# Universal Algebra and Computational Complexity
# Lecture 1

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# Outline

Lecture 1: Decision problems and Complexity Classes

Lecture 2: Nondeterminism, Reductions and Complete problems

Lecture 3: Results and problems from Universal Algebra

# Three themes: problems, algorithms, efficiency

A *Decision Problem* is . . .

- A *YES/NO question*
- parametrized by one or more *inputs*.
    - Inputs must:
        - range over an *infinite* class.
        - be "finitistically described"

What we seek:

- An *algorithm* which correctly answers the question for all possible inputs.

What we ask:

- How *efficient* is this algorithm?
- Is there a better (more efficient) algorithm?

# Directed Graph Reachability problem (PATH)

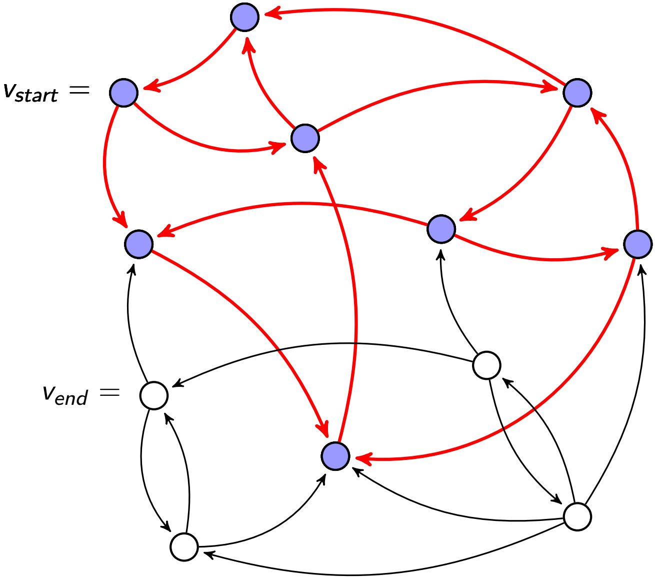
INPUT:

- A finite directed graph $G = (V, E)$
- Two distinguished vertices $v_{start}, v_{end} \in V$.

QUESTION:

- Does there exist in $G$ a *directed path* from $v_{start}$ to $v_{end}$?

# An Algorithm for PATH




Answer: "NO"

# Efficiency of this algorithm

How long does this algorithm take?

- I.e., how many steps . . .
- . . . as a function of the size of the input graph.

I'll give three answers to this.

# First answer – Heuristics

Only action is changing a vertex's color.

Only changes possible are

- white $\Rightarrow$ red
- red $\Rightarrow$ green
- green $\Rightarrow$ blue.

So if $n = |V|$, then the algorithm requires at most $3n$ vertex-color changes.

# Second answer – pseudo-code

Simplifying assumptions:

- $V = \{0, 1, \ldots, n-1\}$
- $E$ is encoded by the adjacency matrix $M_E = [e_{i,j}]$ where

$$e_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{else.} \end{cases}$$

Auxiliary variables:

- $i, j$ will range over $\{0, 1, \ldots, n{-}1\}$.
- For $i < n$ let $c_i$ be a variable recording the color of vertex $i$.
- Let $GreenVar$ be a variable storing whether there are green-colored vertices.

# Second answer – pseudo-code

Algorithm:

- Input $n$, $M_E$, *start* and *end*.
- For $i = 0$ to $n - 1$ set $c_i := white$.
- Set $c_{start} = green$.
- Set $GreenVar := yes$.
- MAIN LOOP: While $GreenVar = yes$ do:
  - For $i = 0$ to $n - 1$; for $j = 0$ to $n - 1$
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$.
  - For $i = 0$ to $n - 1$
    - If $c_i = green$ then set $c_i := blue$
  - Set $GreenVar := no$
  - For $i = 0$ to $n - 1$
    - If $c_i = red$ then (set $c_i := green$ and set $GreenVar := yes$)
- If $c_{end} = blue$ then output YES; else output NO.

