Graphs, Polymorphisms, and Multi-Sorted Structures

Ross Willard

University of Waterloo

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Background

Structure: $\mathbf{A} = (A; (R_i))$.

- Always **finite** and in a **finite relational language**.
- $\mathbf{A}^c = \mathbf{A}_A = (\mathbf{A}, (\{a\})_{a \in A})$; "**A with constants**."

Relations **definable** in **A**.

- I.e., definable by a 1st-order logical formula in the language of **A**.
- We are interested only in **primitive-positive (pp)** formulas:

$$\varphi(\mathbf{x}) \text{ of the form } \qquad \exists \mathbf{y}[\ \bigwedge atomic(\mathbf{u})\ ]$$

$$\uparrow$$

vars from $\mathbf{x}, \mathbf{y}$

- A relation is **ppc-definable** in **A** if it is definable by a pp-formula with parameters (i.e., in $\mathbf{A}^c$).

Let $\mathbf{A}, \mathbf{B}$ be finite structures. Assume for simplicity that

$$\mathbf{B} = (B; R, S), \qquad R \subseteq B^2, \quad S \subseteq B^3.$$

Definition

$\mathbf{B}$ is **ppc-interpretable** in $\mathbf{A}$ if, for some $k \geq 1$, there exist ppc-definable relations U, E, R^*, S^* <u>of $\mathbf{A}$</u> of arities k, $2k$, $2k$, $3k$ such that

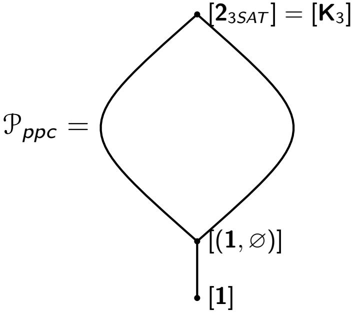
- E is an equivalence relation on U.
- $R^* \subseteq U^2$, $S^* \subseteq U^3$.
- R^*, S^* are invariant under E.
- $(U/E;\ R^*/E,\ S^*/E) \ \cong\ \mathbf{B}$.

Notation: $\mathbf{B} \leq_{ppc} \mathbf{A}$, $\quad \mathbf{B} \equiv_{ppc} \mathbf{A}$.

In particular, $\mathbf{A}^c \equiv_{ppc} \mathbf{A}$.

In the usual fashion, $\leq_{ppc}$ and $\equiv_{ppc}$ determines a poset:

- $[\mathbf{A}] = \{\mathbf{B} : \mathbf{B} \equiv_{ppc} \mathbf{A}\}$.
- $[\mathbf{B}] \leq [\mathbf{A}]$ iff $\mathbf{B} \leq_{ppc} \mathbf{A}$.
- $\mathcal{P}_{ppc} = (\{\text{all finite structures}\}/\equiv_{ppc};\ \leq)$.



$\mathbf{2}_{3SAT} = (\{0,1\};\ R_{000}, R_{100}, R_{110}, R_{111})$
where $R_{abc} = \{0,1\}^3 \setminus \{abc\}$

$\mathbf{K}_3 = (\{0,1,2\};\ \neq)$

$\mathbf{1} = (\{0\};\,)$

Constraint Satisfaction Problems

Fix a finite structure **A**.

$\mathrm{CSP}(\mathbf{A}^c)$

Input: An $=$-free, quantifier-free pp-formula $\varphi(\mathbf{x})$ in the language of $\mathbf{A}^c$ (i.e., allowing parameters).

Question: Is $\exists \mathbf{x} \varphi(\mathbf{x})$ true in $\mathbf{A}^c$?

