A new property of finite NU(4) algebras

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Definition

Suppose

- $\mathbf{B}, \mathbf{C}$ are algebras with $\mathbf{B} \leq \mathbf{C}$.
- $t(x_1, \ldots, x_k)$ is a term.

Say $\mathbf{B}$ is an **absorbing subalgebra** of $\mathbf{C}$ (with respect to t) if

$\forall\ i = 1, \ldots, k,$

$$t(B, \ldots, B, \underset{\substack{\uparrow \\ i}}{C}, B, \ldots, B) \subseteq B$$

Notation: $\mathbf{B} \lhd_t \mathbf{C}$

$\mathbf{B} \leq \mathbf{C}$ and $t(B, \ldots, B, C, B, \ldots, B) \subseteq B$ for all positions of C.

Examples.

1. $\mathbf{C} = (C, +, -, 0, \cdot)$ a ring without 1. Let $t(x, y) := x \cdot y$.

   $\mathbf{B} \lhd_t \mathbf{C}$ iff B is a two-sided ideal of $\mathbf{C}$.

2. $\mathbf{C}$ an algebra with Maltsev term $p(x, y, z)$.

   $\mathbf{B} \lhd_p \mathbf{C}$ iff $B = C$.

3. t a constant-valued term on $\mathbf{C}$.

   $\mathbf{B} \lhd_t \mathbf{C}$ iff $\mathbf{B} \leq \mathbf{C}$.

Absorbing subalgebras play a key role in recent CSP results.

- A. Bulatov, Combinatorial problems raised from 2-semilattices, 2006.
- E. Kiss and M. Valeriote, Tractability and congruence distributivity, 2007.
- L. Barto and M. Kozik, CSPs of bounded width, 2008.
- L. Barto and M. Kozik, Absorbing subalgebras, cyclic terms and the constraint satisfaction problem, 2010.

Lots of basic questions about absorbing subalgebras remain open.

Absorption in idempotent NU algebras.

Recall: a term $t(x_1, \ldots, x_k)$ is a **near-unanimity** (NU) term for **C** if it satisfies

$$t(x, \ldots, x, \underset{\substack{\uparrow \\ i}}{y}, x, \ldots, x) = x$$

for all i and all $x, y \in C$.

(Recall: **C** is **idempotent** if $\{x\} \leq \mathbf{C} \quad \forall x \in C$.)

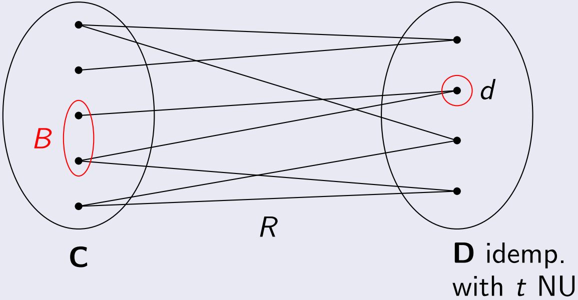
Observation: If **C** is idempotent, the definition of NU term can be expressed as

$$\{x\} \lhd_t \mathbf{C} \quad \forall x \in C.$$

"Propagation" in idempotent NU algebras

Lemma

Suppose $\mathbf{C}, \mathbf{D}$ are algebras with $\mathbf{D}$ idempotent, t is an NU term for $\mathbf{D}$, and $\mathbf{R} \leq_{sd} \mathbf{C} \times \mathbf{D}$. Fix $d \in D$ and define $B = \{x \in C : (x, d) \in R\}$.



Then $\mathbf{B} \lhd_t \mathbf{C}$.

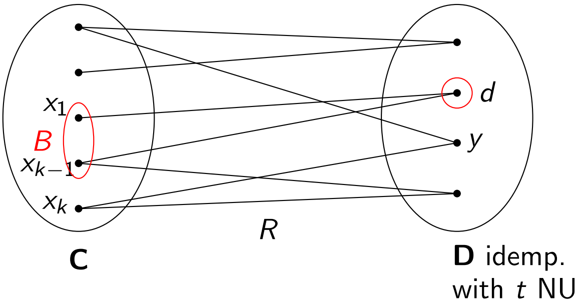
Proof.

(Must show $\mathbf{B} \lhd_t \mathbf{C}$.) $\mathbf{B} \leq \mathbf{C}$ because $\{d\} \leq \mathbf{D}$.

Suppose $x_1, \ldots, x_{k-1} \in B$ and $x_k \in C$. $\exists y \in D$ with columns in R:

$$\begin{matrix} x_1 & x_2 & \cdots & x_{k-1} & x_k \\ d & d & \cdots & d & y \end{matrix} \xrightarrow{t} \begin{matrix} t(\mathbf{x}) \\ d \end{matrix}$$

Apply t. Produces $(t(\mathbf{x}), d) \in R$, hence $t(\mathbf{x}) \in B$. □



Here is a strange new property involving absorption.

