Characterizing $[\alpha, \beta] = 0$ using Kiss terms

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Notation

Suppose $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.

1. $A^{2\times 2} = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} : a, b, c, d \in A \right\}$. $\sim A^4$

2. $R(\alpha, \beta)$ denotes the subuniverse of $\mathbf{A}^{2\times 2}$ consisting of all

$$\begin{array}{ccc} a & \overset{\beta}{\text{———}} & c \\ \alpha \Big| & & \Big| \alpha \\ b & \underset{\beta}{\text{———}} & d \end{array}$$

Observe:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta) \implies (c, c') \in \alpha \cap \beta.$$

More notation

Given $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, let:

$$\operatorname{Const}(\alpha,\beta) := \left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} : x \overset{\alpha}{\equiv} y \right\} \cup \left\{ \begin{bmatrix} u & v \\ u & v \end{bmatrix} : u \overset{\beta}{\equiv} v \right\} \quad \subseteq R(\alpha,\beta)$$

$$M(\alpha,\beta) := \text{the subalgebra of } \mathbf{A}^{2\times 2} \text{ generated by } \operatorname{Const}(\alpha,\beta)$$

$$= \left\{ \begin{bmatrix} t(\mathbf{x},\mathbf{u}) & t(\mathbf{x},\mathbf{v}) \\ t(\mathbf{y},\mathbf{u}) & t(\mathbf{y},\mathbf{v}) \end{bmatrix} : t \text{ a term, } x_i \overset{\alpha}{\equiv} y_i,\ u_j \overset{\beta}{\equiv} v_j \right\}. \quad \leq R(\alpha,\beta)$$

(the "α,β-matrices")

"$[\alpha,\beta]=0$" means

$$\boxed{a = c \iff b = d} \quad \text{for all } \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M(\alpha,\beta).$$

Difference term varieties

Definition

A term $p(x, y, z)$ is a *difference term* for a variety $\mathcal{V}$ if it satisfies

$$p(x, x, y) \approx y \quad \text{throughout } \mathcal{V} \tag{1}$$

$$p(x, y, y) \overset{[\theta,\theta]}{\equiv} x \quad \text{whenever } (x, y) \in \theta \in \operatorname{Con} \mathbf{A} \text{ in } \mathcal{V} \tag{2}$$

$\mathcal{V}$ is a *difference term* (DT) *variety* if it has a difference term.

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$$\overset{y}{\cancel{p(x, x, y)}} \approx y \quad \text{throughout } \mathcal{V} \tag{1}$$

$$\overset{y}{\cancel{p(x, y, y)}} \overset{[\theta,\theta] = \theta}{\equiv} x \quad \text{whenever } (x, y) \in \theta \in \operatorname{Con} \mathbf{A} \text{ in } \mathcal{V} \tag{2}$$

$\mathcal{V}$ is a *difference term* (DT) *variety* if it has a difference term.

Examples of DT varieties

1. Any variety with a Maltsev term.
2. Any CM variety (Herrmann, Gumm).
3. Any SD($\wedge$) variety: they satisfy $[\alpha, \beta] = \alpha \cap \beta$, so $p(x, y, z) := z$ is a difference term.

Some facts (Lipparini, Kearnes, Szendrei 1990s)

1. $[\alpha, \beta] = [\beta, \alpha]$ in DT varieties. (In fact, $[\alpha, \beta] = [\alpha, \beta]_\ell$.)
2. $\{$DT varieties$\}$ is a Maltsev class $\subsetneq$ $\{$Taylor varieties$\}$.

