p exists.*

Proof. If $p \perp r$, then let $b \in \mathrm{dcl}(A)$ and $f : p \to q := \mathrm{tp}(b/A)$ be a constant function. Clearly, $b \in \mathrm{dcl}(Aa)$ and q is almost r-internal, as it is algebraic. If $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ and $\mathrm{tp}(c/A)$ is almost r-internal, then by Lemma 2.2.27, $\mathrm{tp}(c/A) \not\perp r$. Hence, $p \not\perp r$ which is a contradiction. Thus $c \in \mathrm{acl}(Ab)$. By definition, $f : p \to q$ is the r-reduction of p.

If $p \not\perp r$, then by Lemma 7.3.8, there is some $b \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(b/A)$ is almost r-internal. Choose b such that $\mathrm{tp}(b/A)$ has maximal U-rank. Suppose there is some $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(c/A)$ is almost r-internal, but $c \notin \mathrm{acl}(Ab)$. Then $U(bc/A) = U(b/A) + U(c/Ab) > U(b/A)$. By Lemma 2.2.23, $\mathrm{tp}(bc/A)$ is non-algebraic, almost r-internal and $bc \in \mathrm{dcl}(Aa)$ which contradicts the maximality of $U(b/A)$. Hence, the definable function $f : p \to q := \mathrm{tp}(b/A)$ given by $a \mapsto b$ is a non-algebraic r-reduction of p. □

The proof of the above lemma shows that p is non-orthogonal to r if and only if the r-reduction of p is non-algebraic. We can then think of the r-reduction of p as capturing that part of p responsible for its non-orthogonality to r.

Lemma 7.3.9. *Let $r_1, ..., r_n$ be pairwise orthogonal minimal types which are non-orthogonal to $\mathrm{tp}(a/A)$. Let b_i be the r_i-reduction of a over A. Then $\{b_1, ..., b_n\}$ is an A-independent set.*

Proof. Since $r_i \not\perp p$, by the proof of Lemma 7.3.8, $b_i \notin \mathrm{acl}(A)$. Suppose, for sake of contradiction, that $\{b_1, ..., b_n\}$ is not A-independent. After potentially reordering, let k be least such that $b_{k+1} \not\mathop{\smile\!\!\!\!\!\mid}_A b_1, ..., b_k$ and $b_{k+1} \mathop{\smile\!\!\!\!\!\mid}_A b_1, ..., b_{k-1}$, otherwise k is not minimal. Thus, $b_{k+1} \not\mathop{\smile\!\!\!\!\!\mid}_{Ab_1,...,b_{k-1}} b_k$. Similarly, if $b_k \not\mathop{\smile\!\!\!\!\!\mid}_A b_1, ..., b_{k-1}$, then k is not minimal. Hence, $\mathrm{tp}(b_{k+1}/A) \not\perp \mathrm{tp}(b_k/A)$. By definition, $\mathrm{tp}(b_k/A)$ is non-algebraic and almost r_k-internal. By Lemma 2.2.31, $\mathrm{stp}(b_{k+1}/A) \not\perp r_k$. Since $\mathrm{stp}(b_{k+1}/A)$ is non-algebraic and almost r_{k+1}-internal, by Lemma 2.2.31, $r_{k+1} \not\perp r_k$, a contradiction. Thus $\{b_1, ..., b_n\}$ is A-independent. $\square$

We show below that the first level of any type is interalgebraic with the non-algebraic reductions of p to every minimal type, which was known but not made explicit in the literature until [18, Theorem 3.17]. We provide an alternative proof here.

Theorem 7.3.10. *Let $p = \mathrm{tp}(a/A)$ be stationary and non-algebraic. Let $r_1, ..., r_n$ be pairwise orthogonal minimal types such that for any minimal type s which is non-orthogonal to p, s is non-orthogonal to r_i for some $i = 1, ..., n$. For all $i = 1, ..., n$, let b_i be the r_i-reduction of a over A and $k_i = U(b_i/A)$.*

Then $\mathrm{acl}(\ell_{1,A}(a)) = \mathrm{acl}(Ab_1, ..., b_n)$ and p is domination equivalent to the Morley product $r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$.

Proof. By definition of r_i-reductions, for all $i = 1, ..., n$, $\mathrm{stp}(b_i/A)$ is almost r_i-internal. If $\mathrm{stp}(b_i/A)$ is algebraic, then $b_i \in \mathrm{acl}(A)$, so we may assume that for all $i = 1, ..., n$, $r_i \not\perp p$. If $\mathrm{stp}(b_i/A)$ is non-algebraic, since $\mathrm{stp}(b_i/A)$ is almost r_i-internal and $\mathrm{stp}(b_i/A) \not\perp p$, $p \not\perp r_i$ by Lemma 2.2.31.

For all $i = 1, ..., n$, $b_i \in \mathrm{acl}(Aa)$, by definition of r_i-reductions. Since $\mathrm{stp}(b_i/A)$ is semiminimal, $\mathrm{acl}(Ab_1, ..., b_n) \subseteq \mathrm{acl}(\ell_{1,A}(a))$. For the reverse inclusion, let $c \in \ell_{1,A}(a)$. Then either $c \in \mathrm{acl}(A) \subseteq \mathrm{acl}(Ab_1, ..., b_n)$ or $c \in \mathrm{acl}(Aa)$ and $\mathrm{stp}(c/A)$ is r-semiminimal for some minimal type r. Then $p \not\perp \mathrm{stp}(c/A)$ and so, by Lemma 2.2.31, $p \not\perp r$. By assumption, $r \not\perp r_i$ for some $i = 1, ..., n$. Hence, by Lemma 2.2.31, $\mathrm{stp}(c/A)$ is almost r_i-internal. By definition of r_i-reductions, $c \in \mathrm{acl}(Ab_i)$. Thus, for every $c \in \ell_{1,A}(a), c \in \mathrm{acl}(Ab_1, ..., b_n)$. Hence, $\mathrm{acl}(\ell_{1,A}(a)) \subseteq \mathrm{acl}(Ab_1, ..., b_n)$.

We now show the domination equivalence. Since $\mathrm{acl}(Ab_1, ..., b_n) \subseteq \mathrm{acl}(Aa)$, $a \trianglerighteq_A b_1, ..., b_n$. By Fact 7.3.2, $\mathrm{acl}(Ab_1, ..., b_n) = \mathrm{acl}(\ell_{1,A}(a)) \trianglerighteq_A a$. Thus, $a \,\square_A b_1, ..., b_n$. By

Lemma 7.3.9, $b_1, ..., b_n$ is an A-independent tuple, hence $b_1, ..., b_n \models \mathrm{stp}(b_1/A) \otimes \cdots \otimes \mathrm{stp}(b_n/A)$. Thus, $p \sqsupseteq\!\sqsubseteq \mathrm{stp}(b_1/A) \otimes \cdots \otimes \mathrm{stp}(b_n/A)$. Since for all $i = 1, ..., n$, $\mathrm{stp}(b_i/A)$ is almost r_i-internal, by Lemma 6.1.8, $\mathrm{stp}(b_i/A) \sqsupseteq\!\sqsubseteq r_i^{(k_i)}$ where $k_i = U(\mathrm{stp}(b_i/A))$. By repeatedly applying Fact 6.1.3, we obtain that $p \sqsupseteq\!\sqsubseteq r_1^{(k_1)} \otimes \cdots \otimes r_n^{(k_n)}$. $\square$

Corollary 7.3.11. *Let $f : p \to q$ be a semiminimal fibration of stationary finite U-rank types.*

1. *The type p is domination equivalent to q if and only if for some $a \models p$, the algebraic closure of the first level of a is equal to the algebraic closure of the first level of $f(a)$.*
2. *If p and q have 1 level, and some fiber of f is almost internal to the minimal type r, then p is domination equivalent to $r^{(k)} \otimes q$ where $k = U(p) - U(q)$*

Proof. We begin with a set up that is used in the proof of both parts of the corollary. Let $r_1, ..., r_n$ be pairwise orthogonal minimal types such that $r_i \not\perp p$ and for any minimal type s such that $s \not\perp p$, then $s \not\perp r_i$ for some $i = 1, ..., n$. Fix $a \models p$. For all $i = 1, ..., n$, let c_i be the r_i-reduction of a over A. By Theorem 7.3.10, $\mathrm{acl}(\ell_{1,A}(a)) = \mathrm{acl}(Ac_1, ..., c_n)$. For all $i = 1, ..., n$ there exists an r_i-reduction of $f(a)$ over A, say c_i'. Note that $c_i' \in \mathrm{acl}(Ac_i)$ since $c_i' \in \mathrm{dcl}(Aa)$ and $\mathrm{stp}(c_i'/A)$ is almost r_i-internal. By Theorem 7.3.10, $\mathrm{acl}(\ell_{1,A}(f(a))) = \mathrm{acl}(Ac_1', ..., c_n')$ (notice that if a minimal type $s \not\perp q$, then $s \not\perp p$ so we may apply the theorem). For all $i = 1, ..., n$, set $\ell_i := U(c_i/A)$ and $l_i' := U(c_i'/A)$. Notice that for all $i = 1, ..., n$, $l_i > 0$. However, it is possible for some of the $l_i' = 0$ since $l_i' > 0$ if and only if $r_i \not\perp q$. By Theorem 7.3.10, $p \sqsupseteq\!\sqsubseteq r_1^{(l_1)} \otimes \cdots \otimes r_n^{(l_n)}$ and $q \sqsupseteq\!\sqsubseteq r_1^{(l_1')} \otimes \cdots \otimes r_n^{(l_n')}$.

