On McKenzie's method

Ross Willard

Dedicated to László Fuchs on the occasion of his 70th birthday

Abstract

This is an expository account of R. McKenzie's recent refutation of the RS conjecture.

We adopt the usual conventions: if $\mathbf{S}$ is an algebra then S denotes the universe of $\mathbf{S}$, $|S|$ denotes the cardinality of S, and $|S|^+$ denotes the successor cardinal to $|S|$. For an algebra $\mathbf{A}$, let $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ denote the class of all nontrivial subdirectly irreducible members of $\mathsf{V}(\mathbf{A})$, and (following McKenzie) define

$$\kappa(\mathbf{A}) = \sup\{|S|^+ \, : \, \mathbf{S} \in \mathsf{V}_{\mathrm{si}}(\mathbf{A})\}.$$

A long time ago R. Quackenbush asked [6] whether there exists a finite algebra $\mathbf{A}$ satisfying $\kappa(\mathbf{A}) = \omega$. In their book on tame congruence theory [2], D. Hobby and R. McKenzie conjectured that $\kappa(\mathbf{A}) \geq \omega$ implies $\kappa(\mathbf{A}) = \infty$ for finite algebras $\mathbf{A}$ (the 'RS conjecture'). But in 1993 McKenzie [3] discovered a method which allowed him to construct a finite algebra $\mathbf{A}_\omega$ with $\kappa(\mathbf{A}_\omega) = \omega$, and a variant of the method with which he could construct a finite algebra $\mathbf{A}_{\omega_1}$ of finite type with $\kappa(\mathbf{A}_{\omega_1}) = \omega_1$. Subsequently, McKenzie [4], M. Valeriote [7] and C. Latting (unpublished) used McKenzie's second method to construct related examples. However, nowhere has the second method been explicitly described, that is, as a general method from which the cited examples are extracted as particular cases.

It is our wish to give an explicit description of McKenzie's second method, similar to McKenzie's description of his first method [3, Sections 1-2]. To aid the reader, we begin the paper with something easier.

[1] 1991 *Mathematics Subject Classification*: Primary 08B26.

[2] *Key words and phrases.* variety, finitely generated, subdirectly irreducible, RS conjecture

[3] The support of the NSERC of Canada is gratefully acknowledged.

1 M-algebras

An **M-algebra** is any algebra $\mathbf{A}$ whose type includes $\wedge$ (binary) and 0 (nullary) but no other nullary operation symbols, and which satisfies

1. The reduct $\langle A, \wedge \rangle$ is a height-1 meet semilattice with least element 0;
2. 0 is an absorbing element for each fundamental operation F of $\mathbf{A}$; that is, if F is n-ary then $0 \in \{a_1, .., a_n\} \subseteq A$ implies $F(a_1, ..., a_n) = 0$.

Suppose $\mathbf{A}$ is an M-algebra and let $U = A \setminus \{0\}$. Define the binary relation $\gg$ on U as follows: $a \gg b$ if and only if $F(a_1, ..., a_n) = b$ for some fundamental operation F of $\mathbf{A}$ and some $a_i \in U$ such that $a \in \{a_1, ..., a_n\}$. Also let $\ggg$ be the transitive closure of $\gg$ on U. (Note: $\gg$ is reflexive because of $\wedge$.) Observe that if a, b are distinct elements of U then (i) $(a, 0) \in \mathrm{Cg}(a, b)$ because of $\wedge$, and (ii) $(b, 0) \in \mathrm{Cg}(a, 0)$ if and only if $a \ggg b$. Hence:

LEMMA 1.1 *Suppose* $\mathbf{A}$ *is an M-algebra and* $U = A \setminus \{0\}$.

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in U$ *such that every* $a \in U$ *satisfies* $a \ggg b$. *If this is the case, then the monolith* μ *of* $\mathbf{A}$ *is given by* $\mu = 0_A \cup (X \cup \{0\})^2$, *where* $X = \{a \in U \,:\, b \ggg a\}$.
2. $\mathbf{A}$ *is simple if and only if* $\mathbf{A}$ *is subdirectly irreducible and* $\ggg$ *is symmetric.*
3. *(For those who know tame congruence theory): Suppose* $\mathbf{A}$ *is finite and subdirectly irreducible, with* μ *and* X *as in item 1. Then* $\mathrm{typ}(0_A, \mu) = 5$, *and the* $\langle 0_A, \mu \rangle$*-minimal sets are the sets* $\{0, x\}$, $x \in X$.