Some facts (Lipparini, Kearnes, Szendrei 1990s)

1. $[\alpha, \beta] = [\beta, \alpha]$ in DT varieties. (In fact, $[\alpha, \beta] = [\alpha, \beta]_\ell$.)
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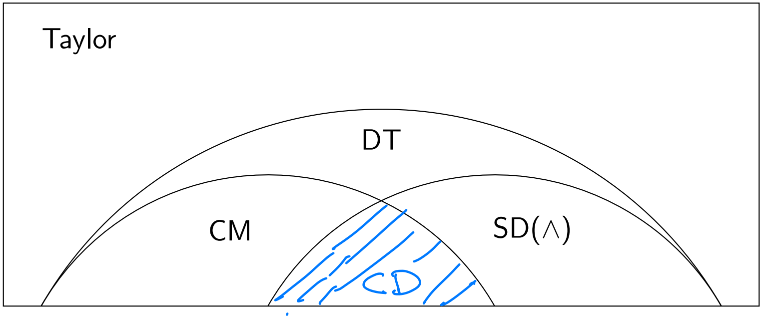

Mantra

If a statement is true for all CM varieties **and** all SD($\wedge$) varieties, then it is probably true for all DT varieties.

Given $\theta \in \operatorname{Con} \mathbf{A}$, let $T(\theta) := \{(a, b, c) : a \overset{\theta}{\equiv} b \overset{\theta}{\equiv} c\}$.

Theorem (Gumm)

Suppose $\mathcal{V}$ is a CM variety with difference term $p(x, y, z)$. Let $\mathbf{A} \in \mathcal{V}$ and $\theta \in \operatorname{Con} \mathbf{A}$. Then $[\theta, \theta] = 0$ if and only if

(G1) p satisfies "the other Maltsev identity" on each θ-block, i.e.,

$$(a, b) \in \theta \implies p(a, b, b) = a.$$

(G2) p restricted to $T(\theta)$ is a homomorphism $\mathbf{T}(\theta) \to \mathbf{A}$.

Given $\theta \in \operatorname{Con} \mathbf{A}$, let $T(\theta) := \{(a,b,c) \,:\, a \overset{\theta}{\equiv} b \overset{\theta}{\equiv} c\}$.

Theorem (Gumm)

Suppose $\mathcal{V}$ is a ~~CM~~ SD(∧) variety with difference term $\cancel{p(x,y,z)}$ z. Let $\mathbf{A} \in \mathcal{V}$ and $\theta \in \operatorname{Con} \mathbf{A}$. Then $[\theta, \cancel{\theta}\ \theta] = 0$ if and only if

(G1) p satisfies "the other Maltsev identity" on each θ-block, i.e.,

$$b = a$$

$$(a,b) \in \theta \implies p(a,b,b) = a.$$

(G2) p restricted to $T(\theta)$ is a homomorphism $\mathbf{T}(\theta) \to \mathbf{A}$.

Fun fact: the theorem is also true in SD(∧) varieties using $p = z$.

So it "should" be true in all DT varieties (and it is: Kearnes 1994).

Definition

A term $q(x, y, z, w)$ is a *Kiss term* for a variety $\mathcal{V}$ if it satisfies

$$q(x, y, x, y) \approx x \quad \text{throughout } \mathcal{V} \tag{3}$$

$$q(x, x, y, y) \approx y \quad \text{throughout } \mathcal{V} \tag{4}$$

$\alpha \cap \beta$

$$q(a, b, c, d) \overset{[\alpha,\beta]}{\equiv} q(a, b, c', d) \quad \text{whenever } \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta)$$

$\alpha \cap \beta$

$$\text{where } \alpha, \beta \in \operatorname{Con} \mathbf{A} \text{ and } \mathbf{A} \in \mathcal{V} \tag{5}$$

Definition

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$$q(a, b, c, d) \overset{[\alpha,\beta]}{\equiv} q(a, b, c', d) \quad \text{whenever } \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta)$$
$$\text{where } \alpha, \beta \in \operatorname{Con} \mathbf{A} \text{ and } \mathbf{A} \in \mathcal{V} \tag{5}$$

