The finite basis problem revisited

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AAA87
Linz, Austria
07.02.2014

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Finite bases

The variety of all groups can be defined by the identities

$$x(yz) \approx (xy)z, \quad x1 \approx x \approx 1x, \quad \text{and} \quad xx^{-1} \approx 1.$$

The symmetric group $\mathbf{S}_3$ satisfies these identities and more; e.g.,

$$x^6 \approx 1, \quad (x^{-1}y^{-1}xy)^3 \approx 1, \quad x^2y^2 \approx y^2x^2.$$

Let $\Sigma = \{\text{these seven identities}\}$.

Fact: All identities true in $\mathbf{S}_3$ can be deduced from Σ.

Equivalently, $\mathcal{V}(\mathbf{S}_3)$ is **axiomatized** by Σ.

We say Σ a **basis** (for $\mathbf{S}_3$ or $\mathcal{V}(\mathbf{S}_3)$).

Finitely based algebras

Definition

An algebra **A** is **finitely based** if $\mathcal{V}(\mathbf{A})$ is finitely axiomatizable.

Examples of finitely based algebras:

- Every 2-element algebra in a finite signature (Lyndon, 1951)

(From now on, all algebras are assumed to be in a finite signature.)

- Every finite group (Oates and Powell, 1965)
- Every finite ring (Kruse, L'vov, 1973)
- Every finite commutative semigroup (Perkins, 1969)
- Every finite lattice (McKenzie, 1970)
- Every finite **A** such that $\mathcal{V}(\mathbf{A})$ is congruence distributive (Baker, 1972)

Non-finitely based algebras

Not every finite algebra is finitely based (Lyndon, 1954).

The following 3-element groupoid is **not** finitely based (Murskii, 1965):

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

The following 6-element semigroup is **not** finitely based (Perkins, 1969):

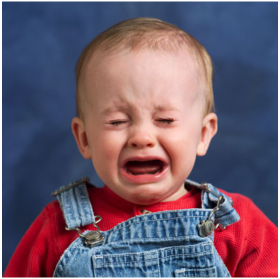
$$\mathbf{B}_2^1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The finite basis problem

Which finite algebras are finitely based?

We'll never know.

Theorem (McKenzie, 1996). There is no algorithm which decides, given a finite algebra **A**, whether **A** is finitely based.



However, this is no reason to cry.

Jónsson's Speculation/Park's Conjecture

Baker's Theorem (1972)

$\mathbf{A}$ finite and $\mathcal{V}(\mathbf{A})$ congruence distributive $\Longrightarrow$ $\mathbf{A}$ is finitely based.

$\Downarrow$ key step

$\exists N < \omega$ such that every subdirectly irreducible $\mathbf{S} \in \mathcal{V}(\mathbf{A})$ has $|S| \leq N$.

Def. We say $\mathcal{V}(\mathbf{A})$ "has a finite residual bound" if it satisfies ↗.

Bjarni Jónsson's Question (early 1970s):

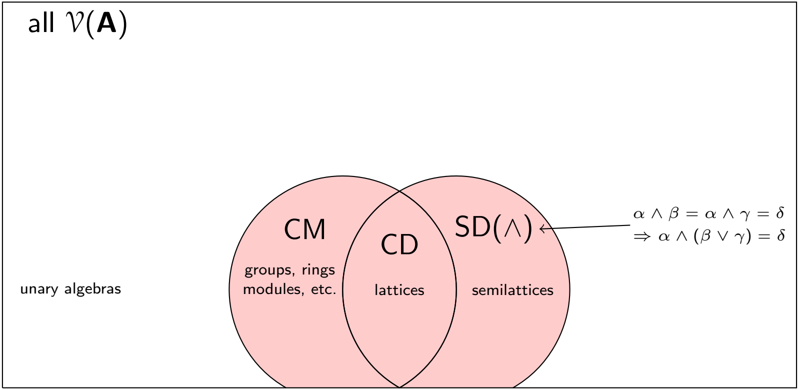
$\mathbf{A}$ finite & $\mathcal{V}(\mathbf{A})$ has a finite residual bound $\overset{?}{\Longrightarrow}$ $\mathbf{A}$ is finitely based.

(Robert Park's conjecture (1976): "YES." Still open!)

Confirmations of Jónsson's Problem/Park's conjecture

The answer for **A** is known to be "YES" if $\mathcal{V}(\mathbf{A})$ is ...