Again, suppose that $\mathbf{A}$ is an M-algebra and $U = A \setminus \{0\}$. Consider an arbitrary power $\mathbf{A}^I$ of $\mathbf{A}$ ($I \neq \emptyset$). Observe that the reduct $\langle A^I, \wedge \rangle$ is also a meet semilattice with least element $\hat{0}$ (the constant 0 function), and

$$\begin{aligned} U^I &= \{f \in A^I \,:\, 0 \notin \mathrm{range}(f)\} \\ &= \{f \in A^I \,:\, f \text{ is maximal in } \langle A^I, \wedge \rangle\}. \end{aligned}$$

Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}^I$ ($I \neq \emptyset$). On $B \setminus \{\hat{0}\}$ define the binary relation $\gg$ by $f \gg g$ if and only if $F^{\mathbf{B}}(h_1, \ldots, h_n) = g$ for some fundamental operation symbol F and some $h_i \in B$ such that $f \in \{h_1, ..., h_n\}$. Also let $\ggg$ be the transitive closure of $\gg$ in $B \setminus \{\hat{0}\}$. Define $B(U) := B \cap U^I$, and note that if $g \in B(U)$ and $f \ggg g$ then $f \in B(U)$.

Now choose any $p \in B(U)$ and let

$$B_p = \{f \in B(U) \ : \ f \ggg p\}.$$

$\mathbf{B}(p)$ denotes the algebra, of the same type as $\mathbf{A}$ and $\mathbf{B}$, whose universe is $B_p \cup \{\hat{0}\}$ and whose fundamental operations are defined as follows:

$$F^{\mathbf{B}(p)}(h_1, ..., h_n) = \begin{cases} F^{\mathbf{B}}(h_1, ..., h_n) & \text{if } F^{\mathbf{B}}(h_1, ..., h_n) \in B_p \\ \hat{0} & \text{otherwise.} \end{cases}$$

Clearly $\mathbf{B}(p)$ is an M-algebra, and the relations $\gg$ and $\ggg$ defined on its nonzero elements prior to Lemma 1.1 coincide with the restrictions to B_p of the relations $\gg$ and $\ggg$ defined on $B \setminus \{\hat{0}\}$ after Lemma 1.1.

The following theorem is essentially the trivial case of McKenzie's second method in [4]. The proof is given in Section 3.

THEOREM 1.2 *Let* $\mathbf{A}$ *be an M-algebra.*

1. *If* $\mathbf{B} \leq \mathbf{A}^I$ $(I \neq \emptyset)$ *and* $p \in B(U)$, *then* $\mathbf{B}(p)$ *is in* $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$.
2. *Conversely, if* $\mathbf{A}$ *is finite then every member of* $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ *is isomorphic to* $\mathbf{B}(p)$ *for some such* $\mathbf{B}$ *and* p.

Example. Suppose $\mathbf{G} = \langle V, E \rangle$ is a graph possibly with loops, which simply means that V is a nonempty set and E is a symmetric relation on V. Fix $0 \notin V$ and define $\mathbf{G}^*$ to be the M-algebra of type $\{\wedge, \cdot, 0\}$ with universe $G^* = V \cup \{0\}$ and $\cdot$ defined by

$$x \cdot y = \begin{cases} x & \text{if } x, y \in V \text{ and } xEy \\ 0 & \text{otherwise.} \end{cases}$$

Thus the reduct $\langle G^*, \cdot \rangle$ is the graph algebra of C. Shallon [5] corresponding to $\mathbf{G}$. Let us call $\mathbf{G}^*$ a **graph M-algebra**, and say that $\mathbf{G}$ and $\mathbf{G}^*$ are **looped** if E is reflexive.

It is worth pointing out that if $\mathbf{G}^*$ is the graph M-algebra corresponding to the graph $\mathbf{G} = \langle V, E \rangle$, then the relation $\gg$ defined on V prior to Lemma 1.1 coincides with the reflexive closure of E. Since E is symmetric, Lemma 1.1

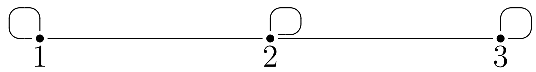

Figure 1: The graph $\mathbf{L}_3$

implies that the following are equivalent: (i) $\mathbf{G}^*$ is subdirectly irreducible; (ii) $\mathbf{G}^*$ is simple; (iii) $\mathbf{G}$ is connected.