1. First suppose that $p \sqsupseteq\!\sqsubseteq q$. Since domination decompositions are unique up to non-orthogonality of minimal types, for all $i = 1, ..., n$, $l_i = l_i'$. In particular, $\mathrm{acl}(Ac_i) = \mathrm{acl}(Ac_i')$ for all $i = 1, ..., n$. Thus,

$$\begin{aligned}\mathrm{acl}(\ell_{1,A}(a)) &= \mathrm{acl}(Ac_0, ..., c_m)\\ &= \mathrm{acl}(Ac_0', ..., c_m')\\ &= \mathrm{acl}(\ell_{1,A}(f(a)))\end{aligned}$$

For the reverse direction, suppose that $\mathrm{acl}(\ell_{1,A}(a)) = \mathrm{acl}(\ell_{1,A}(f(a)))$. By Fact 7.3.2, $a \sqsupseteq\!\sqsubseteq_A \mathrm{acl}(\ell_{1,A}(a))$ and $f(a) \sqsupseteq\!\sqsubseteq_A \mathrm{acl}(\ell_{1,A}(f(a)))$. Hence, $a \sqsupseteq\!\sqsubseteq_A f(a)$ showing that $p \sqsupseteq\!\sqsubseteq q$.

2. Since p and q have 1 level, we have that $\mathrm{acl}(Aa) = \mathrm{acl}(Ac_1, ..., c_n)$ and $\mathrm{acl}(Af(a)) = \mathrm{acl}(Ac_1', ..., c_n')$. Hence, $U(p) = \sum_{i=1}^{n} l_i$ and $U(q) = \sum_{i=1}^{n} l_i'$. On the other hand, by Theorem

6.3.1, $p \sqsubseteq r^{(k)} \otimes q$ for some $0 \leq k \leq U(p) - U(q)$. Thus $r_1^{(l_1)} \otimes r_2^{(l_2)} \otimes \cdots \otimes r_n^{(l_n)} \sqsubseteq r^{(k)} \otimes r_1^{(l'_1)} \otimes r_2^{(l'_2)} \otimes \cdots \otimes r_n^{(l'_n)}$ where $k = \sum_{i=1}^{n} l_i - l'_i = U(p) - U(q)$. □

7.4 Proof of Theorem 7

We now complete the proof of Theorem 7 by giving an intrinsic characterization of the multiplicity of a minimal type r in a semiminimal analysis $\mathbf{f}$ of a stationary finite U-rank type p; intrinsic in that it will not depend on the semiminimal analysis. We begin with some preparatory lemmas.

Lemma 7.4.1. *Let $f : p \to q$ be a fibration of stationary finite U-rank types $p, q \in S(A)$. Suppose $p = \operatorname{tp}(a/A)$ has $m + 1$ levels and some fiber of f is almost internal to a minimal type $r \in S(B)$. For all $0 \leq i \leq m$, let p'_i be the strong type of a over its ith level and let $f'_i : p'_i \to q'_i$ be the definable function given by $a \mapsto f(a)$.*

Then p'_i is domination equivalent to either:

1. *the Morley product $r^{(k_i)}$ where $k_i = U(p'_i)$ if the type q'_i is algebraic, or*
2. *the Morley product $r^{(k_i)} \otimes q'_i$ for some $0 \leq k_i \leq U(p'_i) - U(q'_i)$.*

Moreover, $U(p) - U(q) = \sum_{i=0}^{m} k_i$.

Proof. Since p has ≥ 1 levels, p cannot be algebraic. Fix $a \models p$. If f has algebraic fibers, then for every $i \leq m$, $\operatorname{acl}(\ell_{i,A}(a)) = \operatorname{acl}(\ell_{i,A}(f(a))) \cap \operatorname{acl}(Aa) = \operatorname{acl}(\ell_{i,A}(f(a)))$. Hence, $\operatorname{acl}(\ell_{i,A}(a)a) = \operatorname{acl}(\ell_{i,A}(a)f(a))$. So, $a \sqsubseteq_{\ell_{i,A}(a)} f(a)$ which shows that $\operatorname{stp}(a/\ell_{i,A}(a)) \sqsubseteq r^{(0)} \otimes \operatorname{stp}(f(a)/\ell_{i,A}(a))$. So we may assume that $f : p \to q$ has non-algebraic fibers.

By Lemma 7.1.2, there is a sequence of stationary types $q_0, ..., q_n$ and indecomposable fibrations $\mathbf{g} = (g_j : q_j \to q_{j-1})_{j=1,...,n}$ such that $q_n = p$, $q_0 = q$, $f = g_1 \circ \cdots \circ g_n : p \to q$ and for all $j = 1, ..., n$, some fiber of $g_j : q_j \to q_{j-1}$ is almost r-internal. Let $(b_n, ..., b_0)$ be a realization of $\mathbf{g}$ such that $\operatorname{tp}(b_j/Ab_{j-1})$ is almost r-internal. After re-labeling, we may assume that $b_n = a$ and $b_0 = f(a)$. Then $U(a/Af(a)) = \sum_{i=1}^{n} U(b_j/Ab_{j-1})$.

Claim 1. *For every $j = 1, ..., n$ and $i = 0, ..., m$, if $\operatorname{stp}(b_j/\ell_{i,A}(a))$ is non-algebraic, then either:*

i) $\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(l_{i,j})}$ *where* $l_{i,j} = U(b_j/\ell_{i,j}(a))$ *if* $\operatorname{tp}(b_{j-1}/\ell_{i,j}(a))$ *is algebraic, or*

ii) $\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(l_{i,j})} \otimes \operatorname{stp}(b_j/\ell_{i,A}(a))$ *where* $0 \leq l_{i,j} \leq U(b_j/\ell_{i,A}(a)) - U(b_{j-1}/\ell_{i,A}(a))$.

Proof of claim: Fix $j = 1, ..., n$ and $i = 1, ..., m$. Consider the definable function $g'_j : \operatorname{stp}(b_j/\ell_{i,A}(a)) \to \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$ given by $b_j \mapsto b_{j-1}$. Since $\operatorname{stp}(b_j/Ab_{j-1})$ is almost r-internal, by Lemma 2.2.23 and Fact 2.2.25, $\operatorname{stp}(b_j/\ell_{i,A}(a)b_{j-1})$ is almost r-internal. If $\operatorname{stp}(b_j/\ell_{i,A}(a))$ is algebraic, set $\ell_{i,j} = 0$. Otherwise, if $\operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$ is algebraic, then $\operatorname{stp}(b_j/\ell_{i,A}(a))$ is almost r-internal and non-algebraic. Hence, by Lemma 6.1.8,

$$\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(\ell_{i,j})} \text{ for } \ell_{i,j} = U(b_j/\ell_{i,A}(a)).$$

If $\operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$ is non-algebraic, we would like to apply Theorem 6.3.1 to show that $\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(\ell_{i,j})} \otimes \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$. However, $g'_j : q_j \to q_{j-1}$ may not be a fibration. By Lemma 2.2.6, there is a fibration $h_j : \operatorname{stp}(b_j/\ell_{i,A}(a)) \to \operatorname{stp}(c/\ell_{i,A}(a))$ and a finite-to-one definable function $\pi : \operatorname{stp}(c/\ell_{i,A}(a)) \to \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$ such that $g'_j = \pi \circ h_j$. We have that $\operatorname{stp}(b_j/\ell_{i,A}(a)h_j(b_j)) = \operatorname{stp}(b_j/\ell_{i,A}(a)g'_j(b_j))$ is almost r-internal. By Theorem 6.3.1,

$$\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(\ell_{i,j})} \otimes \operatorname{stp}(h_j(b_j)/\ell_{i,A}(a)),\ 0 \leq l_i \leq U(b_j/\operatorname{acl}(\ell_{i,A}(a))) - U(b_{j-1}/\operatorname{acl}(\ell_{i,A}(a))).$$

Since $\operatorname{acl}(\ell_{i,A}(a)h_j(b_j)) = \operatorname{acl}(\ell_{i,A}(a)b_{j-1})$, $\operatorname{stp}(h_j(b_j)/\ell_{i,A}(a)) \mathrel{\underline{\Box}} \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$. Repeatedly applying Fact 6.1.3, we obtain that $\operatorname{stp}(b_j/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(\ell_{i,j})} \otimes \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$. We have shown that either $i)$ or $ii)$ must hold. ■

Fix $i = 0, ..., m$. Claim 1 shows that $\operatorname{stp}(a/\ell_{i,A}(a)) = \operatorname{stp}(b_n/\ell_{i,A}(a))$ is either domination equivalent to $r^{(l_{i,n})}$, where $l_{i,n} = U(a/\ell_{i,A}(a))$, or it is domination equivalent to $r^{(l_{i,n})} \otimes \operatorname{stp}(b_{n-1}/\ell_{i,A}(a))$ for some $0 \leq l_{i,n} \leq U(b_n/\ell_{i,A}(a)) - U(b_{n-1}/\ell_{i,A}(a))$. In the former case, for all $j = 0, ..., n-1$, we have that $\operatorname{stp}(b_j/\ell_{i,A}(a))$ is algebraic, hence $l_{i,j} = 0$. In the later case, by Fact 6.1.3, either $\operatorname{stp}(a/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(l_{i,n})} \otimes r^{(l_{i,n-1})}$ with $l_{i,n-1} = U(b_{n-1}/\ell_{i,A}(a))$ and $l_{i,j} = 0$ for all $j = 1, ..., n-2$, or $\operatorname{stp}(a/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(l_{i,n})} \otimes r^{(l_{i,n-1})} \otimes \operatorname{stp}(b_{n-2}/\ell_{i,A}(a))$ for some $0 \leq l_{i,n-1} = U(b_{n-1}/\ell_{i,A}(a)) - U(b_{n-2}/\ell_{i,A}(a))$. Continuing, we obtain that for all $i = 0, ..., m$, either

