GOMORY CUTS

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Gomory fractional (GF) cuts were introduced in 1958 with an announcement by Ralph E. Gomory [1] of an "algorithm for integer solutions to linear programs (LPs)" (a full version of the algorithm appears in Ref. 2), which gave a finite cutting-plane algorithm for pure integer programming. In a later paper [3], he also introduced the so called *Gomory mixed-integer (GMI)* cuts. Since then, the field of (mixed-) integer programming has developed immensely, but throughout the years and up to this day, Gomory's fractional cuts, the closely related Chvátal–Gomory (CG) cuts, and the mixed-integer cuts of Gomory play a significant role in much of the research that is done in the field. The purpose of this paper is to do a brief survey of the research that has been carried out around Gomory cuts throughout the years. The interested reader is also referred to some surveys on general cutting planes [4–6], as well as related articles about cutting planes (in the section titled "Cutting Planes") in this encyclopedia.

The organization of this paper is as follows. In what remains of this section, we define what are the GF cuts, the closely related CG cuts, and the GMI cuts. We then introduce mixed-integer rounding (MIR) cuts and explore their relationship with GMI cuts. The section titled "Closures and Rank" is devoted to defining and reviewing some of the results related to rank and closure for each of the classes of cuts. The section titled Special "Cases, Variations, Strengthening, and Others" surveys some variations, special cases, and relations that Gomory cuts have with other types of cuts. The section titled "Implementation and Computational Aspects" is devoted to computational issues, while the section titled "Multirow Gomory Cuts" talks about the recent interest in the so-called *multirow Gomory cuts*.

Throughout this paper, we refer to the following general mixed-integer sets:

$$P_I = \{(x,z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Dz \le b\} \quad (1)$$

and

$$Q_I = \{(x,z) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Dz \le b\}. \quad (2)$$

GOMORY'S FRACTIONAL CUTS

Gomory's fractional cuts are defined for the pure integer case and when all variables are assumed to be nonnegative, that is, they are cuts that are valid for the following set (P_I with $p = 0$):

$$P_I = \{x \in \mathbb{Z}_+^n : Ax \le b\}. \quad (3)$$

In addition, Gomory's fractional cuts were derived in Ref. 1 when the constraints were in equality form. Since, in the context of mixed-integer programming, it is assumed that all data is rational, we may assume that A and b are integral and introduce slack variables to obtain $P_I' = \{(x,s) \in \mathbb{Z}_+^{n+m} : Ax + s = b\} := \{z \in \mathbb{Z}_+^{n+m} : Gz = b\}$, where $z = (x,s)$ and $G = (A, I)$.

We can relax P_I' by multiplying all equations by a vector $u \in \mathbb{R}^m$ to obtain the set $\{z \in \mathbb{Z}_+^{n+m} : uGz = ub\}$. We can further relax this set to obtain the set $P_G = \{z \in \mathbb{Z}_+^{n+m} : uGz \equiv ub \mod 1\}$, which means that uGz and ub differ by an integer. Moreover, let $f(a) := a - \lfloor a \rfloor$ denote the fractional part of a real number a. Now since P_G is expressed in terms of a modular equation, it can be rewritten as $P_G = \{z \in \mathbb{Z}_+^{n+m} : f(uG)z \equiv f(ub) \mod 1\}$, where $f(uG)$ is the vector obtained by taking the fractional part of each component of uG. Notice that $f(uG) \ge 0$ and $z \ge 0$, hence $f(uG)z \ge 0$. Finally, since $f(uG)z \equiv f(ub) \mod 1$ and $f(uG)z \ge 0$, then $f(uG)z$ must be equal to

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$f(ub) + k$ for some integer $k \geq 0$. This argument shows that the following inequality is valid for P_G:

$$f(uG)z \geq f(ub). \tag{4}$$

If we substitute in Equation (4) the identities $f(uG) = uG - \lfloor uG \rfloor$, $f(ub) = ub - \lfloor ub \rfloor$, and $s = b - Ax$, we obtain the *GF cut* on the original x variables:

$$\lfloor uA \rfloor x - \lfloor u \rfloor Ax \leq \lfloor ub \rfloor - \lfloor u \rfloor b. \tag{5}$$

We would like to point out that GF cuts are defined here by using any multiplier $u \in \mathbb{R}^m$, but Gomory's cutting-plane algorithm uses a specific set of multipliers u that are associated with basic feasible solutions of $P' = \{(x, s) \in \mathbb{R}_+^{n+m} : Ax + s = b\}$. For more details on this distinction and its implications, see Ref. 7.

CHVÁTAL–GOMORY CUTS

CG cuts were defined by Chvátal [8] and are also defined for the pure integer case (that is, when $p = 0$). However, unlike in the case of the GF cuts, CG cuts are valid for the following set

$$Q_I = \{x \in \mathbb{Z}^n : Ax \leq b\}, \tag{6}$$

that is, even when there is no nonnegativity assumption on the variables. Similar to Gomory's fractional cuts, we can aggregate the constraints in Q_I by using a multiplier $u \in \mathbb{R}_+^m$ to obtain the relaxation $\{x \in \mathbb{Z}^n : uAx \leq ub\}$. Now, if $uA \in \mathbb{Z}^n$, then $uAx \in \mathbb{Z}$ and hence the following CG cut is valid for Q_I.

$$uAx \leq \lfloor ub \rfloor. \tag{7}$$

If our original feasible set is described as

$$P_I = \{x \in \mathbb{Z}_+^n : Ax \leq b\}, \tag{8}$$

that is, with a nonnegativity assumption on the variables, a slightly different argument can be used to derive the CG cut.

Note that if $Q_I = \{x \in \mathbb{Z}^n : Ax \leq b\ ;\ -x \leq 0\}$, that is, if the nonnegativity constraints are included in the constraint matrix, rather than explicitly imposed, we have that any cut of the form (7) is

$$(uA - \lambda)x \leq \lfloor ub \rfloor,$$

where $(u, \lambda) \in \mathbb{R}_+^{m+n}$ are the CG multipliers.

