The decidable discriminator variety problem

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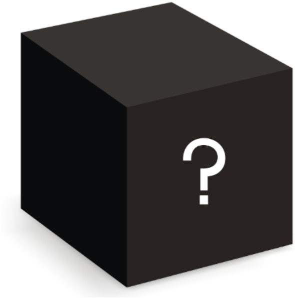
Variations on Homogeneity

A black box



In the box: certain $\forall_1$ classes of structures which are

$$\left.\begin{array}{l} \blacktriangleright\ \text{locally finite} \\ \blacktriangleright\ \text{in a finite signature} \end{array}\right\} \quad \text{“small } \forall_1 \text{ classes”}$$

Which small $\forall_1$ classes are in the box?

Hints

Which small $\forall_1$ $\mathcal{K}$ are in the box?

1. If $\mathcal{K}$ is a finite set of finite structures, then $\mathcal{K}$ is in the box.
2. If every countable member of $\mathcal{K}$ is (hereditarily) homogeneous, then $\mathcal{K}$ is in the box.

   - homogeneous: every isomorphism between finite substructures extends to an automorphism.
   - hereditarily: every substructure is homogeneous.

3. The box is a candidate for the smallest "natural" collection of small $\forall_1$ classes satisfying (1)–(2).

Intuition

The box captures some version of "hereditarily homogeneous modulo finite."

Guess #1

Definition

1. **M** is weakly hereditarily homogeneous if there exists a finite set $A \subseteq M$ such that $\mathbf{M}_A$ is hereditarily homogeneous.
2. A small $\forall_1$ class $\mathcal{K}$ is weakly hereditarily homogeneous if there exists $n \geq 0$ such that every countable member $\mathbf{M} \in \mathcal{K}$ is weakly hereditarily homogeneous via a set $A \subseteq M$ of size $\leq n$.

Getting warm!

- Every class in the box is weakly hereditarily homogeneous.
- But not conversely: the class

$$\{\,\text{graphs having at most one edge}\,\}$$

is **not** in the box.

Guess #2

Definition

A small $\forall_1$ class $\mathcal{K}$ is upwardly weakly hereditarily homogeneous if there exists $n \geq 0$ such that for all $\mathbf{M} \in \mathcal{K}_{fin}$ there exists $A \subseteq M$ with $|A| \leq n$, satisfying:

1. $\mathbf{M}_A$ is hereditarily homogeneous.
2. For all $\mathbf{N} \in \mathcal{K}_{fin}$ and embeddings $\sigma_1, \sigma_2 : \mathbf{M} \hookrightarrow \mathbf{N}$ with $\sigma_1|_A = \sigma_2|_A$, there exists $\alpha \in \mathrm{Aut}\,\mathbf{N}$ with $\alpha \circ \sigma_1 = \sigma_2$.

Getting hot!!

- {graphs with ≤ 1 edge} is **not** UWHH.
- Every class in the box **is** UWHH.
- (I don't know if the converse holds.)

Answer

Suppose $\mathcal{K}$ is a small $\forall_1$ class.

Definition

$\mathcal{K}$ is in the box if there exists a relation $\lhd$ between finite sets and members of $\mathcal{K}_{fin}$ such that for some $n \geq 0$,

1. $A \lhd \mathbf{M}$ implies $A \subseteq M$, $\mathbf{M}_A$ is homogeneous, and $|A| \leq n$.
2. $\lhd$ is invariant under isomorphisms.
3. For all $\mathbf{M} \in \mathcal{K}_{fin}$ there exists $A \lhd \mathbf{M}$.
4. If $A \lhd \mathbf{M}$ and $A \subseteq \mathbf{M}' \leq \mathbf{M}$, then $A \lhd \mathbf{M}'$.
5. If $A \lhd \mathbf{M} \leq \mathbf{N} \in \mathcal{K}_{fin}$ then there exists $B \lhd \mathbf{N}$ with $A \subseteq B$.
6. If $A \subseteq B \lhd \mathbf{N}$ and $\mathbf{M}_1, \mathbf{M}_2 \leq \mathbf{N}$ with $A \lhd \mathbf{M}_1, \mathbf{M}_2$, then every isomorphism $\sigma : \mathbf{M}_1 \cong \mathbf{M}_2$ fixing A pointwise extends to some $\alpha \in \mathrm{Aut}\, \mathbf{N}$ fixing B pointwise.

(Ugh)

$\boxed{\text{Decidable equational classes}}$

Universal algebra

Algebraic structure, or algebra: a structure in a signature with no relation symbols.

Equational theory: a deduction-closed set of **identities**

$$\forall \mathbf{x} : s(x) = t(\mathbf{x})$$

Equational class: $\mathrm{Mod}(T)$ for some equational theory T.

Decidable Equational Class Problem

Problem

For which equational classes $\mathcal{E}$ in finite signature is the 1st-order theory of $\mathcal{E}$ decidable?