Examples of varieties having a Kiss term

1. Any CM variety (Kiss).
2. Any SD($\wedge$) variety: $q(x, y, z, w) := z$ is a Kiss term.
3. Any DT variety (Lipparini):

$$q(x, y, z, w) := p(p(x, z, z), p(y, w, z), z).$$

Theorem (Kiss)

Let $\mathcal{V}$ be a CM variety with Kiss term $q(x, y, z, w)$. Suppose $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$. Then $[\alpha, \beta] = 0$ if and only if

(K1) *q restricted to $R(\alpha, \beta)$ does not depend on its 3rd variable:*

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta) \implies q(a, b, c, d) \overset{[\alpha,\beta]}{=} q(a, b, c', d).$$

(K2) *q restricted to $R(\alpha, \beta)$ is a homomorphism $\mathbf{R}(\alpha, \beta) \to \mathbf{A}$.*

Theorem (Kiss)

Let $\mathcal{V}$ be a CM variety with Kiss term $q(x,y,z,w)$. Suppose $\mathbf{A} \in \mathcal{V}$ *and* $\alpha, \beta \in \operatorname{Con} \mathbf{A}$*. Then* $[\alpha,\beta] = 0$ *if and only if*

(K1) *q restricted to $R(\alpha,\beta)$ does not depend on its 3rd variable:*

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha,\beta) \implies \cancel{q(a,b,c,d)}^{c} = \cancel{q(a,b,c',d)}^{c'}.$$

(K2) *q restricted to $R(\alpha,\beta)$ is a homomorphism $\mathbf{R}(\alpha,\beta) \to \mathbf{A}$.* ✓

Observe: the theorem is also true for SD($\wedge$) varieties with $q = z$.

Proof. $q = z \Rightarrow$ (K2) always.

$\boxed{\text{RTP: } \alpha \cap \beta = 0 \iff \text{(K1)}.}$

($\Rightarrow$) clear.

($\Leftarrow$). Assume (K1), $(a,b) \in \alpha \cap \beta$. $\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \begin{bmatrix} a & b \\ a & a \end{bmatrix} \in R(\alpha,\beta)$

(K1) $\Rightarrow a = b$.

So it "should" also be true for DT varieties.

Theorem (KSW 2016)

Every DT variety $\mathcal{V}$ has a Kiss term q such that in any $\mathbf{A} \in \mathcal{V}$,

$[\alpha, \beta] = 0 \iff$

(K1) q restricted to $R(\alpha, \beta)$ does not depend on z, and

(K2) $q : \mathbf{R}(\alpha, \beta) \to \mathbf{A}$ is a homomorphism.

This theorem was a key step in our proof of Park's Conjecture for DT varieties.

The story gets interesting

Aug 2021: the authors of ALVIN enlist volunteers to help them proofread the forthcoming volumes 2 and 3.

- Our extension of Kiss's Theorem is in Vol. 3.

Oct 2021: Peter Mayr notices that the ALVIN proof of our theorem is bogus. Alerts Ralph Freese.

- Ralph and Peter study our published proof of the theorem. They find that our proof is also bogus!

What we want: $[\alpha,\beta]=0 \iff$

(K1) q restricted to $R(\alpha,\beta)$ does not depend on z

(K2) $q : \mathbf{R}(\alpha,\beta) \to \mathbf{A}$ is a homomorphism.

Easy Lemma

In any algebra with a Kiss term q,

1. $[\alpha,\beta]=0 \implies$ (K1).
2. ((K1) & (K2)) $\implies [\alpha,\beta]=0$.

Pf of 2. Observe

$$q\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = q(abab) = a$$

$$q\left(\begin{bmatrix} a & c \\ a & c \end{bmatrix}\right) = q(aacc) = c$$

$\therefore$ $q = 3^{rd}$ proj. on Const(α,β)

gen's of $M(\alpha,\beta)$

(K2) $\Rightarrow$ $q = 3^{rd}$ proj. on $M(\alpha,\beta)$.

Show $[\alpha,\beta]=0$.

