Groups in action
or
How to count (mod symmetry)

Ross Willard

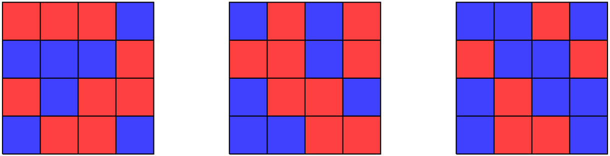
University of Waterloo

University of Northern British Columbia
April 8, 2013

Motivating problem

The newest collectible craze sweeping Northern B.C. is a game played on a 4×4 red-and-blue checkerboard.

The twist: the colour (red/blue) of each square is random.

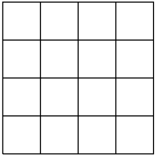


3 different boards

Jennifer is obsessed with this game!

Problem: How many different game boards can she collect?

First solution

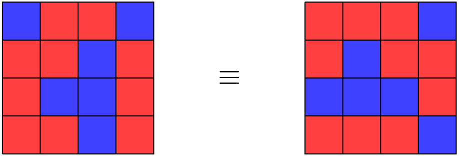


16 squares.

Each square can be  or  .

$\therefore$ There are $2^{16} = 65,536$ distinct boards.

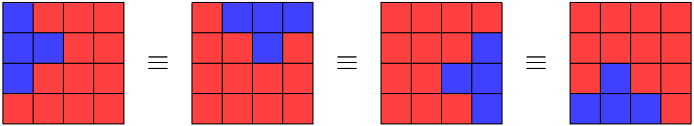
What is wrong with this solution?



Different pictures, same board.

Problem: Different pictures can represent the same board (by rotating), so 2^{16} is too high.

Second solution



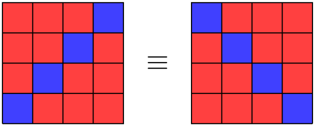
Each board is represented by 4 pictures.

There are 2^{16} distinct pictures.

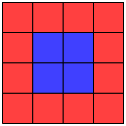
$\therefore$ There are $2^{16}/4 = 16{,}384$ distinct boards.

What is wrong with this solution?

Problem: Some boards are represented by **fewer** than 4 pictures.



Only 2 pictures

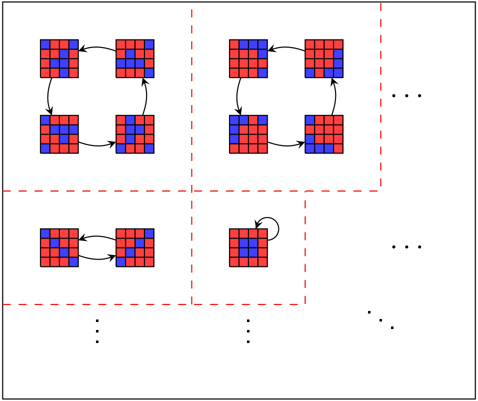


Only 1 picture

So $2^{16}/4 = 16{,}384$ is too low.

(Correct answer: 16,456.)

The Big Picture



Let $X = \{\text{ all } 4 \times 4 \text{ red/blue pictures }\}$

X is **partitioned** into sets (or orbits) of size 4, 2 or 1.

We want to count the number of orbits in this partition.

Generalization: group actions

Suppose G and X are sets.

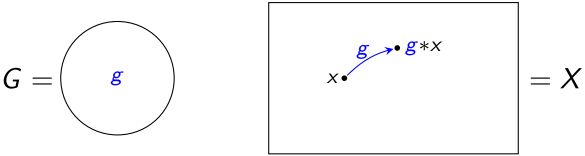
Definition

An operation of G on X is a function $*$ from $G \times X$ to X.

Example

$G = \mathbb{R}$, $X = \mathbb{R}^n$, $* =$ scalar multiplication.