In this case, we do not need to require that $uA \in \mathbb{Z}^n$, because, since $x \geq 0$, $\lfloor uA \rfloor x \leq uAx \leq ub$ and, since $\lfloor uA \rfloor x \in \mathbb{Z}$, the CG cut

$$\lfloor uA \rfloor x \leq \lfloor ub \rfloor \tag{9}$$

is valid for P_I. Equivalently, we require $(uA - \lambda) \in \mathbb{Z}^n$ and any nondominated CG cut will have $\lambda = uA - \lfloor uA \rfloor$, which leads to Equation (9).

It is easy to see that any CG cut of the form (9) can be obtained as a CG cut of the form (7) when the nonnegativity constraints are not explicitly imposed, but are present in the constraint matrix. Conversely, under the nonnegativity assumption, any cut of the form (7) can either be obtained as a cut of the form (9) or is dominated by one.

The name CG cuts comes from the fact that, under the nonnegativity assumption on the variables, any cut of the form (5) can be obtained as a cut of the form (9) and vice versa (see Refs 7 and 9 for more details). This is the reason why these two types of cuts are often said to be equivalent. However, note that the cut (Eq. 7) cannot always be obtained as a cut of the form (5), since the latter requires the nonnegativity assumption.

GOMORY MIXED-INTEGER CUTS

The arguments for the derivation of CG cuts and GF cuts rely on the fact that all variables are required to be integral and therefore the same arguments cannot be directly applied when we have a mixed-integer set

$$P_I = \{(x, z) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Dz \leq b\}. \tag{10}$$

By adding slack variables, we obtain the following set in equality form

$$P_I' = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^{p+m} : Ax + Gy = b\}, \tag{11}$$

where $G = (D, I)$ and $y = (z, s)$. We again multiply the equations by a vector $u \in \mathbb{R}^m$ to obtain the set

$$P'_I(u) = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^{p+m}_+ : \overline{a}x + \overline{g}y = \overline{b}\}, \tag{12}$$

where $\overline{a} = uA$, $\overline{g} = uG$, and $\overline{b} = ub$ and we assume that $f(\overline{b}) > 0$. Let $f_i = f(\overline{a}_i)$ and $f_0 = f(\overline{b})$. We are now ready to derive the GMI cut. Notice that $P'_I(u)$ can be rewritten as

$$\sum_{i:\, f_i \leq f_0} f_i x_i + \sum_{i:\, f_i > f_0} (f_i - 1)x_i + \overline{g}y = \left(\lfloor \overline{b} \rfloor - \sum_{i=1}^{n} \lfloor \overline{a}_i \rfloor x_i - \sum_{i: f_i > f_0} x_i \right) + f_0.$$

Since $k := \left(\lfloor \overline{b} \rfloor - \sum_{i=1}^{n} \lfloor \overline{a}_i \rfloor x_i - \sum_{i: f_i > f_0} x_i \right) \in \mathbb{Z}$, then either $k \leq -1$, in which case

$$-\sum_{i:\, f_i \leq f_0} \frac{f_i}{1 - f_0} x_i + \sum_{i:\, f_i > f_0} \frac{1 - f_i}{1 - f_0} x_i - \frac{\overline{g}}{1 - f_0} y \geq 1 \tag{13}$$

or $k \geq 0$, in which case

$$\sum_{i:\, f_i \leq f_0} \frac{f_i}{f_0} x_i + \sum_{i:\, f_i > f_0} \frac{f_i - 1}{f_0} x_i + \frac{\overline{g}}{f_0} y \geq 1. \tag{14}$$

Since both these cuts are of the form $a^1 w \geq 1$ and $a^2 w \geq 1$, and $w \geq 0$, then the cut $\sum_j \max\{a^1_j, a^2_j\} w_j \geq 1$ is valid for $P'_I(u)$ and hence valid for P'_I.

Therefore, the following inequality, called the *GMI cut*, is valid for P'_I

$$\sum_{i:\, f_i \leq f_0} \frac{f_i}{f_0} x_i + \sum_{i:\, f_i > f_0} \frac{1 - f_i}{1 - f_0} x_i + \sum_{j: \overline{g}_j \geq 0} \frac{\overline{g}_j}{f_0} y_j - \sum_{j: \overline{g}_j < 0} \frac{\overline{g}_j}{1 - f_0} y_j \geq 1 \tag{15}$$

(for this and other derivations of this cut see Refs 2, 4, 9).

It is worth noting that, in the pure integer case, the GMI cut is still valid and is of the form

$$\sum_{i:\, f_i \leq f_0} \frac{f_i}{f_0} x_i + \sum_{i:\, f_i > f_0} \frac{1 - f_i}{1 - f_0} x_i \geq 1. \tag{16}$$

We can compare that cut with the GF cut

$$\sum_{i=1}^{n} \frac{f_i}{f_0} x_i \geq 1$$

and notice that when $f_i > f_0$, we have that $\frac{1-f_i}{1-f_0} < \frac{f_i}{f_0}$. Therefore, the GMI cut dominates the GF cut in the pure integer case.