1. ... Congruence distributive (CD) – (Baker, 1972).
2. ... Congruence modular (CM) – (McKenzie, 1987).
3. ... Congruence meet semi-distributive (SD($\wedge$)) – (W, 2000).



How are these theorems proved?

Fix a finite algebra $\mathbf{A}$.

Let $\mathcal{V}^{(n)} :=$ the variety axiomatized by all the $\leq n$-variable identities of $\mathbf{A}$.

Facts:

1. $\mathcal{V}^{(1)} \supseteq \mathcal{V}^{(2)} \supseteq \mathcal{V}^{(3)} \supseteq \cdots \supseteq \mathcal{V}^{(n)} \supseteq \cdots$ and $\bigcap_{n=1}^{\infty} \mathcal{V}^{(n)} = \mathcal{V}(\mathbf{A})$.
2. Each $\mathcal{V}^{(n)}$ is finitely axiomatizable.
3. $\mathbf{A}$ is finitely based $\iff$ $\mathcal{V}(\mathbf{A}) = \mathcal{V}^{(n)}$ for some (hence all large) n.

Definable relations

Definition

In any algebra **B**, let $D(x, y, z, w)$ denote the "disjointness relation":

$$D(a, b, c, d) \stackrel{\text{def}}{\iff} \mathrm{Cg}^{\mathbf{B}}(a, b) \cap \mathrm{Cg}^{\mathbf{B}}(c, d) = 0_B.$$

Note that

$$\neg\, D(a, b, c, d) \iff \underbrace{\exists x, y \left[x \neq y \ \& \ \bigvee_{\pi} \pi(x, y, a, b) \ \& \ \bigvee_{\pi'} \pi'(x, y, c, d) \right]}_{(*)}$$

where both $\bigvee$'s are infinite disjunctions over **principal congruence formulas** (in the language of **A**).

$(*)$ is a formula of infinitary logic. In general, it cannot be made first-order.

Ein Gedankenexperiment

Suppose ...

1. $\mathcal{V}(\mathbf{A})$ has a finite residual bound;
2. $D(x,y,z,w)$ is definable in $\mathcal{V}(\mathbf{A})$ by a first-order formula.
3. D is definable in $\mathcal{V}^{(n)}$ by the same formula (for large enough n).

Recall that an algebra **B** is **finitely subdirectly irreducible** (**FSI**) if 0_B is meet-irreducible in $\mathrm{Con}\,\mathbf{B}$; equivalently, if

$$\forall x,y,z,w[\,D(x,y,z,w) \to (x=y \ \text{ or } \ z=w)\,].$$

The assumptions imply that

"I am FSI"

is first-order definable in $\mathcal{V}^{(n)}$ (for large n).

"I am FSI" is first-order definable in $\mathcal{V}^{(n)}$ (for large n), say by Φ.

Recall: $\mathcal{V}(\mathbf{A})$ has a finite residual bound. This means:

- $\mathcal{V}(\mathbf{A})_{SI} = \{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ for some **finite** algebras $\mathbf{S}_1, \ldots, \mathbf{S}_k$ (up to $\cong$).
- $\mathcal{V}(\mathbf{A})_{FSI} = \mathcal{V}(\mathbf{A})_{SI}$.

Then:

- "I am in $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$" is first-order definable (absolutely), say by Ψ.
- $\mathcal{V}(\mathbf{A}) \models \Phi \to \Psi$.
- Hence $\mathcal{V}^{(n)} \models \Phi \to \Psi$ for large enough n (Compactness Theorem).
- I.e., $(\mathcal{V}^{(n)})_{FSI} \subseteq \{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$.
- Hence $(\mathcal{V}^{(n)})_{SI} \subseteq (\mathcal{V}^{(n)})_{FSI} \subseteq \mathcal{V}(\mathbf{A})_{SI}$.
- Hence $\mathcal{V}^{(n)} \subseteq \mathcal{V}(\mathbf{A})$.
- Hence $\mathcal{V}(\mathbf{A}) = \mathcal{V}^{(n)}$, so $\mathcal{V}(\mathbf{A})$ is finitely axiomatizable! □

Baker's CD theorem and my extension to SD($\wedge$)

In summary, if

1. $\mathcal{V}(\mathbf{A})$ has a finite residual bound;
2. $D(x, y, z, w)$ is first-order definable in $\mathcal{V}(\mathbf{A})$;
3. The same formula defines D in $\mathcal{V}^{(n)}$ (for large n);

then $\mathbf{A}$ is finitely based.

