Constraint Satisfaction Problems
A Survey

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(with corrections)

CSP = specifications of subpowers of a finite algebra

Fix a finite algebra $\mathbf{A}$.

Definition

A **constraint network over A** is a pair (n, φ) where

- $n \geq 1$
- φ is a quantifier-free formula of the form $\bigwedge_{i \in I} R_i(\mathbf{x}_i)$,

where for each $i \in I$,

- $\mathbf{x}_i$ is a d-tuple of variables from $\{x_1, \ldots, x_n\}$ (for some d)
- R_i is a **subuniverse** of $\mathbf{A}^d$.

The relation **defined by** (n, φ) is

$$\mathrm{Rel}_{\mathbf{A}}(n, \varphi) = \{\mathbf{a} \in A^n : \varphi(\mathbf{a})\}.$$

Example

Let
$$\begin{array}{rcl} \mathbf{A} & = & (\{0,1\};\ x+y+z) \\ R_0 & = & \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \\ R_1 & = & \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}. \end{array}$$

$R_0, R_1 \leq \mathbf{A}^3$. Thus the following is a constraint network over $\mathbf{A}$:

$$(6,\ \underbrace{R_0(x_1,x_2,x_3) \wedge R_1(x_1,x_4,x_5) \wedge R_0(x_2,x_4,x_6) \wedge R_1(x_3,x_5,x_6)}_{\varphi}).$$

We can view φ as asserting (over $\mathbb{Z}_2$)

$$\begin{array}{ccccccccccccl} x_1 & + & x_2 & + & x_3 & & & & & & & = & 0 \\ x_1 & & & & & + & x_4 & + & x_5 & & & = & 1 \\ & & x_2 & & & + & x_4 & & & + & x_6 & = & 0 \\ & & & + & x_3 & & & + & x_5 & + & x_6 & = & 1. \end{array}$$

$\mathrm{Rel}_{\mathbf{A}}(6,\varphi)$ is the solution-set to this linear system.

Variant notations

A **constraint network over A** is a pair (n, φ), $\varphi = \bigwedge_i R_i(\mathbf{x}_i)$ ...

...	may be written as ...
n	$\{x_1, \dots, x_n\}$ $(= V,$ the set of **variables**$)$
φ	$\{(\mathbf{x}_i, R_i) \,:\, i \in I\}$ $(= \mathcal{C})$ • $(\mathbf{x}_i, R_i)$ is called a **constraint** • $\mathbf{x}_i$ is its **scope** • R_i is its **constraint relation**
(n, φ)	$(V, \mathcal{C})$ or $(V, A, \mathcal{C})$
$\mathrm{Rel}_{\mathbf{A}}(n, \varphi)$	$\mathrm{Sol}(V, \mathcal{C})$

Decision Problems

Definition

(n, φ) is k-**ary** if each scope has length $\leq k$.

Definition

$\mathrm{CSP}(\mathbf{A}, k)$

Input: A k-ary constraint network (n, φ) over $\mathbf{A}$.

Question: Is $\mathrm{Rel}_{\mathbf{A}}(n, \varphi) \neq \varnothing$?

Dichotomy Conjecture (Feder & Vardi)

For all $\mathbf{A}$ and k, $\mathrm{CSP}(\mathbf{A}, k)$ is in P or is NP-hard.

Algebraic Dichotomy Conjecture (Bulatov, Krokhin & Jeavons)

If $\mathbf{A}$ has a Taylor operation, then $\underbrace{\mathrm{CSP}(\mathbf{A}, k) \text{ is in P for every } k}_{\mathbf{A} \text{ is tractable}}$.

Taylor operations

Definition

An operation $t : A^n \to A$ is a **Taylor operation** if

1. t is idempotent ($t(x, x, \ldots, x) \approx x$);
2. For each $i = 1, \ldots, n$, t satisfies an identity of the form $t(\mathbf{x}) \approx t(\mathbf{y})$ with $x_i \neq y_i$.

Theorem (Taylor; Barto & Kozik; Hobby & McKenzie)

For a finite algebra $\mathbf{A}$, the following are equivalent:

1. $\mathbf{A}$ has a Taylor (term) operation.
2. $\mathbf{A}$ satisfies some idempotent Maltsev condition not satisfied by $\textsc{Sets}$.
3. $\mathbf{A}$ has an idempotent **cyclic** term $t(x_1, \ldots, x_n)$, i.e.,

$$t(x_1, x_2, \ldots, x_n) \approx t(x_2, \ldots, x_n, x_1).$$

4. $V(\mathbf{A})$ omits type $\mathbf{1}$.

