PMath 911 Topics in Logic: Stability Theory **Assignment # 2**
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Due on Thursday February 14th.

1. Suppose X and Y are disjoint definable sets with ordinal-valued Morley rank. Then

$$\mathrm{dM}(X \cup Y) = \begin{cases} \mathrm{dM}(X) + \mathrm{dM}(Y) & \text{if } \mathrm{RM}(X) = \mathrm{RM}(Y) \\ \mathrm{dM}(X) & \text{if } \mathrm{RM}(X) > \mathrm{RM}(Y) \\ \mathrm{dM}(Y) & \text{if } \mathrm{RM}(Y) > \mathrm{RM}(X) \end{cases}$$

2. Prove the following

(a) Suppose $X \subseteq Y$ are definable sets both of Morley rank α an ordinal. Then $Y \setminus X$ either has Morley rank $< \alpha$ or has Morley rank α and Morley degree less than $\mathrm{dM}(Y)$.

(b) Suppose $X \subset Y$ and $\mathrm{RM}(X) < \mathrm{RM}(Y)$. Then $\mathrm{RM}(Y \setminus X) = \mathrm{RM}(Y)$ and, if $\mathrm{RM}(Y)$ is an ordinal, $\mathrm{dM}(Y \setminus X) = \mathrm{dM}(Y)$.

3. Suppose $\mathcal{M}$ is ω-saturated and X is a definable set such that $\mathrm{RM}(X) = \alpha$ an ordinal, and $\mathrm{dM}(X) = d$. Let $Z_1, \ldots, Z_d \subseteq X$ be disjoint definable sets of Morley rank α such that $X = \bigcup_{i=1}^{d} Z_i$. (We know this exists.) Prove that

(a) $\mathrm{dM}(Z_i) = 1$ for each $i = 1, \ldots, d$

(b) if $X = \bigcup_{i=1}^{d} Y_i$ where $Y_1, \ldots, Y_d$ are pairwise disjoint definable sets of Morley rank α, then for each Y_j there is a unique Z_i such that $\mathrm{RM}(Y_j - Z_i) < \alpha$. Here by $Y_j - Z_i$ I mean the symmetric difference $(Y_j \setminus Z_i) \cup (Z_i \setminus Y_j)$.

Note: Part (b) says that any two decompositions of a definable set of Morley rank α into disjoint definable subsets of Morely rank α are the same "upto definable sets of smaller Morley rank". This is a kind of uniquencess for such decompositions.

4. Let $L_n := \{E_1, \ldots, E_n\}$ be a language with n binary relations symbols. Let T_n be the theory saying:

- Each E_i is an equivalence relation.
- For all $i < n$, E_{i+1} refines E_i. That is, $\forall xy\big(E_{i+1}(x, y) \to E_i(x, y)\big)$.
- For all $i < n$, each E_i-class contains infinitely many E_{i+1}-classes.
- There are infinitely many E_1-classes and each E_n-class is infinite.

Do the following exercises:

(a) Prove that T_n is complete and admits quantifier elimination.

(b) Compute $(\mathrm{RM}, \mathrm{dM})(x = x)$ in T_n.

(c) Let $L_\omega := \bigcup_{n<\omega} L_n$ and let $T_\omega := \bigcup_{n<\omega} T_n$. Prove that T_ω is complete and admits QE, and compute $(\mathrm{RM}, \mathrm{dM})(x = x)$ in T_ω.

5. Show that a theory T in a countable language is ω-stable if and only if the formula $(x = x)$ has bounded (i.e., ordinal-valued) Morley rank.