On the model theory of geodesic differential equations

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Differentially closed fields

Definition

A differentially closed field is an existentially closed model of the theory of differential field (of characteristic 0).

- Blum(1968): Axiomatic of differentially closed fields and elimination of quantifiers in the language $\mathcal{L}_\delta = \{0, 1, +, x, -, \delta\}$ of differential rings.
  In particular, one gets the following description of n-types with parameters in a differential field (K, δ) :

$$\phi : S_n(K) \longrightarrow \{I \subset K\{X_1, \ldots, X_n\} \text{ prime differential ideal}\}$$

- Together with Ritt-Raudenbush Theorem (in fact, a weak form of this theorem), this implies that the theory of differentially closed field is ω-stable.

Why is the theory $\mathbf{DCF}_0$ interesting ?

- The theory $\mathbf{DCF}_0$ is "the least misleading ω-stable theory".
  - Shelah(1973), Rosenlicht(1974): Non-minimality of prime models.
  - Hrushvoski-Scanlon(1999): Morley rank and Lascar rank differ.
- Fruitful applications of the ideas of geometric stability theory to differential algebra.
  - Poizat(1984), Pillay(1998): Reformulation and generalization of Kolchin's differential Galois theory for linear differential equations.
  - Hrushovski (1996): Proof of the Mordell-Lang conjecture for function fields in characteristic 0.
- To summarize, differentially closed fields are both:
  - sufficiently complicated (in contrast with algebraically closed fields) to prevent algebraic geometry to be omnipotent.
  - tame enough so that the ideas of geometric stability theory could be applied to them.

Trichotomy Theorem of $\mathbf{DCF}_0$

- The celebrated theorem of Hrushovski and Zilber (1993) on Zariski geometries led, in various classical stable theories, to a classfication of minimal types.

Theorem (Hrushovski-Sokolovic, 1996)

Let $(\mathcal{U}, \delta_{\mathcal{U}})$ be a differentially closed field. For a minimal type p with parameters in $\mathcal{U}$, exactly one of the following case holds:

(i) *If p is non-locally modular, then p is non-orthogonal to the constants.*

(ii) *If p is locally modular and non-disintegrated, then p is non-orthogonal to the generic type of the "Manin's Kernel" associated to a simple abelian variety A over $\mathcal{U}$, which does not descend to the constants.*

(iii) *The type p is a minimal and disintegrated type.*

Commentary

- The resolution of every differential equation of finite rank can be reduced the successive resolutions of minimal differential equations.
- A differential equation satisfies (i) (or more precisely, is internal to the constants) if one can algebraically parametrized all the solutions of this differential equation, given a finite number of particular solutions $f_1, \ldots, f_k$.
- On the other hand, the case (iii) is satisfied when the solutions of the differential equation tend to be independent: If $f_1, \ldots f_k$ are solutions then every "essentially new" solution will be algebraically independent from $f_1, \ldots, f_k$.
- The case (ii) treats of very specific differential equations – "Manin's Kernels" – that we understand very precisely.

Which differential equations are minimal and disintegrated ?

For differential equations of order 1 (defined over a constant differential field), the situation is very well understood.

- Hrushovski-Itaï (2003): Unlimited families of minimal and disintegrated differential equations $f(y, y') = 0$ of order 1.

Surprisingly, in higher dimension, few examples are known.

- Nagloo-Pillay (2011): Painlevé equations with very generic parameters are strongly minimal and disintegrated.
- Freitag-Scanlon (2014): the minimal differential equation of order 3 satisfied by the j function is strongly minimal and disintegrated.

The purpose of my thesis was to construct unlimited families of minimal and disintegrated differential equations of order > 1.

Autonomous differential equations

Let k be a field of characteristic 0.