Progress

Algebraic Dichotomy Conjecture

If **A** has a Taylor operation, then $\underbrace{\mathrm{CSP}(\mathbf{A}, k) \text{ is in P for every } k}_{\mathbf{A} \text{ is tractable}}$.

Theorem

A is known to be tractable if:

1. $V(\mathbf{A})$ is CM. (Dalmau '05 + IMMVW '07, using Barto '16?)
2. $V(\mathbf{A})$ is SD($\wedge$). (Barto & Kozik '09; Bulatov '09)
3. **A** is Taylor + **conservative**, i.e. $\mathrm{Su}(\mathbf{A}) = \mathcal{P}(A)$. (Bulatov '03)
4. **A** is Taylor and $|A| = 2$ or 3. (Schaefer '78, Bulatov '02)

Definition

Let $\mathbf{A}$ be a finite algebra, $\mathcal{A}$ a set of finite algebras.

1. $\mathrm{CSP}(\mathbf{A}) = \bigcup_k \mathrm{CSP}(\mathbf{A}, k)$. "Global"
2. $\mathrm{CSP}(\mathcal{A}, k) = \bigcup_{\mathbf{A}\in\mathcal{A}} \mathrm{CSP}(\mathbf{A}, k)$. "Uniform"

Can't ask these problems to be in P. (Set of inputs is problematic.)

Definition

Say $\mathrm{CSP}(\mathbf{A})$ [$\mathrm{CSP}(\mathcal{A}, k)$] is "in" P if there is a poly-time algorithm which correctly decides all inputs to $\mathrm{CSP}(\mathbf{A})$ [$\mathrm{CSP}(\mathcal{A}, k)$].

Global Tractability Problem

If $\mathbf{A}$ is tractable, does it follow that $\underbrace{\mathrm{CSP}(\mathbf{A}) \text{ is "in" P}}_{\mathbf{A} \text{ is globally tractable}}$?

Uniform Tractability Question

(For a given Taylor class $\mathcal{A}$): Is $\underbrace{\mathrm{CSP}(\mathcal{A}, k) \text{ "in" P for all } k}_{\mathcal{A} \text{ is uniformly tractable}}$?

Theorem

A is known to be globally tractable if:

1. **A** has a **cube term**. (Dalmau '05 + IMMVW '07)
2. $V(\mathbf{A})$ is SD($\wedge$). (Bulatov '09; Barto '14)
3. **A** is Taylor + conservative. (Bulatov '03)
4. **A** is Taylor and $|A| = 2$ or 3. (Schaefer '78, Bulatov '02)

Theorem (Bulatov '09; Barto '14)

The class $\mathcal{SD}_\wedge$ of all finite algebras generating an SD($\wedge$) variety is uniformly globally tractable.

Open problems

1. If $V(\mathbf{A})$ is congruence modular, is $\mathbf{A}$ globally tractable?

2. Is the class $\mathcal{M}$ of finite Maltsev algebras uniformly tractable?

3. If $\mathbf{A}$ has a difference term, is $\mathbf{A}$ tractable?

4. Suppose $\mathbf{A}$ is idempotent and has a congruence θ such that
   - $\mathbf{A}/\theta \in \mathcal{SD}_\wedge$, and
   - Each θ-block is in $\mathcal{M}$.

   ("SD($\wedge$) over Maltsev.") Is $\mathbf{A}$ tractable?

Standard reductions

CSP($\mathbf{A}, k$) reduces to:

1. CSP($\mathbf{A}\|_U, k$), where U is a minimal range of a unary idempotent term, and $\mathbf{A}\|_U$ is the induced term-minimal algebra defined on U.
2. CSP($(\mathbf{A}\|_U)^{\mathrm{id}}, k$) where $(\mathbf{B})^{\mathrm{id}}$ is the idempotent reduct of $\mathbf{B}$.

(This is the "reduction to the idempotent case.")

3. CSP($\mathbf{A}^{\lceil k/2 \rceil}, 2$)
4. multi-CSP($H(\mathbf{A})_{si}, kd$), where $\mathbf{A}$ is a subdirect product of d subdirectly irreducible homomorphic images.