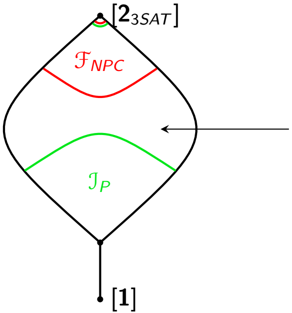
Connection to $\leq_{ppc}$:

Theorem (Bulatov, Jeavons, Krokhin 2005; Larose, Tesson 2009)

If $\mathbf{B} \leq_{ppc} \mathbf{A}$, *then* $\mathrm{CSP}(\mathbf{B}^c) \leq_L \mathrm{CSP}(\mathbf{A}^c)$.

Corollary

- $\mathcal{I}_P = \{[\mathbf{A}] : \mathrm{CSP}(\mathbf{A}^c) \text{ is in } P\}$ is an order ideal of $\mathcal{P}_{ppc}$.
- $\mathcal{F}_{NPC} = \{[\mathbf{A}] : \mathrm{CSP}(\mathbf{A}^c) \text{ is NP-complete}\}$ is an order filter.



The **CSP Dichotomy Conjecture** asserts that this region is empty (if P $\neq$ NP).

The **Algebraic CSP Dichotomy Conjecture** asserts that $\mathcal{I}_P = \mathcal{P}_{ppc} \setminus \{[\mathbf{2}_{3SAT}]\}$ (if P $\neq$ NP).

Connection to algebra

Fix a finite structure $\mathbf{A}$.

Definition

A **polymorphism** of $\mathbf{A}$ is any operation $h : A^n \to A$ which preserves the relations of $\mathbf{A}$ (equivalently, is a homomorphism $h : \mathbf{A}^n \to \mathbf{A}$).

$h : A^n \to A$ is **idempotent** if it satisfies $h(x, x, \ldots, x) = x \ \ \forall x \in A$.

The **polymorphism algebra** of $\mathbf{A}$ is

$$\mathrm{PolAlg}(\mathbf{A}) \ := \ (A; \{\text{all polymorphisms of } \mathbf{A}\}).$$

The **idempotent polymorphism algebra** of $\mathbf{A}$ is

$$\begin{aligned} \mathrm{IdPolAlg}(\mathbf{A}) \ &:= \ (A; \{\text{all idempotent polymorphisms of } \mathbf{A}\}) \\ &= \ \mathrm{PolAlg}(\mathbf{A}^c). \end{aligned}$$

Fix a set Σ of formal identities in operations symbols $\mathsf{F}, \mathsf{G}, \mathsf{H}, \ldots$.

Assume that $\Sigma \vdash \mathsf{F}(x, x, \ldots, x) \equiv x,\ \mathsf{G}(x, x, \ldots, x) \equiv x, \ldots$.

(I.e., Σ is **idempotent**.)

Definition

An algebra $\mathbb{A} = (A; \mathcal{F})$ **satisfies Σ as a Maltsev condition** if there exist (term) operations $f, g, h, \ldots$ of $\mathbb{A}$ such that $(A; f, g, h, \ldots) \models \Sigma$.

Definition

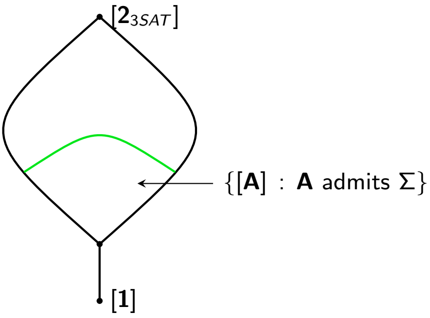
A structure **$\mathbf{A}$ admits** Σ if $\mathrm{IdPolAlg}(\mathbf{A})$ satisfies Σ as a Maltsev condition.

Fix an idempotent set Σ of identities.

Theorem (Bulatov, Jeavons, Krokhin)

Suppose $\mathbf{B} \leq_{ppc} \mathbf{A}$. *If* $\mathbf{A}$ *admits* Σ, *then so does* $\mathbf{B}$.

