A Characterization of Congruence Permutable Locally Finite Varieties

Matthew A. Valeriote*
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
Canada L8S 4K1

Ross Willard†
Department of Mathematics
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213-3890

July 28, 1989

Abstract

A variety $\mathcal{V}$ of universal algebras is said to be congruence permutable if for every algebra $\mathbf{A}$ of $\mathcal{V}$ and every pair of congruences α, β from $\mathbf{A}$ we have $\alpha \circ \beta = \beta \circ \alpha$. We show that if $\mathcal{V}$ is locally finite (i.e., every finitely generated member of $\mathcal{V}$ is finite) then congruence permutability is equivalent to a local property of the finite members of $\mathcal{V}$, expressible in the language of tame congruence theory. This answers a question of R. McKenzie and D. Hobby.

*Support of NSERC is gratefully acknowledged
†Research supported by an Ontario Graduate Scholarship

1 Introduction

By an **algebra** we mean simply any structure $\mathbf{A} = \langle A, f_i(i \in I)\rangle$ consisting of a nonvoid set A, called the **universe**, and a system of finitary operations f_i over A. A **congruence** of an algebra $\mathbf{A}$ is an equivalence relation θ on the universe of $\mathbf{A}$ which is compatible with the operations of $\mathbf{A}$, in the sense that an algebraic structure can be defined naturally on the set A/θ of θ equivalence classes.

The **relational product** of two congruences α and β of $\mathbf{A}$ is the relation

$$\alpha \circ \beta = \{\langle a, b\rangle \,:\, \langle a, c\rangle \in \alpha \text{ and } \langle c, b\rangle \in \beta \text{ for some element } c\}.$$

$\mathbf{A}$ has **permuting congruences**, or is **congruence permutable**, if for all congruences α and β of $\mathbf{A}$, $\alpha \circ \beta = \beta \circ \alpha$. A variety (i.e., a class of algebras defined by a set of equations) $\mathcal{V}$ is congruence permutable if every algebra in $\mathcal{V}$ is congruence permutable.

Many familiar algebraic structures, such as groups, rings and modules, are congruence permutable and many deep results in universal algebra have been proved about algebras and varieties having permuting congruences. One of the earliest and most important results dealing with permutability was proved by A. I. Maltsev in [5]. He showed that a variety $\mathcal{V}$ is congruence permutable if and only if there is some term $t(x, y, z)$ in the language of $\mathcal{V}$ such that

$$\mathcal{V} \models t(x, x, y) \approx t(y, x, x) \approx y.$$

For example, the appropriate terms for groups and rings are $x \cdot y^{-1} \cdot z$ and $x - y + z$ respectively.

Since Maltsev's work on congruence permutability, other properties of varieties, such as congruence distributivity and congruence modularity, have been shown to have similar sorts of characterizations using terms and equations. Characterizations of this sort are now called Maltsev conditions; see Taylor [7] or Neumann [6] for the general definition.

In the early 1980's, Ralph McKenzie and his student David Hobby made important advances in the study of finite algebras and locally finite varieties. (A variety is locally finite if every finitely generated member is finite.) They developed a theory called *tame congruence theory* which amongst other things demonstrates that the lattice of congruences of any finite algebra determines to a great degree the structure of that algebra. A brief overview of this theory will be given in the next section.

One of the many interesting aspects of the work of McKenzie and Hobby can be found in Chapter 8 of [3]. There they show that congruence distributivity and congruence modularity in locally finite varieties can be characterized in the language of tame congruence theory. It is left open whether or not the condition stated in Exercise 8.8 (1) provides a tame congruence theoretic characterization of locally finite congruence permutable varieties. We answer that question here in the affirmative.

2 Tame Congruence Theory

One of the key steps in the development of tame congruence theory was the realization that locally the behaviour of finite algebras is quite limited. This is made precise in the following definitions and theorems. The reader may wish to refer to [3] for further details and proofs. (For the basic theory of universal algebra, consult [1].)