Baker proved that (2), (3) hold if $\mathcal{V}(\mathbf{A})$ is CD.

- Via a combinatorial analysis of failures of $D(x, y, z, w)$, using Jónsson operations and their identities.

My extension to SD($\wedge$) varieties shamelessly plagiarizes Baker's proof.

- Jónsson operations/identities are replaced by operations/identities characterizing SD($\wedge$) due to Kearnes, Szendrei (1998) and Lipparini (1998), improving Hobby, McKenzie (1988) and Czédli (1983).

Prologue to McKenzie's CM theorem: the commutator

All algebras admit a **commutator operation** $[-,-]$ defined on congruences.

Properties:

- $[\alpha, \beta] \leq \alpha \wedge \beta$.
- $[\alpha, \beta] = \alpha \wedge \beta$ in CD (or SD($\wedge$)) varieties.
- $[\alpha, \beta] = 0_B \iff \alpha, \beta$ are "independent" (in some sense).
- $[\alpha, \alpha] = 0_B \iff \alpha$ is "abelian."

Definition

Let $C(x, y, z, w)$ denote the "**commutator disjointness relation**":

$$C(a, b, c, d) \stackrel{\text{def}}{\iff} [\mathrm{Cg}(x, y), \mathrm{Cg}(z, w)] = 0.$$

C is analoguous to, but weaker than, D.

McKenzie's CM theorem

Theorem (McKenzie, 1987)

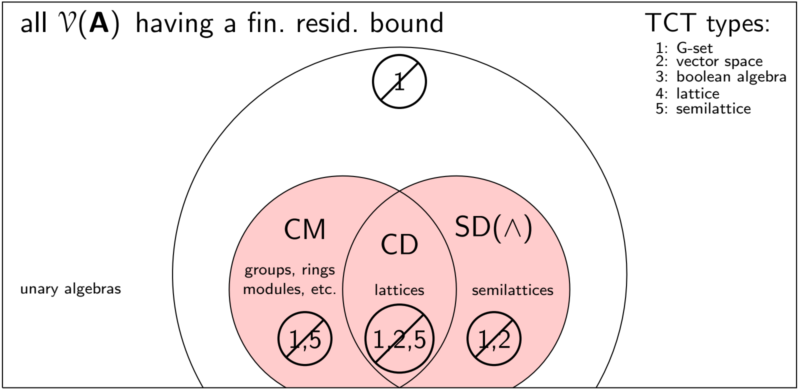
Suppose $\mathcal{V}(\mathbf{A})$ is congruence modular. If $\mathcal{V}(\mathbf{A})$ has a finite residual bound, then $\mathcal{V}(\mathbf{A})$ is finitely axiomatizable.

Remarks on the proof. Assuming $\mathcal{V}(\mathbf{A})$ is CM and has a finite residual bound, McKenzie proved that:

1. **Step 1**: C is first-order definable in $\mathcal{V}(\mathbf{A})$.
2. **Steps 2-10**: Various relations derived from C are first-order definable in $\mathcal{V}^{(n)}$ (for large enough n).
   - Hence certain commutator facts about $\mathcal{V}(\mathbf{A})$ can be lifted to $\mathcal{V}^{(n)}$.
3. **Step 11**: $(\mathcal{V}^{(n)})_{SI} \subseteq \mathcal{V}(\mathbf{A})_{SI}$. Hence $\mathcal{V}(\mathbf{A}) = \mathcal{V}^{(n)}$.

In search of a common generalization

$\mathcal{V}(\mathbf{A})$ for which Jónsson's speculation/Park's conjecture is confirmed:



It would be **very nice** to confirm Jónsson's speculation/Park's conjecture for any $\mathcal{V}(\mathbf{A})$ omitting type 1. Unfortunately, we haven't done that.

Abelian congruences

Let **B** be an algebra and $\theta \in \mathrm{Con}\mathbf{B}$. The **trace** of θ is

$$Tr(\theta) = \mathrm{Sg}^{\mathbf{B}^{2\times 2}}\left(\left(\begin{array}{cc} a & a \\ b & b \end{array}\right), \left(\begin{array}{cc} a & b \\ a & b \end{array}\right) \ : \ a \overset{\theta}{\equiv} b\right) \leq \mathbf{B}^{2\times 2}.$$

Definition

We say θ is **abelian** if for all $\left(\begin{array}{cc} x & y \\ z & w \end{array}\right) \in Tr(\theta)$,