Let $\begin{bmatrix} a & a \\ b & d \end{bmatrix} \in M(\alpha,\beta)$

$\Rightarrow$ $\begin{bmatrix} b & d \\ a & a \end{bmatrix} \in M(\alpha,\beta)$

$\begin{bmatrix} b & b \\ a & a \end{bmatrix} \in M(\alpha,\beta)$

(K1) $\Rightarrow$

$$\underset{\substack{\parallel \\ d}}{q(bada)} = \underset{\substack{\parallel \\ b}}{q(baba)}$$

$\Rightarrow$ $b=d$. ✓

Hard implication (not yet proved): $[\alpha, \beta] = 0 \implies$ (K2).

Reduction ($\approx$ Kiss)

Assume $\mathbf{A}$ has a Kiss term and $[\alpha, \beta] = 0$. To prove (K2), it is enough to prove the existence of $M(\alpha, \beta) \subseteq \Delta \leq \mathbf{R}(\alpha, \beta)$ satisfying

(R1) For all $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta, \quad a = c \iff b = d.$

(R2) For all $\begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha, \beta)$ there exists c with $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta.$

Observe:

- $\Delta := M(\alpha, \beta)$ makes (R1) true.
- $\Delta := R(\alpha, \beta)$ makes (R2) true.

Why this is enough

Assume q is a Kiss term, $[\alpha,\beta]=0$, and $M(\alpha,\beta)\subseteq \Delta \leq \mathbf{R}(\alpha,\beta)$ with

(R1) $a=c \iff b=d \quad \forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$

(R2) For all $\begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ there ~~exists~~ $\exists!$ c with $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Pf. Claim 1. $q(abcd) = c \quad \forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Pf. $\underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\text{given}}, \underbrace{\begin{bmatrix} b & b \\ b & b \end{bmatrix}, \begin{bmatrix} c & c \\ d & d \end{bmatrix}, \begin{bmatrix} d & b \\ d & b \end{bmatrix}}_{\text{in } M(\alpha,\beta)} \in \Delta$. Apply q:

$$\begin{bmatrix} q(abcd) & q(cbcb) \\ q(bbdd) & q(dbdb) \end{bmatrix} = \begin{bmatrix} q(abcd) & c \\ d & d \end{bmatrix} \in \Delta \overset{(R1)}{\Longrightarrow} q(abcd) = c.$$

Claim 2. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta \subseteq R(\alpha,\beta) \Longrightarrow c = c'$.

Pf. Def. of Kiss term $\Longrightarrow$ $\underbrace{q(abcd)}_{\substack{\| \text{ Claim 1} \\ c}} \overset{[\alpha,\beta]=0}{\equiv} \underbrace{q(abc'd)}_{\substack{\| \\ c'}}$

$\boxed{\text{Note: strengthens (R2).}}$

$\Delta \leq \mathbf{R}(\alpha,\beta)$, and

(R2!) $\forall \begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha,\beta) \quad \exists!\ c$ with $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Claim 3. $\forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$,

$q(abcd) =$ the unique c' s.t. $\begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta$.

Pf. Like pf of Claim 2: let $c' =$ unique... as in (R2!)

By def.,

$$q(abcd) \overset{[\alpha,\beta]=0}{\equiv} \underbrace{q(abc'd)}_{=\ c'}$$

by Claim 1.

Claim 3 $\Longrightarrow$ graph$(q|_{R(\alpha,\beta)})$ is pp-def. from $R(\alpha,\beta)$ and Δ.

$\Longrightarrow$ $q|_{R(\alpha,\beta)}$ is a hom, ie. (K2) ∎

When does such Δ exist?

Recall: given $[\alpha, \beta] = 0$, we just need $M(\alpha,\beta) \subseteq \Delta \leq \mathbf{R}(\alpha,\beta)$ with

(R1) $a = c \iff b = d \quad \forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$

(R2) For all $\begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ there exists c with $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Δ exists ...