MIXED-INTEGER ROUNDING CUTS

In this section, we introduce MIR cuts. Notice that this class of cuts is the topic of another completely independent article of this encyclopedia (see ***Mixed-Integer Rounding***). However, MIR cuts are so closely related to GMI cuts that it is worth devoting some attention to this relationship in more detail. First, we need to point out that there are multiple cuts that are called *MIR* cuts. We use here the definition from Ref. 10. See also Refs 9,11–13 for other references on MIR cuts. Consider the general mixed-integer set

$$Q_I = \{(x, z) \in \mathbb{Z}^n \times \mathbb{R}^p : A'x + D'z \leq b'\}. \tag{17}$$

Suppose we multiply the inequalities in Equation (17) by two different vectors $\mu^1, \mu^2 \in \mathbb{R}^{m'}_+$, where $\mu^1 D' = \mu^2 D'$ and $(\mu^2 - \mu^1)A' \in \mathbb{Z}^n$ then the inequality

$$(\mu^2 - \mu^1)A'x + \frac{\mu^1 A'x + \mu^1 D'z - \mu^1 b'}{1 - ((\mu^2 b' - \mu^1 b') - \lfloor \mu^2 b' - \mu^1 b' \rfloor)} \leq \lfloor \mu^2 b' - \mu^1 b' \rfloor \tag{18}$$

is called the *MIR inequality* and is valid for Q_I. Since MIR cuts are the subject of the article titled ***Mixed-Integer Rounding*** in this encyclopedia, we will refrain from proving that the MIR inequality is valid for Q_I. We will instead point out that Q_I does not explicitly require the variables to be nonnegative. However, if $A' = [A, -I, \mathbf{0}]^T$, $D' = [D, \mathbf{0}, -I]^T$, and $b' = [b, \mathbf{0}, \mathbf{0}]^T$ (that is, the nonnegativity constraints are included in $A'x + D'z \leq b'$), then $Q_I = P_I$ and the MIR inequalities (Eq. 18) can be derived as *GMI inequalities* (Eq. 15) and vice versa.

To see how, let us assume that the nonnegativity constraints are contained in the inequalities $A'x + D'z \leq b'$. Therefore, the multipliers μ^1 and μ^2 can be written as $\mu^1 = [\mu_o^1, \mu_x^1, \mu_z^1]^T$ and $\mu^2 = [\mu_o^2, \mu_x^2, \mu_z^2]^T$, where μ_o^i is the multiplier for the constraints $Ax + Dz \leq b$, μ_x^i for the $-x \leq 0$ constraints and μ_z^i for the $-z \leq 0$ constraints, for $i = 1, 2$. Notice that even if we vary μ^1 and μ^2, as long as $\mu^2 - \mu^1$ does not change, the only term that varies in Equation (18) is the numerator of the fraction. Therefore, since $A'x + D'z - b' \leq 0$, and $\mu^1 \geq 0$, if we increase μ^1 and μ^2 by the same amount, we get a weaker inequality. So, we may assume that for every i, either $\mu_i^1 = 0$ or $\mu_i^2 = 0$. So we may define $\mu = \mu^1 - \mu^2 \in \mathbb{R}^{m'}$ with no sign restriction and $\mu^1 = \mu^+$ and $\mu^2 = (-\mu)^+$, with Equation (18) becoming

$$-\mu A'x + (\mu)^+ \frac{A'x + D'z - b'}{1 - (-\mu b' - \lfloor -\mu b' \rfloor)} \leq \lfloor -\mu b' \rfloor. \tag{19}$$

Now, notice that $\lfloor -\mu b' \rfloor = -\lceil \mu b' \rceil$, so we can rewrite Equation (19) as

$$(\mu b' - \lfloor \mu b' \rfloor)\mu A'x + (\mu)^+(b' - A'x - D'z) \geq (\mu b' - \lfloor \mu b' \rfloor)\lceil \mu b' \rceil. \tag{20}$$

Moreover, if we expand μ, A', D', and b', we get

$$(\mu_o b - \lfloor \mu_o b \rfloor)(\mu_o A - \mu_x)x + (\mu_o)^+(b - Ax - Dz) + (\mu_x)^+ x + (\mu_z)^+ z \geq (\mu_o b - \lfloor \mu_o b \rfloor)\lceil \mu_o b \rceil. \tag{21}$$

But recall that $\mu A' \in \mathbb{Z}^n$ and $\mu D' = 0$. This means that $\mu_o A - \mu_x \in \mathbb{Z}^n$ and that $\mu_o D - \mu_z = 0$. Now $\mu_o A - \mu_x \in \mathbb{Z}^n$ implies that $\mu_x = \hat{a} + t$ for some integral vector $t \in \mathbb{Z}^n$, where $\hat{a} = \mu_o A - \lfloor \mu_o A \rfloor$. If $t_i \geq 1$, then $(\mu_x)_i \geq 1$ and hence $(\mu_x)_i^+ = (\mu_x)_i$. So decreasing t_i to 0 decreases the coefficient of x_i. If $t_i < -1$, then $(\mu_x)_i \leq -1$ and hence $(\mu_x)_i^+ = 0$. So the coefficient of x_i can be decreased by increasing t_i. Hence $t_i \in \{0, -1\}$ for any nondominated MIR inequality.

Defining $\mu_o b - \lfloor \mu_o b \rfloor = f_0$, we can rewrite Equation (21) as

$$(\mu_x)^+ x - f_0 \mu_x x + f_0 \hat{a} x + f_0 \lfloor \mu_o A \rfloor x + (\mu_o)^+ \times (b - Ax - Dz) + (\mu_o D)^+ z \geq f_0 \lceil \mu_o b \rceil. \tag{22}$$

When $t_i = 0$, we have that $(\mu_x)_i = \hat{a}_i$ so the coefficient of x_i in the first three terms of Equation (22) is $\hat{a}_i$. When $t_i = -1$, we have that $(\mu_x)_i = \hat{a} - 1$, so the coefficient of x_i in the first three terms of Equation (22) is f_0. Therefore, it is easy to see that any nondominated MIR inequality will have the form

$$\min\{\mu_o A - \lfloor \mu_o A \rfloor, f_0 \mathbf{1}\} x + f_0 \lfloor \mu_o A \rfloor x + (\mu_o)^+ \times (b - Ax - Dz) + (\mu_o D)^+ z \geq f_0 \lceil \mu_o b \rceil. \tag{23}$$

It is not hard to see, after some algebraic manipulations, that the MIR inequality (Eq. 23) is exactly the GMI inequality (Eq. 15) that we obtain if we do as follows:

1. add slack variables s to $Ax + Dz \leq b$;
2. use μ_o as a multiplier for $Ax + Dz + s = b$ and obtain Equation (15);
3. multiply Equation (15) by $f_0(1 - f_0)$, add it to f_0 times $\mu_o Ax + \mu_o Dz + \mu_o s = \mu_o b$ and substitute out the slack variables.