1. $\operatorname{stp}(f(a)/\ell_{i,A}(a))$ is algebraic and so $\operatorname{stp}(a/\ell_{i,A}(a)) \mathrel{\underline{\Box}} r^{(l_{i,n})} \otimes \cdots \otimes r^{(l_{i,n'})}$ such that:
   (a) for $n' \leq n$ the least integer k such that $U(b_{k-1}/\ell_{i,A}(a))$ is algebraic and so $l_{i,n'} = U(b_{n'}/\ell_{i,A}(a))$,
   (b) for all $j = n', ..., n$, $0 \leq l_{i,j} \leq U(b_j/\operatorname{acl}(\ell_{i,A}(a))) - U(b_{j-1}/\operatorname{acl}(\ell_{i,A}(a)))$, and
   (c) for all $j = 1, ..., n'-1$, $l_{i,j} = 0$, or

2. $\operatorname{stp}(a/\ell_{i,A}(a)) \sqsubset r^{(l_{i,n})} \otimes \cdots \otimes r^{(l_{i,1})} \otimes \operatorname{stp}(f(a)/\ell_{i,A}(a))$ such that for all $1 \leq j \leq n$, $0 \leq l_{i,j} \leq U(b_j/\ell_{i,A}(a)) - U(b_{j-1}/\ell_{i,A}(a))$.

Setting $p'_i := \operatorname{stp}(a/\ell_{i,A}(a))$, $q'_i := \operatorname{stp}(f(a)/\ell_{i,A}(a))$ and $f'_i : p'_i \to q'_i$ to be the definable function given by $a \mapsto f(a)$, we have shown that, $p'_i \sqsubset r^{(k_i)}$ if q'_i is algebraic, or $p'_i \sqsubset r^{(k_i)} \otimes q'_i$ and in both cases, $k_i = \sum_{j=1}^{n} l_{i,j}$. If $p'_i \sqsubset r^{(k_i)}$, then the almost r-internal fibers of the definable function $f'_i : p'_i \to q'_i$ are exactly p'_i. Hence, p'_i is almost r-internal, and non-algebraic by assumption, thus $k_i = U(p_i)$ by Lemma 6.1.8. This is exactly condition 1) of the conclusion for the lemma. If q'_i is non-algebraic, then $p'_i \sqsubset r^{(k_i)} \otimes q'_i$ and $k_i = \sum_{j=1}^{n} l_{i,j}$. For all $j = 1, ..., n$, we have that $0 \leq l_{i,j} \leq U(b_j/\operatorname{acl}(\ell_{i,A}(a))) - U(b_{j-1}/\operatorname{acl}(\ell_{i,A}(a)))$. Thus,

$$0 \leq k_i \leq \sum_{j=1}^{n} U(b_j/\operatorname{acl}(\ell_{i,A}(a))) - U(b_j/\operatorname{acl}(\ell_{i,A}(a))) = U(b_n/\operatorname{acl}(\ell_{i,A}(a))) - U(b_0/\operatorname{acl}(\ell_{i,A}(a))).$$

But, $b_n = a$ and $b_0 = f(a)$. Hence, $0 \leq k_i \leq U(p'_i) - U(q'_i)$, which is condition 2) of the conclusion of the lemma.

For the moreover statement, we first show that for a fixed $j = 1, ..., n$, we have that $U(p_j) - U(p_{j-1}) = \sum_{i=1}^{l} l_{i,j}$ where either $\operatorname{stp}(b_j/\ell_{i,A}(a)) \sqsubset r^{(l_{i,j})}$ or $\operatorname{stp}(b_j/\ell_{i,A}(a)) \sqsubset r^{(l_{i,j})} \otimes \operatorname{stp}(b_{j-1}/\ell_{i,A}(a))$, as in Claim 1.

We induct on $m+1$, the number of levels of p. Suppose that $f : p \to q$ is simply a definable function, not necessarily a fibration. This will make the inductive step easier. Then, by Lemma 2.2.6, there is $q' \in S(A)$, a fibration $f' : p \to q'$ and a finite-to-one definable function $\pi : q' \to q$ such that $f = \pi \circ f'$. Since the fibers of f are almost r-internal and π is finite to-one, the fibers of f' are almost r-internal and $U(p) - U(q) = U(p) - U(q')$.

Suppose that p has 1 level, hence $m = 0$. By definition, $\ell_{0,A}(a) = A$. By Remark 7.3.5, q' has either 1 level or 0 levels. If q' is algebraic, then the fibers of f' are p itself. Hence, p is almost r-internal. By Lemma 6.1.8, $p \sqsubset r^{(k_0)}$ with $k_0 = U(p)$. Since q' is algebraic we have that $U(p) - U(q) = U(p) - U(q') = U(p) = k_0$. So we may assume that q' has 1 level. Since p and q' have 1 level, by Corollary 7.3.11, $p \sqsubset r^{(k_0)} \otimes q$ where $k_0 = U(p) - U(q)$.

Suppose we have shown that for any definable function, $g : p' \to q'$, of stationary finite U-rank types over B which has an almost r-internal fiber where $p' = \operatorname{tp}(b/B)$ is non-algebraic and has m levels, then $U(\operatorname{stp}(b/Bg(b))) = \sum_{i=0}^{m-1} l_i$ where $\operatorname{stp}(b/\ell_{i,B}(b)) \sqsubset r^{(l_i)}$

if $\mathrm{stp}(g(b)/\ell_{i,B(a)})$ is algebraic, otherwise $\mathrm{stp}(b/\ell_{i,B}(b)) \sqsupseteq r^{(l_i)} \otimes \mathrm{stp}(g(b)/\ell_{i,B}(b))$ for some $0 \leq l_i \leq U(b/\mathrm{acl}(\ell_{i,B}(b))) - U(g(b)/\mathrm{acl}(\ell_{i,B}(b)))$

Assume that p has $m+1$ levels. By Remark 7.3.5, $\mathrm{stp}(b_j/\ell_{1,A}(a))$ has less than or equal to m levels. Let $g'_j : \mathrm{stp}(b_j/\ell_{1,A}(a)) \to \mathrm{stp}(b_{j-1}/\ell_{1,A}(a))$ be the definable function given by as $b \mapsto b_{j-1}$. By the inductive hypothesis, $U\left(\mathrm{stp}\left(b_j/\ell_{1,A}(a)b_{j-1}\right)\right) = \sum_{i=1}^{m} l_{i,j}$. By Fact 7.3.2, $b_j \underset{\ell_{1,A}(b_j)}{\perp} \ell_{1,A}(a)$. Hence, $U\left(\mathrm{stp}(b_j/\ell_{1,A}(b_j)b_{j-1})\right) = \sum_{i=1}^{m} l_{i,j}$.

Since $g_j : q_j \to q_{j-1}$ is an indecomposable fibration, Corollary 6.3.3 shows that either $l_{0,j} = 0$ or $l_{0,j} = U(b_{j+1}/Ab_j)$. If $l_{0,j} = 0$, then $q_j \sqsupseteq q_{j-1}$. So, by Corollary 7.3.11, $\mathrm{acl}(\ell_{1,A}(b_j)) = \mathrm{acl}(\ell_{1,A}(b_{j-1}))$. Hence,

$$U(\mathrm{stp}(b_j/Ab_{j-1})) = U(\mathrm{stp}(b_j/\ell_{1,A}(b_{j-1})b_{j-1})) = U(\mathrm{stp}(b_j/\ell_{1,A}(b_j)b_{j-1})) = \sum_{i=1}^{m} l_{i,j} = \sum_{i=0}^{m} l_{i,j}$$

where the last equality is because $l_{0,j} = 0$.

If $l_{0,j} = U(b_j/Ab_{j-1})$, then $\mathrm{acl}(\ell_{1,A}(b_{j-1})) \subsetneq \mathrm{acl}(\ell_{1,A}(b_j))$ by Corollary 7.3.11.

Claim 2. *Let $f : p \to q$ be an indecomposable fibration of stationary finite U-rank types $p, q \in S(A)$. If p is not domination equivalent to q, then for some (equivalently any) $a \models p$, $a \in \mathrm{acl}(\ell_{1,A}(a)f(a))$.*

Proof of claim: Let $r_1, ..., r_n$ be pairwise orthogonal minimal types such that, for some for some $0 < m_1, ..., m_n$, $p \sqsupseteq r_1^{(m_1)} \otimes \cdots \otimes r_n^{(m_n)}$. For all $i = 1, ..., n$, let c_i be the r_i-reduction of a over A. In particular, $c_i \in \mathrm{dcl}(Aa)$, and by Lemma 7.3.10, $\mathrm{acl}(\ell_{1,A}(a)) = \mathrm{acl}(Ac_1, ..., c_n)$. For all $i = 1, ..., n$, there exists an r_i-reduction of $f(a)$ over A, say c'_i. By Lemma 7.3.10, $\mathrm{acl}(\ell_{1,A}(f(a))) = \mathrm{acl}(Ac'_1, ..., c'_n)$.