For (R2), $c := p(abd)$.

- In Maltsev varieties, $\Delta = M(\alpha,\beta)$ works.
- (Kiss) In CM varieties, View $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ as $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) \subseteq \alpha \times \alpha$

Let $\Delta_{\alpha,\beta} :=$ (horizontal) trans. closure of $M(\alpha,\beta)$

$= Cg^{\tilde{\alpha}}(M(\alpha,\beta))$. $\Delta := \Delta_{\alpha,\beta}$ works.

- In SD($\wedge$) varieties, $\Delta = R(\alpha,\beta)$ works! (Exercise)
- In DT varieties...??

Definition (Moorhead, 2021)

Suppose $\alpha, \beta \in \operatorname{Con} \mathbf{A}$. Let $\Delta(\alpha, \beta)$ denote the

$$\boxed{\text{“horizontal } \underline{\text{and}} \text{ vertical transitive closure” of } M(\alpha, \beta)}$$

i.e., the smallest subset $\Delta \subseteq R(\alpha, \beta)$ satisfying

- $M(\alpha, \beta) \subseteq \Delta$
- $\begin{bmatrix} a & x \\ b & y \end{bmatrix}, \begin{bmatrix} x & c \\ y & d \end{bmatrix} \in \Delta \implies \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta.$
- $\begin{bmatrix} a & c \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ b & d \end{bmatrix} \in \Delta \implies \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta.$

$\Delta(\alpha,\beta)$ "is" a congruence on $\underset{\sim}{\alpha}$ (columns)

" " " β (rows)

"2-dim. congruence"

$\Delta(\alpha, \beta) =$ "horizontal <u>and</u> vertical transitive closure" of $M(\alpha, \beta)$

Easy facts

1. $\Delta(\alpha, \beta) \leq \mathbf{R}(\alpha, \beta)$ always.
2. In CM varieties, $\Delta(\alpha, \beta) = \Delta_{\alpha,\beta}$.

<u>Pf.</u> $\Delta_{\alpha,\beta} = \Delta^{\alpha,\beta}$ ("vertical trans. closure of $M(\alpha,\beta)$")

3. In DT varieties, $[\alpha, \beta] = 0 \implies \Delta(\alpha, \beta)$ satisfies (R1), i.e., $[\alpha, \beta]_H = 0$.

$\begin{bmatrix} a & = & c \\ b & = & d \end{bmatrix} \in \Delta(\alpha,\beta)$

(Because $[\alpha, \beta] = [\alpha, \beta]_H = [\alpha, \beta]_\ell$ in DT varieties.)

Does $\Delta(\alpha,\beta)$ satisfy (R2)??

Theorem (KSW)

In DT varieties, $[\alpha,\beta]=0 \implies \Delta(\alpha,\beta)$ satisfies (R2), i.e.,

$$\forall \begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha,\beta), \quad \exists\, c \text{ with } \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta(\alpha,\beta).$$

Proof idea

Let $\mathbf{A}=(A,\cdot)$ be a semilattice and $[\alpha,\beta]=0$, i.e., $\alpha\cap\beta=0$.

Claim: $\Delta(\alpha,\beta)=R(\alpha,\beta)$.

Pf. Let $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$.

$$= \begin{bmatrix} ac & c \\ ac & c \end{bmatrix} \cdot \begin{bmatrix} c & c \\ d & d \end{bmatrix}$$

Can show $\begin{bmatrix} a & ac \\ ab & abc \end{bmatrix} = \begin{bmatrix} ac & c \\ acd & cd \end{bmatrix} \in M(\alpha,\beta)$.

Claim: $abc = acd$. $\implies \begin{bmatrix} a & c \\ ab & cd \end{bmatrix} \in \Delta(\alpha,\beta)$

Pf. $abc = abcc \overset{\alpha}{\equiv} aacd = acd$

$abc \overset{\beta}{\equiv} adc = acd$.

