Universal algebra at a crossroads

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Outline

1. What is universal algebra?
2. Digraphs and their homomorphism problems
3. The Dichotomy Conjecture
4. Where the story gets exciting
5. Epilogue

Part 1 – What is universal algebra?

Examples

universe

Objects: 'algebras' $\mathbb{A} = (A\,;\, \underbrace{\circ, f, \ldots}_{\text{operations}})$

Any group

$(G; \cdot, {}^{-1}, 1)$

Constructions

- 'subalgebras' — subgroups
- 'quotient algebras' — G/N
- 'direct products' (and 'powers') — $G \times H$ and G^n

Varieties

Examples

'Variety' — a class of algebras axiomatized by 'identities'

{all groups}

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot 1 = x$, $1 \cdot x = x$

$x \cdot x^{-1} = 1$, $x^{-1} \cdot x = 1$

'The variety generated by $\mathbb{A}$' (smallest variety containing $\mathbb{A}$)

- denoted $\mathrm{var}(\mathbb{A})$

$$\mathrm{var}(S_3)$$

$$= \{G : \exists N \triangleleft G \text{ where } N \text{ and } G/N \text{ are abelian, } N \models x^3 = 1, \text{ and } G/N \models x^2 = 1\}$$

axiomatized by group axioms plus

$$x^6 = 1 \quad \text{and} \quad x^2y^2 = y^2x^2$$

Terms

Examples

$x\cdot(y\cdot z)$

x^6 i.e.

$((((x\cdot x)\cdot x)\cdot x)\cdot x)\cdot x$

$x\cdot x^{-1}$

'Terms' — formal expressions which
can occur in identities

Terms play many roles in universal algebra, including:

- Building new operations (in many variables) in an algebra $\mathbb{A}$.

$$\boxed{\text{Often, the set of \underline{all} `term operations' of } \mathbb{A} \text{ is more important than the `starting operations' of } \mathbb{A}.}$$

Part 2 – directed graphs

'Directed graph' (or 'digraph'): $\mathbf{A} = (\overset{\text{always \underline{finite}}}{\overset{\downarrow}{A}}\,;\ \underset{\underset{\text{binary relation, i.e. } E \subseteq A^2}{\uparrow}}{E})$

Examples:

$\mathbf{P}_2 = (\{0,1\}\,;\,\leq)$

$\leq\ = \{(0,0),(0,1),(1,1)\}$

$\mathbf{C}_3 = (\{0,1,2\}\,;\,\underline{\mathrm{succ}})$

$\{(0,1),(1,2),(2,0)\}$

$\mathbf{K}_3 = (\{0,1,2\}\,;\,\neq)$

Digraph homomorphisms

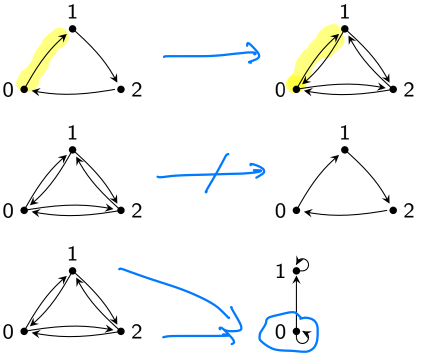
Let $\mathbf{A} = (A\,;\, E_{\mathbf{A}})$ and $\mathbf{B} = (B\,;\, E_{\mathbf{B}})$ be digraphs.

A **homomorphism from A to B** is a function $h : A \to B$ satisfying

$$h(E_{\mathbf{A}}) \subseteq E_{\mathbf{B}}.$$

Examples:

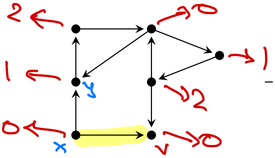
1. $\mathrm{id} : \mathbf{C}_3 \to \mathbf{K}_3$
2. $\nexists\, h : \mathbf{K}_3 \to \mathbf{C}_3$
3. $\mathbf{K}_3 \stackrel{?}{\to} \mathbf{P}_2$



A game

— Fix a target digraph **H**.
— Ask: "How hard is it to decide whether some other digraph has a homomorphism <u>to</u> **H**?"

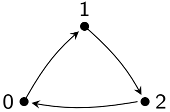
Example: $\mathbf{H} = \mathbf{C}_3$. Given a digraph **A**, can I tell whether $\mathbf{A} \xrightarrow{\exists} \mathbf{C}_3$?