By a **polynomial operation** of the algebra $\mathbf{A}$ we mean an operation on A (the universe of $\mathbf{A}$) of the form $t^{\mathbf{A}}(x_1, \ldots, x_n, c_1, \ldots, c_m)$, where t is a term of $\mathbf{A}$ and $c_1, \ldots, c_m$ are elements in A. We denote the set of all polynomials of $\mathbf{A}$ by Pol $\mathbf{A}$, and the set of all unary ($n = 1$) polynomials of $\mathbf{A}$ by $\mathrm{Pol}_1\mathbf{A}$. Two algebras $\mathbf{A}$ and $\mathbf{A}'$ having the same universe are called **polynomially equivalent** if Pol $\mathbf{A}$ = Pol $\mathbf{A}'$.

Definition 2.1 Let $\mathbf{A}$ be a finite algebra and let α and β be congruences of $\mathbf{A}$.

(1) We say that a function $f : A \to A$ **collapses** β into α and write $f(\beta) \subset \alpha$ if $\langle f(a), f(b)\rangle \in \alpha$ for all $\langle a, b\rangle \in \beta$.

(2) By a **congruence quotient** of $\mathbf{A}$ we mean a pair $\langle \alpha, \beta\rangle$ of congruences of $\mathbf{A}$ such that $\alpha < \beta$. A congruence quotient $\langle \alpha, \beta\rangle$ of $\mathbf{A}$ is called a **prime quotient** iff β *covers* α in Con $\mathbf{A}$ (the lattice of congruences of $\mathbf{A}$ ordered by inclusion). The relation of covering between two elements of Con $\mathbf{A}$ is written $\alpha \prec \beta$.

(3) Let $\langle \alpha, \beta\rangle$ be a congruence quotient of $\mathbf{A}$ and let

$$U_{\mathbf{A}}(\alpha, \beta) = \{f(A) : f \in \mathrm{Pol}_1\mathbf{A} \text{ and } f(\beta) \not\subset \alpha\}$$

and $M_{\mathbf{A}}(\alpha, \beta)$ be the set of all minimal members of $U_{\mathbf{A}}(\alpha, \beta)$ relative to the ordering of inclusion. A member of $M_{\mathbf{A}}(\alpha, \beta)$ is called an $\langle\alpha, \beta\rangle$-**minimal set** of $\mathbf{A}$.

Definition 2.2 Let $\mathbf{A}$ be a finite algebra and suppose that $\alpha \prec \beta \in \mathrm{Con}\,\mathbf{A}$. By an $\langle\alpha, \beta\rangle$-**trace** in $\mathbf{A}$ we mean any set $N \subset A$ such that for some $U \in M_{\mathbf{A}}(\alpha, \beta)$, $N \subset U$ and N is of the form $(x/\beta) \cap U$ for some $x \in U$ such that $(x/\alpha) \cap U \neq (x/\beta) \cap U$. The **body** and the **tail** of an $\langle\alpha, \beta\rangle$-minimal set U with respect to $\langle\alpha, \beta\rangle$ are defined by

$$\text{body} = \bigcup\{\langle\alpha, \beta\rangle\text{-traces contained in } U\},$$

$$\text{tail} = U - \text{body}.$$

For U a nonvoid subset of an algebra $\mathbf{A}$, we let $(\mathrm{Pol}\,\mathbf{A})|_U$ denote the set of all $f|_U$ where $f \in \mathrm{Pol}\,\mathbf{A}$ and U is closed under f. The (non-indexed) algebra $\mathbf{A}|_U$ having universe U and fundamental operations $(\mathrm{Pol}\,\mathbf{A})|_U$ is called the **algebra induced by A on** U.

In tame congruence theory we focus on the algebras induced on the minimal sets, bodies and traces of finite algebras. We close this section by stating the features of these algebras we will need in the next section.