Visualization:

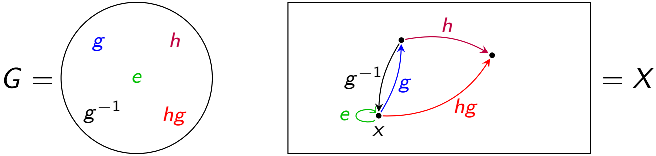


Now suppose G is a *group*[1].

Definition

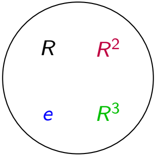
An operation $*$ of G on X is a group action if it satisfies the following (natural) conditions: for all $g, h \in G$ and $x, y \in X$,

(A1) $e * x = x$.
(A2) *If* $g * x = y$, *then* $g^{-1} * y = x$.
(A3) $h * (g * x) = (hg) * x$.



[1] Elements of G can be composed; G contains an identity element e; every element has an inverse.

Our motivating problem is an **example** of a group action.

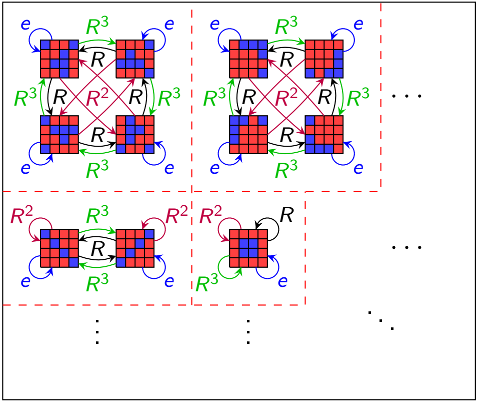


$$G = \{e, R, R^2, R^3\},$$

the cyclic group of order 4

R gives rotation by 90°,
R^2 gives rotation by 180°, etc.

$$X = \{\text{ all } 4 \times 4 \text{ red/blue pictures }\}$$



Orbits and Symmetry sets

Similarly, in any group action, the set X is partitioned into orbits.

Notation

For $x \in X$, we use $\mathcal{O}_x$ to denote the orbit containing x.

Definition

If $g \in G$ and $x \in X$, we say that

g is a symmetry of x, or x is an invariant of g,

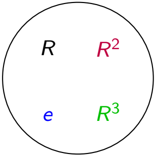
if $g * x = x$.

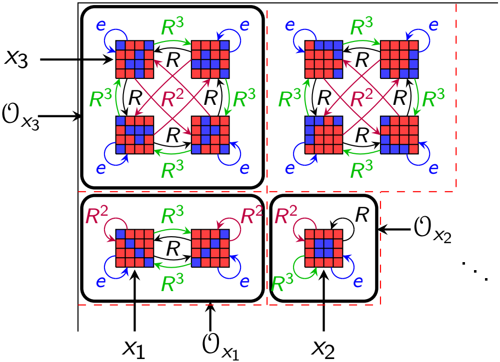
Definition

Given $x \in X$, the symmetry set (or *stabilizer*) of x is

$$G_x := \{g \in G : g * x = x\}.$$

Orbits and Symmetry sets - Example





$$G_{x_1} = \{e, R^2\}$$
$$G_{x_2} = \{e, R, R^2, R^3\}$$
$$G_{x_3} = \{e\}$$

Note: big $\mathcal{O}_x \equiv$ small G_x.

Orbit-Symmetry Set Theorem

For any group action, for any $x \in X$, $\quad |\mathcal{O}_x| \cdot |G_x| = |G|$.

Invariant sets

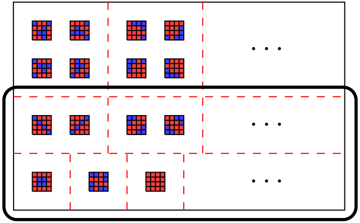
Definition

Given $g \in G$, the invariant set of g is

$$X_g \ := \ \{x \in X \ : \ g * x = x\}.$$

Example: our motivating problem

$X_{R^2} =$



What is X_R? What is X_{R^2}?

Burnside's Lemma

Invariant sets give us a slick way to compute the number of orbits.