Let $\mathbf{L}_3$ be the looped graph pictured in Figure 1, with vertex-set $U = \{1, 2, 3\}$, and let $\mathbf{A} = \mathbf{L}_3^*$. The following proposition should be compared to [5, Theorems 1 and 2′].

PROPOSITION 1.3 *1. $\mathsf{V}(\mathbf{A})$ contains every looped graph M-algebra.*

2. An algebra is in $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ if and only if it is the graph M-algebra of some connected looped graph.

3. $\mathsf{V}(\mathbf{A})$ is semisimple; i.e., every member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ is simple.

Proof. Suppose $\mathbf{G} = \langle V, E\rangle$ is a looped graph, let $I = V^2$, and recall that $U = \{1, 2, 3\} = A \setminus \{0\}$. For each $v \in V$ define $f_v \in U^I$ by

$$f_v(x, y) = \begin{cases} 1 & \text{if } v = x \\ 3 & \text{if } v = y \text{ and } \neg xEy \\ 2 & \text{otherwise.} \end{cases}$$

Note that $u \neq v$ implies $f_u \neq f_v$, and that $f_u \cdot f_v \in U^I$ (working in $\mathbf{A}^I$) if and only if uEv, in which case $f_u \cdot f_v = f_u$. Let $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}^I}(\{f_v : v \in V\})$. Clearly $B(U) := B \cap U^I = \{f_v : v \in V\}$. Let $\theta = 0_B \cup (B \setminus B(U))^2$; then $\mathbf{G}^* \cong \mathbf{B}/\theta \in \mathsf{V}(\mathbf{A})$. This proves the first item.

In light of the remarks preceding this Proposition, it remains only to show that every member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ is a looped graph M-algebra. Let $\mathbf{S}$ be a member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$. Then $\mathbf{S} \cong \mathbf{B}(p)$ for some $\mathbf{B} \leq \mathbf{A}^I$ and $p \in B(U)$, by Theorem 1.2. By examining the operation $\cdot$ in $\mathbf{A}^I$, we see that $\mathbf{B}(p)$ satisfies $\forall x (x \cdot x = x)$ and $\forall x \forall y (x \cdot y \neq 0 \;\Rightarrow\; [x \cdot y = x \;\&\; y \cdot x \neq 0])$. Hence $\mathbf{B}(p)$ (and therefore $\mathbf{S}$) is a looped graph M-algebra. ∎

2 Adding Conditions

Let $\mathbf{A}$ be a finite M-algebra, $U = A \setminus \{0\}$, $\mathbf{B} \leq \mathbf{A}^I$, and $p \in B(U)$, so that $\mathbf{B}(p)$ is defined. In the spirit of McKenzie's exposition of his first method [3, Sec. 1] we introduce two kinds of "conditions" which we shall wish to impose on $\mathbf{B}(p)$.

First, suppose that $\bar{x}$ is a finite sequence of distinct variables, that $\Sigma(\bar{x})$ is a conjunction of equations $x_i = x_j$ between pairs of these variables, that $s(\bar{x}), t(\bar{x})$ are terms in these variables, and that each variable in $\bar{x}$ occurs explicitly in s or t. The "$\mathcal{C}$-condition corresponding to s, t, Σ" is the following first-order sentence, denoted by $\mathcal{C}_\Sigma^{s=t}$:

$$(\forall \bar{x})(s(\bar{x}) = t(\bar{x}) \neq 0 \quad \Rightarrow \quad \Sigma(\bar{x})).$$

We say that $(\mathbf{B}, p)$ **satisfies** $\mathcal{C}_\Sigma^{s=t}$ if this sentence is true in $\mathbf{B}(p)$.

Secondly, suppose $\Phi(\bar{x})$ is an n-ary predicate on U. The "$\mathcal{B}$-condition corresponding to Φ," denoted by $\mathcal{B}_\Phi$, is the following condition on $\mathbf{B}$:

> There do not exist $f_1, ..., f_n \in B(U)$ having the property that $\Phi(f_1(i), \ldots, f_n(i))$ is true at every coordinate $i \in I$.

We say that $(\mathbf{B}, p)$ **satisfies** $\mathcal{B}_\Phi$ when the above condition holds.

