Meditation on Isaev's algebra

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AMS Sectional Meeting
Honolulu, Hawai'i
March 3, 2012

All algebras have finite signature.

Definition

Inherently non-finitely based (INFB)

- A finite algebra **A** is **INFB** if for every n there exists an infinite, finitely generated model of the n-variable identities of **A**.

From McNulty and Shallon, "Inherently nonfinitely based algebras" (1982):

Congruence Modular Problem (1982)

Does there exist a finite **A** belonging to a CM variety which is INFB?

Isaev's Answer (1989)

Yes (even with an abelian group reduct): "Isaev's algebra"

Definition (McN, Szekely, W 2008)

Inherently non-finitely based at the finite level (INFB@FL)

- A finite algebra **A** is **INFB@FL** if for every n, for some $p > n$ there exist arbitrarily large finite p-generated models of the n-variable identities of **A**.

Notes:

1. INFB@FL $\Rightarrow$ INFB.
2. **A** INFB@FL $\Rightarrow$ the pseudovariety generated by **A** is non-finitely based.

Finite Level Problem

Is every finite INFB algebra also INFB@FL?

Evidence for positive solution to FL problem:

1. True for semigroups (follows from Sapir 1987)
2. True for any algebra shown to be INFB by the "shift automorphism method" (McN, Szekeley, W 2008)

Obvious "next" question:

Isaev Problem

Is Isaev's algebra INFB@FL?

Isaev Meta-Problem

Does anyone understand Isaev's freakin' algebra?

(Deep breath . . .)

Let $\mathbf{T} = T_2(\mathbb{Z}_2)$ denote the ring of upper-triangular matrices over $\mathbb{Z}_2$.

I.e.,

$$\mathbf{T} \;=\; \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} : * \in \mathbb{Z}_2 \right\}.$$

The Jacobson radical $J = J(\mathbf{T})$ is

$$J \;=\; \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} : * \in \mathbb{Z}_2 \right\}.$$

We have $J^2 = \{0\}$ and $\mathbf{T}/J \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Finite basis for **T**

Let $\mathcal{W}_2$ denote the variety of all rings of characteristic 2.

Theorem (Polin, 1980)

$HSP(\mathbf{T})$ is the class of rings $R \in \mathcal{W}_2$ characterized by

- *R/J is a Boolean ring (i.e., is in $HSP(\mathbb{Z}_2)$), and*
- *$J^2 = \{0\}$,*

where $J = J(R)$ is the Jacobson radical of R.

Corollary

$HSP(\mathbf{T})$ is axiomatized relative to $\mathcal{W}_2$ by

$$\begin{aligned}
(xy - yx)u(zw - wz) &\approx 0 \\
(xy - yx)u(z^2 - z) &\approx 0 \\
(x^2 - x)u(zw - wz) &\approx 0 \\
(x^2 - x)u(z^2 - z) &\approx 0.
\end{aligned}$$

Normal forms for $HSP(\mathbf{T})$

Let $X = \{x_1, x_2, x_3, \dots, \}$ be a set of variables.

Given a monomial $\mathbf{w} = x_{i_1} x_{i_2} \cdots x_{i_k}$ over X, an occurrence $x_i x_j$ of two consecutive variables in $\mathbf{w}$ is **bad** if $i \geq j$.

$\mathbf{w}$ is **bad** if it has two <u>disjoint</u> bad occurrences of consecutive variables. Otherwise, $\mathbf{w}$ is good.

Examples:

- $x_3x_5x_2x_3x_5$ is good. $x_3\boxed{x_5x_2}x_3x_5$
- $x_1x_2x_1x_2x_3x_4x_4x_5$ is bad. $x_1\boxed{x_2x_1}x_2x_3\boxed{x_4x_4}x_5$
- $x_1x_4x_3x_2x_3$ is good. $x_1\,x_4\,x_3\,x_2\,x_3$
- $x_1x_4x_3x_2x_2x_3$ is bad. $x_1\,x_4\,x_3\,x_2\,x_2\,x_3$

Suppose **w** is bad.

For example, $\mathbf{w} = Ax_4x_3Bx_1x_1C$.

Using $(xy - yx)u(z^2 - z) \approx 0$, we deduce

$$x_4x_3Bx_1x_1 \ \approx \ x_3x_4Bx_1x_1 + x_4x_3Bx_1 + x_3x_4Bx_1,$$

hence

$$Ax_4x_3Bx_1x_1C \ \approx \ Ax_3x_4Bx_1x_1C + Ax_4x_3Bx_1C + Ax_3x_4Bx_1C.$$

Any bad monomial can replaced by a sum of three "less bad" monomials (and ultimately by a sum of good monomials).

Proposition (McN, W)

The good monomials over X form a vector space basis for the free algebra on X in $HSP(\mathbf{T})$.

(Another deep breath . . .)

Consider the 2-sorted variety $\mathcal{W}_2^*$ defined as follows:

A member of $\mathcal{W}_2^*$ is of the form

$$\mathcal{A} = (\mathbf{A}, \mathbf{U}, \cdot)$$

where

- First sort: $\mathbf{A} = (A, +, 0)$ is a vector space over $\mathbb{Z}_2$.
- Second sort: $\mathbf{U} = (U, +, 0)$ is a vector space over $\mathbb{Z}_2$.
- The operation $\cdot$ is bilinear $A \times U \to U$.

The operation $\cdot$ encodes an additive homomorphism $\mathbf{A} \to End(\mathbf{U})$ (via $a \mapsto L_a$ where $L_a(u) = a \cdot u$).