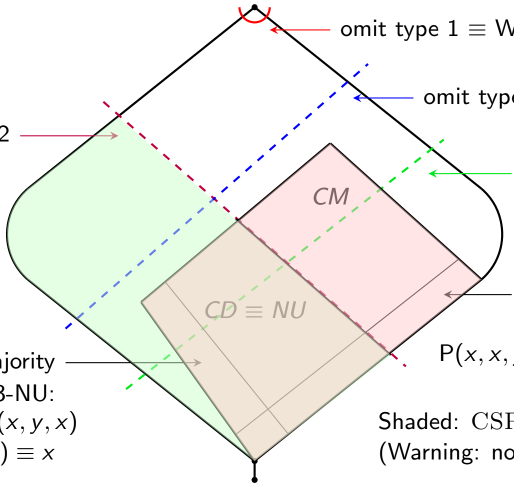
Hence $\{[\mathbf{A}] : \mathbf{A} \text{ admits } \Sigma\}$ *is an order ideal of* $\mathcal{P}_{ppc}$.



In fact, $\mathbf{A} \equiv_{ppc} \mathbf{B}$ iff $\mathbf{A}, \mathbf{B}$ admit the same (finite) idempotent sets of identities. $\leq_{ppc}$ has a similar characterization.

In this way, $\mathcal{P}_{ppc}$ is "stratified" by idempotent Maltsev conditions arising in universal algebra.

$\mathcal{P}_{ppc} =$



Shaded: $\mathrm{CSP}(\mathbf{A}^c)$ proved in P
(Warning: not to scale!)

Where are you favorite structures (relative to these Maltsev conditions)?

Aims of this talk

My goals of this lecture are to:

1. Say some things about **bipartite graphs** and where they fit in the picture.
2. Argue that **multi-sorted structures** are not evil.
3. Give a connection between (1) and (2).

Multi-sorted structures

Multi-sorted structure: $\mathbf{A} = (A_0, A_1, \ldots, A_n; (R_i))$.

- $0, 1, \ldots, n$ are the **sorts**; A_k is the **universe of sort** k.
- Each R_i is a **sorted relation**: e.g., $R_1 \subseteq A_2 \times A_0 \times A_0$.

(Sorted) Relations **definable** in $\mathbf{A}$.

- Adapt 1st-order logic in the usual way (every variable has a specified sort; an equality relation for each sort).

Ppc-interpretations of one 2-sorted structure in another, i.e., $\mathbf{B} \leq_{ppc} \mathbf{A}$.

- each universe B_i of $\mathbf{B}$ is realized as a U_i/E_i where U_i, E_i are (sorted) ppc-definable relations of $\mathbf{A}$.
- each sorted R relation of $\mathbf{B}$ is realized as $R^*/$"the appropriate E_i's."

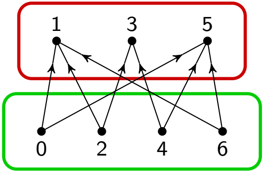
Example

Let **A** be the (1-sorted) structure $(A;\ E_0, E_1)$ pictured at right, where E_0, E_1 are the indicated equivalence relations on A.

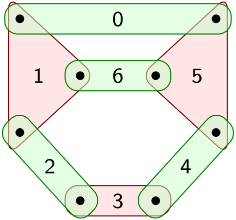
Let $\mathbf{B} = (B_0, B_1;\ R)$ be the 2-sorted structure pictured below.

$B_1 =$

$B_0 =$



$\mathbf{B} = (B_0, B_1;\ R)$
$R \subseteq B_0 \times B_1$



$\mathbf{A} = (A;\ E_0, E_1)$

$E_0 =$ (green) blocks

$E_1 =$ (red) blocks

Claim: $\mathbf{B} \leq_{ppc} \mathbf{A}$.

