PMath 911 Topics in Logic: Stability Theory **Assignment # 1**
Winter 2013, Rahim Moosa

Due on Thursday January 31st.

1. Suppose T is a complete theory. Show that for any $\mathcal{M} \models T$, $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$.

2. Each of the following theories is complete and admits quantifier elimination (proved last term). Suppose $\mathcal{M} \models T$ and $A \subseteq M$. Describe all the complete 1-types in $\mathcal{M}$ over A, and say how many there are (in terms of $|A|$).

(a) T is the theory of infinite sets in the language $L = \emptyset$.
(b) $L = \{E\}$ where E is a binary relation symbol and T is the theory which says that E is an equivalence relation with infinitely many infinite equivalence classes.
(c) Let F be a field and let T the theory of infinite F-vector spaces, in the language $L = \{0, +, -, \lambda_f : f \in F\}$ where the λ_f are unary function symbols interpreted as scalar multiplication by f.

3. Suppose κ is an infinite cardinal. For each of the theories in Question 3, describe the κ-saturated models.

4. Suppose $\mathcal{M}$ is an L-structure, $A \subseteq M$, and $b = (b_1, \ldots, b_n)$ and $b' = (b'_1, \ldots, b'_n)$ are n-tuples from M.

(a) Show that $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$ if and only if the partial map $f : A \cup \{b_1, \ldots, b_n\} \to M$ given by $f \restriction A = \mathrm{id}$ and $f(b_i) = b'_i$ for all $i = 1, \ldots, n$, is a partial elementary map.
(b) Suppose $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$ and f is the pem given by part (a). Suppose $c, c' \in M$. Show that $\mathrm{tp}(bc/A) = \mathrm{tp}(b'c'/A)$ if and only if c' realised $\mathrm{tp}(c/Ab)^f$.

5. Suppose $\mathcal{M}$ is an L-structure, $\emptyset \neq A \subseteq M$, and $b, b' \in M^n$. Prove that if $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$ then $\mathrm{tp}\big(b/\operatorname{dcl}(A)\big) = \mathrm{tp}\big(b'/\operatorname{dcl}(A)\big)$.

6. Suppose $K \models \mathrm{ACF}$ and $A \subseteq K$. Then $\operatorname{acl}(A) = \mathbb{F}(A)^{\mathrm{alg}}$. That is, the model-theoretic algebraic closure of a set is equal to the field-theoretic algebraic closure of the field generated by that set. On the other hand

$$\operatorname{dcl}(A) = \begin{cases} \mathbb{Q}(A) & \text{if } \operatorname{char}(K) = 0 \\ \mathbb{F}_p(A)^{\mathrm{per}} & \text{if } \operatorname{char}(K) = p > 0 \end{cases}$$

where $k^{\mathrm{per}} := \{a \in k^{\mathrm{alg}} : a^{p^n} \in k, \text{ some } n\}$ is the perfect closure of a field k in positive characteristic.
Hint: Pass to an $|A|$-saturated and strongly $|A|$-homogeneous elementary extension.

7. What is $\operatorname{acl}(A)$ and $\operatorname{dcl}(A)$ for $A \subset V$, where V is a model of the theory of infinite vector spaces over a field F (F is fixed in advance), in the language of F-vector spaces?