Therefore, under the nonnegativity assumption, MIR inequalities and GMI inequalities are said to be equivalent. However, note that Equation (15) was derived using the nonnegativity of the variables, while Equation (18) was not. In that sense, the relationship between MIR inequalities and GMI inequalities is analogous to the relationship between CG cuts and GF cuts. For more details, see Refs 7, 14, 15.

CLOSURES AND RANK

The concepts of the *elementary closure* and *rank* were introduced by Chvátal [16] for CG cuts and are important topics, which have received much attention since they are theoretical concepts that give an indication of the strength of a family of cuts or alternatively a measure of its complexity. We now define these concepts for Gomory cuts and survey a few of the results that are known for ranks and closures of polyhedra.

For each of the cuts presented in the previous section, let P denote the polyhedron obtained by dropping the integrality requirements on the sets from which the cuts were derived (P_I with $p = 0$ for GF, Q_I with $p = 0$ for CG, P_I for GMI). Notice that all the cuts presented in the previous section were derived by aggregating the set of linear constraints in P with some multipliers $u \in \mathbb{R}^m$ ($u \in \mathbb{R}^m_+$ in the case of CG cuts). This observation means that, once the multipliers are fixed, the derivation of the cuts is unique and it makes sense to define $GF(u, P)$, $CG(u, P)$, and $GMI(u, P)$ as the Gomory fractional/CG/GMI cut obtained by using u as multipliers to aggregate the linear constraints in P. This immediately gives rise to families of these inequalities $F_{GF}(P) := \{GF(u, P) : u \in \mathbb{R}^m\}$, $F_{CG}(P) := \{CG(u, P) : u \in \mathbb{R}^m_+\}$, and $F_{GMI}(P) := \{GMI(u, P) : u \in \mathbb{R}^m\}$ obtained by considering all possible values of u.

We can now define

$$P^1_{CG} = \{x \in P : x \text{ satisfies all inequalities in } F_{CG}(P)\} \tag{24}$$

as the *rank-1 CG-closure* of P, and one can similarly define P^1_{GF} and P^1_{GMI}, which are the *rank-1 GF-closure* and *rank-1 GMI closure* of P. The rank-1 closure is also called the *elementary closure* of a given family of cuts derived from P and valid for P_I.

It is also useful to apply the same concept recursively and define for $k \geq 1$:

$$P^k_{CG} = \{x \in P^{k-1}_{CG} : x \text{ satisfies all inequalities in } F_{CG}(P^{k-1}_{CG})\} \tag{25}$$

as the *rank-k CG-closure* where $P^0_{CG} = P$. Similarly define the *rank-k GF closure* P^k_{GF} and the *rank-k GMI closure* P^k_{GMI}. With that, we can also define the CG rank of an inequality being k if it is valid for P^k_{CG} but not for P^{k-1}_{CG}. Finally, we say that the CG rank of a class of inequalities is the maximum CG rank among all inequalities in that class and we say that the rank of a polyhedron P is the highest rank among all inequalities defining $conv(P \cap \mathbb{Z}^n)$ (we can derive similar definitions for GF/GMI ranks).

Relationships Between Closures

As mentioned earlier, GF cuts are equivalent to CG cuts in the presence of nonnegativity constraints and, therefore, the immediate result is that $P^1_{CG} = P^1_{GF}$. Similarly, in the mixed-integer case, the rank-1 GMI closure is the same as the rank-1 MIR closure, under the nonnegativity assumption. In addition, if the variables are nonnegative, the rank-1 GMI closure is the same as the rank-1 split closure of P, obtained from another important family of cuts: split cuts [17,18]. Because of this fact, GMI, MIR, and split cuts are considered "equivalent" under the nonnegativity assumption. For more details on MIR and split cuts, see the articles titled ***Mixed-Integer Rounding*** and ***Split Cuts*** in this encyclopedia. As shown before, in the pure integer case, the GMI cut dominates the GF cut, which means that $P^1_{GMI} \subseteq P^1_{CG}$. In addition, the inclusion is strict, so not only does the individual GMI cut derived using a specific set of multipliers dominate the CG cut using the same multiplier but the class of rank-k GMI cuts is also "stronger" than the class of rank-k CG cuts. In fact, when compared to several different classes of cuts, the rank-1 GMI closure is stronger than most of the other rank-1 closures. For more details on these and other related results see the detailed study on the relationship between several elementary closures by Cornuéjols and Li [7].

Separation and Polyhedrality of Closures

Note that the way rank-1 CG/GMI closures are defined, they consist of infinitely many inequalities. Therefore, one natural question is whether there exists a finite set of those inequalities that suffice to describe P^1_{CG}, P^1_{GF}, and P^1_{GMI}.

The answer is positive in both cases. In the pure integer case, P^1_{CG} (and hence P^1_{GF}) was shown to be polyhedral by Chvátal [16] and Schrijver [19,20]. In addition, Bockmayr and

Eisenbrand [21] show that, in fixed dimension, the number of inequalities needed to describe P^1_{CG} is polynomial in the size of the input. In the pure and mixed-integer case, P^1_{GMI} was shown to be polyhedral by Cook *et al*. [18], in terms of the split-closure (equivalent to P^1_{GMI} under nonnegativity). Alternative proofs of this fact were later given by Andersen *et al*. [22,23], Vielma [24], and Dash *et al*. [15].