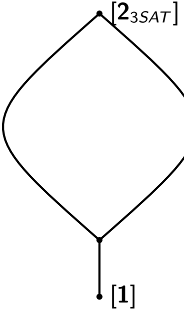
Proof: define $U_0 = U_1 = A$ and $(x, y) \in R^* \iff \exists z[xE_0z \ \&\ zE_1y]$.

Then $\mathbf{B} \cong (A/E_0, A/E_1;\ R^*/E_0 \times E_1)$.

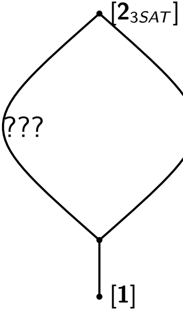
Just as in the 1-sorted case, $\leq_{ppc}$ gives a poset:

$$\mathcal{P}^{+}_{ppc} = (\{\text{all finite } \underline{\underline{\text{multi-sorted}}} \text{ structures}\}/\equiv_{ppc};\ \leq).$$

$\mathcal{P}_{ppc} =$



$\mathcal{P}^{+}_{ppc} =$



Fact: $\mathcal{P}^{+}_{ppc} = \mathcal{P}_{ppc}$.

I.e., for every multi-sorted $\mathbf{B}$ there exists a 1-sorted $\mathbf{A} \equiv_{ppc} \mathbf{B}$.

Moral: Multi-sorted structures have no value.

Let's be immoral.

$\mathrm{CSP}(\mathbf{A}^c)$ can be defined for a multi-sorted $\mathbf{A}$.

- Inputs are now multi-sorted quantifier-free pp-formulas.

The BJK-LT connection to $\leq_{ppc}$ is remains true for multi-sorted $\mathbf{A}, \mathbf{B}$:

$$\text{If } \mathbf{B} \leq_{ppc} \mathbf{A}, \text{ then } \mathrm{CSP}(\mathbf{B}^c) \leq_L \mathrm{CSP}(\mathbf{A}^c)$$

Polymorphisms of multi-sorted $\mathbf{A}$ are more complicated.

Definition (Bulatov, Jeavons 2003)

Let $\mathbf{A} = (A_0, A_1, \ldots, A_n; (R_i))$. An m-**ary polymorphism** of $\mathbf{A}$ is a tuple $(f^0, \ldots, f^n)$ of m-ary operations $f^k : A_k^m \to A_k$ which "jointly preserve" the relations of $\mathbf{A}$. E.g., if $R_1 \subseteq A_1 \times A_0$, then

$$\forall (a_1, b_1), \ldots, (a_m, b_m) \in R_1, \text{ need } (f^1(\mathbf{a}), f^0(\mathbf{b})) \in R_1.$$

Polymorphism "algebra"

Fix $\mathbf{A} = (A_0, A_1, \ldots, A_n; (R_i))$.

Let $\mathrm{Pol}(\mathbf{A}) = \{$all polymorphisms $\vec{f} = (f^0, f^1, \ldots, f^n)$ of $\mathbf{A}\}$.

Define

$$\begin{aligned} \mathbb{A}_0 &= (A_0; (f^0 : \vec{f} \in \mathrm{Pol}(\mathbf{A})) \\ \mathbb{A}_1 &= (A_1; (f^1 : \vec{f} \in \mathrm{Pol}(\mathbf{A})) \\ &\vdots \\ \mathbb{A}_n &= (A_n; (f^n : \vec{f} \in \mathrm{Pol}(\mathbf{A})). \end{aligned}$$

$\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_n$ are (ordinary) algebras with a common language.

Definition (Bulatov, Jeavons 2003)

The **polymorphism "algebra"** of $\mathbf{A}$ is the tuple $(\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_n)$ of algebras defined above.

Similarly for $\mathrm{IdPolAlg}(\mathbf{A})$.

Fix an idempotent set Σ of formal identities.

Definition

Let **A** be a multi-sorted structure and $\mathrm{IdPolAlg}(\mathbf{A}) = (\mathbb{A}_0, \ldots, \mathbb{A}_n)$ its corresponding idempotent polymorphism "algebra."