Theorem (McKenzie, Valeriote 1989)

In the locally finite case, this problem is solved modulo two special cases:

1. Modules over a finite ring.
2. "Discriminator varieties."

What is a *discriminator variety*?

$$\boxed{\boxed{\text{Discriminator varieties}}}$$

Recipe

1. Start with a $\forall_1$-class of structures.
2. Replace each n-ary basic relation R with an $n+2$-ary operation f_R defined by
$$f_R(\mathbf{x}, y, z) = \begin{cases} y & \text{if } R(\mathbf{x}) \\ z & \text{else.} \end{cases}$$
3. Also add $f_=$.
4. Denote the resulting $\forall_1$-class of algebras $\mathcal{K}^*$.
5. Let $T_e(\mathcal{K}^*)$ be the equational theory of $\mathcal{K}^*$.
6. $\mathcal{D}(\mathcal{K}) := \mathrm{Mod}(T_e(\mathcal{K}^*))$ is a typical discriminator variety.
   - Note: $\mathcal{K}^*$ is the class of simple algebras in $\mathcal{D}(\mathcal{K})$.

Example

Start with $\mathcal{K} = \{\mathbf{2}\}$ where $\mathbf{2} = (\{0,1\}, 0, 1)$.

$\mathcal{K}^* = \{\mathbf{2}^*\}$ where $\mathbf{2}^* = (\{0,1\}, f_=, 0, 1)$,

$$f_=(x, y, z, w) \;=\; \begin{cases} z & \text{if } x = y \\ w & \text{else.} \end{cases}$$

Note: $\mathbf{2}^*$ is the 2-element boolean algebra. Hence

$$\begin{aligned} \mathcal{D}(\{\mathbf{2}\}) &= \mathrm{Mod}(T_e(\{\mathbf{2}^*\})) \\ &= \mathrm{Mod}(T_e(\{\text{the 2-element boolean algebra}\})) \\ &= \{\text{all boolean algebras}\} \end{aligned}$$

Take-aways

1. Discriminator varieties correspond to $\forall_1$ classes:

$$\begin{array}{ccccc} \text{(loc. fin., fin. sign.)} & & & & \text{(small)} \\ \text{discrim. varieties} & & \leftrightsquigarrow & & \forall_1 \text{ classes} \\ \mathcal{D}(\mathcal{K}) & \Longleftrightarrow & \mathcal{K}^* & \equiv & \mathcal{K} \end{array}$$

2. Discriminator varieties are (equational) classes of "generalized boolean algebras."

The Decidable Discriminator Variety problem

The question

> Which (loc. fin., fin. sign.) discriminator varieties have decidable 1st-order theory?

can be reformulated

> Which (small) $\forall_1$ classes $\mathcal{K}$ are such that $\mathcal{D}(\mathcal{K})$ has decidable 1st-order theory?

Conjecture

Answer to 2nd question: the ones in the box!

Evidence

Theorem (W)

Suppose $\mathcal{K}$ is in the box.

1. $\{$graphs$\}$ does **not** interpret[1] into $\mathcal{D}(\mathcal{K})$.
2. If $\mathrm{Th}_{\forall_1}(\mathcal{K})$ is decidable (e.g., if $\mathcal{K}$ is finitely axiomatizable), then $\mathrm{Th}(\mathcal{D}(\mathcal{K}))$ is decidable.

Moreover

In classes studied to date[2], no counter-examples found to:

1. $\mathcal{K}$ not in the box $\stackrel{?}{\Longrightarrow}$ $\{$graphs$\}$ interprets into $\mathcal{D}(\mathcal{K})$.
2. $\mathcal{K}$ in the box $\stackrel{?}{\Longrightarrow}$ $\mathcal{K}$ finitely axiomatizable.

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[1] "right totally" as per Hodges

[2] unary algebras (W '93), lattices (W '94), dihedral groups (Delić '05)

Ingredients in the proof

- Every member of $\mathcal{D}(\mathcal{K})$ has a representation as the algebra of global sections of some Hausdorff sheaf over a Stone space, with stalks from $\mathcal{K}^*$.
- Assuming $\mathcal{K}$ is in the box, one can obtain a (non-effective) Feferman-Vaught analysis of the countable members of $\mathcal{D}(\mathcal{K})$ (via their representations).
- This translates the theory of $\mathcal{D}(\mathcal{K})$ to the theory of boolean algebras with countably many ideals (decidable by Rabin).
- If $\mathrm{Th}_{\forall_1}(\mathcal{K})$ is decidable, then the translation can be made effective.

Help!

Recall: $\mathcal{K}$ in the box $\Longrightarrow$ $\mathcal{K}$ UWHH.

1. Does $\mathcal{K}$ UWHH $\stackrel{?}{\Longrightarrow}$ $\mathcal{K}$ in the box?
2. What are generic obstacles to UWHH? To being in the box?
   - In all examples I know, there is a witnessing pair $\mathbf{M} < \mathbf{N}$ of countably infinite structures and a finite set A such that $\mathrm{Aut}(\mathbf{M}_A)$ has an infinite orbit that gets "badly split" in $\mathbf{N}_A$.
3. Does UWHH (or being in the box) imply finite axiomatizability?
4. Does anyone give a rip??

Thank you!