Since $\mathrm{tp}(a/A) \not\sqsupseteq \mathrm{tp}(f(a)/A)$, by Corollary 7.3.11, $\mathrm{acl}(\ell_{1,A}(f(a))) \subsetneq \mathrm{acl}(\ell_{1,A}(f(a)))$. So, there is some $i = 1, ..., n$, such that $c_i \in \mathrm{acl}(\ell_{1,A}(a)) \setminus \mathrm{acl}(\ell_{1,A}(f(a)))$. Since we have the equality $\mathrm{acl}(\ell_{1,A}(f(a))) = \mathrm{acl}(\ell_{1,A}(a)) \cap \mathrm{acl}(Af(a))$ by Lemma 7.3.4, we have that $c_i \notin \mathrm{acl}(Af(a))$. By definition, we have that $c_i \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(Af(a))$ and since the fibration $f : \mathrm{tp}(a/A) \to \mathrm{tp}(f(a)/A)$ is indecomposable, $a \in \mathrm{acl}(Af(a)c_i) \subseteq \mathrm{acl}(\ell_{1,A}(a)f(a))$. ■

By Claim 2, we have that $b_j \in \mathrm{acl}(\ell_{1,A}(b_j)b_{j-1}) \subseteq \mathrm{acl}(\ell_{1,A}(a)b_{j-1}) \subseteq \mathrm{acl}(\ell_{i,A}(a)b_{j-1})$ for all $i = 1, ..., n$. Hence, $\mathrm{stp}(b_j/\ell_{i,A}(a))$ is interalgebraic with $\mathrm{stp}(b_{j-1}/\ell_{i,A}(a))$ which gives the equivalence $\mathrm{stp}(b_j/\ell_{i,A}(a)) \sqsupseteq \mathrm{stp}(b_{j-1}/\ell_{i,A}(a))$ and thus $l_{i,j} = 0$ for all $i = 1, ..., n$. Hence, we have the equality $U(\mathrm{stp}(b_j/Ab_{j-1})) = l_{0,j} + 0 = \sum_{i=0}^{m} l_{i,j}$.

Recall that k_i is the exponent such that $\operatorname{stp}(a/\ell_{i,A}(a)) \sqsubseteq r^{(k_i)}$ if $\operatorname{stp}(f(a)/\ell_{i,A}(a))$ is algebraic, or $\operatorname{stp}(a/\ell_{i,A}(a)) \sqsubseteq r^{(k_i)}) \otimes \operatorname{stp}(f(a)/\ell_{i,A}(a))$. Moreover, we have shown that $k_i = \sum_{j=1}^{n} l_{i,j}$. Hence, $U(p) - U(q) = \sum_{j=1}^{n} U(\operatorname{stp}(b_j/Ab_{j-1})) = \sum_{j=1}^{n}\sum_{i=0}^{m} l_{i,j} = \sum_{i=0}^{m} k_i$.

We have shown that $U(a/Af(a)) = \sum_{i=0}^{m} k_i$ when $\operatorname{tp}(a/A)$ has $m+1$ levels, concluding the induction. $\square$

We are now ready to prove that multiplicity does not depend on the semiminimal analysis:

Theorem 7.4.2. *Let $p \in S(A)$ be a stationary, non-algebraic, finite U-rank type with $m+1$ levels. Fix $a \models p$ and consider, for each $i = 0, ..., m$, the type of a over the algebraic closure of its ith level, say $p'_i := \operatorname{stp}(a/\ell_{i,A}(a))$. Given any minimal type r, the multiplicity of r in any semiminimal analysis of p is equal to the sum over i of the U-ranks of the r-reductions of p'_i for $i = 0, ..., m$.*

In particular, this quantity does not depend on the semiminimal analysis.

Proof. Let $r \in S(B)$ be a minimal type, by taking a non-forking extension we may assume that $B \supseteq A$. Fix $\mathbf{f} = (f_j : p_j \to p_{j-1})_{j=1,...,n}$, a semiminimal analysis for p and a realization $(a_n, ..., a_0)$ of $\mathbf{f}$ such that $\operatorname{acl}(Aa) = \operatorname{acl}(Aa_n)$. For all $j = 0, ..., n$, fix a minimal type s_j such that $\operatorname{tp}(a_j/Aa_{j-1})$ is s_j-semiminimal. Recall that if $r \not\perp s_j$, then $\operatorname{tp}(a_{j,A}/Aa_{j-1})$ is almost r-internal by Lemma 2.2.31. Let $J = \{1 \leq j \leq n \mid \operatorname{tp}(a_j/Aa_{j-1}) \text{ is almost } r\text{-internal}\}$. By definition, the multiplicity of r in $\mathbf{f}$ is the sum over J of $U(p_j) - U(p_{j-1})$. Recall that for every $j \in J$, $U(p_j) - U(p_{j-1}) = U(a_j/Aa_{j-1})$. We now relate these U-ranks to the U-ranks of r-reductions through the use of domination. First note that by definition of levels, since $\operatorname{acl}(Aa_n) = \operatorname{acl}(Aa)$, $\operatorname{acl}(\ell_{i,A}(a)) = \operatorname{acl}(\ell_{i,A}(a_n))$ for all $i = 1, ..., m$. In particular, p_n also has $m+1$ levels and $p \sqsubseteq p_n$. For every $j \in J$, let $m_j + 1$ be the number of levels for p_j. By Remark 7.3.5, $m_j + 1 \leq m + 1$.

Fix some $j \in J$. Recall that if $j > 1$, then by Theorem 6.3.1 $p_j \sqsubseteq r^{(k_{0,j})} \otimes p_{j-1}$ for some $0 \leq k_{0,j} \leq U(a_j/Aa_{j-1})$. If $j = 1$, then p_{j-1} is algebraic, hence p_j is almost r-internal and by Lemma 6.1.8, $p_j \sqsubseteq r^{(k_{0,j})}$ for $k_{0,j} = U(p_j)$. In Lemma 7.4.1, we showed that $U(a_j/Aa_{j-1}) = \sum_{i=0}^{m} k_{i,j}$ where:

i) for all $0 \leq i \leq m_j$ where $\operatorname{tp}(a_{j-1}/\ell_{i,A}(a_j))$ is algebraic, $\operatorname{stp}(a_j/\ell_{i,A}(a_j)) \sqsubseteq r^{(k_{i,j})}$ and $0 \leq k_{i,j} \leq U(\operatorname{stp}(a_j/\ell_{i,A}(a)))$,

ii) for all $0 \leq i \leq m_j$ where $\text{tp}(a_{j-1}/\ell_{i,A}(a_j))$ is non-algebraic, we have that

$$\text{stp}(a_j/\ell_{i,A}(a_j)) \sqsupset r^{(k_{i,j})} \otimes \text{stp}(a_j/\ell_{i,A}(a_j))$$

and $0 \leq k_{i,j} \leq U(\text{stp}(a_j/\ell_{i,A}(a))) - U(\text{stp}(a_{j-1}/\ell_{i,A}(a)))$,

iii) for all $m_j < i \leq m$, $k_{i,j} = 0$.

Thus the multiplicity of r in $\mathbf{f}$ is $\sum_{j\in J} U(a_j/Aa_{j-1}) = \sum_{j\in J}\sum_{i=1}^{m} k_{i,j}$.

However, we also showed that we can express the domination decomposition of types in terms of their reductions to minimal types. By Theorem 7.3.10, for every $i = 0, ..., m$, $\text{stp}(a/\ell_{i,A}(a))$ is domination equation to the Morley product of the image p'_i of its r-reduction and some type q'_i which is orthogonal to r. Since p'_i is almost r-internal, we have the equivalence $\text{stp}(a/\ell_{i,A}(a)) \sqsupset r^{(l_i)} \otimes q'_i$ where l_i is the U-rank of the r-reduction of $\text{stp}(a/\ell_{i,A}(a))$.

We show that for all $i = 1, ..., m$, $l_i = \sum_{j\in J} k_{i,j}$. Indeed if this equality holds, then the multiplicity of r in $\mathbf{f}$ is $\sum_{j\in J} U(a_j/Aa_{j-1}) = \sum_{j\in J}\sum_{i=1}^{m} k_{i,j} = \sum_{i=1}^{m}\sum_{j\in J} k_{i,j} = \sum_{i=0}^{m} l_i$, the sum of the U-rank of the images under the r-reduction of $p'_i = \text{stp}(a/\ell_{i,A}(a))$.

Fix $i = 1, ..., m$. Since $\mathbf{f} = (f_j : p_j \to p_{j-1})_{j=1,...,n}$ is a semiminimal analysis of p, by Corollary 7.1.4, $p \sqsupset s_1^{(k_{0,1})} \otimes \cdots \otimes s_n^{(k_{0,n})}$ such that for every $j = 1, ..., n$ the fibers of f_j are almost s_j-internal and $0 \leq k_{0,i} \leq U(p_j) - U(p_{j-1})$. For $\text{stp}(a/\ell_{i,A}(a))$, we obtain a sequence of definable functions $(f_j : \text{stp}(a_j/\ell_{i,A}(a_j)) \to \text{stp}(a_{j-1}/\ell_{i,A}(a_j))_{j=1,...,n}$. To apply Corollary 7.1.4 to $\text{stp}(a/\ell_{i,A}(a))$, we need a sequence of fibrations, not just definable maps. Note that for every $j = 1, ..., n$, since $\text{tp}(a_j/Aa_{j-1})$ is almost s_j-internal, by Lemma 2.2.23 and Fact 2.2.25, $\text{stp}(a_j/\ell_{i,A}(a_j)a_{j-1})$ is also almost s_j-internal. However, some fibers may now be algebraic.