- During my talk, all the algebraic differential equations will be autonomous, i.e., defined over some constant differential field.
- An algebraic differential equation with parameters in k (and with n variables) is a differential equation of the form

$$P(x_1, \ldots, x_n, \delta(x_1), \cdots, \delta(x_n), \ldots, \delta^k(x_1), \ldots, \delta^k(x_n)) = 0$$

  where P is a polynomial with coefficients in k.
- In more modern terms, we will describe systems of differential equations as pair (X, v) where X is an (algebraic) variety over k and v is a vector field on X.

Geodesic differential equations

Let $\Sigma \subset \mathbb{R}^n$ be a smooth algebraic subset of the Euclidean space. The geodesic equation associated to Σ can be described alternatively as:

- *Newton's law of motion*: The system of differential equations describing the movement of a particle, constrained to move without friction on Σ.
- *Hamiltonian formalism*: The pair $(T\Sigma, v)$ where $T\Sigma$ is the tangent space of Σ and v is the vector field associated to the free Hamiltonian

  $$H_g(x, y) = \frac{1}{2} g_x(y, y).$$

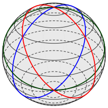
  where g is the restriction of the Euclidean metric to Σ.

We are interested in the model-theoretic behavior of the fibres of:

$$H_g : (T\Sigma, v) \longrightarrow (\mathbb{P}^1, 0).$$

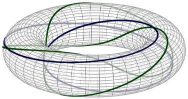
Curvature and geodesics

- The Euclidean sphere $\mathbb{S}^2$ has constant positive curvature.



- Every particle follows a periodic trajectory
- The equation is algebraically integrable.

- The flat torus $\mathbb{T}^2$ has zero curvature.



- The periodic trajectories are dense but there also exist non-periodic trajectories.
- The equation reduces to a linear differential equation.

- Negative curvature: "Chaotic" hyperbolic dynamic.
  - (Anosov 1967) Given any particles p, q, there is a particle, which (up to ϵ) follows the past of the particle p and the future of q.

Main Theorem

Theorem

Let $\Sigma = X(\mathbb{R}) \subset \mathbb{R}^n$ be a smooth, connected and compact algebraic subset of dimension 2. Denote by (TX, v) the geodesic differential equation of Σ and consider the Hamiltonian rational integral:

$$H_g : (TX, v) \longrightarrow (\mathbb{P}^1, 0).$$

Each non-zero fibre of H_g is absolutely irreducible. Moreover, if Σ has negative curvature, then the generic type q of any fibre of H_g satisfies one of the two following cases:

(i) *Either, the type q is minimal and disintegrated.*

(ii) *Or there exists an absolutely irreducible curve (C, v_C) endowed with a vector field v_C (whose generic type $p_{(C,v_C)}$ is disintegrated) such that q and $p^{(3)}_{(C,v_C)}$ are interalgebraic (over $\mathbb{R}$).*

Remarks

- In fact, even in particular cases, we don't know if it is actually the case (i) or (ii) that occurs. We expect that the answer to this question does not depend on the chosen Riemannian manifold and that one always gets a minimal disintegrated type.
- This strengthening would provide unlimited families of disintegrated minimal types of order > 1 in the theory $\mathbf{DCF}_0$.
- Nevertheless, this result already implies that the generic type of a geodesic differential equation in negative curvature has a trivial forking geometry (since it is dominated by a trivial minimal type in both cases).

Strategy of the proof

Pick a smooth and compact algebraic subset Σ of the Euclidean space with negative curvature. Denote by (X, v) a fibre of the geodesic differential equation and by q its generic type.

- Model-theoretic arguments.
  - The type q is of order 3 defined over a constant differential field.
  - It is sufficient to prove that type q is orthogonal to the constants and semi-minimal (using Hrushovski-Sokolovic Trichotomy).
- Geometric arguments. We translate these two properties of the type q in terms of:
  - rational integrals of the D-variety (X, v) and its powers $(X, v)^n$.
  - algebraic foliations $\mathcal{F}$ on X invariant by the Lie derivative of the vector field v.
- Dynamical arguments. Use Anosov property to prevent the existence of these "invariant algebraic structures".