A admits Σ if $\{\mathbb{A}_0, \ldots, \mathbb{A}_n\}$ satisfies Σ as a Maltsev condition.

The characterizations of $\equiv_{ppc}$ and $\leq_{ppc}$ remain true for multi-sorted $\mathbf{A}, \mathbf{B}$.

- $\mathbf{A} \equiv_{ppc} \mathbf{B}$ iff $\mathbf{A}, \mathbf{B}$ admit the same idempotent sets of identities.
- $\mathbf{B} \leq_{ppc} \mathbf{A}$ iff every such Σ admitted by **A** is admitted by **B**.

Immoral Moral: Nothing bad will happen if we embrace multi-sorted structures.

Bipartite graphs in $\mathcal{P}_{ppc}$

Question: How "dense" in $\mathcal{P}_{ppc}$ are graphs, digraphs, posets, etc?

Theorem (Kazda (2011))

Let **D** *be a finite digraph. If* **D** *admits the* **Maltsev** *identities*

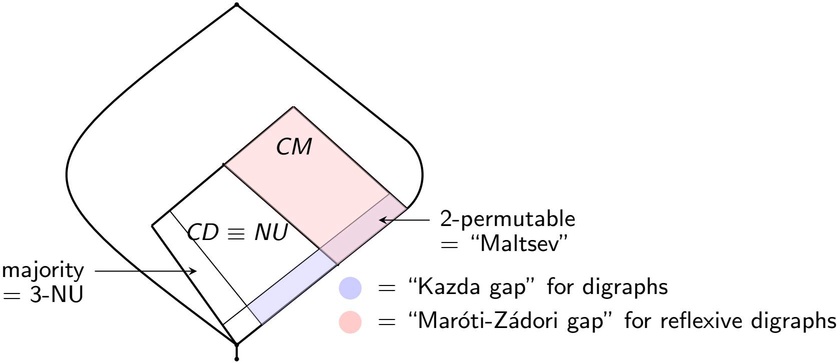
$$\mathsf{P}(x,x,y) \equiv y \equiv \mathsf{P}(y,x,x)$$

for 2-permutability, then **D** *admits the* **majority** *(or* **3-NU***) identities*

$$\mathsf{M}(x,x,y) \equiv \mathsf{M}(x,y,x) \equiv \mathsf{M}(y,x,x) \equiv x.$$

Theorem (Maróti, Zádori (2012))

Let **P** *be a reflexive digraph (e.g., a poset). If* **P** *admits identities for congruence modularity, then* **P** *admits the k-ary* **near unanimity** *(NU) identities for some $k \geq 3$.*



Theorem (Bulín, Delić, Jackson, Niven (?))

For every finite structure $\mathbf{A}$ *there is a directed graph* $\mathcal{D}(\mathbf{A})$ *such that*

1. $\mathrm{CSP}(\mathcal{D}(\mathbf{A})) \equiv_L \mathrm{CSP}(\mathbf{A})$.
2. $\mathbf{A} \leq_{ppc} \mathcal{D}(\mathbf{A})$.
3. *The "Kazda gap" is essentially all that separates* $\mathcal{D}(\mathbf{A})$ *from* $\mathbf{A}$.

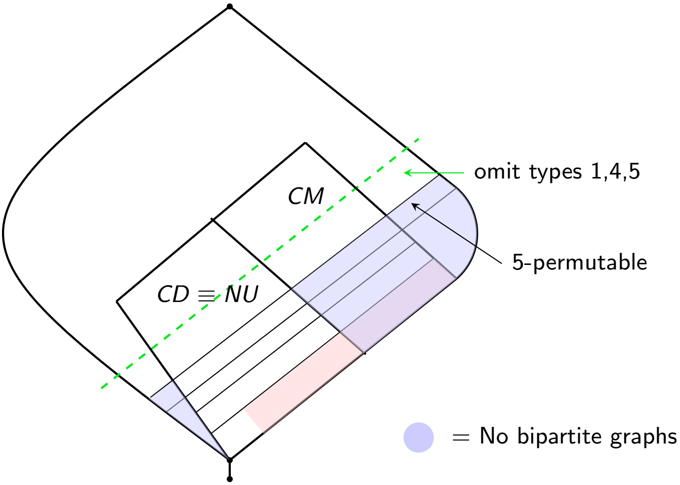
What about (symmetric, irreflexive) graphs?