THEOREM 2.3 *Let $\mathbf{A}$ be a finite algebra and let $\langle\alpha, \beta\rangle$ be a prime quotient of $\mathbf{A}$. If N_1 and N_2 are $\langle\alpha, \beta\rangle$-traces then $\alpha|_{N_i}$ is a congruence of $\mathbf{A}|_{N_i}$ for $i = 1, 2$ and the algebras $(\mathbf{A}|_{N_1})/(\alpha|_{N_1})$ and $(\mathbf{A}|_{N_2})/(\alpha|_{N_2})$ are isomorphic. Furthermore these quotient algebras are polynomially equivalent to exactly one algebra (up to isomorphism) from the following list:*

(1) *a faithful G-set, for some finite group G,*

(2) *a vector space,*

(3) *a two-element Boolean algebra,*

(4) *a two-element lattice,*

(5) *a two-element semilattice.*

We say that the **type** *of the prime quotient $\langle\alpha, \beta\rangle$ is equal to i if the algebra $(\mathbf{A}|_{N_1})/(\alpha|_{N_1})$ is polynomially equivalent to an algebra in the ith entry of this list. We denote this type by $\mathrm{typ}(\alpha, \beta)$.*

Definition 2.4

(1) Let $\langle \delta, \gamma \rangle$ be any congruence quotient of a finite algebra $\mathbf{A}$. We define $\mathrm{typ}\{\delta, \gamma\}$ to be the set

$$\{\mathrm{typ}(\alpha, \beta) \, : \, \delta \leq \alpha \prec \beta \leq \gamma\}.$$

(2) For a finite algebra $\mathbf{A}$ we define $\mathrm{typ}\{\mathbf{A}\}$ to be $\mathrm{typ}\{0_A, 1_A\}$.

(3) For a class $\mathcal{K}$ of algebras we define $\mathrm{typ}\{\mathcal{K}\}$ to be the set

$$\bigcup \{\mathrm{typ}\{\mathbf{A}\} : \mathbf{A} \in \mathcal{K} \text{ and } \mathbf{A} \text{ is finite}\}\,.$$

We say that a finite algebra **omits type** i if $i \notin \mathrm{typ}\{\mathbf{A}\}$. A class $\mathcal{K}$ omits type i if every finite member of $\mathcal{K}$ does so.

THEOREM 2.5 *Let $\langle \alpha, \beta \rangle$ be a prime congruence quotient of a finite algebra $\mathbf{A}$ and let $U \in M_{\mathbf{A}}(\alpha, \beta)$ and B be the $\langle \alpha, \beta \rangle$-body of U.*

(i) *If $\mathrm{typ}(\alpha, \beta) = \mathbf{2}$ then $\mathbf{A}|_B$ is a Maltsev algebra and hence is congruence permutable.*

(ii) *If $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{3}, \mathbf{4}\}$ then B is a two element set and so $\mathbf{A}|_B$ is congruence permutable.*

3 Permutability

In this section we prove that a finite algebra $\mathbf{A}$ is congruence permutable if it omits types **1** and **5** and satisfies the condition given in the next definition. Of course the converse is not true since any simple algebra is trivially congruence permutable.

Definition 3.1 Let $\mathbf{A}$ be a finite algebra and let $\langle \alpha, \beta \rangle$ be a prime quotient of $\mathbf{A}$. We say that this quotient satisfies the HM condition if whenever $\langle a, b \rangle \in \beta - \alpha$ then there is some u in A and some $\langle \alpha, \beta \rangle$-trace N such that $\langle u, b \rangle \in \alpha$ and $\{a, u\} \subseteq N$. A finite algebra satisfies the HM condition if every one of its prime quotients does, and a class of algebras satisfies the HM condition if every finite algebra in it does so.

The following theorem was originally proved by P. Idziak in [4] under the assumption of congruence modularity. R. McKenzie subsequently noticed that this assumption was not necessary.