Notation: if $x \in A$, then $\widehat{x} = (x, x, \ldots, x) \in A^n$.
$\widehat{A} = \{\widehat{x} : x \in A\}$.

Definition

Suppose $\mathbf{A}$ is an algebra and t is a term.

Say that $a \in A$ is an **absorption-forced constant** for $\mathbf{A}$ with respect to t if the following holds: $\forall\, n \geq 1$, $\forall\, \mathbf{B} \leq \mathbf{C} \leq \mathbf{A}^n$, if

- $\widehat{A} \subseteq C$,
- $\mathbf{B} \lhd_t \mathbf{C}$, and
- $\mathbf{B} \leq_{sd} \mathbf{A}^n$,

then $\widehat{a} \in B$.

$a \in A$ is an absorption-forced constant for $\mathbf{A}$ with respect to t if ...

$$[\ \mathbf{C} \le \mathbf{A}^n \ \text{ and } \ \widehat{A} \subseteq C \ \text{ and } \ \mathbf{B} \lhd_t \mathbf{C} \ \text{ and } \ \mathbf{B} \le_{sd} \mathbf{A}^n\ ] \ \text{ imply } \ \widehat{a} \in B.$$

Examples

Example 1: Suppose 0 is an absorbing element for t in $\mathbf{A}$ in classical sense, i.e., $t(\ldots, 0, \ldots) = 0$.

Then 0 is the unique absorption-forced constant for $\mathbf{A}$ with respect to t.

- To prove uniqueness, use $\mathbf{B} = \{(x, y) \in A^2 \ : \ x = 0 \text{ or } y = 0\}$.

Example 2: Suppose t is a majority term for $\mathbf{A}$.

Then every $a \in A$ is an absorption-forced constant for $\mathbf{A}$ with respect to t.

- Proof: $\mathbf{B} \lhd_t \mathbf{C} \le \mathbf{A}^n$ and $\mathbf{B} \le_{sd} \mathbf{A}^n$ implies $\mathbf{B} = \mathbf{C}$.

Last year, while working on the "$\mathrm{NU} \stackrel{?}{\Rightarrow}$ linear datalog" conjecture, Marcin Kozik realized he could prove the conjecture, **provided**

> Every finite NU algebra has an absorption-forced constant with respect to its NU term.

Call the above statement the **Kozik conjecture**.

In December I proved:

Theorem (W)

The Kozik conjecture is true for NU terms of arity 4.

I have a marvelous proof of this, which this talk is too short to contain.

Last month Kozik visited McMaster University. During the visit, he and Libor Barto proved:

Theorem (Barto, Kozik)

Kozik's conjecture is true for NU terms of all arities.

Corollary

The "NU $\Rightarrow$ linear datalog" conjecture is true.

In fact:

Theorem (Barto, Kozik)

Every finite algebra has a **universal absorption-forced constant**, *i.e., an element $a \in A$ which is an absorption-forced constant with respect to every idempotent term of the algebra!*

Fix $\mathbf{A}$. Must show that $\exists a \in A$ such that $\forall$ idempotent t, $\forall n \geq 1$,

$$[\ \mathbf{C} \leq \mathbf{A}^n \ \text{ and } \ \widehat{A} \subseteq C \ \text{ and } \ \mathbf{B} \lhd_t \mathbf{C} \ \text{ and } \ \mathbf{B} \leq_{sd} \mathbf{A}^n\ ] \ \text{ imply } \ \widehat{a} \in B.$$

- WLOG, can assume $\mathbf{A}$ is idempotent.
- For each $n \geq 1$, let $Ab(n)$ denote the set of elements $a \in A$ which work for subalgebras of $\mathbf{A}^n$.

  Since $Ab(n) \supseteq Ab(n+1)$, suffices to show $Ab(n) \neq \varnothing$ for all $n \geq 1$.
- Fix n. Let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_q$ be a list of all the subalgebras $\mathbf{B} \leq_{sd} \mathbf{A}^n$ for which there exist $\mathbf{C}$ and t satisfying the hypotheses above.
- Main Claim: $\exists \mathbf{S} < \mathbf{A}$ such that $\mathbf{B}_i \cap \mathbf{S}^n \leq_{sd} \mathbf{S}^n$ for all i.
- Induct.

In conclusion: given a finite algebra $\mathbf{A}$, define

$$Ab(\mathbf{A}) = \{a \in A \,:\, a \text{ is a universal absorption-forced constant for } \mathbf{A}\}.$$

Easy to show that $Ab(\mathbf{A}) \leq \mathbf{A}$.

Some questions:

1. What characterizes this subalgebra?
2. How can we easily determine it?
3. What is it good for?

Thank you