$n$ loops

$n^2$ cases

$O(n^3)$ steps if $n = |V|$

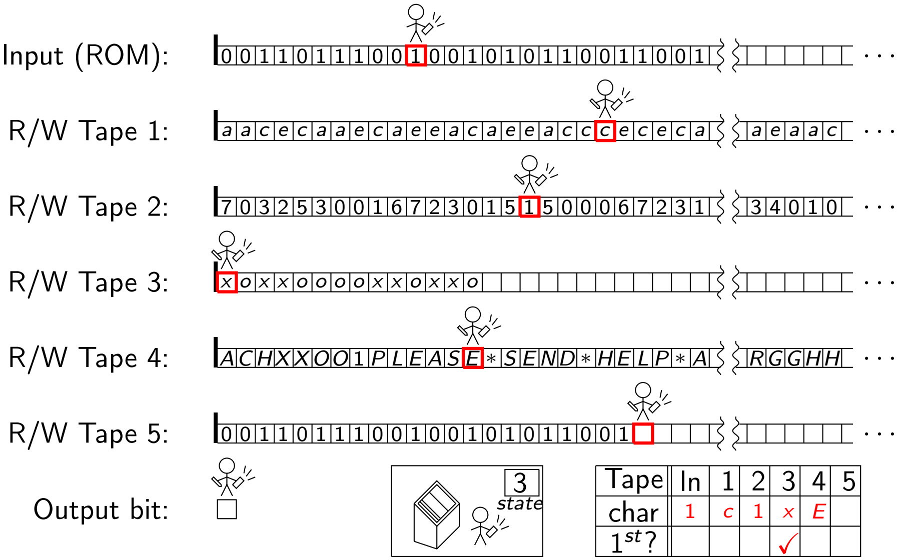
# Third answer – machine implementation

Again assume $V = \{0, 1, \ldots, n-1\}$.

Assume also that $v_{start} = 0$ and $v_{end} = 1$.

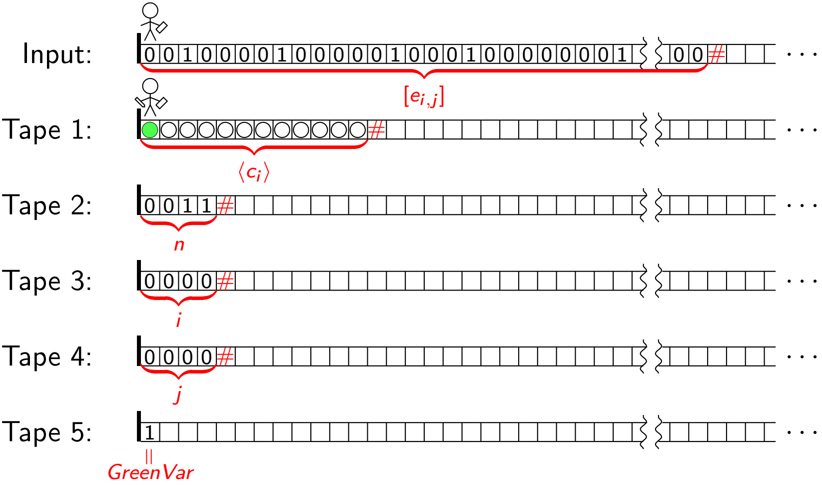
Assume the adjacency matrix is presented as a binary string of length $n^2$.

Implement the algorithm on a *Turing machine*.




| Tape | In | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| char | 1 | c | 1 | x | E | |
| $1^{st}$? | | | | ✓ | | |

# Implementing the algorithm for *PATH*




Main loop: For $i, j = 0$ to $n - 1$ ...

# Pseudo-code revisited

Point: overhead needed to keep track of $i, j, c_i, c_j$.

Thus:

- While $GreenVar = yes$ do: — $n$ loops
  - For $i = 0$ to $n-1$; for $j = 0$ to $n-1$ — $n^2$ cases
    - if $e_{i,j} = 1$ and $c_i = green$ and $c_j = white$ then set $c_j := red$. — $O(n \log n)$ steps

SUMMARY: on an input graph $G = (V, E)$ with $|V| = n$, our algorithm decides the answer to $PATH$ using:

| | |
|---|---|
| Heuristics | $3n$ color changes |
| Pseudo-code | $O(n^3)$ operations |
| Turing machine | $O(n^4 \log n)$ steps (Time)<br>$O(n)$ memory cells (Space) |

# Turing machine complexity

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be given.

## Definition

A decision problem $D$ (with a specified encoding of its inputs) is:

1. in $TIME(f(N))$ if there exists a Turing machine solving $D$ in at most $O(f(N))$ steps on inputs of length $N$.
2. in $SPACE(f(N))$ if there exists a Turing machine solving $D$ requiring at most $O(f(N))$ memory cells (not including the input tape) on inputs of length $N$.

# Complexity of *PATH*

Recall that our Turing machine solves *PATH* on graphs with $n$ vertices in

- Time: $O(n^4 \log n)$ steps
- Space: $O(n)$ memory cells.

Since "length $N$ of input" $= n^2$ (when $n = |V|$), this at least proves

**Theorem**

$$\begin{aligned} PATH &\in TIME(N^{2+\epsilon}) \\ PATH &\in SPACE(\sqrt{N}) \end{aligned}$$

(Question: can we do better?...)