Burnside's Lemma

Let $*$ be a group action of a finite group G on a set X. Then

$$\#\text{ of orbits} = \frac{1}{|G|}\sum_{g\in G}|X_g|.$$

That is, the number of orbits of the action is the average size of the invariant set X_g, as g ranges over the group.

Example: the Motivating Problem

$X = \{\text{all } 4 \times 4 \text{ red/blue pictures}\}, \quad G = \{e, R, R^2, R^3\}.$

To use Burnside, we need to know the sizes of the invariant sets.

$$X_e = X, \quad \text{so } |X_e| = 2^{16}.$$

$$X_R = \{x \in X : x \text{ is invariant under } 90^\circ \text{ rotation}\}$$

$$= \left\{ \begin{array}{|c|c|c|c|} \hline a & b & c & a \\ \hline c & u & u & b \\ \hline b & u & u & c \\ \hline a & c & b & a \\ \hline \end{array} : a, b, c, u \in \{r, b\} \right\}$$

4 independent choices from $\{r, b\}$, so $|X_R| = 2^4$.

Similarly,

$$X_{R^2} = \{x \in X : x \text{ is invariant under } 180^\circ \text{ rotation}\}$$

$$= \left\{ \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline f & u & v & e \\ \hline e & v & u & f \\ \hline d & c & b & a \\ \hline \end{array} : a, b, c, d, e, f, u, v \in \{r, b\} \right\}$$

so $|X_{R^2}| = 2^8$.

$$X_{R^3} = \{x \in X : x \text{ is invariant under } 270^\circ \text{ rotation}\}$$

$$= X_R, \quad \text{so } |X_{R^3}| = 2^4.$$

Summary:

$$|X_e| = 2^{16}, \quad |X_R| = |X_{R^3}| = 2^4, \quad |X_{R^2}| = 2^8.$$

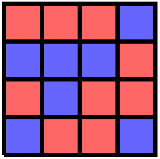
By Burnside's Lemma,

$$\begin{aligned}
\#\text{ of orbits} &= \frac{1}{|G|}\sum_{g\in G}|X_g| \\
&= \frac{1}{4}(|X_e| + |X_R| + |X_{R^2}| + |X_{R^3}|) \\
&= (2^{16} + 2^4 + 2^8 + 2^4)/4 \\
&= 16{,}456.
\end{aligned}$$

Thus there are 16,456 different game boards for Jennifer to collect.

New problem

An internet company from Prince George makes stained-glass windows. They are world-famous for their *Random*™ line of square, 4×4 tiled windows such as the one below:

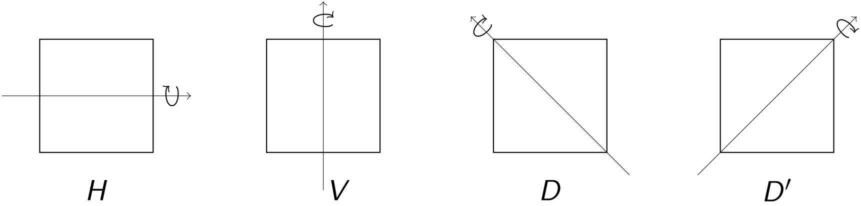


Problem: How many 4×4 windows of this kind can the company make, using just red and blue glass?

Similar to original problem, except there is one new dimension of symmetry: "flipping" (front-to-back).

We can model this problem using:

- The same set X (of 4×4 red/blue pictures).
- The dihedral group $D_4 = \{e, R, R^2, R^3, H, V, D, D'\}$.



D_4 acts naturally on X.

$\#$ of distinct windows $=$ $\#$ of orbits under the action of D_4.

We can use Burnside's Lemma: $\# \text{ of orbits } = \frac{1}{|D_4|} \sum_{g \in D_4} |X_g|$.

We have already calculated $|X_e|$, $|X_R|$, $|X_{R^2}|$ and $|X_{R^3}|$.