THEOREM 3.2 *If a finite algebra* $\mathbf{A}$ *fails to be congruence permutable then there are congruences* α, β *and* γ *of* $\mathbf{A}$ *such that* $\gamma \prec \alpha$, $\gamma \prec \beta$ *and* α *and* β *do not permute.*

PROOF. Choose a pair of nonpermuting congruences of $\mathbf{A}$ whose meet is maximal amongst all such pairs and call this meet γ. Next, choose a pair $\langle \alpha, \beta \rangle$ of nonpermuting congruences, minimal in the lattice $(\mathrm{Con}\,\mathbf{A})^2$ amongst all those nonpermuting pairs whose meet is γ.

Following the proof given by Idziak we now show that α and β cover γ. Let α_0 and β_0 be any congruences satisfying $\gamma \leq \alpha_0 < \alpha$ and $\gamma \leq \beta_0 < \beta$, and let $\theta = \alpha_0 \vee \beta_0$. It will suffice to show that necessarily $\theta = \gamma$, so assume for the sake of argument that $\theta > \gamma$. Let $\alpha' = \alpha \vee \theta$ and $\beta' = \beta \vee \theta$; then α' and β' permute by the maximality of γ. Note also that $\langle \alpha_0, \beta \rangle$, $\langle \alpha, \beta_0 \rangle$ and $\langle \alpha_0, \beta_0 \rangle$ are permuting pairs, by the minimality of $\langle \alpha, \beta \rangle$.

Now

$$\alpha' = \alpha \vee \theta = \alpha \vee \alpha_0 \vee \beta_0 = \alpha \vee \beta_0 = \beta_0 \circ \alpha$$

and similarly

$$\beta' = \beta \vee \theta = \beta \vee \alpha_0 \vee \beta_0 = \beta \vee \alpha_0 = \beta \circ \alpha_0$$

so

$$\begin{aligned} \alpha \circ \beta &\subset \alpha' \circ \beta' \\ &= \beta' \circ \alpha' \\ &= (\beta \circ \alpha_0) \circ (\beta_0 \circ \alpha) \\ &= \beta \circ (\beta_0 \circ \alpha_0) \circ \alpha \\ &= \beta \circ \alpha. \end{aligned}$$

This implies that α and β permute, which is a contradiction. ∎

LEMMA 3.3 *Let* $\mathbf{A}$ *be a finite algebra with congruences* α, β, γ *and* θ *with* $\alpha \wedge \beta = \gamma$, $\alpha \vee \beta = \theta$, $\beta \prec \theta$ *and* $\mathrm{typ}(\beta, \theta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. *Further, suppose that the prime quotient* $\langle \beta, \theta \rangle$ *satisfies the* HM *condition. Then* α *and* β *permute.*

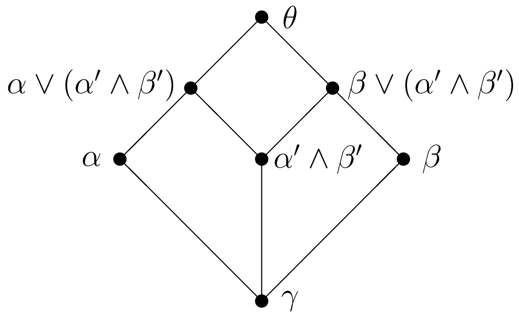

Figure 1: The lattice $\mathbf{D}_1$.

PROOF. It will suffice to show that $\theta - \beta \subset \alpha \circ \beta$. Let $\langle a, b\rangle \in \theta - \beta$ and let $u \in A$ and $N \subseteq A$ be an element of A and a $\langle \beta, \theta\rangle$-trace so that a, u are in N and $\langle u, b\rangle \in \beta$. Choose a $\langle \beta, \theta\rangle$-minimal set U which contains N and let B be the body of U. Since the type of the quotient $\langle \beta, \theta\rangle$ is **2**, **3** or **4**, by Theorem 2.5 the induced algebra $\mathbf{A}|_B$ is congruence permutable.