Now suppose $\mathcal{X}$ is a fixed collection of such $\mathcal{B}$- and $\mathcal{C}$-conditions. For each condition $\mathcal{C}_\Sigma^{s=t}$ in $\mathcal{X}$ let $\mathrm{C}_\Sigma^{s=t}$ be a new n-ary operation symbol, where n is the number of variables occurring explicitly in s, t. Similarly, for each condition $\mathcal{B}_\Phi$ in $\mathcal{X}$ let B_Φ be a new $(n+3)$-ary operation symbol, where n is the arity of Φ. Finally, let J and J$'$ be new 3-ary operation symbols, let S_2 be a new 5-ary operation symbol, let ν be a new unary operation symbol, and let $\tau(\mathcal{X})$ be the expansion of the type of $\mathbf{A}$ to include all of the new operation symbols mentioned above. For each $\mathbf{D}$ of the same type as $\mathbf{A}$ define $\mathbf{D}^\otimes$ to be the expansion of $\mathbf{D}$ to $\tau(\mathcal{X})$ given by

$$\begin{array}{rcll}
\mathrm{B}_\Phi(\bar{x}, u, z, w) & = & 0 & \text{for each } \mathcal{B}_\Phi \in \mathcal{X} \\
\mathrm{C}_\Sigma^{s=t}(\bar{x}) & = & s(\bar{x}) \wedge t(\bar{x}) & \text{for each } \mathcal{C}_\Sigma^{s=t} \in \mathcal{X} \\
\mathrm{J}(x, y, z) & = & x \wedge y & \\
\mathrm{J}'(x, y, z) & = & x \wedge y \wedge z & \\
\mathrm{S}_2(x, y, u, z, w) & = & 0 & \\
\nu(x) & = & x. &
\end{array}$$

If $\mathcal{K} \subseteq \mathsf{V}(\mathbf{A})$ then let $\mathcal{K}^\otimes = \{\mathbf{D}^\otimes : \mathbf{D} \in \mathcal{K}\}$.

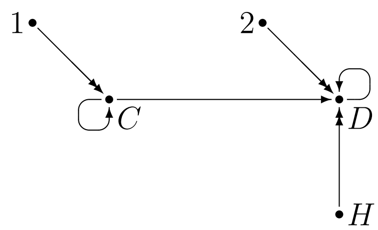

Figure 2: The $\gg$ relation of $\mathbf{A}$

THEOREM 2.1 *[McKenzie's Method, weak version] Suppose $\mathbf{A}$ is a finite M-algebra and that $\mathcal{X}$ is a collection of $\mathcal{B}$- and $\mathcal{C}$-conditions. Let $\mathcal{K}_1$ be the class of all algebras $\mathbf{B}(p)$ where $\mathbf{B} \leq \mathbf{A}^I$, $p \in B(U)$, and $(\mathbf{B}, p)$ satisfies all of the conditions in $\mathcal{X}$, and let $\mathcal{K}_2 = \{\mathbf{B}(p) \in \mathcal{K}_1 : B_p = B(U)\}$.*

There exists a finite algebra $\mathbf{A}_1^+$, of type $\tau(\mathcal{X})$, having $\mathbf{A}$ as a subreduct, and such that

1. $\mathcal{K}_2^{\otimes} \subseteq \mathsf{V}_{\mathrm{si}}(\mathbf{A}_1^+)$;
2. *Conversely, every member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A}_1^+) \setminus \mathsf{HS}(\mathbf{A}_1^+)$ is isomorphic to a member of $\mathcal{K}_1^{\otimes}$.*

Remarks: (i) The algebra $\mathbf{A}_1^+$ is not an M-algebra. (ii) The operation ν is unnecessary, and was not used by McKenzie. We include it only to simplify the proof of Theorem 2.1. (iii) Though Theorem 2.1 is enough for our purposes, it does not describe McKenzie's method in sufficient generality to encompass his argument in [4] that $\kappa(\mathbf{A})$ is not a computable function of $\mathbf{A}$. At the end of this section we shall state a stronger version (Theorem 2.7), which is adequate. The proof can be found in Section 3.