5. CSP($\mathbf{A}^{+}, k$) where $\mathbf{A}^{+} = (A; \mathrm{Pol}(\mathrm{Su}(\mathbf{A}^{k})))$.

Conditioning the input – local consistency

Let (n, φ) be a 2-ary constraint network over $\mathbf{A}$.

At essentially no cost, one can assume that (n, φ) is "determined" by a "(2,3)-minimal" constraint network.

Definition

A 2-ary constraint network (n, φ) is a **(2,3)-system**[1] provided for all $i, j \in \{1, 2, \ldots, n\}$:

1. φ has exactly one constraint $R_{i,j}(x_i, x_j)$ with scope (x_i, x_j).
2. $R_{j,i} = (R_{i,j})^{-1}$.
3. For all k, $R_{i,j} \subseteq R_{i,k} \circ R_{k,j}$.

The "associated potatoes" are $A_i := \mathrm{proj}_1(R_{i,j}), \;\; i = 1, \ldots, n$.

Fact

There is a poly-time algorithm which, given a 2-ary constraint network over $\mathbf{A}$, outputs an equivalent (2,3)-system over $\mathbf{A}$.

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[1] There is no standard terminology.

Conditioning the input – absorption

Definition

Suppose **A** is a finite idempotent algebra and $\mathbf{B} \leq \mathbf{A}$.

1. **B** is an **absorbing subalgebra** if there exists a term operation $t(x_1, \ldots, x_m)$ of **A** such that

$$t(B, \ldots, B, A, B, \ldots, B) \subseteq B$$

   for all possible positions of A.

2. **A** is **absorption-free** if it has no proper absorbing subalgebra.

Given a (2,3)-system (n, φ) over an idempotent **A**, Barto & Kozik show how to "shrink" the associated potatoes to absorption-free algebras, though losing (2,3)-systemhood and equivalency.

In some situations this has proven to be useful.

Miklós magic

Lemma (Maróti '09)

Suppose $\mathbf{A}$ is idempotent and has a term operation $t(x, y)$ such that:

1. $\mathbf{A} \models t(x, t(x, y)) \approx t(x, y)$.
2. $t(a, x)$ is non-surjective, for all $a \in A$.
3. There exists a proper subalgebra $\mathbf{C} < \mathbf{A}$ such that if $t(x, a)$ is surjective then $a \in C$.

Then $\mathrm{CSP}(\mathbf{A}, k)$ can be reduced to multi-$\mathrm{CSP}(\mathcal{B} \setminus \{\mathbf{A}\}, \ell)$, where

- $\mathcal{B}$ is the closure of $\{\mathbf{A}\}$ under H, S, and "idempotent unary polynomial retracts."
- $\ell = \max(k, |A|)$.

This may seem random, but it is useful (and the proof is beautiful).

Moving forward

Suppose (n, φ) is a k-ary constraint network over $\mathbf{A}$, and $R = \mathrm{Rel}_{\mathbf{A}}(n, \varphi) \leq \mathbf{A}^n$.

Definition

A **compact k-frame** for R is a subset $F \subseteq R$ such that

1. $\mathrm{proj}_J(F) = \mathrm{proj}_J(R)$ for all $J \subseteq \{1 \ldots, n\}$ with $|J| \leq k$.
2. $|F| \leq |A|^k \cdot \binom{n}{k}$.

Every relation definable by a k-ary constraint network over $\mathbf{A}$ has a compact k-frame, and is determined by any one of its k-frames.

Speculation: Is it possible to mimic the few subpowers algorithm without having few subpowers?

To carry this out, we would need a notion of "compact k-representation" extending compact k-frames with more data.

The following problem seems central:

Functional Dependency Problem

Suppose

- $\mathbf{A}$ is finite, idempotent, Taylor.
- F is a compact k-frame for a relation $R \leq \mathbf{A}^n$ defined by some k-ary constraint network over $\mathbf{A}$.
- $X \subseteq \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, n\} \setminus X$.

What additional data would enable us to efficiently decide whether $\mathrm{proj}_{X \cup \{\ell\}}(R)$ is the graph of a function $f : \mathrm{proj}_X(R) \to \mathrm{proj}_\ell(R)$?

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