# Another problem: Boolean Formula Value (*FVAL*)

INPUT:

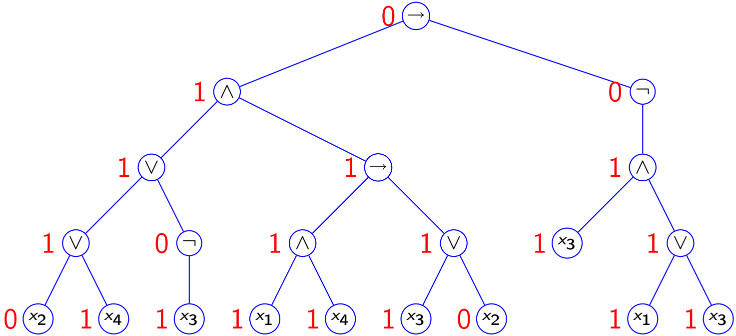
- A boolean formula $\varphi$ in propositional variables $x_1, \ldots, x_n$.
- A sequence $\mathbf{c} = (c_1, \ldots, c_n) \in \{0, 1\}^n$.

QUESTION:

- Is $\varphi(\mathbf{c}) = 1$?

# An algorithm for *FVAL*

$\varphi = ((((x_2 \vee x_4) \vee (\neg(x_3))) \wedge ((x_1 \wedge x_4) \to (x_3 \vee x_2))) \to (\neg(x_3 \wedge (x_1 \vee x_3))))$, $\quad \mathbf{c} = (1,0,1,1)$.




Seems to use *TIME*($N$) and *SPACE*($N$).
But space can be re-used. In this example, 3 memory bits suffice.

# Complexity of *FVAL*

In general, a bottom-up computation, always computing a larger subtree first, can be organized to need only $O(\log|\varphi|)$ intermediate values.

A careful implementation on a Turing machine yields:

**Theorem (Nancy Lynch, 1977)**

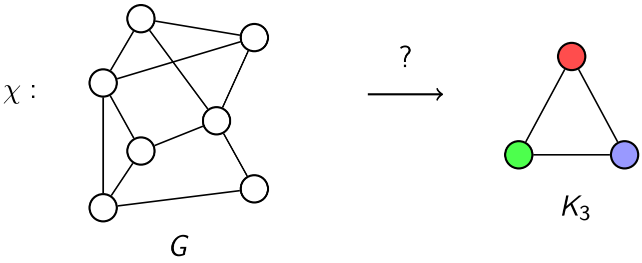
$$\begin{aligned} FVAL &\in TIME(N^{2+\epsilon}) \\ FVAL &\in SPACE(\log N). \end{aligned}$$

# A third problem: Graph 3-Colorability ($3COL$)

INPUT: a finite graph $G = (V, E)$.

QUESTION: Is it possible to color the vertices **red**, **green** or **blue**, so that no two adjacent vertices have the same color?

Equivalently: does there exist a homomorphism



# An algorithm for $3COL$

Brute force search algorithm:

- For each function $\chi : V \to K_3$:
  - Test if $\chi$ works.

$3^{|V|} = 2^{O(\sqrt{N})}$ loops

$O(N^2)$ time,
$O(\sqrt{N})$ space

**Theorem**

*This at least proves:*

$$3COL \in TIME(2^{O(\sqrt{N})})$$
$$3COL \in SPACE(\sqrt{N})$$

(Question: can we do better?. . . )

# A fourth problem: Clone membership (*CLO*)

INPUT:

- A finite algebra $\mathbf{A}$.
- An operation $g : A^k \to A$.

QUESTION: Is $g$ a term operation of $\mathbf{A}$?

All known algorithms essentially generate the full $k$-generated free algebra in $\mathbf{V}(\mathbf{A})$,

$$\mathbf{F}_k \leq \mathbf{A}^{(A^k)}$$

and test whether $g \in F_k$.

In the worst case this could require as much as $O(|A^{(A^k)}|) = 2^{(|A|^k)^{1+\epsilon}}$ time and space.

I.e., exponential in the size of the input. (More on this in Lecture 3.)

## Definition

1. $P = PTIME = \bigcup_{k=1}^{\infty} TIME(N^k) = TIME(N^{O(1)})$.
2. $PSPACE = \bigcup_{k=1}^{\infty} SPACE(N^k) = SPACE(N^{O(1)})$.

Problems known to be in $P$ are said to be *feasible* or *tractable*.

## Definition

3. $EXPTIME = \bigcup_{k=1}^{\infty} TIME(2^{N^k}) = TIME(2^{N^{O(1)}})$.
4. $L = LOGSPACE = SPACE(\log(N))$.

$$L \subseteq P \subseteq PSPACE \subseteq EXPTIME$$

$$FVAL \in L, \quad PATH \in P, \quad 3COL \in PSPACE, \quad CLO \in EXPTIME$$

# Tomorrow

$$\begin{array}{ccccccc} L & \subseteq & P & \subseteq & PSPACE & \subseteq & EXPTIME \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ FVAL & & PATH & & 3COL & & CLO \end{array}$$

In tomorrow's lecture I will:

- Introduce "nondeterministic" versions of these 4 classes.
- Introduce problems which are "hardest" for each class.