Recall that $\mathbf{T}$ is the ring of 2×2 upper triangular matrices over $\mathbb{Z}_2$.

$\mathbf{T}$ is also a vector space over $\mathbb{Z}_2$ and acts naturally on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition

$\mathcal{T}$ is the 2-sorted algebra $(\mathbf{T}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \cdot) \in \mathcal{W}_2^*$.

There is a close connection between the equational theories of $\mathbf{T}$ and $\mathcal{T}$. In particular,

Normal forms for $HSP(\mathcal{T})$

The (k, ℓ)-generated free algebra in $HSP(\mathcal{T})$ with generators $\{x_1, \ldots, x_k\}$ and $\{u_1, \ldots, u_\ell\}$ has the following normal forms:

- $\{x_1, \ldots, x_k\} =: X$ is a basis for the first sort.
- The elements "$\mathbf{w}$" $\cdot\, u_t$ where $u_t \in \{u_1, \ldots, u_\ell\}$ and $\mathbf{w}$ is a good monomial over X form a basis for the second sort.

Here "$x_{i_1} x_{i_2} \cdots x_{i_n}$" $\cdot\, u \overset{df}{=} x_{i_1} \cdot (x_{i_2} \cdot (\cdots (x_{i_n} \cdot u) \cdots))$.

Corollary

The following identities form an infinite equational basis for $HSP(\mathcal{T})$ *relative to* $\mathcal{W}_2^*$*:*

$$\begin{aligned}
\text{“}(xy - yx)v_1 v_2 \cdots v_k(zw - wz)\text{”} \cdot u &\approx 0 \\
\text{“}(xy - yx)v_1 v_2 \cdots v_k(z^2 - z)\text{”} \cdot u &\approx 0 \\
\text{“}(x^2 - x)v_1 v_2 \cdots v_k(zw - wz)\text{”} \cdot u &\approx 0 \\
\text{“}(x^2 - x)v_1 v_2 \cdots v_k(z^2 - z)\text{”} \cdot u &\approx 0.
\end{aligned}$$

Proof.

These identities suffice for the “reduction to good monomials.” □

Refinement (McN, W)

Suppose $n \geq 3$, $\mathcal{A} = (\mathbf{A}, \mathbf{U}, \cdot) \in \mathcal{W}_2^*$, and $\text{span}(X) = A$.

If the above equations with $k \leq n-2$ hold true in $\mathcal{A}$ at instances with $x, y, z, w, v_1, v_2, \ldots, v_k \underline{\in X}$, then $\mathcal{A}$ satisfies all the (n, n)-variable identities of $\mathcal{T}$.

This gives a strategy for showing that $\mathcal{T}$ is INFB.

Strategy

For every $n \geq 3$, show there exists

- a vector space $\mathbf{U}$ over $\mathbb{Z}_2$ with nonzero $u \in U$;
- a finite set $\Lambda = \{L_x : x \in X\} \subseteq End(\mathbf{U})$;

such that

1. The members of Λ satisfy

$$\begin{aligned} (L_xL_y - L_yL_x)L_{v_1}\cdots L_{v_k}(L_zL_w - L_wL_z) &= 0 \qquad (k \leq n-2) \\ (L_xL_y - L_yL_x)L_{v_1}\cdots L_{v_k}(L_z^2 - L_z) &= 0 \\ (L_x^2 - L_x)L_{v_1}\cdots L_{v_k}(L_zL_w - L_wL_z) &= 0 \\ (L_x^2 - L_x)L_{v_1}\cdots L_{v_k}(L_z^2 - L_z) &= 0; \end{aligned}$$

2. The orbit of u under Λ is infinite.

Then $(\mathrm{span}(\Lambda), \mathrm{Orbit}_\Lambda(u), \cdot)$ is an infinite, finitely generated model of the (n, n)-variable identities of $\mathcal{T}$.

We can do that!

Corollary (McN, W)

$\mathcal{T}$ *is INFB.*

An easy modification of our construction gives

Corollary

$\mathcal{T}$ *is INFB@FL.*

What about Isaev's algebra? . . . Deep breath . . .

Let $\mathbf{I}$ be the "1-sortification" of $\mathcal{T} = (\mathbf{T}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \cdot\,)$. I.e.,

- Universe is $I = T \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.
- Operations can be taken to be
  - addition: $(x_1, u_1) + (x_2, u_2) := (x_1 + x_2,\ u_1 + u_2)$.
  - multiplication: $(x_1, u_1)(x_2, u_2) := (0,\ x_1 \cdot u_2)$.
  - decomposition: $d((x_1, u_1), (x_2, u_2)) := (x_1, u_2)$.

Because 1-sortification preserves everything[1], $\mathbf{I}$ (like $\mathcal{T}$) is INFB@FL.

Definition

Isaev's algebra is $\mathbf{I}$ with d thrown away.

The construction showing that $\mathcal{T}$ is INFB@FL, translated to $\mathbf{I}$, never uses d. Hence it applies equally to Isaev's algebra. I.e., Isaev's algebra is INFB@FL.

[1]that makes sense to preserve

Problems

1. Find a "softer" proof that Isaev's algebra is INFB (or INFB@FL).
2. Determine which finite algebras in $\mathcal{W}_2^*$ are finitely based; INFB; INFB@FL.
3. Similarly for $\mathcal{W}_K^*$ where K is any finite field.

Mahalo!