By Lemma 2.2.6, there is a definable fibration $g_n : \text{stp}(a_n/\ell_{i,A}(a)) \to \text{stp}(b_{n-1}/\ell_{i,A}(a)))$ and a finite-to-one definable function $\pi_n : \text{stp}(b_{n-1}/\ell_{i,A}(a)) \to \text{stp}(a_{n-1}/\ell_{i,A}(a))$ such that $f_n = \pi_n \circ g_n$. Since $\text{acl}(\ell_{i,A}(a)a_{n-1}) = \text{acl}(\ell_{i,A}(a)b_{n-1})$ and $\text{stp}(a_n/\ell_{i,A}(a)a_{n-1})$ is almost s_n-internal, $\text{stp}(a_n/\ell_{i,A}(a)b_{n-1})$ is also almost s_n-internal. For all $j = 1, ..., n-1$, inductively applying Lemma 2.2.6 to $\pi_j \circ f_j$ for all $j = 1, ..., n$, we obtain a sequence of definable fibrations $\mathbf{g} = (g_j : \text{stp}(b_j/\ell_{i,A}) \to \text{stp}(b_{j-1}/\ell_{i,A}))_{j=1,...,n}$ such that $\text{stp}(b_j/\ell_{i,A}(a)b_{j-1})$ is almost s_j-internal. If for some $j = 1, ..., n$, $g_j : q_j \to q_{j-1}$ has algebraic fibers, we can replace $g_{j+1} : q_{j+1} \to q_j$ with the definable function $g_j \circ g_{j+1} : q_{j+1} \to q_{j-1}$ and apply Lemma

2.2.6. Indeed, the fibers of this definable function are $\mathrm{tp}(a_{j+1}/\mathrm{acl}(\ell_{i,A}(a))a_{j-1})$. Since $\mathrm{acl}(\ell_{i,A}(a)a_j) = \mathrm{acl}(\ell_{i,A}(a)a_{j-1})$ and $\mathrm{stp}(a_{j+1}/\ell_{i,A}(a)a_j)$ are almost s_{j+1}-internal, the type $\mathrm{tp}(a_{j+1}/\ell_{i,A}(a)a_{j-1})$ is almost s_j-internal. The usual argument shows that the fibers of the fibration obtained by applying Lemma 2.2.6 to $g_j \circ g_{j+1} : q_{j+1} \to q_{j-1}$ are also s_{j+1}-internal.

Since $\mathrm{stp}(b_j/\ell_{i,A}(a)b_{j-1}) \sqsupseteq \mathrm{stp}(a_j/\ell_{i,A}(a)a_{j-1})$, by interalgebraicity of a_j and b_j, Corollary 7.1.4, gives that $\mathrm{stp}(a/\ell_{i,A}(a)) \sqsupseteq s_1^{(k_{i,1})} \otimes \cdots \otimes s_n^{(k_{i,n})}$, where $k_{i,j} = 0$ if the resulting fiber is algebraic. For all $j \in J$, we have that $s_j \not\perp r$, hence $s_j \sqsupseteq r$ by Fact 6.1.4. Repeatedly applying Fact 6.1.3, we obtain that $\mathrm{stp}(a/\ell_{i,A}(a)) \sqsupseteq r^{\left(\sum_{j\in J} k_{i,j}\right)} \otimes \bigotimes_{j\notin J} s_j^{(k_{i,j})}$. By uniqueness of domination decompositions, we have that $l_i = \sum_{j\in J} k_{i,j}$.

We have shown that the multiplicity of r in $\mathbf{f}$ is the sum over $0 \leq i \leq m$ of the U-ranks of the reductions of $\mathrm{stp}(a/\ell_{i,A}(a))$. Since the r-reductions of $\mathrm{stp}(a/\ell_{i,A}(a))$ are independent of the choice of $\mathbf{f}$, we have shown that the multiplicity of r in one semiminimal analysis of p is the multiplicity of r in any semiminimal analysis of p. □

In particular, Theorem 7.4.2 implies Theorem 7.

7.5 Comparison to r-weight

Because of Theorem 7.4.2, we can now define the multiplicity of any minimal type in any finite U-rank type as follows:

Definition 7.5.1. Let p be a non-algebraic stationary finite U-rank type and let r be a minimal type. The *multiplicity of r in p*, denoted by $\mathrm{mult}_r(p)$ is the multiplicity of r in some (equivalently any) semiminimal analysis for p as defined in Theorem 7. If p is algebraic, we set $\mathrm{mult}_r(p) = 0$.

It follows (by Proposition 7.2.3) that $\mathrm{mult}_r(p) > 0$ if and only if p is not hereditarily non-orthogonal to r.

In this section, we point out that when p is what is called r-simple, the multiplicity of r in p recovers what is called r-weight. So, our notion of multiplicity can be viewed as a generalisation of r-weight beyond the r-simple setting. Let us recall r-simplicity and r-weight. Our exposition follows that of Hrushovki's thesis [30].

A type being r-simple requires the extra assumption that r is non-orthogonal to a parameter set A. This means that r is non-orthogonal to some complete type over A. If we work under Condition 6.1.10, where there exists a non-locally modular minimal type $t \in S(\emptyset)$ such that any non-locally modular minimal type is non-orthogonal to t, by Theorem 6.1.11, r is domination equivalent to a minimal type $s \in S(A)$. Since both r and s are minimal types, r is non-orthogonal to s by Fact 6.1.6. Hence, every minimal type satisfies the assumption r is non-orthogonal to A for any parameter set A. For the remainder of this section we will assume our theory satisfies Assumption 6.1.10, so we may drop the assumption that r is non-orthogonal to the parameter set.

Definition 7.5.2. Let $B \supseteq A$, $p \in S(A)$ and let $r \in S(B)$ be a minimal type. We say that p is r-*simple* if there is some $C \supseteq A \cup B$, $a \models p|_C$ and a set Y of independent realizations of r such that $\mathrm{stp}(a/CY)$ is hereditarily orthogonal to r.

Since we work in a totally transcendental theory, if p is r-simple, we can choose Y to be a finite independent tuple of realizations of p (see the discussion following Definition 7.1.6 in [63]). By definition, every algebraic type is r-simple.

Example 7.5.3. Let $p \in S(A)$ be stationary and finite U-rank. Let $r \in S(B)$ be a minimal type.

1. If p is hereditarily orthogonal to r, then p is r-simple.

2. If p is almost r-internal, then p is r-simple.

3. Let $T = \mathrm{DCF}_0$, F be an algebraically closed field of constants and p the generic type over F of the algebraic vector field $(\mathbb{A}^2, s)$ where $s(x,y) = (x, y, x^3(x-1), xy)$. Let r be the generic type of the constants over $\emptyset$. Then p is not r-simple, but r has multiplicity in p greater than 0.

Proof. 1. Suppose that p is hereditarily orthogonal to r. Let $Y = \emptyset$ and $C = A$. Then for any $a \models p$, we have that $\mathrm{stp}(a/CY) = \mathrm{stp}(a/A)$ is hereditarily orthogonal to r.

2. Let $C \supseteq A \cup B$, $a \models p|_C$ and $c_1, ..., c_n \models r$ be such that $a \in \mathrm{acl}(Cc_1, ..., c_n)$. By Lemma 2.2.26, we may assume that $c_1, ..., c_n$ is a C-independent tuple. Then $\mathrm{stp}(a/Cc_1, ..., c_n)$ is algebraic, hence for any $D \supseteq Cc_1, ..., c_n$, $a' \models \mathrm{stp}(a/Cc_1, ..., c_n)$, and $c \models r|_D$, we have that $a' \underset{D}{\downarrow} d$. By definition, $\mathrm{stp}(a/Cc_1, ..., c_n)$ is hereditarily orthogonal to r, and thus p is r-simple.

3. Jaoui, Jimenez and Pillay [40, Example 5.6] show that $p \perp \mathcal{C}$. There is also a fibration $\pi : p \to q$ projection onto the x co-ordinate where q is the generic type of $(\mathbb{A}, x \mapsto (x, x^3(x-1)))$. For every $a \models p$, the fiber $\operatorname{tp}(a/F\pi(a))$ is non-algebraic and almost $\mathcal{C}$-internal. Thus for every $C \supseteq F$, $a \models p|_C$ and $c_1, ..., c_n \models r$, $a \underset{F}{\perp\!\!\!\perp} Cc_1, ..., c_n$. In particular, $a \underset{F\pi(a)}{\not\perp\!\!\!\perp} Cc_1, ..., c_n$, so $a \notin \operatorname{acl}(\pi(a)Cc_1, ..., c_n)$.

Since $\operatorname{tp}(a/F\pi(a))$ is almost $\mathcal{C}$-internal, it is almost r-internal. By Lemma 2.2.23 and Fact 2.2.25, $\operatorname{stp}(a/C\pi(a)c_1, ..., c_n)$ is almost r-internal. Since $\operatorname{stp}(a/C\pi(a)c_1, ..., c_n)$ is non-algebraic, by Corollary 2.2.27, $\operatorname{stp}(a/C\pi(a)c_1, ..., c_n) \not\perp r$. Hence, for every $C \supseteq F$, $a \models p|_C$ and $c_1, ..., c_n \subseteq r(\mathcal{U})$, $\operatorname{stp}(a/Cc_1, ..., c_n)$ has some extension which is non-orthogonal to r.

