Algebra Universalis, 36 (1996) 573–575 0002-5240/96/040573-03$1.50 + 0.20/0

Mailbox

Problem 17 of Grätzer and Kisielewicz

S. BURRIS AND R. WILLARD

If $\mathbf{A}$ is an algebra and $n < \omega$ then $p_n(\mathbf{A})$ denotes the number of n-ary term operations of $\mathbf{A}$ that depend on all n variables. Let $\mathscr{C}$ denote the class of all algebras of type $\langle 2, 2\rangle$ in which both basic operations are commutative and idempotent. In [6, Problem 17] G. Grätzer and A. Kisielewicz ask whether there exists $\mathbf{A} \in \mathscr{C}$ satisfying $p_2(\mathbf{A}) = 2$ and such that for all $\mathbf{B} \in \mathscr{C}$, if $p_2(\mathbf{B}) = 2$ then $p_n(\mathbf{B}) \geq p_n(\mathbf{A})$ for all n. J. Dudek [4] has shown that if the answer is yes, then the algebra $\mathbf{A}$ can be taken to be the two-element distributive lattice $\mathbf{D}_2$; and more recently has shown [5] that to answer the question it suffices to compare the p_n-sequence of $\mathbf{D}_2$ to the p_n-sequences of two specific four-element members of $\mathscr{C}$. One of these 'test members' is the idempotent commutative groupoid $\mathbf{N}_2$ whose universe is $\{0, 1, 2, 3\}$ and whose binary operation $\circ$ is given by the following table:

$\circ$	0	1	2	3
0	0	2	3	3
1	2	1	3	3
2	3	3	2	3
3	3	3	3	3

(To consider $\mathbf{N}_2$ as a member of $\mathscr{C}$, let $\circ$ also be the second operation.) Using a computer, we can show that $p_4(\mathbf{N}_2) = p_4(\mathbf{D}_2) = 114$ but $p_5(\mathbf{N}_2) = 2586 < 6894 = p_5(\mathbf{D}_2)$, which solves Problem 17 negatively. The purpose of this note is to explain the tricks we use to make the computation of $p_5(\mathbf{N}_2)$ feasible.

Presented by J. Berman.
Received November 3, 1995; accepted in final form April 24, 1996.
1991 Mathematics Subject Classification. Primary 08A40; Secondary 08-04.
Key words and phrases. p_n sequence, free spectra, minimal extension property.
This research was supported by the NSERC of Canada.

Fix $n \geq 2$ and define

$$\Gamma_n = \{\mathbf{a} \in \{0, 1, 2\}^n : \exists j(a_j = 1) \ \& \ \exists i(a_i = 0 \ \& \ \forall j(a_j = 1 \rightarrow i < j))\}.$$

For $i = 2, \ldots, n$ define $\mathbf{e}^i = (e_1^i, \ldots, e_n^i) \in \{0, 1\}^n$ so that $e_j^i = 1$ iff $j = i$. Also define $\mathbf{e}^1 = (0, 1, 1, \ldots, 1) \in \{0, 1\}^n$. Note that $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\} \subseteq \Gamma_n$.

CLAIM. *If $s(\mathbf{x})$ and $t(\mathbf{x})$ are terms built from $\circ$ and $x_1, \ldots, x_n$ such that $\mathbf{N}_2 \nvDash s(\mathbf{x}) \approx t(\mathbf{x})$, then there exists $\mathbf{a} \in \Gamma_n$ such that $s(\mathbf{a}) \neq t(\mathbf{a})$.*

Indeed, suppose that $s(\mathbf{e}^i) = t(\mathbf{e}^i)$ for all $i = 1, \ldots n$. Clearly $s(\mathbf{e}^1) = 1$ iff x_1 does not occur in s, while for $i > 1$ $s(\mathbf{e}^i) = 0$ iff x_i does not occur in s. Similar remarks hold for t, and hence s and t mention the same variables. This implies that $s(\mathbf{a}) = t(\mathbf{a})$ whenever $a_i = 3$ for some i such that x_i occurs in s or t (since 3 is an absorbing element for $\mathbf{N}_2$); or whenever $\{a_1, \ldots, a_n\}$ is a subset of either $\{0, 2, 3\}$ or $\{1, 2, 3\}$ (since the latter are subuniverses for which the corresponding subalgebras are semilattices).