$\mathbf{A} = (A\,;\, E_{\mathbf{A}})$

$\exists\,?$ - - - - - - - - - - - - - - →

NO

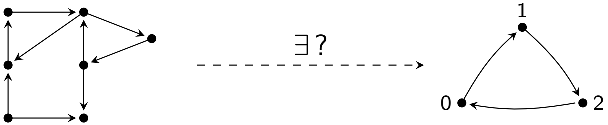


$\mathbf{C}_3$

A game

— Fix a target digraph $\mathbf{H}$.
— Ask: "How hard is it to decide whether some other digraph has a homomorphism to $\mathbf{H}$?"

Example: $\mathbf{H} = \mathbf{C}_3$. Given a digraph $\mathbf{A}$, can I tell whether $\mathbf{A} \overset{\exists}{\to} \mathbf{C}_3$?

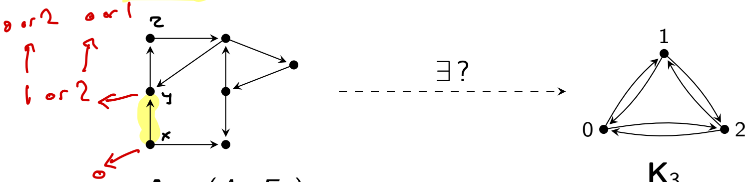


$\mathbf{A} = (A\,;\, E_{\mathbf{A}})$ $\qquad$ $\mathbf{C}_3$

Summary: local analysis leads quickly to a homomorphism, or to a proof that none exists.

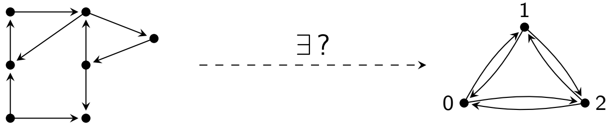
So the question $\square \overset{\exists ?}{\to} \mathbf{C}_3$ is **easy** (e.g., solvable in polynomial time).

Example: $\mathbf{H} = \mathbf{K}_3$. Given a digraph $\mathbf{A}$, can I tell whether $\mathbf{A} \xrightarrow{\exists} \mathbf{K}_3$?



$\mathbf{A} = (A\,;\, E_{\mathbf{A}})$ $\qquad$ $\mathbf{K}_3$

Example: $\mathbf{H} = \mathbf{K}_3$. Given a digraph $\mathbf{A}$, can I tell whether $\mathbf{A} \overset{\exists}{\to} \mathbf{C}_3$?



$\mathbf{A} = (A\,;\, E_{\mathbf{A}})$ $\mathbf{K}_3$

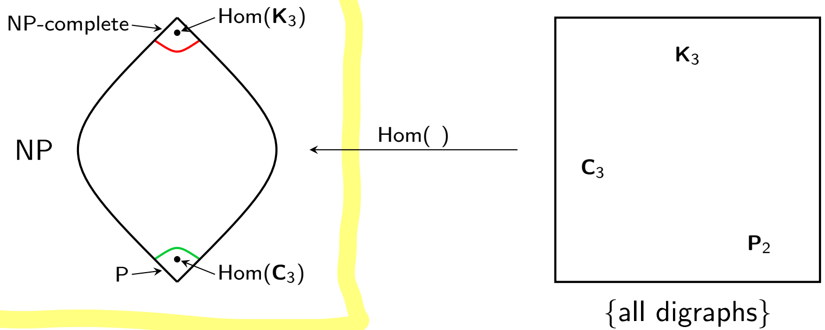
Local analysis isn't helpful.

In the worst case, may need to consider $\approx 2^{|A|}$ cases.

This is as bad as **brute force search** through all $3^{|A|}$ maps $A \to \{0, 1, 2\}$.

Part 3 – the Dichotomy Conjecture

For fixed **H**, $\boxed{\text{Hom}(\mathbf{H}) \text{ is the computational problem } \square \overset{\exists ?}{\to} \mathbf{H}.}$

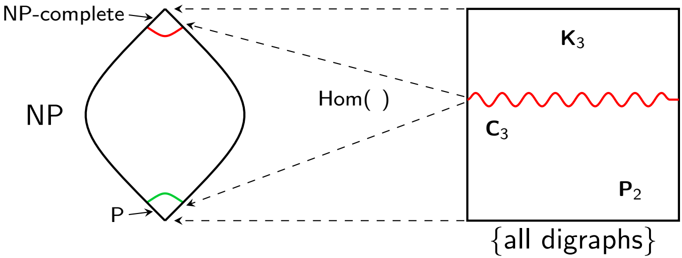


Dichotomy Conjecture (Feder & Vardi, 1998).

For every digraph **H**, Hom(**H**) is in P or is NP-complete.

Dichotomy Conjecture (Feder & Vardi, 1998).