Since a and u belong to N, then $\langle a, u\rangle \in \theta|_B$ and so $\langle a, b\rangle \in \theta|_B \circ \beta$. From Lemma 2.4 of [3] we know that since $\theta = \alpha \vee \beta$, then $\theta|_B = \alpha|_B \vee \beta|_B$ in Con $\mathbf{A}|_B$. As $\mathbf{A}|_B$ is congruence permutable, it follows that this join is equal to $\alpha|_B \circ \beta|_B$ and so $\langle a, b\rangle \in (\alpha|_B \circ \beta|_B) \circ \beta$ which is contained in $\alpha \circ \beta$ as required. ∎

THEOREM 3.4 *Let* $\mathbf{A}$ *be a finite algebra that omits types* **1** *and* **5** *and is such that every prime quotient of* $\mathbf{A}$ *satisfies the* HM *condition. Then* $\mathbf{A}$ *is congruence permutable.*

PROOF. By Theorem 3.2 it suffices to show that congruences α and β permute whenever they both cover their meet in Con $\mathbf{A}$. Let α and β be such a pair and let γ and θ be their meet and join respectively.

According to Lemma 3.3 it will be enough to show that θ covers at least one of α and β. If θ covers neither, then choose congruences α' and β' such that $\alpha < \alpha' < \theta$ and $\beta < \beta' \prec \theta$. If $\alpha' \wedge \beta' \neq \gamma$ then the congruences α, β and $\alpha' \wedge \beta'$ generate a sublattice of Con $\mathbf{A}$ isomorphic to the lattice $\mathbf{D}_1$ as pictured in Figure 1. This contradicts Lemma 6.4 of [3] since $\mathbf{A}$ omits type **1**. Thus $\alpha' \wedge \beta' = \gamma$. By Lemma 3.3, using the congruences γ, α, α', β' and θ

we conclude that β' permutes with both α and α'. But then $\{\gamma, \alpha, \alpha', \beta', \theta\}$ forms a permuting nonmodular sublattice of Con $\mathbf{A}$, which is impossible. ∎

This last theorem plus some results from [3] yield several tame congruence theoretic characterizations of locally finite congruence permutable varieties, which we list in the next Corollary. Condition **(vi)** below was proposed by McKenzie and Hobby in Exercise 8.8 (1) of [3].

COROLLARY 3.5 *Let $\mathcal{V}$ be a locally finite variety. The following are equivalent:*

(i) *$\mathcal{V}$ is congruence permutable,*

(ii) *The finite members of $\mathcal{V}$ are congruence permutable,*

(iii) *For any finite algebra $\mathbf{A}$ in $\mathcal{V}$, the covers of 0_A in Con $\mathbf{A}$ permute,*

(iv) *There is a term $t(x, y, z)$ such that $\mathcal{V} \models t(x, x, y) \approx t(y, x, x) \approx y$,*

(v) *$\mathcal{V}$ omits types $\mathbf{1}$ and $\mathbf{5}$ and satisfies the* HM *condition,*

(vi) *$\mathcal{V}$ omits types $\mathbf{1}$, $\mathbf{4}$ and $\mathbf{5}$ and satisfies the* HM *condition,*

(vii) *$\mathcal{V}$ omits types $\mathbf{1}$, $\mathbf{4}$ and $\mathbf{5}$ and satisfies the* HM *condition, and for all finite $\mathbf{A}$ in $\mathcal{V}$ and all prime quotients $\langle\alpha, \beta\rangle$ of $\mathbf{A}$, the $\langle\alpha, \beta\rangle$-minimal sets have empty tails.*

PROOF. The equivalence of **(i)**, **(ii)** and **(iv)** is well known, and their equivalence to **(iii)** follows from Theorem 3.2. The implications **(vii)** ⇒ **(vi)** ⇒ **(v)** are trivial, while **(v)** ⇒ **(ii)** follows from Theorem 3.4. To prove **(i)** ⇒ **(vii)**, assume $\mathcal{V}$ is congruence permutable and let $t(x, y, z)$ be a term as in condition **(iv)**. Using Theorems 9.14 and 8.5 of [3] we conclude that $\mathcal{V}$ omits types $\mathbf{1}$, $\mathbf{4}$ and $\mathbf{5}$ and has empty tails.