Let's count the pictures stabilized by the new group operations:

V:

a	b	b	a
c	d	d	c
e	f	f	e
g	h	h	g

$\therefore \quad |X_V| = 2^8$

Similarly, $|X_H| = 2^8$

D:

a	b	c	d
b	u	w	e
c	w	v	f
d	e	f	g

$\therefore \quad |X_D| = 2^{10}$

Similarly, $|X_{D'}| = 2^{10}$

Summary:

$$|X_e| = 2^{16}, \quad |X_R| = |X_{R^3}| = 2^4, \quad |X_{R^2}| = 2^8,$$
$$|X_H| = |X_V| = 2^8, \quad |X_D| = |X_{D'}| = 2^{10}.$$

Thus by Burnside's Lemma,

$$\begin{aligned} \#\text{ of distinct windows} &= \frac{1}{8}\sum_{g \in D_4} |X_g| \\ &= (2^{16} + 2 \cdot 2^{10} + 3 \cdot 2^8 + 2 \cdot 2^4)/8 \\ &= 8{,}548. \end{aligned}$$

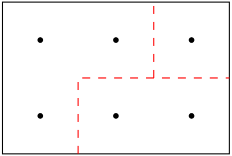
Proof of Burnside's Lemma

Given: G a finite group, X a set, $*$ a group action of G on X.

Goal: to count the number of orbits.

First observation: $$\#\text{ of orbits} = \sum_{x \in X} \frac{1}{|\mathcal{O}_x|}.$$

Proof by example: if $X =$



then

$$\sum_{x \in X} \frac{1}{|\mathcal{O}_x|} = \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \frac{1}{1} + \left(\frac{1}{2} + \frac{1}{2}\right) = 1 + 1 + 1 = 3.$$

Next, create a G-by-X table of all the symmetries.

	x_1	x_2	x_3	$\cdots$	x	$\cdots$	x_{n-1}	x_n
g_1		✓	✓					✓
g_2	✓				✓			
g_3			✓				✓	
$\vdots$								
g		✓			?			
$\vdots$								
g_m	✓						✓	

Put ✓in the (g, x) position if $g * x = x$. (Leave blank otherwise.)

Question: How many ✓s are in the table?

First Answer: by rows:

	⋯	x	⋯
⋮			
g		✓	
⋮			

In Row g, there is a ✓ for each invariant of g (i.e., each $x \in X_g$).

So # of ✓s in Row $g = |X_g|$.

Hence the total of ✓s in the table is $\sum_{g \in G} |X_g|$.

Second Answer: by columns

	$\cdots$	x	$\cdots$
$\vdots$			
g		✓	
$\vdots$			

In Col x, there is a ✓ for each symmetry of x (i.e., each $g \in G_x$).

So # of ✓s in Col $x = |G_x|$.

Hence the total of ✓s in the table is $\sum_{x \in X} |G_x|$.

Equating the two answers:

$$\sum_{g \in G} |X_g| \;=\; \sum_{x \in X} |G_x|.$$

Recall:

$$|\mathcal{O}_x| \cdot |G_x| = |G|, \qquad \text{so} \quad |G_x| = \frac{|G|}{|\mathcal{O}_x|}.$$

Thus

$$\sum_{g \in G} |X_g| \;=\; \sum_{x \in X} \frac{|G|}{|\mathcal{O}_x|},$$

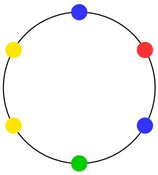
so

$$\frac{1}{|G|} \sum_{g \in G} |X_g| \;=\; \sum_{x \in X} \frac{1}{|\mathcal{O}_x|} \;=\; \#\text{ of orbits.} \qquad \square$$

Homework

Jennifer secretly spends most of her workday hours making bracelets.

Each bracelet consists of 6 beads equally spaced around a circle. Each bead can be green, red, blue or yellow.



Question: How many different bracelets must Jennifer make to have a complete set?

Thank you!