On the other hand, $(\pi : p \to q, q \to q_0)$, for some algebraic type q_0, is a semiminimal analysis for p and the fibers of $\pi : p \to q$ are almost r-internal. Hence, the multiplicity of r in p is $U(p) - U(q) = 1 > 0$. $\square$

Being r-simple has some nice properties.

Fact 7.5.4. *[63, Lemma 7.1.4] Let $r \in S(B)$ be minimal and $p = \operatorname{tp}(a/A)$ be stationary and r-simple.*

1. *If $\operatorname{tp}(b/A)$ is stationary and r-simple, then $\operatorname{tp}(a, b/A)$ is r-simple.*
2. *If $B \supseteq A$ and $b \in \operatorname{acl}(Ba)$, then $\operatorname{tp}(b/B)$ is r-simple.*

We can now look at a local version of weight, which is the minimal possible size for Y witnessing that a type is r-simple.

Definition 7.5.5. Let $B \supseteq A$ and $p \in S(A)$, $r \in S(B)$ be stationary types. If p is r-simple, then the *r-weight* of p is the minimal κ such that there is $C \supseteq A \cup B, a \models p|_C$ and $\bar{c}$ a C-independent tuple of realizations of $r|_B$ with size κ where $\operatorname{stp}(a/C\bar{b})$ hereditarily orthogonal to r. We denote the r-weight of p by $\operatorname{wt}_r(p)$. For $\operatorname{wt}_r(\operatorname{tp}(a/A))$ we write $\operatorname{wt}_r(a/A)$.

Since we work in a superstable theory, $\operatorname{wt}_r(p)$ exists and is finite. Note that $\operatorname{wt}_r(p) = 0$ if and only if p hereditarily orthogonal to r.

We show that on r-simple non-algebraic finite U-rank types, r-weight and multiplicity of r agree. We first prove the preliminary case when p is itself almost r-internal. This will be used in the proof of the general case.

Lemma 7.5.6. *Let $p \in S(A)$ be stationary and non-algebraic and let $r \in S(B)$ be minimal. If p is almost r-internal, then the multiplicity of r in p is the same as the r-weight of p.*

Proof. Suppose that p is almost r-internal. Then p is r-simple, hence the r-weight of p exists. Let $m = \mathrm{wt}_r(p)$ and let $C \supseteq A \cup B$, $a \models p|_C$, and $c_1, ..., c_m \models (r|_C)^{(m)}$ be such that $\mathrm{stp}(a/Cc_1, ..., c_m)$ is hereditarily orthogonal to r. By definition of r-weight, we have that m is minimal among all C. Since $\mathrm{tp}(a/A)$ is almost r-internal, by Lemma 2.2.23 and Fact 2.2.25, $\mathrm{stp}(a/Cc_1, ..., c_m)$ is almost r-internal. If $\mathrm{stp}(a/Cc_1, ..., c_m)$ is non-algebraic, by Corollary 2.2.27, $\mathrm{tp}(a/Cc_1, ..., c_m) \not\perp r$, a contradiction. Hence, $a \in \mathrm{acl}(Cc_1, ..., c_m)$.

Suppose for some $i = 1, ..., m$, $c_i \notin \mathrm{acl}(Ca)$. Without loss of generality, we may assume that $i = 1$. By minimality of r, $c_1 \underset{C}{\downarrow} a$. Hence, $a \models p|_{Cc_1}$. Taking $D = Cc_1$ we have that $\mathrm{stp}(a/Dc_2, ..., c_m)$ is hereditarily orthogonal to r, contradicting minimality of m. Thus, $c_1, ..., c_m \in \mathrm{acl}(Ca)$ which shows that $\mathrm{acl}(Ca) = \mathrm{acl}(Cc_1, ..., c_m)$.

Since $a \models r|_C$ and $c_1, ..., c_m \models (r|_C)^{(m)}$, we have shown that $a \,\square_C\, c_1, ..., c_m$, hence $p \,\square\, r^{(m)}$. Since p is almost r-internal, by Lemma 6.1.8, $m = U(p)$. Since p is almost r-internal, its r-reduction is a definable function to p. Hence, by Theorem 7.4.2, $\mathrm{mult}_r(p) = m$. We have shown that $\mathrm{mult}_r(p) = \mathrm{wt}_r(p)$. □

Proposition 7.5.7. *Let $r \in S(B)$ be minimal and $p \in S(A)$ be stationary, non-algebraic and finite U-rank. If p is r-simple, then the r-weight of p is equal to the multiplicity of r in p.*

Proof. Fix $a \models p$. Suppose that p has $m + 1$ levels. For all $i = 0, ..., m$, let $q_i = \mathrm{stp}(b_i/\ell_{i,A}(a))$ be the, possibly algebraic type such that b_i is the r-reduction of a over $\mathrm{acl}(\ell_{i,A}(a))$. By Theorem 7.4.2, $\mathrm{mult}_r(p) = \sum_{i=0}^{m} U(q_i)$. By definition, the r-reduction of q_i is a definable function to q_i. Since q_i is almost r-internal, by Theorem 7.4.2, $\mathrm{mult}_r(q_i) = U(q_i)$. Note that for all $i = 0, ..., m$, q_i is almost r-internal, and so r-simple. By Lemma 2.2.23 and Fact 2.2.25, every extension of q_i is also r-internal and thus r-simple. Hence, r-weight of q_i and its extensions is defined.

If q_i is algebraic, then $U(q_i) = 0 = \mathrm{wt}_r(q_i)$ since algebraic types are orthogonal to every type. Otherwise, since q_i is non-algebraic and almost r-internal, by Lemma 7.5.6, $\mathrm{mult}_r(q_i) = \mathrm{wt}_r(q_i)$. Since q_i is itself semiminimal, q_i has 1 level and hence, by Theorem 7.4.2, $\mathrm{mult}_r(q_i) = U(q_i)$. Hence for all $i = 0, ..., m$, $U(q_i) = \mathrm{wt}_r(q_i)$.

Thus we have that:

$$\begin{aligned}\mathrm{mult}_r(p) &= \sum_{i=0}^{m} U(q_i)\\ &= \sum_{i=0}^{m} \mathrm{wt}_r(q_i)\\ &= \sum_{i=0}^{m} \mathrm{wt}_r(b_i/\ell_{i,A}(a)).\end{aligned}$$

On the other hand, since p has $m+1$ levels, $\mathrm{acl}(Aa) = \mathrm{acl}(\ell_{m+1,A}(a))$.

By Theorem 7.3.10, for all $i = 0, ..., m$ we have that $\mathrm{acl}(\ell_{1,\ell_{i,A}(a)}(a)) = \mathrm{acl}(\ell_{i,A}(a)b_{i,0}, ..., b_{i,n_i})$ where $b_{i,0}$ is the r-reduction of a over $\mathrm{acl}(\ell_{i,A}(a))$ and $b_{i,j}$ is the $r_{i,j}$-reduction of a over $\mathrm{acl}(\ell_{i,A}(a))$, for some pairwise orthogonal minimal types $r_{i,1}, ..., r_{i,n_i}$. By Lemma 7.3.4, $\mathrm{acl}(\ell_{i+1,A}(a)) = \mathrm{acl}(\ell_{1,\ell_{i,A}(a)}(a))$. Hence, $\mathrm{acl}(\ell_{i+1,A}(a)) = \mathrm{acl}(\ell_{i,A}(a)b_{i,0}, ..., b_{i,n_i})$. Let n be the size of the maximal pairwise orthogonal collection of $r_{0,1}, ..., r_{0,n_0}, ..., r_{m,1}, ..., r_{m,n_m}$ and enumerate the resulting set as $r_1, ..., r_n$. Let $b_{i,j}$ be the r_j-reduction of a over $\ell_{i,A}(a)$, then $\mathrm{acl}(Ab_i, b_{i,1}, ..., b_{i,n})$ (note that some of the $b_{i,j}$ may be algebraic). Hence, $\mathrm{acl}(Aa) = \mathrm{acl}(Ab_0, b_{0,1}, ..., b_{0,n}, ..., b_m, b_{m,1}, ..., b_{m,n})$.

Claim. *If p and q are interalgebraic over A, then* $\mathrm{wt}_r(p) = \mathrm{wt}_r(q)$.

Proof of claim: This is well-known, but we include a proof for completeness. By symmetry, it suffices to show that $\mathrm{wt}_r(q) \leq \mathrm{wt}_r(p)$. Let $\mathrm{wt}_r(p) = n$ and let $C \supseteq A \cup B$, $a' \models p|_C$, and $(c_1, ..., c_n) \models (r|_C)^{(n)}$ such that $\mathrm{stp}(a'/Cc_1, ..., c_n)$ is hereditarily orthogonal to r. Let $\tau \in \mathrm{Aut}_A(\mathcal{U})$ be such that $\tau(a) = a'$. Set $b' = \tau(b)$. Then $\mathrm{acl}(Aa') = \mathrm{acl}(Ab')$. Hence, $b' \models q|_C$.

Let $D \supseteq \mathrm{acl}(Cc_1, ..., c_n)$, $b'' \models \mathrm{stp}(b'/Cc_1, ..., c_n)|_D$ and $c \models r|_D$. Since a' and a'' have the same type over $Cc_1, ..., c_n$, there exists $\sigma \in \mathrm{Aut}_C(\mathcal{U})$ such that $\sigma(a') = a''$. Let $b'' = \sigma(b')$. Then $\mathrm{acl}(Aa'') = \mathrm{acl}(Ab'')$.