Some things we know.

- (Bulatov) If **G** is a non-bipartite graph, then $[\mathbf{G}] \equiv_{ppc} [\mathbf{2}_{3SAT}]$.
- (Using Rival) If **G** is bipartite with girth ≥ 6, then $[\mathbf{G}] \equiv_{ppc} [\mathbf{2}_{3SAT}]$.
- Trees and complete bipartite graphs admit the majority identities and hence are low in $\mathcal{P}_{ppc}$.
- (Kazda) Bipartite graphs suffer the "Kazda gap."
- (Feder, Hell, Larose, Siggers, Tardif [2013?]) Characterize bipartite graphs admitting the k-NU identities, $k \geq 3$.

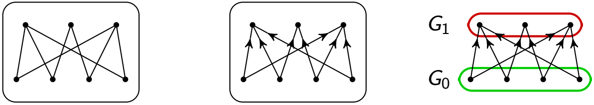
A new gap (W)

If **G** *is bipartite and admits the* **Hagemann-Mitschke** *identities for 5-permutability, then* **G** *admits an NU polymorphism of some arity.*


omit types 1,4,5
CM
5-permutable
CD ≡ NU
= No bipartite graphs

Useful tool: reduction to 2-sorted structures.

Definition:



$\mathbf{G}$ bipartite — $\vec{\mathbf{G}}$ strongly bipartite — $\mathbf{G}^\sharp$ 2-sorted

Lemma (W)

Let Σ be an idempotent set of identities such that

1. Every identity in Σ mentions at most two variables;
2. The 2-element connected graph admits Σ.

Let $\mathbf{G}$ be a connected bipartite graph and let $\vec{\mathbf{G}}$ and $\mathbf{G}^\sharp$ be the corresponding strongly bipartite and 2-sorted digraphs respectively.

If any of $\mathbf{G}$, $\vec{\mathbf{G}}$ or $\mathbf{G}^\sharp$ admit Σ, then all admit Σ.

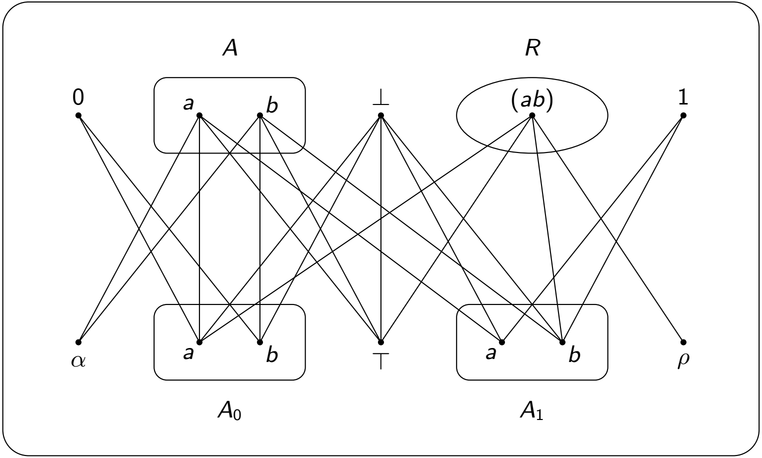
Proof: $\mathbf{G}^\sharp \leq_{ppc} \vec{\mathbf{G}} \leq_{ppc} \mathbf{G}$. A recipe shows $\mathbf{G}^\sharp$ admits $\Sigma \Rightarrow \mathbf{G}$ admits Σ.