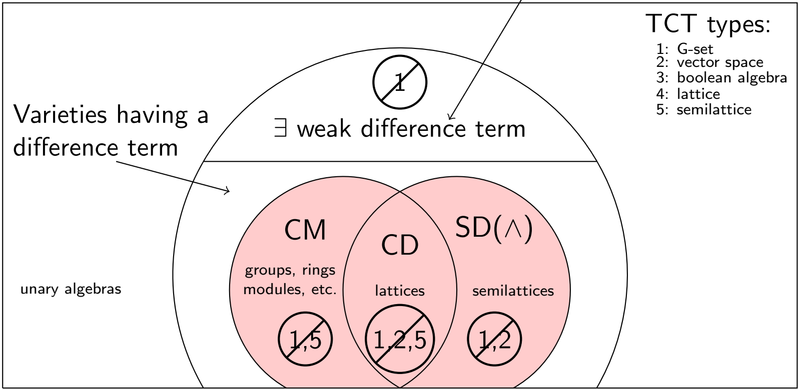
$$x = y \iff z = w.$$

(This is equivalent to $[\theta, \theta] = 0_B$.)

Difference terms

Def: a **weak difference term** (for $\mathcal{V}$) is a term $p(x, y, z)$ satisfying $p(a, a, b) = b = p(b, a, a)$ whenever $(a, b) \in$ an abelian congruence

Def: a **difference term** (for $\mathcal{V}$) is a weak difference term satisfying $p(x, x, y) \approx y$.



Confirmation of Jónsson/Park for DT varieties

Theorem (Kearnes, Szendrei, W)

Suppose $\mathcal{V}(\mathbf{A})$ has a difference term. If $\mathcal{V}(\mathbf{A})$ has a finite residual bound, then $\mathcal{V}(\mathbf{A})$ is finitely axiomatizable.

Remarks on the proof.

1. **Step 1**: we show $C(x, y, z, w)$ is first-order definable in $\mathcal{V}(\mathbf{A})$. (syntactic analysis + TCT + Baker-McNulty-Wang trick)
2. **Steps 2-10**: We then shamelessly attempt to plagiarize McKenzie's proof in the CM case, as far as possible.
3. At the **very last step** of McKenzie's CM proof, we are stuck: he uses definability of "$\mathrm{Cg}(x, y)$ is a non-abelian atom" which we don't have.

Damn!

Apply the KISS principle

Actually

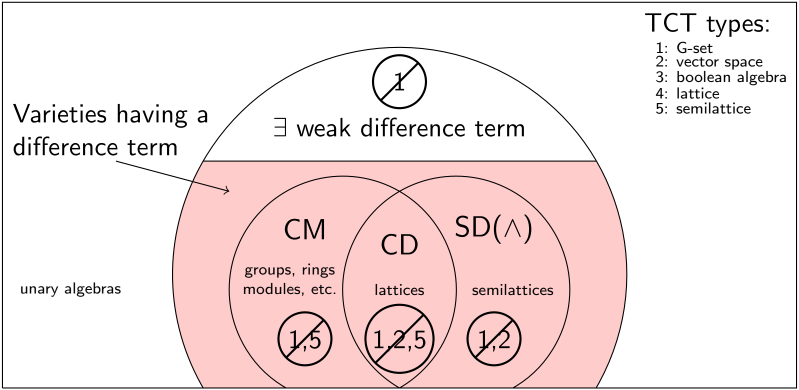
E.W. Kiss, Three remarks on the modular commutator, *AU* **29** (1992).

- Kiss proves that every CM variety has a "4-ary difference term," with interesting properties.
- We prove the same for varieties with a difference term.
- We use this to lift first-order definability of $C(x, y, z, w)$ from $\mathcal{V}(\mathbf{A})$ to $\mathcal{V}^{(j)}$ (something McKenzie wasn't able to do).
- Then some new tricks finish the argument.

Looking backward

At AAA66 in Klagenfurt, in my lecture "The finite basis problem," I posed this challenge:

"**Challenge**. Prove [the Jónsson/Park] conjecture for . . . varieties having a difference term. (Reward: fame and glory.)" ✓



Looking forward

Updated Challenge: Prove the Jónsson/Park conjecture for varieties having a weak difference term (i.e., omitting type 1). (Reward: glory . . .)

Also from my AAA66 lecture:

"**Challenge 2**: Find a counter-example to [the Jónsson/Park conjecture]. (Reward: 50 euros for the first counter-example found.)"

Updated Challenge 2: now 87 euros! 

Thank you!