If $b'' \not\perp_D c$, since $b'' \in \mathrm{acl}(Aa'')$, $a'' \not\perp_D c$ and $a'' \models \mathrm{tp}(a'/Cc_1, ..., c_n)|_D$, contradicting that $\mathrm{stp}(a'/Cc_1, ..., c_n)$ is hereditarily orthogonal to r. Hence, $\mathrm{stp}(b'/Cc_1, ..., c_n)$ is hereditarily orthogonal to r, where $b \models p|_C$ and $(c_1, ..., c_n) \models (r|_C)^{(n)}$ showing that $\mathrm{wt}_r(q) \leq n = \mathrm{wt}_r(p)$. ■

So it suffices to show that $\mathrm{mult}_r(p) = \mathrm{wt}_r(b_0, b_{0,1}, ..., b_{0,n}, ..., b_m, b_{m,1}, ..., b_{m,n}/A)$.

Fix $i = 0, ..., m$. Since $\mathrm{stp}(b_{i,j}/\ell_{i,A}(a))$ is almost r_j-internal, by Lemma 2.2.23 and Fact 2.2.25, every extension s of the type is almost r_j-internal. If the extension s is algebraic, then $s \perp r$. Otherwise, since s is almost r_j-internal and $r_j \perp p$, by Lemma 2.2.31, $s \perp r$. Hence, $\mathrm{stp}(b_{i,j}/\ell_{i,A}(a))$ is hereditarily orthogonal to r, showing that it is r-simple with r-weight of zero. By Lemma 7.3.9, $b_i, b_{i,1}, ..., b_{i,n}$ is $\mathrm{acl}(\ell_{i,A}(a))$-independent. By this independence together with additivity of r-weight (see for example [63, Lemma 7.1.11]), $\mathrm{wt}_r(b_i, b_{i,1}, ..., b_{i,n}/\ell_{i,A}(a)) = \mathrm{wt}_r(b_i/\ell_{i,A}(a))$. So we compute

$$\begin{aligned}
\mathrm{mult}_r(p) &= \sum_{i=0}^{m} \mathrm{wt}_r(b_i/\ell_{i,A}(a)) \\
&= \sum_{i=0}^{m} \mathrm{wt}_r(b_i b_{i,1}, ..., b_{i,n}/\ell_{i,A}(a)) \\
&= \sum_{i=0}^{m} \mathrm{wt}_r(b_i b_{i,1}, ..., b_{i,n}/A b_0, b_{0,1}, ..., b_{0,n}, ..., b_{i-1}, b_{i-1,1}, ..., b_{i-1,n}) \\
&= \mathrm{wt}_r(b_0, ..., b_m, b_{0,1}, ..., b_{0,n}, ..., b_{m,1}, ..., b_{m,n}/A) \\
&= \mathrm{wt}_r(p).
\end{aligned}$$

Hence, $\mathrm{mult}_r(p) = \mathrm{wt}_r(p)$. $\square$

7.6 Equivalence of semiminimal analyses

In this final section, we begin to investigate how far semiminimal analyses of a given finite U-rank type are from being uniquely determined by the type. We focus on some examples and propose a conjecture. But first, let us make explicit the notion of equivalence up to which we are comparing analyses.

Definition 7.6.1. Let $p \in S(A)$ be stationary. Let $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ and $\mathbf{g} = (g_i : q_i \to q_{i-1})_{i=1,...,m}$ be two semiminimal analyses for p. We say that the analyses are *equivalent* if $n = m$ and there is some $(a_n, ..., a_0)$ realizing $\mathbf{f}$ and $(b_n, ..., b_0)$ realizing $\mathbf{g}$ such that $\mathrm{acl}(Aa_i) = \mathrm{acl}(Ab_i)$ for all $i = 0, ..., n$.

In general, two arbitrary semiminimal analyses of p need not be equivalent. For example, let $r = \mathrm{tp}(a/A)$ and $s = \mathrm{tp}(b/A)$ be two orthogonal minimal types. Let $\pi_r : r \otimes s \to r$ denote the projection onto the r-coordinate. Then $(\pi_r : r \otimes s \to r, r \to p_0)$ and $(\pi_s : r \otimes s \to s, s \to p_0)$, where p_0 is any algebraic type, are two semiminimal analyses for $r \otimes s$. However, we have $(a, b) \models r \otimes s$ with $a = \pi_r(a, b)$ and $b = \pi_s(a, b)$. Since r and s are orthogonal minimal types, $\operatorname{acl}(Aa) \neq \operatorname{acl}(Ab)$.

This is a special case of non-equivalence when the type p is a Morley product of the first steps of the two analyses. This is not the only way to produce examples of non-equivalence.

Example 7.6.2. Let $T = \mathrm{DCF}_0$ and $(\mathcal{U}, \delta)$ be a sufficiently saturated model of DCF_0 with $\mathcal{C}$ its field of constants. Let F be a small algebraically closed differential field of constants. Let r be the generic type of the constants over $\emptyset$. Fix $b \models r|_F$. Let q be the generic type over F of $\log_\delta^{-1}(\mathcal{C})$ (see Example 2.2.35). Let $u \models q$ be such that $\log_\delta(u) = b$. Fix $a \models r|_{Fu,b}$. Let $p = \operatorname{tp}(a, b, u/F)$ and p_0 be any algebraic type.

Letting π_c denote the projection onto the c-coordinates, we then have the sequences $(\pi_u : p \to q, \delta\log_{\mathbb{G}_m} : q \to r|_F, r|_F \to p_0)$ and $(\pi_{a,b} : p \to r|_F^{(2)}, \pi_a : r|_F^{(2)} \to r|_F, r|_F \to p_0)$ are two non-equivalent semiminimal analyses for p where the fibers are all almost internal to r. Moreover, there is no finite-to-one definable function from p to $q \otimes r|_F^{(2)}$.

Proof. We first show that $(\pi_u : p \to q, \delta\log_{\mathbb{G}_m} : q \to r|_F, r|_F \to p_0)$ is a semiminimal analysis for p where every fiber is almost internal to r. The fiber $\operatorname{tp}(abu/Fu) = \operatorname{tp}(a/Fu) = r|_{Fu}$ is minimal and almost r-internal by Lemma 2.2.23 and Fact 2.2.25. The fibers of $r \to p_0$ are exactly $r|_F$ which is itself r-internal and minimal. The fibers of $\delta\log_{\mathbb{G}_m} : q \to r$ are almost $\mathcal{C}$-internal and non-algebraic since $U(q) = 2$ and $U(r|_F) = 1$, $U(q) - U(r|_F) = 1$ (see for example [44, Proposition 5.3]).

For the second sequence, the fiber $\operatorname{tp}(abu/Fab) = \operatorname{tp}(u/Fb)|_a$ which is a non-forking extension of the fiber $\operatorname{tp}(u/Fb) = \operatorname{tp}(u/F\log_\delta(u))$ which is almost r-internal. Hence, $\operatorname{tp}(abu/Fab)$ is almost r-internal by Lemma 2.2.23 and Fact 2.2.25. Similarly, the fiber $\operatorname{tp}(ab/Fa) = \operatorname{tp}(b/Fa) = r|_{Fa}$ which is almost r-internal by Lemma 2.2.23 and Fact 2.2.25. The fibers of the function $r|_F \to p_0$ are exactly $r|_F$ which is itself r-internal.

For the non-equivalence, since r is minimal, $a \models r|_F$ and $a \mathop{\not\perp\!\!\!\perp}_F u$, we get that $a \notin \operatorname{acl}(Fu)$. Hence, $\operatorname{acl}(Fu) \neq \operatorname{acl}(Fab)$ showing these analyses are not equivalent.

Finally, note that $\operatorname{acl}(Fabu) = \operatorname{acl}(Fau)$. Hence, $U(p) = 3$ and $U(q \otimes r|_F^{(2)}) = 4$. Thus, there cannot be a finite-to-one definable function from p to $q \otimes r|_F^{(2)}$. However, the definable function from $p \to q \otimes r|_F$ is given by $abu \mapsto au$ is finite-to-one. □

In the above example, we saw that p was not interalgebraic with the Morley product of the first steps of the non-equivalent composition analyses, i.e., $\operatorname{acl}(Fuab) \neq \operatorname{acl}(Fua)$. However, p is eventually interalgebraic with two independent steps of the analyses. This is an example of what we call Morley product branching:

Definition 7.6.3. We say that two semiminimal analyses $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ and $\mathbf{g} = (g_i : q_i \to q_{i-1})_{i=1,...,m}$ agree up to *Morley product branching* if for some realizations $(a_n, ..., a_0)$ of $\mathbf{f}$ and $(b_m, ..., b_0)$ of $\mathbf{g}$, there is $0 \leq \ell \leq \min\{n, m\}$ such that:

1. $\mathrm{acl}(Aa_{n-i}) = \mathrm{acl}(Ab_{m-i})$ for $0 \leq i \leq \ell - 1$.
2. $\mathrm{acl}(Aa_{n-\ell+1}) = \mathrm{acl}(Aa_{n-\ell}b_{m-\ell}) = \mathrm{acl}(Ab_{m-\ell+1})$, and
3. $a_{n-\ell} \underset{A}{\perp} b_{m-\ell}$.

This is not the only other possibility for non-uniqueness.