Now the number of inequalities required to describe P^1_{CG}, P^1_{GF}, and P^1_{GMI} is usually very large and, therefore, one natural question that arises is to solve the separation problem, that is, given a point $x^* \in P$, either show that $x^* \in P^1_{CG}/P^1_{GF}/P^1_{GMI}$ or provide a hyperplane separating x^* from $P^1_{CG}/P^1_{GF}/P^1_{GMI}$. Unfortunately, the separation problem cannot be solved in polynomial time unless $P = NP$: Separation over P^1_{CG} is shown to be strongly NP-complete by Eisenbrand [25]. Even if P is contained in the nonnegative orthant, separation over P^1_{CG} (equivalently over P^1_{GF}), and over P^1_{GMI}, was shown to be strongly NP-complete by Caprara and Letchford [26]. In spite of that the above, in the case of set covering, Bienstock and Zuckerberg [27] have recently shown that one can approximate the rank-k CG closure to an arbitrary fixed precision in polynomial time (for a fixed k).

Bounds on the Rank

Another interesting question that has received significant research attention relates to obtaining bounds on the rank of polyhedra. These bounds give an idea on how much does each family of cutting planes approximate the convex hull of integer solutions.

For pure integer programs (IPs), it is well known [8,19] that all valid inequalities have finite CG rank. However, the rank of certain inequalities can be arbitrarily large, even in dimension 2 (see Chvátal [8]). This implies that, even if one could generate and add all possible rank-1 cuts, this procedure needs to be repeated an arbitrarily large number of times to obtain the convex hull, which gives an indication of how hard it is to solve integer programs by using only CG cuts.

Cook *et al*. [28] and Gerards [29] show that there is an upper bound on the CG-rank that only depends on the coefficients of the constraint matrix (not on the right-hand side); see Refs 9 and 20. There are, however, special cases that have some more reasonable bounds on the CG rank of polyhedra. For problems where P is contained in the 0/1 cube, Bockmayr and Eisenbrand [30] give a bound of $O(n^3 \log n)$ on the CG rank of a polyhedron, where n is the number of integer variables. This bound was later improved to $O(n^2 \log n^{n/2})$ by Bockmayr *et al*. [31] and then to $O(n^2 \log n)$ by Eisenbrand and Schulz [32]. In that same paper, they give a different upper bound of $n + ||c||_1$ on the rank of any inequality $cx \leq \delta$ (with $c \in \mathbb{Z}^n$) valid for the integer hull of P. A very similar bound was obtained by Chvátal *et al*. [33] for monotone polyhedra.

In that same paper, Chvátal *et al*. [33] also give an example of a polyhedron P with a CG rank of n (where n is the number of variables), thereby showing that the lower bound on the CG-rank of polyhedra is n. Several other results are known about rank of polyhedra. For instance, Hartmann [34] gives conditions under which an inequality has rank greater than 1 and Eisenbrand and Schulz [32] give a lower bound on the rank of value $(1 + \epsilon)n$ for some $\epsilon > 0$.

These results on the rank are for CG cuts in the pure integer case and do not immediately translate to the mixed-integer case. For instance, Cook *et al*. [18] give an example of a simple mixed-integer program (MIP) that has infinite GMI-rank, which means that even if we add all possible GMI cuts at every round of cut generation, we still do not get even a finite algorithm for solving a very simple MIP.

However, like in the case of the CG rank, the GMI-rank becomes much smaller when the polyhedra are contained in the 0/1 cube. In particular, Cornuéjols and Li [35] show that for a mixed-integer program where all the n integer variables are restricted to be binary the GMI rank of the polyhedron P is bounded above by n. This result actually follows from the result of Balas [17] on facial disjunctive programs and is best possible as there exists a lower bound of n given by Cornuéjols and Li [35] themselves. The example they use to provide the lower bound is exactly the same example proposed for the CG-rank by Chvátal *et al*. [33]. Notice that, in particular, these results also imply that, for pure 0–1 integer programs, the GMI-rank of polyhedra has an upper bound of n, in contrast with the CG rank for which only an upper bound of $O(n^2 \log n)$ is currently known.

SPECIAL CASES, VARIATIONS, STRENGTHENING, AND OTHERS

The importance of CG and GMI cuts led several researchers to consider different special cases, variations, and ways to strengthen these cuts. This section highlights a few of these endeavors.

$\mathbb{Z}$*half*-Cuts and mod-*k* Cuts

The term $\mathbb{Z}half$-CG cuts ($\mathbb{Z}half$-cuts for short) was introduced by Caprara and Fischetti [36]. The idea of $\mathbb{Z}half$-cuts is to restrict the multipliers $u \in \mathbb{R}^m_+$ used to aggregate the linear constraints of the LP relaxation to be in the set $\mathbb{Z}half$. The importance of this special case of Gomory cuts is that many classes of inequalities for combinatorial optimization problems are $\mathbb{Z}half$-cuts, for example, comb inequalities for the traveling salesman problem (TSP). In Ref. 36, it was shown that, even though the CG multipliers are very restricted, separation of $\mathbb{Z}half$-cuts remains strongly NP-hard. In Ref. 26, the authors show that, even under the nonnegativity assumption, separation of $\mathbb{Z}half$-cuts remains strongly NP-hard. Recently, Letchford *et al*. [37] have shown that, even if we restrict ourselves to $\{0, 1\}$ problems and allow some additional structure like set-packing, the separation of $\mathbb{Z}half$-cuts remains strongly NP-hard.

Nevertheless, computationally, Andreello *et al*. [38] show that $\mathbb{Z}half$-cuts can significantly decrease the solution time of branch-and-cut codes for some specific combinatorial optimization problems. Moreover, there are some positive theoretical results involving $\mathbb{Z}half$-cuts.

