Varieties with a difference term and Jónsson's problem

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What you need to know

A variety is **finitely based** if it is axiomatizable by finitely many identities.

An algebra $\mathbf{A}$ is **finitely based** if $\mathsf{V}(\mathbf{A})$ is.

A variety $\mathcal{V}$ is **residually small** if there is a cardinal upper bound to the sizes of the subdirectly irreducible (s.i.) members of $\mathcal{V}$.

$\mathcal{V}$ has a **finite residual bound** if the bound can be chosen to be finite.

In 1967, B. Jónsson proved that if $\mathbf{A}$ is finite and $\mathsf{V}(\mathbf{A})$ is congruence distributive (CD), then $\mathsf{V}(\mathbf{A})_{si} \subseteq \mathsf{HS}(\mathbf{A})$.

In 1972, K. Baker proved that if $\mathbf{A}$ is finite, $\mathsf{V}(\mathbf{A})$ is CD, and the language of $\mathbf{A}$ is finite, then $\mathbf{A}$ is finitely based.

"In the early 1970s, Bjarni Jónsson asked ..."

1. If **A** is finite and $\mathrm{V}(\mathbf{A})_{si} \subseteq \mathrm{HS}(\mathbf{A})$, must **A** be finitely based? (Taylor '75; publ. '77)
2. If **A** is finite and $\mathrm{V}(\mathbf{A})$ has a finite residual bound, must **A** be finitely based? (Baker '76; McKenzie '77)
3. If **A** is finite and $\mathrm{V}(\mathbf{A})$ is residually small, must **A** be finitely based? (McKenzie '87)
4. If $\mathcal{V}$ is a variety and $\mathcal{V}_{fsi}$ is definable by a first-order sentence, must $\mathcal{V}$ be finitely based? (Oberwolfach '76)

"Jónsson's Problem"

(All algebras/varieties in a finite language.)

Jónsson's Problem

If **A** is finite, has a finite language, and V(**A**) has a finite residual bound, must **A** be finitely based?

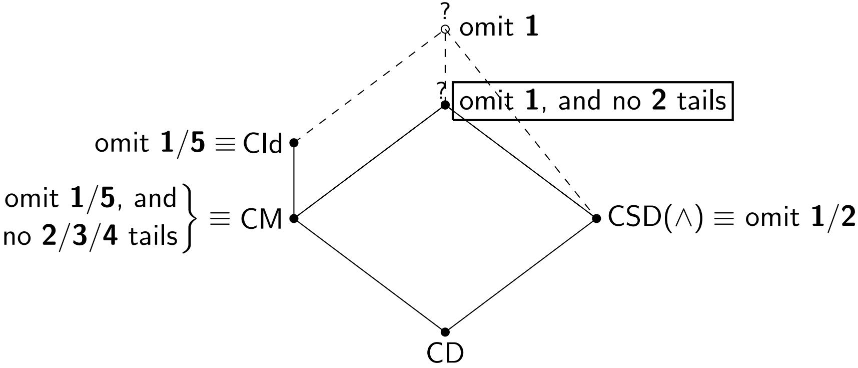
Park's Conjecture

"YES" (1976 PhD thesis)

Confirmations:

1. YES if V(**A**) is congruence distributive (Baker, '72).
2. YES if V(**A**) is congruence modular (McKenzie, '87)
   - YES if V(**A**) satisfies any nontrivial congruence identity (using Hobby/McKenzie)
3. YES if V(**A**) is congruence SD($\wedge$). (W, '00).

Confirmations of Jónsson's Problem



We want a confirmation which generalizes all of these results.

Theorem (Kearnes '95)

Let $\mathcal{V}$ be a locally finite variety. TFAE:

1. *$\mathcal{V}$ omits type* $\mathbf{1}$ *and has no type-*$\mathbf{2}$ *tails.*
2. *$\mathcal{V}$ has a* **difference term**, *i.e., a term $p(x, y, z)$ such that*
   - *$\mathcal{V}$ models $p(x, x, y) \approx y$.*
   - *$p(x, y, z)$ is a Maltsev operation on each block of any abelian congruence in any member of $\mathcal{V}$.*

Notes:

- In a CM variety, the final Gumm term $p(x, y, z)$ is a difference term.
- In a CSD($\wedge$) variety, $p(x, y, z) := z$ is a difference term.
- "Having a difference term" is characterized by an idempotent Maltsev condition, equivalent to CSD($\wedge$) + Maltsev. (Kearnes, Szendrei '98)

Our result (July '13)

Theorem (Kearnes, Szendrei, W)

Jónsson's Problem has an affirmative answer for varieties having a difference term.

I.e., if $\mathcal{V}$ is a variety in a finite language, $\mathcal{V}$ omits type $\mathbf{1}$, $\mathcal{V}$ has no type-$\mathbf{2}$ tails, and $\mathcal{V}$ has a finite residual bound, then $\mathcal{V}$ is finitely based.

Elements in the proof:

1. Give a new syntactic characterization of "having a difference term."
2. Prove that "$[\mathrm{Cg}(x,y), \mathrm{Cg}(z,w)] = 0$" is first-order definable in $\mathcal{V}$.
3. Extend Kiss's "4-ary difference term" characterization of $[\alpha, \beta] = 0$.
4. Mimic, as far as possible, McKenzie's proof in the CM case.

The syntactic characterization

Lemma

Let $\mathcal{V}$ be a variety. Let $p(x, y, z)$ be a term. TFAE:

1. *p is a difference term for $\mathcal{V}$.*
2. *$\mathcal{V} \models p(x, x, y) \approx y$, and $\exists$ finitely many pairs (f_i, g_i) of idempotent 3-ary terms such that the following are valid in $\mathcal{V}$:*

$$f_i(x, y, x) \approx g_i(x, y, x) \text{ for all } i, \text{ and}$$

$$\bigwedge_i [f_i(x, x, y) = g_i(x, x, y) \leftrightarrow f_i(x, y, y) = g_i(x, y, y)] \ \rightarrow \ p(x, y, y) = x.$$

Mmmm, Ralph's plate sure looks good ...

Proof that $[\mathrm{Cg}(x, y), \mathrm{Cg}(z, w)] = 0$ is definable

It's syntactic.

We do not use a Ramsey argument; we do use the trick used by Baker, McNulty, Wang in the CSD($\wedge$) case.

Details: http://www.math.uwaterloo.ca/~rdwillar/.

Thank you!