Example 1 [McKenzie]. Let $U = \{1, H, 2, C, D\}$ and let $\mathbf{A}$ be the M-algebra of type $\{\wedge, \cdot, 0\}$ with universe $U \cup \{0\}$ and with $\cdot$ defined by $1 \cdot C = C$, $H \cdot C = 2 \cdot D = D$, and $x \cdot y = 0$ in all other cases. Note that in $\mathbf{A}$, or more generally in any M-algebra of the same type as $\mathbf{A}$, the relation $\gg$ on the nonzero elements is particularly simple: $a \gg b$ if and only if $a = b$ (corresponding to $a \wedge a = a$) or $c \cdot a = b$ or $a \cdot c = b$ for some $c \neq 0$. Let us suppress the first case and distinguish the second and third cases by writing $a \to b$ or $a \twoheadrightarrow b$ respectively. See Figure 2 for a picture of the $\gg$ relation of $\mathbf{A}$ drawn according to this convention.

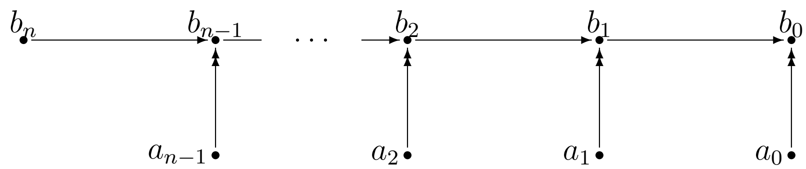

Figure 3: The $\gg$ relation of $\mathbf{S}_n$

For each $n < \omega$ let $\mathbf{S}_n$ be the unique M-algebra of type $\{\wedge, \cdot, 0\}$ having universe $\{0\} \cup \{a_i : i < n\} \cup \{b_i : i \leq n\}$ and whose $\gg$ relation is the one pictured in Figure 3. Also let $\mathbf{S}_\omega = \bigcup_{n<\omega} \mathbf{S}_n$.

Finally, let

$$\mathcal{X} = \{\mathcal{B}_{x \in \{1,2\}},\ \mathcal{C}^{x \cdot y = z \cdot w}_{y=w}\},$$

let $\mathcal{K}_1$ be the class of all algebras $\mathbf{B}(p)$ where $\mathbf{B} \leq \mathbf{A}^I$, $p \in B(U)$, and $(\mathbf{B}, p)$ satisfies the two conditions in $\mathcal{X}$, and let $\mathcal{K}_2 = \{\mathbf{B}(p) \in \mathcal{K}_1 : B_p = B(U)\}$.

PROPOSITION 2.2 *Both $\mathcal{K}_1$ and $\mathcal{K}_2$ consist, up to isomorphism, of the algebras $\mathbf{S}_n$, $n \leq \omega$.*

Proof. Suppose first that $n \leq \omega$. Let $N = \min(n+1, \omega)$ and define $a_i \in A^N$ $(i < n)$ and $b_i \in A^N$ $(i < N)$ by

$$a_i(j) = \begin{cases} 1 & \text{if } j < i \\ H & \text{if } j = i \\ 2 & \text{if } i < j < N \end{cases}$$

$$b_i(j) = \begin{cases} C & \text{if } j < i \\ D & \text{if } i \leq j < N \end{cases}$$

Let $V = \{a_i : i < n\} \cup \{b_i : i < N\}$, $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}^N}(V)$, and $p = b_0$. Then $B(U) = B_p = V$, $\mathbf{B}(p) \cong \mathbf{S}_n$, and $(\mathbf{B}, p)$ satisfies the conditions in $\mathcal{X}$.

Conversely, suppose $\mathbf{B} \leq \mathbf{A}^I$ and $p \in B(U)$ are such that $(\mathbf{B}, p)$ satisfies the conditions in $\mathcal{X}$. If $|B_p| = 1$ then $\mathbf{B}(p) \cong \mathbf{S}_0$. For the remainder of this proof assume that $|B_p| \geq 2$. Let $X = \{C, D\}^I \cap B_p$ and $Y = \{1, H, 2\}^I \cap B_p$, and note that

1. If $f, g \in B_p$ then $f \to g$ implies $f, g \in X$ while $f \twoheadrightarrow g$ implies $g \in X$ and $f \in Y$.

Recall that for every $f \in B_p$ there exists a directed $\gg$-path from f to p. Since $|B_p| \geq 2$, item 1 implies that

2. $p \in X$ and $B_p = X \cup Y$.

Thus to prove that $\mathbf{B}(p) \cong \mathbf{S}_n$ for some $n \leq \omega$ it suffices (consult Figure 3) to show the following:

3. There do not exist $f_0, ..., f_n \in X$ satisfying $f_i \rightarrow f_{i+1}$ for $i < n$ and $f_n \rightarrow f_0$.

4. $f \rightarrow h$ and $g \rightarrow h$ imply $f = g$.

