My favourite open problems in universal algebra

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AMS Spring Southeastern Sectional Meeting
College of Charleston
March 10, 2017

Problem #1

The Restricted Quackenbush Question

R. Quackenbush, 1971

Let **A** be a finite algebra in a finite signature.

If $V(\mathbf{A})$ contains arbitrarily large finite subdirectly irreducible algebras, must $V(\mathbf{A})$ contain an infinite subdirectly irreducible algebra?

(**A** finite, finite signature.)

If $V(\mathbf{A})$ contains arbitrarily large finite subdirectly irreducible algebras, must $V(\mathbf{A})$ contain an infinite subdirectly irreducible algebra?

The story

R. Quackenbush, "Equational classes generated by finite algebras," *Algebra Universalis* **1** (1971), 265–266.

Bob proved that (without the finite signature assumption) if $V(\mathbf{A})$ has an infinite SI, then $V(\mathbf{A})$ must also contain arbitrarily large finite SIs.

Bob asked whether the opposite implication holds:

- for general finite algebras; ("Unrestricted Quackenbush")
- for finite algebras in finite signatures; ("Restricted Quackenbush")
- for groupoids, semigroups, and groups.

McKenzie 1993 (publ. 1996) answered Unrestricted Quackenbush: NO

But it's still possible the answer to Restricted Quackenbush is YES.

The evidence

Restricted Quackenbush is known to have a YES answer in many cases:

- algebras generating CD varieties (vacuously): Foster & Pixley 1964.
- groups: Ol'shanskii 1969.
- semigroups: Golubov & Sapir 1979; McKenzie 1983.
- algebras generating CM varieties: Freese & McKenzie 1981.
- algebras **A** for which $1, 5 \notin \mathrm{typ}(V(\mathbf{A}))$: Hobby & McKenzie 1988.
- algebras generating SD($\wedge$) varieties: Kearnes & W 1999.
- strongly nilpotent algebras: Kearnes & Kiss 2003.

($\mathbf{A}$ finite, finite signature.)

If $V(\mathbf{A})$ contains arbitrarily large finite subdirectly irreducible algebras, must $V(\mathbf{A})$ contain an infinite subdirectly irreducible algebra?

Naive argument for a "yes" answer:

In all examples we've seen, the finite SIs come in tidy families that, if unbounded in size, lead "continuously" to infinite SIs.

Naive argument for a "no" answer:

Remember what Ralph did to us in '93.

What do you think?

Problem: What if $V(\mathbf{A})$ omits type 1? (Surely the answer is YES?)

Problem #2

Definition. A variety is . . .

- *residually large* if there is no cardinal bounding the sizes of its SIs.

The Recognizing Residual Largeness Question

1990s?

Among finite algebras in finite signatures, is

$$\{\mathbf{A} : V(\mathbf{A}) \text{ is residually large}\}$$

recursively enumerable?

The story

Definition. A variety . . .

- *has a finite residual bound* if $\exists n < \omega$ such that every SI has size $\leq n$.
- is *residually finite* if it has no infinite SI.
- is *residually small* if there is a cardinal bounding the sizes of its SIs.

D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, 1988

Conjectured that if $\mathbf{A}$ is finite (no restriction on signature) and $V(\mathbf{A})$ does not have a finite residual bound, then $V(\mathbf{A})$ is residually large.

This came to be known as the "RS Conjecture."

It was the focus of much work in the 1980s and early 1990s.

The RS program

1. Find "bad configurations" which, if present, produce residual largeness.
2. Prove that the bad configurations are complete: $V(\mathbf{A})$ is residually large iff $V(\mathbf{A})_{fin}$ realizes a bad configuration.
3. Prove that if $V(\mathbf{A})_{fin}$ omits the bad configurations, then SIs must be finite with bounded size.

Expectation: testing whether $V(\mathbf{A})_{fin}$ realizes a bad configuration (i.e., is residually large) should be decidable.

Unfortunately, Ralph in 1993 ruined everything by:

1. Refuting the RS conjecture (even in finite signature).
2. Proving that "testing residual largeness" is undecidable.

But it's still possible that "testing residually largeness" is r.e.

Evidence:

- CM varieties: decidable — Freese & McKenzie 1981
- Varieties omitting types 1,5: decidable — Hobby & McKenzie 1988
- SD($\wedge$) varieties: r.e. — McKenzie 2000
- Varieties omitting type 1: r.e. — Kearnes (unpubl.)

What do you think?

Problem: What is the next case to tackle?

Problem #3

Definition. A variety ...

- *has a finite residual bound* if $\exists n < \omega$ such that every SI has size $\leq n$.

The Recognizing Finite Residual Bound Question

2000s?

Among finite algebras in finite signatures, is

$$\{\mathbf{A} : V(\mathbf{A}) \text{ has a finite residual bound}\}$$

recursively enumerable?