That $\mathcal{V}$ satisfies the HM condition is the content of Exercise 8.8 (1)(ii) in [3]; for completeness we provide a proof. Let $\mathbf{A}$ be a finite member of $\mathcal{V}$ and let $\langle\alpha, \beta\rangle$ be a prime quotient of $\mathbf{A}$. From Lemmas 5.22 and 5.24 (2) of [3] we gather that β is equal to the binary relation

$$\alpha \circ \{N^2 : N \text{ is an } \langle\alpha, \beta\rangle\text{-trace}\} \circ \alpha.$$

Thus if $\langle a, b\rangle \in \beta - \alpha$ then there are elements c and d belonging to some $\langle \alpha, \beta\rangle$-trace N such that $\langle a, c\rangle$ and $\langle d, b\rangle$ belong to α and $\langle c, d\rangle \in \beta - \alpha$.

Consider the unary polynomial $p(x) = t^{\mathbf{A}}(x, c, a)$. Since $p(c) = a$ and $p(d) \equiv_\alpha t^{\mathbf{A}}(d, c, c) = d \equiv_\alpha b$ it follows that p does not collapse the relation $\beta|_N$ into α. So by Exercise 2.19 (6) of [3] we conclude that $p(N) = N'$ is an $\langle \alpha, \beta\rangle$-trace. This trace contains a and the element $u = p(d)$ which is α-related to b. Thus we have verified the HM condition for the quotient $\langle \alpha, \beta\rangle$. ∎

4 Conclusion

In Chapter 9 of [3] McKenzie and Hobby show that for certain subsets I of $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ the property $\mathrm{typ}\{\mathcal{V}\} \cap I = \emptyset$ for a locally finite variety $\mathcal{V}$ is equivalent to an algebraic condition on the congruence lattices of the members of $\mathcal{V}$, which in turn can be characterized by a Maltsev condition. One example is the following result (Theorem 9.14 in [3]): a locally finite variety omits types **1**, **4** and **5** if and only if it is n-permutable for some n, if and only if for some n it satisfies the Maltsev condition for n-permutability given in [2]. (A variety is n-permutable if for every algebra $\mathbf{A}$ in the variety and for every pair of congruences α and β of $\mathbf{A}$, $\alpha \circ^n \beta = \beta \circ^n \alpha$, where $\alpha \circ^n \beta$ means $\alpha \circ \beta \circ \alpha \circ \cdots$ with $n - 1$ occurrences of $\circ$.)

Our Corollary 3.5 can be seen as a refinement of this last result of McKenzie and Hobby to the case $n = 2$. It would be interesting to obtain similar results for other fixed values of n. In particular, we pose the following problem.

PROBLEM: *Find a tame congruence theoretic characterization of the locally finite 3-permutable varieties.*

McKenzie and Hobby introduced a class of Maltsev conditions which they call "special " (see Chapter 9 in [3]). All of the Maltsev conditions we have considered in this paper fall into this class.

PROBLEM: *Does every special Maltsev condition for locally finite varieties have a tame congruence theoretic description?*

References

[1] Stanley Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer-Verlag, 1981.

[2] J. Hagemann and A. Mitschke. On n-permutable congruences. *Algebra Universalis*, 3:8–12, 1973.

[3] David Hobby and Ralph McKenzie. *The Structure of Finite Algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, 1988.

[4] P. M. Idziak. Varieties with decidable finite algebras II: Permutability. *Algebra Universalis*, 26:247–256, 1989.

[5] A. I. Maltsev. On the general theory of algebraic systems. *Mat. Sb. (N.S.)*, 35:3–20, 1954.

[6] W. D. Neumann. On Mal'cev conditions. *J. Austral. Math. Soc.*, 17:376–384, 1974.

[7] W. Taylor. Characterizing Mal'cev conditions. *Algebra Universalis*, 3:351–397, 1973.