5. $f \rightarrow g$ and $f \rightarrow h$ imply $g = h$.

6. $f \twoheadrightarrow h$ and $g \twoheadrightarrow h$ imply $f = g$.

7. $f \twoheadrightarrow g$ and $f \twoheadrightarrow h$ imply $g = h$.

8. For each $f \in X$, if there exists $g \rightarrow f$ then there exists $h \twoheadrightarrow f$ and conversely.

These may be proved as follows. Assume that $f \in Y$ and $g, h \in X$ and $f \cdot g = h$ (so that $f \twoheadrightarrow h$ and $g \rightarrow h$). Then

$$\begin{array}{rcl} g^{-1}(D) & = & f^{-1}(2) \\ h^{-1}(D) & = & f^{-1}(2) \cup f^{-1}(H). \end{array}$$

$f^{-1}(H) \neq \emptyset$ because $(\mathbf{B}, p)$ satisfies the condition $\mathcal{B}_{x \in \{1,2\}}$, and so $g^{-1}(D) \subsetneq h^{-1}(D)$. This proves item 3. The condition $\mathcal{C}^{x \cdot y = z \cdot w}_{y = w}$ implies item 4. Item 5 follows from item 4 and the fact that there is a directed $\rightarrow$-path from every point in $X \setminus \{p\}$ to p.

Again assuming $f \cdot g = h$, the displayed equations show that g and h uniquely determine f. This fact and item 4 imply item 6. The displayed equations also show that f uniquely determines h, which proves item 7. Item 8 follows automatically from the definitions of $\rightarrow$ and $\twoheadrightarrow$. ∎

COROLLARY 2.3 *Let $\mathbf{A}_1^+$ be the algebra given by Theorem 2.1 for the algebra $\mathbf{A}$ and set of conditions $\mathcal{X}$ of Example 1. Then $\mathbf{A}_1^+$ is finite, is of finite type, and satisfies $\kappa(\mathbf{A}_1^+) = \omega_1$.*

Figure 4: Expanding $\mathbf{S}_\eta$ to $\bar{\mathbf{S}}_\eta$.

PROOF. The type of $\mathbf{A}_1^+$ is finite because both the type of $\mathbf{A}$ and the set $\mathcal{X}$ are finite. The rest follows from Theorem 2.1 and Proposition 2.2. ▮

REMARK: The algebra $\mathbf{A}_1^+$ produced in the proof of Theorem 2.7 has 11 elements. By arguing more carefully, McKenzie did better: his $\mathbf{A}_1^+$ has just 8 elements.

Example 2 [Valeriote]. Let $\bar{\mathbf{A}}$ be the expansion of the algebra $\mathbf{A}$ from the previous example obtained by adding two new binary operations $\circ$ and $\star$ defined as follows: $1 \circ C = H \circ D = C$, $2 \circ D = D$, and $x \circ y = 0$ in all other cases; $1 \star C = 1$, $H \star C = H$, $2 \star D = 2$, and $x \star y = 0$ in all other cases. The effect of the new operations is to make $\ggg$ symmetric in $B(U)$ for any $\mathbf{B} \leq \bar{\mathbf{A}}^I$. Also let $\bar{\mathcal{X}} = \{\mathcal{B}_{x\in\{1,2\}},\ \mathcal{C}_{y=w}^{x\cdot y=z\cdot w},\ \mathcal{C}_{y=w}^{x\circ y=z\circ w}\}$, let $\bar{\mathcal{K}}_1$ be the class of algebras $\mathbf{B}(p)$ where $\mathbf{B} \leq \bar{\mathbf{A}}^I$, $p \in B(U)$, and $(\mathbf{B}, p)$ satisfies the conditions in $\bar{\mathcal{X}}$, and let $\bar{\mathcal{K}}_2 = \{\mathbf{B}(p) \in \bar{\mathcal{K}}_1 : B_p = B(U)\}$.

