Testing expressibility is hard

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Outline

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Part 0 - Apology

I'm sorry

Part 1 - The Definition

Let Γ be a constraint language (always finite, on a finite domain D).

Let R be a relation (on D), say of arity $k > 0$.

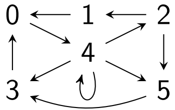
Definition

Γ can **express** R if there exist a CSP $\mathcal{P} = (X, D, \mathcal{C})$ over Γ (the **gadget**) and $x_1, \ldots, x_k \in X$ such that

$$\mathrm{proj}_{(x_1,\ldots,x_k)}(\mathrm{Sol}(\mathcal{P})) = R.$$

Example: $\Gamma = \{\rightarrow, U\}$, where

$D = \{0, 1, \ldots, 5\}$,



$U = \{0, 3\}$

Can Γ can express $R = \{3, 4, 5\}$?

Answer: Yes, via the gadget $\mathcal{P}$:

$$(a \rightarrow b) \ \& \ (b \rightarrow x) \ \& \ (x \rightarrow c) \ \& \ U(a) \ \& \ U(c).$$

Solutions to $\mathcal{P}$:

a	b	x	c
0	4	3	0
0	4	4	3
0	4	5	3
3	0	4	3

Part 2 - The Problem

Definition. $\textsc{Expr}(\Gamma)$ is the set of all relations expressible by Γ.

Expressibility Problem

Input:

- a constraint language Γ.
- a relation R.

Question:

- Is $R \in \textsc{Expr}(\Gamma)$?

Basic question: How hard is this problem to answer? To witness?

(Jeavons, Cohen, Gyssens; CP 1996, *Constraints* 1999):

Polymorphisms and 'Indicator problems'

1. If $R \notin \mathrm{Expr}(\Gamma)$, then this is witnessed by a **polymorphism** of Γ of a specified arity which does not preserve R.

2. If $R \in \mathrm{Expr}(\Gamma)$, then R is expressed by a canonical gadget called an **indicator problem**.

Hence we can (in principle) test whether $R \in \mathrm{Expr}(\Gamma)$ by either:

- Searching among all operations on D of a specified arity for one which witnesses $R \notin \mathrm{Expr}(\Gamma)$, **or**
- Building the indicator problem for (Γ, R), finding all of its solutions, and comparing the solution set to R.

How **practical** is this algorithm?

Answer: it's crap.

1. If $R \notin \text{EXPR}(\Gamma)$:
   general theory promises a witnessing polymorphism, but of arity $|R|$ (and hence table size $|D|^{|R|}$).

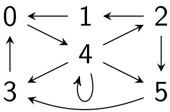
2. If $R \in \text{EXPR}(\Gamma)$:
   general theory offers the indicator problem for (Γ, R), but it has
   - $|D|^{|R|}$ variables
   - For each $S \in \Gamma$, $|S|^{|R|}$ constraints using S.

The size of the witness (polymorphism or gadget) guaranteed by theory is exponential in $|R|$. In particular, the JCG algorithm is co-*NEXPTIME*.

However: Practice does not always match theory.

Consider again $\Gamma = \{\rightarrow, U\}$:

$D = \{0, 1, \ldots, 5\}$,



$U = \{0, 3\}$

1. The relation $R = \{3, 4, 5\}$ **can** be expressed by Γ. Witnessing gadget:
   - Theory: $6^3 = 216$ variables and $10^3 + 2^3 = 1\,008$ constraints.
   - Practice: 4 variables and 5 constraints.
2. The relation $\nrightarrow \ := \ D^2 \setminus \rightarrow$ **cannot** be expressed by Γ. Witnessing polymorphism:
   - Theory: arity 26 (and thus table size $6^{26} \approx 1.7 \times 10^{20}$).
   - Practice: arity 1.

This is our problem:

- General theory seems to require witnesses of size $\exp(|R|)$.
- The Jeavons/Cohen/Gyssen algorithm is co-*NExpTime*.

But:

- Examples suggest that much smaller witnesses might suffice.
- Perhaps there is a better (polynomial-time?) algorithm for testing expressibility.

Positive evidence for better theory: boolean case ($D = \{0, 1\}$)

Theorem (Dalmau, 2000)

For each **fixed** *constraint language* Γ *over* $D = \{0, 1\}$, *testing membership in* $\textsc{Expr}(\Gamma)$ *is in P.*

Theorem (Creignou, Kolaitis & Zanuttini, 2008)

There is a uniform polynomial-time algorithm for testing expressibility over the boolean domain.

Consequences (boolean domain):

- Polynomial-sized gadgets if $R \in \textsc{Expr}(\Gamma)$;
- (Conjecture) $O(\log |R|)$-arity polymorphisms if $R \notin \textsc{Expr}(\Gamma)$.

What about the non-boolean case?

Conjecture (Vardi, AIM 2008): It's VERY VERY BAD. To wit,

- The general expressibility problem is co-*NExpTime*-complete, even on domain size 3.

(If true: no poly-sized witnesses in general; JCG algorithm can't be improved.)