In order to introduce these results, let us first introduce the mod-*k* cuts, which generalize $\mathbb{Z}half$-cuts. The mod-*k* cuts are CG cuts where the multipliers $u \in \mathbb{R}^m_+$ used to aggregate the linear constraints of the LP relaxation are restricted to be in the set $\left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\right\}$ for some integer $k > 1$. It is not hard to see that all nondominated CG cuts can be written as a mod-*k* cuts for some k. Clearly, the NP-hardness results for $\mathbb{Z}half$-cuts immediately imply that the separation of mod-*k* cuts is also NP-hard. This particular class of cuts was studied by Caprara *et al*. [39] where they showed that if we only restrict ourselves to maximally violated mod-*k* cuts, then we can solve the separation problem in polynomial time. This positive result obviously extends to $\mathbb{Z}half$-cuts and generalizes a result of Applegate *et al*. [40] and Fleischer and Tardos [41] for the separation of maximally violated comb inequalities for the TSP. Later on, Letchford [42] extended their results to show that *totally tight* mod-*k*-cuts (of which maximally violated mod-*k* cuts are a subclass) can also be separated in polynomial time. Though Caprara *et al*. [39] had already derived this result for totally tight mod-*k* cuts as well, in Caprara *et al*.'s result one must specify in advance which class of mod-*k* cuts one wishes to separate (i.e, we must know what *k* is in advance). Letchford's results show that this is actually not necessary.

Another generalization of $\mathbb{Z}half$-cuts are binary clutter inequalities [43] for which separation is again strongly NP-hard, but polinomially solvable under certain conditions (this, in particular, gives other conditions under which $\mathbb{Z}half$-cuts can be separated in polynomial time).

Projected CG Cuts

Notice that CG cuts are only defined for pure integer programs. However, if one considers the general mixed integer set

$$P_I = \{(x, z) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : Ax + Dz \leq b\}, \quad (26)$$

one can still use CG cuts to obtain important cuts for P_I. The idea is as follows. Let P be the polyhedron obtained by dropping the integrality constraints from P_I, that is,

$$P = \{(x, z) \in \mathbb{R}^n_+ \times \mathbb{R}^p_+ : Ax + Dz \leq b\}. \quad (27)$$

Now, define $P(x)$ as the projection of P in the x-space. Finally, define $P_I(x) := P(x) \cap \mathbb{Z}^n$. Then, since $P_I(x)$ is a pure integer set, one can generate CG cuts for it. Moreover, due to the structure of how $P(x)$ is obtained, it is easy to see that any CG cut generated for $P_I(x)$ is a cut that is valid for P_I. For more details on this, see Ref. 44. In particular, the authors of that paper also show that such cuts correspond to a specific class of split cuts (and hence to a specific class of GMI cuts). Another interesting result related to projected CG cuts is that, if all the continuous variables have a zero coefficient in the objective function, then one can optimize over the MIP by using a finite number of rounds of projected CG cuts.

Other Variations

There have been several other variations, strengthenings, and studies of Gomory cuts. We mention a few of them as follows:

- k-cuts [45] are GMI cuts obtained from integer multiples of tableau rows. The above paper analyzes when k-cuts are better than the simple GMI cuts obtained by a tableau row and give a study when k-cuts seem to help in practice.
- In reduce-and-split cuts [46], the authors observe in the above paper that the coefficients of the continuous variables in the GMI cut are an important factor to determine the distance from the cut to the point being cut off. Since this distance is one of the measures of "cut quality," the authors give an algorithm to try to reduce the coefficients of the continuous variables and show that by doing so there is a significant improvement over only using GMI cuts from tableau rows.
- Balas and Perregard [47] give a precise correspondence between lift-and-project cuts [48] for mixed 0−1 programs, split cuts [18], and GMI cuts. Such correspondence gives rise to the upper bound on the GMI rank of n for pure 0/1 IPs and to an algorithm that tries to obtain better cuts than the GMI cut from a tableau row by looking at GMI cuts obtained from a (possibly infeasible) basis of the tableau.
- Letchford and Lodi [42] show a simple way to strengthen CG cuts, obtaining a cut with CG rank 2.
- Köppe and Weismantel [49] use basis reduction to generate strong GMI cuts.
- Ceria *et al*. [50] consider GMI cuts obtained by aggregating the rows of the optimal tableau from the LP relaxation with integer multipliers.
- Glover and Sherali [51] give a closed-form expression for a cut obtained by repeating the CG procedure several times, thus obtaining a closed form for some CG cuts whose rank is higher than 1.

These are just a few examples and the literature of papers that fall into this same category is enormous and would not fit into the available space we have. Therefore, we refrain from making an exhaustive list of the same.

Gomory Cuts as Group Cuts

In this section, we introduce Gomory's corner and group relaxations, which are important relaxations for MIPs. The *corner relaxation* is a common relaxation of the feasible region of an MIP, obtained from the system of inequalities defining the tableau rows of the LP relaxation and relaxing the nonnegativity constraints on the basic variables.

Specifically, consider the feasible region for an MIP:

$$\begin{aligned} &Ax = b \\ &x \geq 0 \\ &x_i \in \mathbb{Z}, \forall i \in I \subseteq \{1, \ldots, n\}. \end{aligned} \tag{28}$$

The tableau rows from the LP relaxation yield the system:

$$\begin{aligned} &x_B = f + Rx_N \\ &x_B, x_N \geq 0 \\ &x_i \in \mathbb{Z}, \forall i \in I \subseteq \{1, \ldots, n\}, \end{aligned} \tag{29}$$

where $f = A_B^{-1}b$ and $R = -A_B^{-1}A_N$. For simplicity, we assume that $I = \{1, \ldots, n\}$ and that

$|B| = m$. If we drop the nonnegativity constraints corresponding to variables x_B, we get the *corner relaxation*:

$$\left\{ x : \begin{array}{l} x_B = f + \sum_{j \in N} r^j x_j \\ x_N \geq 0 \\ x_B \in \mathbb{Z}^m \\ x_i \in \mathbb{Z}, \forall i \in N \end{array} \right\}, \quad (30)$$

where r^j are the columns of R.

If we add one variable for every possible element of the group generated by the r^j vectors, we get the master group relaxation. If $m = 1$, we call it the master cyclic group relaxation. For a more thorough discussion on the group problem, corner polyhedron, and master group relaxations, the reader is referred to Refs 52–54 and the article titled ***Inequalities from Group Relaxation*** in this encyclopedia.