Therefore, if $\mathbf{N}_2 \nvDash s(\mathbf{x}) \approx t(\mathbf{x})$, there must exist $\mathbf{a} \in \{0, 1, 2\}^n$ such that $\{0, 1\} \subseteq \{a_1, \ldots, a_n\}$ and $s(\mathbf{a}) \neq t(\mathbf{a})$. Since the map which switches 0 and 1 but leaves 2 and 3 fixed is an automorphism of $\mathbf{N}_2$, we can assume that the first occurrence of 0 in $\mathbf{a}$ precedes the first occurrence of 1, i.e., $\mathbf{a} \in \Gamma_n$.

The Claim has the practical effect that the free algebra on 5 generators in the variety generated by $\mathbf{N}_2$ can be computed as a 5-generated subalgebra of $(\mathbf{N}_2)^{90}$, whereas the "standard" construction of the free algebra would be as a 5-generated subalgebra of $(\mathbf{N}_2)^{1024}$. This gives a better-than-tenfold reduction in the space required to represent the elements of the free algebra, as well as in the time needed to construct and compare them. Burris wrote a C program to compute the relevant subalgebra of $(\mathbf{N}_2)^{90}$. In the notation of [2], the program successively computes $E^0(X) = X, E^1(X), E^2(X), \ldots$ where X is the set consisting of the five generators while $E^{n+1}(X)$ is the union of $E^n(X)$ with the set of all products of pairs of members of $E^n(X)$. The sets $E^n(X)$ are represented as (linked) lists so that $E^n(X)$ is an initial segment of $E^{n+1}(X)$. This allows certain efficiencies in computing $E^{n+1}(X)$ when $n > 0$, namely: one only has to compute the products $\mathbf{x} \circ \mathbf{y}$ where $\mathbf{x} \in E^n(X)$ and $\mathbf{y} \in E^n(X) \backslash E^{n-1}(X)$ and either $\mathbf{x} = \mathbf{y}$ or $\mathbf{x}$ precedes $\mathbf{y}$ (by commutativity of $\circ$). The program halts as soon as it discovers $E^{n+1}(X) = E^n(X)$. Running on a SPARCclassic, the program took 36 minutes to determine $|\mathbb{F}_{\mathscr{V}(\mathbf{N}_2)}(5)| = 3281$. Essentially the same program computed $|\mathbb{F}_{\mathscr{V}(\mathbf{N}_2)}(4)| = 170$ in 3 seconds and $|\mathbb{F}_{\mathscr{V}(\mathbf{N}_2)}(3)| = 19$ instantly. Using $|\mathbb{F}_{\mathscr{V}(\mathbf{N}_2)}(2)| = 4$ and formula (C2) from [6] one obtains $p_3(\mathbf{N}_2) = 10$, $p_4(\mathbf{N}_2) = 114$ and $p_5(\mathbf{N}_2) = 2586$ as claimed. (In a similar fashion, the values of $p_n(\mathbf{D}_2)$, $n \leq 5$, can be computed from the cardinalities of the free distributive lattices of rank ≤ 5. These cardinalities are well known; e.g., see [3] or [1, p. 63].)

Acknowledgement

We wish to thank J. Dudek for directing us to the problem considered in this note.

REFERENCES

[1] G. BIRKHOFF, *Lattice Theory*, 3rd ed., Amer. Math. Soc., Providence (RI), 1967.
[2] S. BURRIS and H. P. SANKAPPANAVAR, *A Course in Universal Algebra*, Springer, New York, 1981.
[3] R. CHURCH, *Numerical analysis of certain free distributive structures*, Duke Math. J. *6* (1940), 732–734.
[4] J. DUDEK, *A characterization of distributive lattices*, in *Contributions to lattice theory (Szeged 1980)*, Colloq. Math. Soc. János Bolyai, 33, North-Holland, Amsterdam, pp. 325–335.
[5] J. DUDEK, *The minimal extension of sequences II. On Problem 17 of Grätzer and Kisielewicz*, manuscript, 1995.
[6] G. GRÄTZER and A. KISIELEWICZ, *A survey of some open problems on p_n-sequences and free spectra of algebras and varieties*, in *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith (Eds.), Heldermann, Berlin, 1992, pp. 57–88.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
e-mail: snburris@thoralf.uwaterloo.ca
e-mail: rdwillar@flynn.uwaterloo.ca