For every digraph $\mathbf{H}$, Hom($\mathbf{H}$) is in P or is NP-complete.



The Dichotomy Conjecture raises some questions:

1. Is it true?
2. If yes, then where is the dividing line?

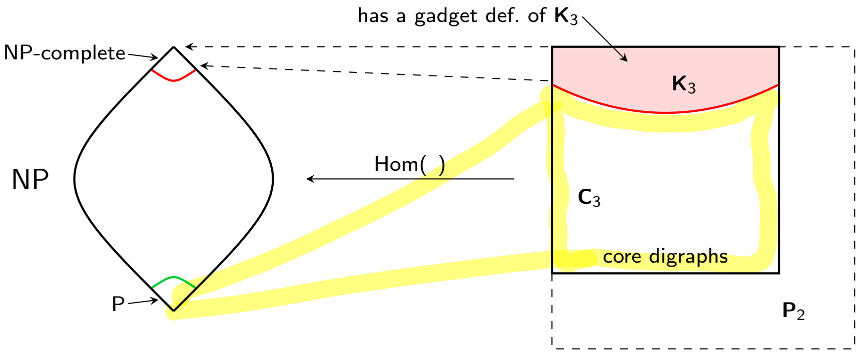
Two simplifications

1. Reduction to 'cores'
   - Every digraph has a 'core'
   - **Fact:** $\mathrm{Hom}(\mathbf{H}) = \mathrm{Hom}(\mathrm{core}(\mathbf{H}))$.

2. 'Gadget definitions'
   - A precise way to encode a given digraph "in" $\mathbf{H}$. (See footnote)[1]
   - **Fact:** Assume $\mathbf{H}$ is a core.
     If $\mathbf{H}$ has a gadget definition of $\mathbf{K}_3$, then $\mathrm{Hom}(\mathbf{H})$ is NP-complete.

Take-away: we only need to consider cores; among cores, we only need to consider those which do not have a gadget definition of $\mathbf{K}_3$.

[1] Using first-order formulas built from variables, $\exists$, $\wedge$, the edge relation $E_{\mathbf{H}}$ of $\mathbf{H}$, and constants naming the elements of H, to define subsets $S \subseteq H^n$ and $\theta, E \subseteq H^{2n}$ such that θ is an equivalence relation on S and $(S/\theta\,;\,E/\theta) \cong$ the given digraph.



Refined Dichotomy Conjecture ($\approx$ Feder-Vardi '98, Bulatov-Jeavons-Krokhin '01)

If $\mathbf{H}$ is core and has no gadget definition of $\mathbf{K}_3$, then Hom($\mathbf{H}$) is in P.

Part 4 - Where the story gets exciting

Universal algebraists (help) conquer the Dichotomy Conjecture!

“The Boys,” from the Hindustan Times, 02 Aug 2019

The boys at Dagstuhl, 2018

After 20 years of intense collaboration between computer scientists and universal algebraists, Andrei Bulatov and Dmitriy Zhuk independently confirm the Refined Dichotomy Conjecture in 2017.

How is universal algebra relevant?

Polymorphisms – "multi-variable symmetries"

Fix a digraph $\mathbf{H} = (H\,;E)$.

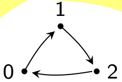
A 'polymorphism of $\mathbf{H}$' is any function $f : H^n \to H$ (for some $n \geq 1$) which "preserves E", meaning

$$(a_1, b_1), \ldots, (a_n, b_n) \in E \implies (f(\mathbf{a}), f(\mathbf{b})) \in E.$$

Example: $\mathbf{P}_2 = (\{0,1\}\,;\leq)$

- Polymorphisms are the monotone boolean functions $\{0,1\}^n \to \{0,1\}$.
- Examples:
  - $\max(x,y)$, $\min(x,y)$
  - $\max(x_1,\ldots,x_n)$, $\min(x_1,\ldots,x_n)$ for any n

Polymorphisms (more examples)



1. $\mathbf{C}_3$ has interesting polymorphisms, e.g.,

$$f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4 \pmod 3$$

idempotent

2. $\mathbf{K}_3$ has no polymorphisms at all, other than projections followed by a permutation:

$$f_{\sigma,i}(x_1, \ldots, x_n) = \sigma(x_i)$$

The idem. polymorphism algebra of a core digraph

Definitions

A function $f : H^n \to H$ is **idempotent** if $f(x, x, \ldots, x) = x \quad \forall x \in H$.