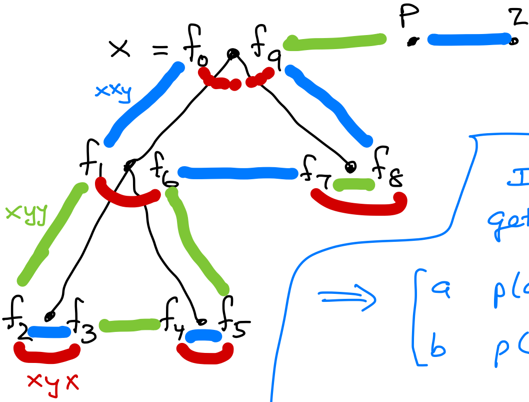
$\begin{bmatrix} ab & cd \\ b & d \end{bmatrix} \in \Delta(\alpha,\beta)$

Proof idea: General ~~SD(∧)~~ DT varieties.

Let $\mathcal{V}$ be ~~SD(∧)~~ DT, $\mathbf{A} \in \mathcal{V}$, and $[\alpha, \beta] = 0$, ~~i.e., $\alpha \cap \beta =$~~ 0.

Claim: ~~$\Delta(\alpha, \beta) = R(\alpha, \beta)$~~. $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta) \implies \ldots$

Pf. Use Maltsev condition for SD(∧).

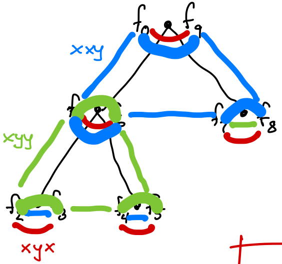


$f_i(x,y,z)$

In DT case, get $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$

$\implies \begin{bmatrix} a & p(acc) \\ b & p(bdd) \end{bmatrix} \in \Delta(\alpha,\beta)$.

Use diff term tricks.



— xxy

— xyy

— xyx

Given $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$

and $\alpha \cap \beta = 0$.

aim to show

$\forall$ node-pair (f_i, f_j),

$$L_{ij} = \begin{bmatrix} f_i(aac) & f_j(aac) \\ f_i(bbd) & f_j(bbd) \end{bmatrix} \in \Delta(\alpha,\beta)$$

and

$$R_{ij} = \begin{bmatrix} f_i(acc) & f_j(acc) \\ f_i(bdd) & f_j(bdd) \end{bmatrix} \in \Delta(\alpha,\beta).$$

At leaves,

$L_{23}, L_{45}, R_{78} \in \Delta(\alpha,\beta)$

Hard part: show

$L_{ij} \in \Delta(\alpha,\beta)$

$\Updownarrow$

$R_{ij} \in \Delta(\alpha,\beta)$

$$L_{09} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta(\alpha,\beta).$$

Putting everything together, we have proved:

Corollary (KSW)

Kiss's characterization of $[\alpha, \beta] = 0$ extends to DT varieties.

Related results

Theorem (KSW)

Suppose q is a Kiss term for $\mathcal{V}$. For any $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$,

1. $\Delta(\alpha, \beta) = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha, \beta) : q(a, b, c, d) \overset{[\alpha,\beta]}{\equiv} c \right\}$.
2. $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha, \beta) \implies \begin{bmatrix} a & q(abcd) \\ b & d \end{bmatrix} \in \Delta(\alpha, \beta)$.

Final comment

Kiss's proof that $\Delta_{\alpha,\beta}$ satisfies (R2) in CM varieties was high-level, using the modular law in $\mathrm{Con}\,\boldsymbol{\alpha}$ and properties of the commutator deducible from a difference term.

Our proof that $\Delta(\alpha, \beta)$ satisfies (R2) in DT varieties is syntactic, using the Maltsev condition for DT varieties.

Question

Are there properties of congruences (or perhaps of 2-dimensional congruences) in DT varieties that could lead to a nicer proof?

References

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Thank you!