Part 3 - Our solution

We confirm most of the AIM conjecture:

Theorem 1 (Exponential witnesses are unavoidable)

For infinitely many n there exist constraint languages Γ_0, Γ_1 and relation R, all over a 22-element domain, such that:

- $|R| = n$, and $\text{arity}(R) = O(\log n)$;
- $R \in \text{EXPR}(\Gamma_0)$, yet every witnessing gadget has $\geq 2^{n/3}$ variables;
- $R \notin \text{EXPR}(\Gamma_1)$, yet every witnessing polymorphism has arity $\geq n/3$.

Fine print: $|\Gamma_0| = |\Gamma_1| = O(n)$ and each $S \in \Gamma_0 \cup \Gamma_1$ has arity $O(\log n)$.

Theorem 2 (The JCG algorithm is essentially best possible)

There exists $d > 3$ such that testing expressibility on d-element domains is co-*NEXPTIME*-complete.

Part 4 - The Proof (hints)

1. Warning: it's complicated.

2. Following a suggestion of Vardi, we encode a class of **tiling problems** into the **complement** of the expressibility problem.

3. Did I mention that it's complicated?

Definition. A **tiling problem** is a special, succinctly presented CSP whose specification includes:

- A finite set Δ of **tile types**.
- A positive integer N, which determines an $N \times N$ grid.

$N = 5$

5	8	6	9	7
4	7	5	8	6
3	6	4	7	5
2	5	3	6	4
1	4	2	5	3

E.g.: $\Delta = \{1, 2, \ldots, 9\}$

Constraints:

H: $\boxed{s}\boxed{t} \quad \Rightarrow \quad t - s \in \{-2, 3\}$

V: $\begin{matrix}\boxed{t}\\ \boxed{s}\end{matrix} \quad \Rightarrow \quad s < t$

- **Constraints** on horizontally and vertically adjacent tile types.

Question: Can one cover the grid with tiles subject to the constraints?

- Optional input: An **initial condition** (prescribed tiles on first row).

Fix $\mathcal{D} = (\Delta, H, V)$.

- $\mathcal{D}$ is called a **domino system**.

"Exponential Tiling-by-$\mathcal{D}$ Problem," or $\textsc{ExpTile}(\mathcal{D})$

Input:

initial condition $\mathbf{w} \in \Delta^+$.

Question:

Can $\mathcal{D}$ tile the $2^m \times 2^m$ grid satisfying $\mathbf{w}$, where $m = |w|$?

Two facts about exponential tiling.

1. (Amusing exercise): there exists a domino system $\mathcal{D}_0$ such that:

   For all $m \geq m_0$ there exists $\mathbf{w}_m \in \Delta^m$ such that $\mathcal{D}_0$ can **almost tile** the $2^m \times 2^m$ grid satisfying initial condition $\mathbf{w}_m$ (i.e., cannot tile it, but can tile $N \times N$ for any $N < 2^m$).

   and

   For all $m \geq m_0$ there exists $\mathbf{x}_m \in \Delta^m$ such that $\mathcal{D}_0$ can tile the $2^m \times 2^m$ grid satisfying initial condition $\mathbf{x}_m$, but **cannot tile it with a repeated row**.

2. (Cf. Grädel *et al*) There exists a **universal** domino system $\mathcal{D}_1$ for *NEXPTIME*, i.e., such that $\textsc{ExpTile}(\mathcal{D}_1)$ is *NEXPTIME*-complete.

Key Construction of our paper

For every domino system $\mathcal{D} = (\Delta, H, V)$ there exists a finite set D and a log-space construction

$$\begin{array}{c} \mathbf{w} \in \Delta^+ \\ (|\mathbf{w}| = m) \end{array} \quad \mapsto \quad \left\{ \begin{array}{ccl} \Gamma & - & \text{constraint language with domain } D \\ R & - & \text{relation on } D \text{ of arity } k = O(\log m) \\ \mathbf{a} & - & \text{element in } D^k \setminus R \end{array} \right.$$

such that the **smallest** relation expressible from Γ and containing R . . .

- is either R or $R \cup \{\mathbf{a}\}$;
- is $R \cup \{\mathbf{a}\} \;\Leftrightarrow\; \mathcal{D}$ can tile $2^m \times 2^m$ with initial condition $\mathbf{w}$.

Hence $R \in \mathrm{EXPR}(\Gamma) \Leftrightarrow \mathcal{D}$ cannot tile $2^m \times 2^m$ with initial condition $\mathbf{w}$.

We also connect properties of tilings (‘almost,’ ‘repeated rows’) to possible sizes of witnesses to expressibility. (Theorems 1 & 2 follow.)

Part 5 - Questions we have not answered

1. Can non-existence of sub-exponential-sized witnesses be pushed down to smaller domains? To 3-element domains?

2. Can co-*NExpTime*-completeness be pushed down to small domains? To 3-element domains?

3. Does there exist a **fixed** constraint language Γ such that deciding membership in $\mathrm{E{\scriptstyle XPR}}(\Gamma)$:

   - Requires exponential-sized witnesses?
   - Is co-*NExpTime*-complete?

(Our construction $\mathbf{w} \mapsto (\Gamma, R, \mathbf{a})$ has Γ depending on $\mathbf{w}$.)

Thank you!