Given a core $\mathbf{H}$, $\mathbb{A}\mathrm{lg}(\mathbf{H})$ is an algebra $(H\,; \ldots)$ with the property

$$\{\text{term operations of } \mathbb{A}\mathrm{lg}(\mathbf{H})\} = \{\text{idempotent polymorphisms of } \mathbf{H}\}$$

rock-paper-scissors.

Example: $\mathbb{A}\mathrm{lg}(\mathbf{C}_3) = (\{0, 1, 2\}\,; \underbrace{-(x + y)}, \underbrace{s(x, y, z, w)}, \sqcup)$

$$s(x,y,z,w) = \begin{cases} x & \text{if } z = w \\ y & \text{else} \end{cases}$$

The idem. polymorphism algebra of a core digraph

Definitions

A function $f : H^n \to H$ is **idempotent** if $f(x, x, \dots, x) = x \ \ \forall x \in H$.

Given a core **H**, $\mathbb{Alg}(\mathbf{H})$ is an algebra $(H\,; \dots)$ with the property

$$\{\text{term operations of } \mathbb{Alg}(\mathbf{H})\} \ = \ \{\text{idempotent polymorphisms of } \mathbf{H}\}$$

Crazy-magic universal algebra (1960s + Hell-Nešetřil '97 + Jeavons et al '98 + $\cdots$)

Assume **H** is core. Then $\mathbb{Alg}(\mathbf{H})$ determines everything:

1. The relative difficulty of Hom(**H**).
2. Whether or not **H** has a gadget definition of $\mathbf{K}_3$.

Characterizing "no gadget definition of $\mathbf{K}_3$"

core and

Theorem (Taylor '77 + Hobby-McKenzie '88 + Bulatov-Jeavons-Krokhin '05 + Maróti-McKenzie '08 + Siggers '10 + Barto-Kozik '12)

For a core $\mathbf{H}$, TFAE:

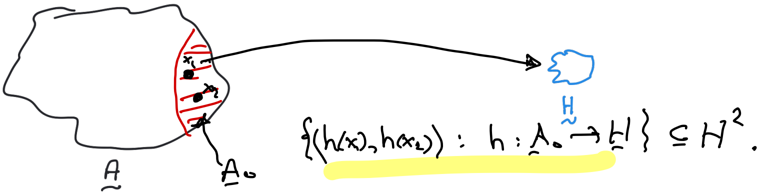
1. $\mathbf{H}$ does **not** have a gadget definition of $\mathbf{K}_3$.
2. The variety generated by $\mathbb{Alg}(\mathbf{H})$ satisfies the weakest structural nontriviality condition of universal algebra.
3. For some $n \geq 2$, $\mathbf{H}$ has an n-ary **cyclically invariant** idempotent polymorphism $f(x_1, \ldots, x_n)$, i.e. satisfying
$$f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1).$$
4. $\mathbf{H}$ has a 6-ary idempotent polymorphism $g(x_1, \ldots, x_6)$ satisfying
$$g(x, x, y, y, z, z) = g(y, z, z, x, x, y).$$

Proof of the Refined Dichotomy Conjecture

Fix a core $\mathbf{H}$. Assume no gadget definition of $\mathbf{K}_3$.

- So $\mathbf{H}$ has a cyclically invariant idempotent polymorphism ...
- ... which is a term operation of $\mathbb{Alg}(\mathbf{H})$.

Let $\mathbf{A}$ be an arbitrary digraph; we want to decide whether $\mathbf{A} \overset{\exists ?}{\to} \mathbf{H}$.



The solution set to a subproblem like this is ...

- a subalgebra of $\mathbb{Alg}(\mathbf{H})^k$, and so is ...
- in $\mathrm{var}(\mathbb{Alg}(\mathbf{H}))$.

By analyzing how subalgebras of powers in $\mathrm{var}(\mathbb{Alg}(\mathbf{H}))$ interact, Bulatov and Zhuk were able to give polynomial-time algorithms solving $\mathrm{Hom}(\mathbf{H})$.

Part 5 - Epilogue

The gravy train of the Refined Dichotomy Conjecture attracted many of the world's best universal algebraists during 2000-2017.

It was lucrative <u>and</u> fun.

Many algebraists are staying on the train. Current directions:

- Hom($\mathbf{H}$) for countably infinite $\mathbf{H}$ with huge symmetric group
- 'Promise CSP' (variants such as Hom($\mathbf{K}_3, \mathbf{K}_5$))
- 'Absorption Theory' (byproduct of the study of the RDC)

But is it universal algebra?

In these new directions,

- Relational structures (e.g., digraphs) are primary.
- Algebras are secondary.

I'm old. I miss that old-timey universal algebra.

What <u>is</u> universal algebra?

Thank you!