Theorem (Feder, Vardi (1990's))

For every finite structure $\mathbf{A}$ *there is a bipartite graph* $\mathcal{B}(\mathbf{A})$ *such that* $\mathrm{CSP}(\mathcal{B}(\mathbf{A})^c) \equiv_P \mathrm{CSP}(\mathbf{A})$.

The construction, assuming $\mathbf{A} = (A; R)$ is a digraph.

$\mathcal{B}(\mathbf{A}) =$

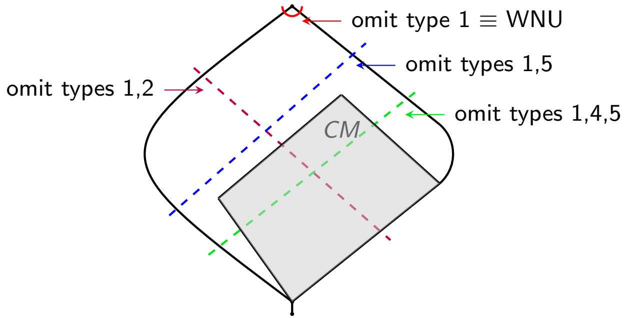


Question: How close are **A** and $\mathcal{B}(\mathbf{A})$ in $\mathcal{P}_{ppc}$?

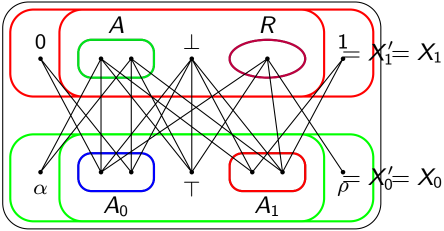
Theorem (Payne, W)

Given a finite structure **A**, *let* $\mathcal{B}(\mathbf{A})$ *be the associated bipartite graph.*

1. $\mathbf{A} \leq_{ppc} \mathcal{B}(\mathbf{A})$.
2. *For each of the six order ideals* $\mathcal{I}$ *of* $\mathcal{P}_{ppc}$ *associated with omitting types, if one of* **A**, $\mathcal{B}(\mathbf{A})$ *belongs to* $\mathcal{I}$, *then so does the other.*
3. $\mathcal{B}(\mathbf{A})$ *never admits the Gumm identities for CM.*



Sketch of the proof of (1).



Let $\mathbf{X} = (X_0, X_1;\ \vec{E}) = \mathcal{B}(\mathbf{A})^\sharp$.

Let $\mathbf{X}' = (\mathbf{X} \setminus \{\alpha, \rho, 0, 1\},\ A_0, A_1, A, R)$.

Let $\mathbf{X}''$ be the induced 4-sorted structure with universes A_0, A_1, A, R.

Then $\mathbf{A} \equiv_{ppc} \mathbf{X}'' \leq_{ppc} \mathbf{X}' \leq_{ppc} \mathbf{X} = \mathcal{B}(\mathbf{A})^\sharp \leq_{ppc} \mathcal{B}(\mathbf{A})$.

Show $\mathbf{X}''$ admits $\Sigma(n) \Rightarrow \mathbf{X}$ admits $\Sigma(n+4)$, for relevant Σ.

Problems

1. Are $\mathbf{A}$ and $\mathcal{B}(\mathbf{A})$ "essentially the same" modulo the 5-perm $\Rightarrow$ NU and Kazda gaps?
2. Find a better map $\mathbf{A} \longmapsto \mathcal{B}'(\mathbf{A})$ *à la* BDJN.
3. Prove or disprove: CM $\Rightarrow$ NU for bipartite graphs.
4. For each "omitting-types" order ideal $\mathcal{I}$ of $\mathcal{P}_{ppc}$, characterize the bipartite graphs in $\mathcal{I}$.

Hvala!