Now we call a function ψ valid for the group relaxation if, for every feasible solution x for the group relaxation, we have that $\sum \psi(r)x_r \geq 1$. Valid functions give rise to valid inequalities for Equation (30) and as a consequence valid inequalities to the original MIP. Also, GMI cuts can be obtained by extreme valid functions (the analogous to a facet-defining inequality), which shows that GMI cuts are important in the context of group cuts. We further discuss the importance of GMI cuts in the context of group cuts in the next section.

IMPLEMENTATION AND COMPUTATIONAL ASPECTS

We now turn our attentions to some of the computational aspects and results that are related to Gomory cuts.

Even though there was interest in Gomory cuts after the 1958 paper [1], they were thought to be computationally ineffective for quite some time [12,55]. Among the many criticisms, Gomory cuts were considered to converge very poorly and their use inside a branch-and-cut framework was questioned since it would require some complex bookkeeping because cuts generated in the branch-and-cut tree are not valid for the whole problem. Much to the surprise of the community, Balas *et al*. [56] showed in 1996 that GMICs can be used effectively within a branch-and-cut framework for 0–1 MIPs. A more detailed and accurate description of the history of the above paper can be found in Ref. 57. In particular, in that paper, they show a way to lift Gomory cuts for 0–1 MIPs so that they become globally valid regardless of where in the branch-and-cut tree are they generated. They also highlight some computational strategies, some of which are still used today, like adding multiple cutting planes at a time to the LP relaxation.

The importance of that paper in reigniting the interest in Gomory cuts as an important tool to solve general MIPs is remarkable. Indeed, currently, the ability to generate GMICs is widely considered an essential part of most MIP solvers [58]. This fact can be evidenced by a series of papers by Bixby *et al*. [59,60] where, among other things, they test which of the features in CPLEX [61] are more important for the solution of MIPs. They tested the average performance degradation on the solution times for a series of benchmark instances. Their conclusions are that “The clear winner in these tests was cutting planes. Disabling this feature resulted in a deterioration of a *factor* of almost 54 in overall performance, a remarkable difference.”, and among the cutting planes “Gomory cuts are the clear winner by this measure.”.

Though it is hard to say what is the exact influence of such a paper in the research efforts of the following years, we believe it is safe to say the Balas *et al*. paper [56] generated a large interest in Gomory cuts decades after their introduction.

In addition to the already mentioned variations and improvements over Gomory cuts, for which several computational tests have been performed, recently there have been some other papers focusing on computational aspects of Gomory cuts, which we highlight below.

Strength of Gomory Cuts

The practical success of Gomory cuts described above comes mostly by using heuristics for generating such cuts inside commercial codes. While their practical success shows the quality of these heuristics, it remains to be answered if we could do a better job in using these (or other) cutting planes. In other words, the natural questions are as follows: How well are we using Gomory cuts? Could we do much better by trying to generate different Gomory cuts or other classes of cuts? In this section, we review some of the efforts to answer such questions.

One answer to those questions has recently been studied by Fischetti and Lodi [62]. In the above paper, they study the effect of rank-1 CG cuts by modeling the separation problem as an MIP. They obtain good results, in particular, being able to solve an open instance of MIPLIB 2003 [63] for the first time. Their approach was later extended [64,65] to pose the separation of GMI cuts (equivalently MIR/split cuts) as an MIP and use them to evaluate the rank-1 GMI closure. Their results also show that there is still much to gain by using rank-1 Gomory cuts if one is able to separate them well or identify the useful ones. Indeed, recently, Dash and Goycoolea [66] proposed a heuristic for generating rank-1 GMI cuts that seems to perform very well in practice.

The papers mentioned above show that it may be worth spending time to generate "better" Gomory cuts. However, one may switch focus and try to generate cuts that are different than Gomory cuts in hopes of improving the practical performance of an MIP solver. However, even though several other classes of cutting planes for integer programs have been proposed throughout the years, the practical success of several of those inequalities when compared to GMI cuts is limited. This has led researchers to investigate if there is an intrinsic reason for this to happen. Indeed, when compared to group cuts, Dash and Günük [67] show that interpolated subadditive cuts are dominated by GMI cuts (in a probabilistic sense). In addition, they show in Ref. 68 that after adding GMI cuts for the tableau rows of the LP relaxation, it is often the case that there are no other violated cyclic group cuts from those same tableau rows. In particular, in those instances, one would not gain anything by attempting to add other cyclic group cuts from the same tableau rows in addition to the GMI cuts.

These results are in line with the computational results of Fischetti and Saturni [69], who show that the performance of GMI is seldom improved when using other cyclic group cuts obtained by interpolation (something that Cornuéjols *et al*. [45] also observe for k-cuts instead of cyclic group cuts). Another indication that GMI are the most important cuts among all the cyclic group cuts comes from Gomory *et al*.'s shooting experiment [70], where they computationally show that a random sample of facets of the cyclic group have most of its elements as GMI inequalities.

In contrast, Fischetti and Monaci [71] study the corner relaxation and show that often one can have a significant bound improvement over GMIs if the whole corner/group relaxation is considered (that is considering all the rows of those relaxations simultaneously—not just a single row at a time as in the case of the cyclic group).

Another empirical study of the strength of GMI cuts is performed in Ref. 72, where they show that adding several rounds of GMI cuts based on the initial tableau rows of the LP relaxation often approximates the intersection of the convex hull of integer points of each of those tableau rows well. In other words, even if one could use any cut valid for the convex hull of integer points satisfying one of the tableau rows, there would often be a negligible gain in terms of bounds, thereby showing the strength of GMI cuts even outside the context of group cuts.

All these computational results seem to indicate that, if one is to significantly improve the performance of Gomory cuts, one either needs to consider cuts that are valid for relaxations with several constraints (as opposed to one constraint as is the case of Gomory cuts) or find a way to identify important single-constraint relaxations and generate Gomory cuts for it. In fact, in the section "Multirow Gomory Cuts," we discuss recent efforts toward the first alternative, namely, the study of multirow Gomory cuts.

