Near unanimity constraints have bounded pathwidth duality

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This talk is about variations of 3-SAT...

For example,

- 2-SAT: $x \vee y = 1, \quad \overline{x} \vee y = 1, \quad \overline{x} \vee \overline{y} = 1$
- Horn (3-)SAT: $\overline{x} \vee \overline{y} \vee z = 1, \quad \overline{x} \vee \overline{y} \vee \overline{z} = 1, \quad x = 1$
- 3-XORSAT: $x \oplus y \oplus z = 0, \quad x \oplus y \oplus z = 1$

Unlike 3-SAT, these variations are all tractable (in P).

However, they have different complexities:

- 2-SAT is NL-complete.
- Horn SAT is P-complete.
- 3-XORSAT is $\oplus$L-complete.

These variations are called **constraint satisfaction problems** (CSPs).

Definition

Let $\mathcal{R}$ be a finite set of nonempty boolean relations (i.e., on $\{0, 1\}$).

CSP($\mathcal{R}$) is the variant of 3-SAT in which clauses are replaced by $\mathcal{R}$-**constraints**:

- i.e., assertions that tuples of variables belong to specific relations in $\mathcal{R}$.

A celebrated result:

Boolean Dichotomy Theorem (Schaefer, 1978)

For any finite set $\mathcal{R}$ *of boolean relations,* $\mathrm{CSP}(\mathcal{R})$ *is* NP*-complete or in* P.

The cases in P were further delineated by Allender et al, who showed that there are precisely 5 possible complexities.

They also characterized the sets $\mathcal{R}$ of each complexity. For example:

Corollary of (Allender, Bauland, Immerman, Schnoor, Vollmer, 2009)

(Assume $\oplus$L $\not\subseteq$ NL.) *Let $\mathcal{R}$ be a finite set of nonempty boolean relations, and suppose $\mathcal{R}$ is "nontrivial." The following are equivalent:*

1. $\mathrm{CSP}(\mathcal{R}) \in$ NL.
2. $\mathcal{R}$ *does not "interpret"* Horn SAT *or* 3-XORSAT.
3. $\exists$ *function* $f : \{0,1\}^n \to \{0,1\}$ (*for some* $n \geq 1$) *which*
   - *preserves each relation in* $\mathcal{R}$ ("polymorphism")
   - *satisfies*

$$f(a,\ldots,a,\underset{\substack{\uparrow \\ i}}{b},a,\ldots,a) = a \qquad \forall a,b \in \{0,1\}, \quad \forall 1 \leq i \leq n$$

"near unanimity"

This paper is a contribution to ongoing efforts to extend the Schaefer Dichotomy and its refined version to **larger (non-boolean) domains**...

...specifically, the following conjectured extension of the Allender *et al* corollary:

Conjecture (Larose, Tesson, 2009)

(Assume $\mathsf{Mod}_p\mathsf{L} \not\subseteq \mathsf{NL}$ for all primes p.)

Let $\mathcal{R}$ be a finite set of nonempty relations on a finite domain, and suppose $\mathcal{R}$ is "not reducible" (i.e., *core*). The following are equivalent:

1. $\mathrm{CSP}(\mathcal{R}) \in \mathsf{NL}$.
2. $\mathcal{R}$ does not "interpret" Horn SAT or 3-LinEq(F) for any finite field F.

Conjecture (restated)

($\mathsf{Mod}_p\mathsf{L} \not\subseteq \mathsf{NL}$) Core $\mathcal{R}$ over general domains, the following are equivalent:

1. $\mathrm{CSP}(\mathcal{R}) \in \mathsf{NL}$.
2. $\mathcal{R}$ does not "interpret" Horn SAT or 3-LinEq(F) for any finite field F.

Remarks

- (1) ⇒ (2) is "obvious."
  - It follows from the complexity assumption, the notion of "interpret," and the known complexities of Horn SAT and 3-LinEq(F).
- Universal algebraists know a "polymorphism characterization" of (2) over general domains. It is related to, but strictly weaker than, $\mathcal{R}$ having a near unanimity polymorphism (NUP).

Our main result

We show that $\mathcal{R}$ having a NUP $\Rightarrow \mathrm{CSP}(\mathcal{R}) \in \mathsf{NL}$.

- This is consistent with, but does not prove, the conjecture.

Remarks on the proof

- Intricate and complicated
  - "[It] is a bit of a mess." (anon. referee)
- Heavily indebted to Dalmau, Krokhin (2008), who proved the result in the case of 3-ary NUPs.
- Like DK, we prove $\mathrm{CSP}(\mathcal{R}) \in \mathrm{NL}$ by establishing a technical property called *bounded pathwidth duality*.
- Even stealing everything we can, our proof introduces significant new complications.
- In particular, we needed to show that a certain algebraic property, which is obvious for 3-ary NUPs, has a surprising (to algebraists) weak analogue in NUPs of higher arities.

Thank you