Example 7.6.4. Let $T = \mathrm{DCF}_0$. Let F a small algebraically closed differential field and let $\mathcal{C}$ be the field of constants for $\mathcal{U}$, a sufficiently saturated model of DCF_0. Let q be a non-algebraic and non-minimal stationary type such that q is almost $\mathcal{C}$-internal and there are pairwise A independent realizations, $a_1, a_2, a_3 \models q$ such that $a_3 \underset{A}{\not\perp} a_1a_2$ and $a_3 \notin \mathrm{acl}(Aa_1a_2)$ i.e. q has degree of non-minimality 2 (such a type exists by Section 4.2 of [26]).

Let $p = \mathrm{tp}(a_1a_2a_3/F)$. Then p is almost $\mathcal{C}$-internal by Lemma 2.2.23. Let $p_0 \in S(F)$ be algebraic and $f : p \to p_0$ be a definable function. Then $(f : p \to p_0)$ is a $\mathcal{C}$-analysis for p.

On the other hand, define $g_2 : p \to q$ as the projection map given by $a_1a_2a_3 \mapsto a_1$ and let $p_0 \in S(F)$ be algebraic and $g_1 : q \to p_0$ be a definable function. Since $a_3 \notin \mathrm{acl}(Fa_1a_2)$ the fibers of g_1 are non-algebraic. Since p is almost $\mathcal{C}$-internal, the fibers of both $g_2 : p \to q$ and $g_1 : q \to p_0$ are almost $\mathcal{C}$-internal. Since q is non-algebraic, $(g_2 : p \to q, g_1 : q \to p_0)$ is a $\mathcal{C}$-analysis of p by fibrations, and is also a semiminimal analysis as the fibers are all almost internal to the generic type of the constants over $\emptyset$.

Clearly these analyses are not-equivalent as they are different lengths and $a \notin \mathrm{acl}(Fa_1a_0)$ for any $a_0 \models p_0$. Thus, they do not differ by Morley-product branching.

Note that the above example is analyzable in the constants. We conjecture that such an example must always be so:

Conjecture 7.6.5. *Let $p \in S(A)$ be stationary, non-algebraic and finite U-rank. Let $\mathbf{f} = (f_i : p_i \to p_{i-1})_{i=1,...,n}$ and $\mathbf{g} = (g_i : q_i \to q_{i-1})_{i=1,...,m}$ be two semiminimal analyses of p. If the two analyses are non-equivalent then either:*

1. $\mathbf{f}$ *and* $\mathbf{g}$ *agree up to Morley product branching, or*

2. *p is analyzable in the family of all non-locally modular minimal types.*

Note that Example 7.6.2 shows that the cases in the conjecture are not mutually exclusive.

We can prove a preliminary case of the conjecture by placing some additional assumptions on the analyses.

Proposition 7.6.6. *Let $p \in S(A)$ be a stationary finite U-rank type. Suppose that p has a semiminimal analysis $\mathbf{f} = (f_2 : p_2 \to p_1, f_1 : p_1 \to p_0)$ such that the fibers of $f_2 : p_2 \to p_1$, $f_1 : p_1 \to p_0$ all satisfy exchange. Let $\mathbf{g} = (g_i : q_i \to q_{i-1})_{i=1,\dots,n+1}$ be another semiminimal analysis for p. If $\mathbf{f}$ and $\mathbf{g}$ are non-equivalent then either:*

1. *$\mathbf{f}$ and $\mathbf{g}$ agree up to Morley product branching, or*

2. *p is analyzable in the family of all A-conjugates of a single non-locally modular minimal type.*

Proof. Fix $a \models p$ and choose $(b_n, ..., b_0)$ realizing $\mathbf{g}$ such that $\operatorname{acl}(Aa) = \operatorname{acl}(Ab_{n+1})$. Let (a_2, a_1, a_0) be a realization of $\mathbf{f}$ such that $\operatorname{acl}(Aa) = \operatorname{acl}(Aa_2)$. For all $i = 0, ..., n+1$, we have that $b_i \in \operatorname{acl}(Aa) = \operatorname{acl}(Aaa_1)$. Since the fibers of $f_2 : p_2 \to p_1$ satisfy exchange, $b_i \in \operatorname{acl}(Aa_1)$ or $a \in \operatorname{acl}(Ab_i a_1)$.

First we show that if $b_n \in \operatorname{acl}(Aa_1)$, then the two analyses must be equivalent. If $b_n \in \operatorname{acl}(Aa_1) \setminus \operatorname{acl}(A)$, then since p_1 satisfies exchange we have that $\operatorname{acl}(Ab_n) = \operatorname{acl}(Aa_1)$. Let $c \in \operatorname{acl}(Ab_n) \setminus \operatorname{acl}(A)$, then $c \in \operatorname{acl}(Aa_1) \setminus \operatorname{acl}(A)$. By exchange, $a_1 \in \operatorname{acl}(Ac)$. Hence, $b_n \in \operatorname{acl}(Aa_1) \subseteq \operatorname{acl}(Ac)$. Therefore q_n satisfies exchange. For all $i = 1, ..., n-1$, since $b_i \in \operatorname{acl}(Ab_n)$ and $U(b_n/A) > U(b_{n-1}/A)$, we have that $b_n \notin \operatorname{acl}(Ab_i)$. As q_n satisfies exchange, $b_i \in \operatorname{acl}(A)$. Thus $n = 1$ and $\operatorname{acl}(Ab_1) = \operatorname{acl}(Aa_1)$ showing the two analyses are equivalent.

Thus, we may assume that $b_n \notin \operatorname{acl}(Aa_1)$. Hence, by exchange $a \in \operatorname{acl}(Aa_1 b_n)$.

Since p_1 satisfies exchange, by Remark 6.2.3, p is either minimal or almost internal to t, a non-locally modular minimal type t.

First suppose that p_1 is minimal. If $a_1 \mathop{\not\downarrow}_A b_n$, then by minimality of p_1, $a_1 \in \operatorname{acl}(Ab_n)$. But then $a \in \operatorname{acl}(Aa_1 b_n) = \operatorname{acl}(Ab_n)$, hence $\operatorname{acl}(Ab_{n+1}) = \operatorname{acl}(Aa) = \operatorname{acl}(Ab_n)$, contradicting that $U(q_{n+1}) > U(q_n)$. So we must have $a_1 \mathop{\downarrow}_A b_n$. Since $a \in \operatorname{acl}(Aa_1 b_n)$, we have that $\operatorname{acl}(Aa) = \operatorname{acl}(Aa_1 b_n)$ and $a_1 \mathop{\downarrow}_A b_n$ which is case (2) of the conclusion of the statement.

So we may assume that p_1 is not minimal, and hence is almost internal to t, a non-locally modular minimal type. Let r_2 be a minimal type such that $\operatorname{tp}(a/Aa_1)$ is almost r_2-internal. We will show that either $a \in \operatorname{acl}(Aa_1b_1)$ and $a_1 \underset{A}{\perp} b_1$, which is case (2) of the conclusion, or $t \not\perp r_1$ and thus p is analyzable in the family of all A-conjugates of the non-locally modular minimal type t.

If $b_1 \in \operatorname{acl}(Aa_1) \setminus \operatorname{acl}(A)$, then since p_1 satisfies exchange, $a_1 \in \operatorname{acl}(Ab_1) \subseteq \operatorname{acl}(Ab_n)$. Thus, $a \in \operatorname{acl}(Aa_1b_n) = \operatorname{acl}(Ab_n)$, contradicting $U(q_{n+1}) > U(q_n)$. Hence, $a \in \operatorname{acl}(Aa_1b_1)$. If $a_1 \underset{A}{\perp} b_1$, then case (2) of the conclusion holds.

So we may assume that $a_1 \underset{A}{\not\perp} b_1$ and show that $r_1 \not\perp t$. By definition, $p_1 \not\perp q_1$. Since p_1 is non-algebraic and almost internal to the minimal type t, by Lemma 2.2.31, $q_1 \not\perp t$. On the other hand, since $f_1 : q_1 \to q_0$ is semiminimal, the fiber $q_1 = \operatorname{tp}(b_1/A)$ is non-algebraic and almost internal to a minimal type s. Since q_1 is non-algebraic, almost s-internal and $q_1 \not\perp t$, by Lemma 2.2.31, $s \not\perp t$.

Since $b_1 \in \operatorname{dcl}(Aa)$ and $\operatorname{tp}(a/Aa_1)$ is almost r_2-internal, by Lemma 2.2.23 and Fact 2.2.25, $\operatorname{tp}(b_1/Aa_1)$ is almost r_2-internal. Since $b_1 \notin \operatorname{acl}(Aa_1)$, $\operatorname{tp}(b_1/Aa_1)$ is non-algebraic. Thus, by Corollary 2.2.27, $\operatorname{tp}(b_1/Aa_1) \not\perp r_2$. Now, since $\operatorname{tp}(b_1/A)$ is almost s-internal, by Lemma 2.2.23 and Fact 2.2.25, $\operatorname{tp}(b_1/Aa_1)$ is almost s-internal. We have shown that $\operatorname{tp}(b_1/Aa_1)$ is non-algebraic and almost internal to the minimal type s_2 and non-orthogonal to r_2. By Lemma 2.2.31, $r_2 \not\perp s$. Since $s \not\perp t$, s is minimal, and, by Fact 2.2.11, non-orthogonality is transitive on a minimal type, hence $r_2 \not\perp t$. Hence, $\operatorname{tp}(a/Aa_1)$ is almost t-internal. Thus p is analyzable in the family of A-conjugates of the non-locally modular minimal type t. $\square$

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