Numerical Issues and Pure Cutting-Plane Algorithm

When designing a computer code to solve a mixed-integer program, several issues start becoming relevant to the accuracy and effectiveness of such code. For a more complete survey on these issues and the progress of mixed-integer programming computations, we refer the reader to Ref. 58.

We now focus on two of these computational issues that are faced by Gomory cuts. The first one is the issue of cut validity. Note that the derivation of Gomory cuts is purely algebraic and hence relies on the accuracy of numerical computations. However, algebraic calculations in computers are usually performed in floating-point arithmetic, which has intrinsic errors. Usually these errors are very small, but even a very small error can cause a big difference in the coefficient obtained in a Gomory cut. For example, the CG cut for $x_1 + 2x_2 \leq 2.005$ is (assuming $x_1, x_2 \in \mathbb{Z}_+$) is $x_1 + 2x_2 \leq 2$. However, if there is an error of 0.01 in the calculations, we may obtain $x_1 + 2x_2 \leq 1.995$ and hence the cut $x_1 + 2x_2 \leq 1$, which may be an invalid cut. Since the generation of an invalid cut may lead to an MIP solver cutting off the optimal solution and returning the incorrect solution, one needs to be careful when implementing Gomory cuts using floating-point arithmetic.

The usual approach taken by MIP solvers is to implement Gomory cuts with care using heuristic rules to try to avoid such numerical problems. However, as reported by Margot [73], modern cutting-plane generators still generate invalid cuts once in a while. Neumaier and Shcherbina [74], Cook *et al*. [75], and Zanette *et al*. [76] propose methods to generate Gomory cuts that will not suffer from these numerical problems by having a structured way to control the error in the floating-point calculations.

Another issue that is often faced when using Gomory cuts (and that is somewhat related to the issue of accuracy) is that, after adding a few rounds of cuts, the cut coefficients tend to start getting large and the determinant of the basis matrix in the optimal tableau also tends to grow. This can lead to numerical problems in the LP solver and is one of the reasons why commercial solvers tend not to add too many rounds of Gomory cuts and also why there is a common knowledge that pure cutting-plane algorithms using Gomory cuts are ineffective.

Recently, however, the above has been discussed again in Ref. 76, where they show that pure cutting- plane algorithms can actually perform well, provided certain care is taken. They show that, if one is to perform Gomory's cutting-plane algorithm with the lexicographic dual simplex, a significant improvement can be obtained in the final bound. Moreover, the authors also show that the problem of obtaining large cut coefficients and basis determinants becomes much less pronounced.

In Ref. 77, the authors extend their study to understand what is the relationship between cutting-plane methods and some enumerative schemes, proposing different variants of Gomory's cutting-plane method. This line of research is extremely recent and shows that there are still several issues (both computational and theoretical) that we do not fully understand with respect to Gomory cuts and the cutting-plane method.

MULTIROW GOMORY CUTS

We now focus on an extension of Gomory cuts that consider multiple-row relaxations of an MIP. Recently, there has been a significant amount of work devoted to these so-called *multirow Gomory cuts*. The idea is to extend the fact that GMI cuts are, among the cyclic group cuts, the ones that have the best coefficients for the continuous variables. Indeed, Dash and Günlük [68] observe that this is one of the reasons why GMI cuts are (computationally) the most important cuts among all cyclic group cuts. The extension of such an idea to multiple rows is to consider a tableau of the linear programming relaxation of an IP, derive the cuts with the best coefficients for the continuous variables, and lift back the integer variables.

To achieve this goal, we follow the same steps that were developed in the section titled "Gomory Cuts as Group Cuts" to obtain the corner and group relaxations. If we now consider Equation (30) and further drop all integrality constraints from x_N and the nonnegativity constraints for x_B, we get the following set:

$$\left\{ x : \begin{array}{l} f + \sum_{j \in N} r^j x_j \in \mathbb{Z}^m \\ x_N \geq 0. \end{array} \right\}. \tag{31}$$

So if we obtain a facet-defining inequality for Equation (31) and then lift the integer variables back to obtain an inequality for Equation (30), then we get an inequality with good coefficients for any continuous variable, thereby emulating the same property that GMI cuts have in the case $m = 1$.

Andersen *et al*. [78] show that facet-defining inequalities for Equation (31) when $m = 2$ are intersection cuts and are related to lattice-free convex sets. Borozan and Cornuéjols [79] show that minimal inequalities for a semiinfinite extension of Equation (31) obtained by adding one variable for all possible real coefficients $r^j \in \mathbb{R}^m$ correspond to maximal lattice-free convex sets. Cornéjols and Margot [80] study which of these inequalities define facets of Equation (31), while the papers [81–83] study the lifting of the integer variables to obtain cuts valid for Equation (30). Other recent papers related to multirow Gomory cuts are Refs 84, 85.

Extensions of multirow Gomory cuts are studied in Refs 86–88 by considering tighter relaxations than Equation (31) and its semiinfinite extension. In addition, the papers [89,90] show computational experiments with multirow Gomory cuts and there are several other groups performing computational experiments as well. Finally, the papers [91,92] study the GMI rank of the multirow Gomory cuts. One particularly interesting result is that some of these cuts have an infinite GMI rank, thereby showing that a lot can be gained by using these cuts in addition to GMI cuts.

Summing up, this is one particular area of research that is very recent and very active. Indeed the earliest of the 15 papers cited above is dated 2007 and we know of several different research groups actively researching questions related to multirow Gomory cuts.

CONCLUSION

In this article, we tried to do a very brief review of Gomory cuts, trying to be at the same time thorough and concise to fit into the allowed space. While this topic is as old as the field of integer programming, this remains a very active research topic, with much research still being carried out today. We hope that we have been able to provide a glimpse into the topic, stating the main results and with enough references so that the interested reader can explore further.

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Abstract: Gomory cuts have played an important role in integer and mixed-integer programming for over 50 years. This article reviews these much studied cuts, surveying some of the research that has been done over the years as well as looks into new research directions that have appeared in the last few years.

Keywords: gomory cuts; cutting planes; mixed-integer programming