Is "$V(\mathbf{A})$ has a finite residual bound" r.e.?

Ralph proved that "testing for finite residual bound" is undecidable ...

... but it's still possible that "testing for finite residual bound" is r.e.

Evidence:

- CM varieties: decidable — Freese & McKenzie 1981
- Varieties omitting types 1,5: decidable — Hobby & McKenzie 1988
- SD($\wedge$) varieties: r.e. — W 2000
  - Reason: given $\mathbf{A}$ and n, can decide whether $V(\mathbf{A})$ is residually $\leq n$.

Problem: Among Taylor algebras in finite signatures, can we decide, given $\mathbf{A}$ and n, whether $V(\mathbf{A})$ is residually $\leq n$?

What do you think?

Problem #4

Definition. A variety is a *Pixley variety* if its signature is finite, it has arbitrarily large finite SIs, but no infinite SI.

Pixley varieties exist: e.g., the variety axiomatized by

$$f(g(x)) \approx x \approx g(f(x)).$$

The Pixley-meets-Taylor Problem

2017?

Does there exist a Taylor Pixley variety?

The story

K. Kaarli & A. Pixley, “Affine complete varieties,” *Algebra Universalis* **24** (1987), 74–90.

Kalle and Alden asked if there is a CD Pixley variety.

Keith and I defined “Pixley variety” (1999)

Does there exist a Taylor Pixley variety?

The evidence

There is no Pixley variety which is ...

- SD($\wedge$) — Kearnes & W 1999
- CM (or satisfies a nontriv. congruence ident.) — Kearnes & W (unpub)

What do you think?

Problem: prove that there is no difference term Pixley variety.

Problem #5

Definition. A variety is *finitely based* if it can be axiomatized by finitely many identities.

An algebra is *finitely based* if the variety it generates is.

Jónsson's Finite Basis Problem
a.k.a. Park's Conjecture

B. Jónsson, early 1970s

If **A** is a finite algebra in a finite signature and $V(\mathbf{A})$ has a finite residual bound, must **A** be finitely based?

(**A** finite, finite signature.)

If $V(\mathbf{A})$ has a finite residual bound, must **A** be finitely based?

The story

Reports that Bjarni posed (a version of) this problem in the 1970s:

- Taylor 1975: If every SI in $V(\mathbf{A})$ is in $HS(\mathbf{A})$, is **A** finitely based?
- Baker 1976: "the conjecture of Jónsson" that $V(\mathbf{A})$ having a finite residual abound implies **A** finitely based.
- McKenzie 1977: "Jónsson once asked whether" $V(\mathbf{A})$ having a finite residual bound implies **A** finitely based.
- McKenzie 1987: "Jónsson wondered, in the early 1970s, whether" $V(\mathbf{A})$ residually small implies **A** finitely based.

(**A** finite, finite signature.)

If $V(\mathbf{A})$ has a finite residual bound, must **A** be finitely based?

The evidence

The answer is YES for finite algebras belonging to:

- CD varieties — Baker 1977
- CM varieties — McKenzie 1987
- Varieties omitting types 1,5 — Hobby & McKenzie 1988
- SD($\wedge$) varieties — W 2000
- Difference term varieties — Kearnes, Szendrei & W 2016

I've also offered 87 euros for a counter-example: still uncollected.

What do you think?

Problem: resolve the question for varieties omitting type 1.

Problem #6

The Eilenberg-Schützenberger Question

S. Eilenberg & M. P. Schützenberger, 1976

Suppose $\mathbf{A}$ is a finite algebra in a finite signature. If there exists a finitely based variety $\mathcal{V}$ with the property that $\mathcal{V}$ and $V(\mathbf{A})$ have exactly the same finite members, does if follow that $\mathbf{A}$ is finitely based?

(**A** finite, finite signature)

If there exists a finitely based variety $\mathcal{V}$ such that $\mathcal{V}_{fin} = V(\mathbf{A})_{fin}$, does if follow that **A** is finitely based?

The story

S. Eilenberg & M. P. Schútzenberger, "On pseudovarieties," *Adv. Math.* **19** (1976), 413–418.

They posed the question for monoids, but also noted that it could be posed for general algebras.

R. Cacioppo (1993) noted that a counter-example must be "inherently nonfinitely based."

George McNulty popularized this question amongst algebraists and reformulated it in terms of "equational complexity."

(**A** finite, finite signature)

If there exists a finitely based variety $\mathcal{V}$ such that $\mathcal{V}_{fin} = V(\mathbf{A})_{fin}$, does if follow that **A** is finitely based?

The evidence

The answer is YES for:

- semigroups (Sapir 1987)
- finitely based algebras (groups, algebras generating CD varieties, etc.)

That's it???

Surely the answer in general is NO. (?)

Problem: Find a counter-example.

- Incentive: $100 (Canadian dollars).

Problem: Is the answer YES for algebras generating CM varieties?

Thank you!