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A note on indecomposable lattices

ROSS WILLARD

It is easy to show that the class of directly indecomposable lattices with 0 and 1 is definable by a single first-order sentence. In this note we prove that the class of indecomposable lattices with 0 cannot be so defined. Indeed, neither the indecomposables nor the decomposables in the variety of distributive lattices with 0 are definable by any class of $\mathscr{L}_{\infty\omega}$ sentences.

DEFINITION. Let L be the set of all countable, compact subsets of $\mathbb{R}$, and let $\mathbf{L}$ be the corresponding distributive lattice (with 0) under union and intersection.

CLAIM 1. $\mathbf{L}$ is indecomposable.

Proof. Otherwise, there would exist ideals I, J of $\mathbf{L}$ such that

(i) $I \neq \{\varnothing\}$, $J \neq \{\varnothing\}$.
(ii) $b \cap c = \varnothing$ for all $b \in I$, $c \in J$.
(iii) For every $a \in L$ there exist $b \in I$, $c \in J$ such that $a = b \cup c$.

Let $U = \bigcup I$ and $V = \bigcup J$. Then it follows from (i)–(iii) that U, V partition $\bigcup L = \mathbb{R}$, and that $a \in L$ implies $a \cap U$, $a \cap V \in L$. Since $\mathbb{R}$ is connected, at least one of U, V is not closed. Suppose U is not closed and pick $x \in \bar{U} \setminus U$; also pick $y_n \in U$ $(n \geq 0)$ such that $|x - y_{n+1}| < \frac{1}{2}|x - y_n|$, and let $a = \{y_n : n \geq 0\} \cup \{x\}$. One can check that $a \in L$ but $a \cap U \notin L$, contradicting a previous remark.

Recall that $\mathscr{L}_{\infty\omega}$ formulas are defined like first-order formulas, except that conjunctions and disjunctions of arbitrary sets of formulas are allowed, provided that the resulting formula has only finitely many free variables.

CLAIM 2. $\mathbf{L}$ and $\mathbf{L} \times \mathbf{L}$ satisfy the same $\mathscr{L}_{\infty\omega}$ sentences.

Proof. It suffices to show that player $\exists$ has a winning strategy for the infinite

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back-and-forth (or "Ehrenfeucht") game played on **L** and $\mathbf{L} \times \mathbf{L}$. (This is a version of C. Karp's Theorem 2 in [2], in the style of [1], Chapter XI.) Put another way, it is enough to display a binary relation $\approx$ between finite sequences of elements from L and finite sequences from $L \times L$, satisfying:

(i) If $\langle a_1, \ldots, a_n \rangle \approx \langle b_1, \ldots, b_m \rangle$, then $m = n$ and the rule $a_i \mapsto b_i$ determines a well-defined partial isomorphism from **L** to $\mathbf{L} \times \mathbf{L}$.
(ii) (*Basis*) $\langle\ \rangle \approx \langle\ \rangle$.
(iii) (*Back*) If $\langle a_1, \ldots, a_n \rangle \approx \langle b_1, \ldots, b_n \rangle$, then for all $b_{n+1} \in L \times L$ there is an $a_{n+1} \in L$ such that $\langle a_1, \ldots, a_{n+1} \rangle \approx \langle b_1, \ldots, b_{n+1} \rangle$.
(iv) (*Forth*) If $\langle a_1, \ldots, a_n \rangle \approx \langle b_1, \ldots, b_n \rangle$, then for all $a_{n+1} \in L$ there is a $b_{n+1} \in L \times L$ such that $\langle a_1, \ldots, a_{n+1} \rangle \approx \langle b_1, \ldots, b_{n+1} \rangle$.

We choose the following relation: $\langle a_1, \ldots, a_n \rangle \approx \langle b_1, \ldots, b_n \rangle$ iff there exist real numbers $r_0, \ldots, r_7$ satisfying $r_{2i} < r_{2i+1}$ $(i < 4)$ and $r_1 < r_2$, and there exist order-preserving homeomorphisms $\phi : [r_0, r_1] \to [r_4, r_5]$, $\psi : [r_2, r_3] \to [r_6, r_7]$ such that for $1 \le j \le n$, $a_j \subseteq [r_0, r_1] \cup [r_2, r_3]$ and $b_j = \langle \phi(a_j \cap [r_0, r_1]), \psi(a_j \cap [r_2, r_3]) \rangle$.

Properties (i) and (ii) are easily verified. For (iii), if $r_0, \ldots, r_7$, ϕ and ψ witness $\langle a_1, \ldots, a_n \rangle \approx \langle b_1, \ldots, b_n \rangle$ and $b_{n+1} = \langle b^1_{n+1}, b^2_{n+1} \rangle \in L \times L$ is given, pick $r'_0, \ldots, r'_7$ such that $r'_{2i} < r_{2i}$ and $r'_{2i+1} > r_{2i+1}$ $(i < 4)$, $r'_1 < r'_2$, and $b^1_{n+1} \subseteq [r'_4, r'_5]$, $b^2_{n+1} \subseteq [r'_6, r'_7]$. Extend ϕ and ψ to homeomorphisms $\phi' : [r'_0, r'_1] \to [r'_4, r'_5]$, $\psi' : [r'_2, r'_3] \to [r'_6, r'_7]$ and let $a_{n+1} = (\phi')^{-1}(b^1_{n+1}) \cup (\psi')^{-1}(b^2_{n+1})$. Then $a_{n+1} \in L$ and $r'_0, \ldots, r'_7$, ϕ' and ψ' witness $\langle a_1, \ldots, a_{n+1} \rangle \approx \langle b_1, \ldots, b_{n+1} \rangle$. The argument for (iv) is similar once it is observed that any $a_{n+1} \in L$ is not dense in $[r_2, r_3]$, i.e. there exist r'_2, r'_3 satisfying $r_2 < r'_2 < r'_3 < r_3$ and $a_{n+1} \cap [r'_2, r'_3] = \varnothing$.

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Department of Pure Mathematics,
University of Waterloo,
Waterloo, Ontario