Recall the algebras $\mathbf{S}_n$ $(n \leq \omega)$ from Example 1, and let $\mathbf{S}_{-\omega}$ and $\mathbf{S}_{\mathbb{Z}}$ be new algebras of the same type, which are nearly identical in form to $\mathbf{S}_\omega$ except that: (i) in $\mathbf{S}_{-\omega}$ the $\rightarrow$-connected elements form a one-way infinite chain with initial point but no final point (instead of the other way around), and (ii) in $\mathbf{S}_{\mathbb{Z}}$ the $\rightarrow$-connected elements form a two-way infinite chain. By arguments similar to those in Example 1, one can show (cf. Valeriote [7])

PROPOSITION 2.4 *Both $\bar{\mathcal{K}}_1$ and $\bar{\mathcal{K}}_2$ consist, up to isomorphism, of the algebras $\bar{\mathbf{S}}_n$ $(n \leq \omega)$, $\bar{\mathbf{S}}_{-\omega}$, and $\bar{\mathbf{S}}_{\mathbb{Z}}$, where $\bar{\mathbf{S}}_\eta$ is the unique expansion of $\mathbf{S}_\eta$ determined by Figure 4.*

COROLLARY 2.5 *Let $\bar{\mathbf{A}}_1^+$ be the algebra given by Theorem 2.1 for the algebra $\bar{\mathbf{A}}$ and set of conditions $\bar{\mathcal{X}}$ of Example 2. Then $\bar{\mathbf{A}}_1^+$ is finite, is of finite type, and satisfies $\kappa(\bar{\mathbf{A}}_1^+) = \omega_1$. Moreover, every member of $\mathsf{V}_{\mathrm{si}}(\bar{\mathbf{A}}_1^+) \setminus \mathsf{HS}(\bar{\mathbf{A}}_1^+)$ is simple.*

REMARK: Valeriote showed that $\bar{\mathbf{A}}_1^+$ can be chosen so that the members of $\mathsf{V}_{\mathrm{si}}(\bar{\mathbf{A}}_1^+) \cap \mathsf{HS}(\bar{\mathbf{A}}_1^+)$ are also simple.

Example 3 [McKenzie]. Here we indicate how McKenzie's second method can be used to produce an example of a finite algebra $\mathbf{A}_\omega$ of infinite type satisfying $\kappa(\mathbf{A}_\omega) = \omega$.

Suppose that U_0 is a finite set disjoint from $\{0, 1\}$ and $(\Phi_n(\bar{x}) \ : \ n < \omega)$ is an infinite family of finitary relations on U_0 with the following property: for each $n < \omega$ there exists a set $R_n \subseteq (U_0)^I$ such that (i) there exists exactly one solution $\bar{f}$ to $\Phi_n(\bar{f})$ in R_n, and (ii) for all $m \neq n$ there exist no solutions $\bar{f}$ to $\Phi_m(\bar{f})$ in R_n.

For each n let F_n be an operation symbol of the same arity as $\Phi_n(\bar{x})$, let $U = U_0 \cup \{1\}$, and let $\mathbf{A}$ be the M-algebra of type $\{\wedge, 0\} \cup \{\mathrm{F}_n \ : \ n < \omega\}$ whose universe is $U \cup \{0\}$ and whose operations F_n are defined by

$$\mathrm{F}_n(\bar{x}) = \begin{cases} 1 & \text{if } \Phi_n(\bar{x}) \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\mathcal{X} = \{\mathcal{B}_{\Phi_n(\bar{x}) \& \Phi_m(\bar{y})} \ : \ n < m < \omega\} \cup \{\mathcal{C}^{\mathrm{F}_n(\bar{x}) = \mathrm{F}_n(\bar{y})}_{\& x_i = y_i} \ : \ n < \omega\}$.

PROPOSITION 2.6 *Let $\mathbf{A}$ and $\mathcal{X}$ be as described in Example 3, and let $\mathbf{A}_1^+$ be the algebra given by Theorem 2.1. Then $\mathbf{A}_1^+$ is finite, is of infinite type, and satisfies $\kappa(\mathbf{A}_1^+) = \omega$.*

REMARK: McKenzie showed [3, §1] that the hypotheses of this example can be achieved with $|U_0| = 2$. The proof of Theorem 2.7 in Section 3 then yields an algebra $\mathbf{A}_1^+$ having 7 elements. McKenzie actually did better; his algebra $\mathbf{A}_\omega$, built using his first method, has only four elements.

Here is the promised strengthening of Theorem 2.1. We first extend the notion of $\mathcal{C}$-condition as follows. Let $\mathbf{A}$ be a fixed finite M-algebra. Suppose $\bar{x}$ is an n-tuple of distinct variables, S is a finite nonempty set of terms in these variables, Σ is a conjunction of equations between pairs of these variables, and $\Phi(\bar{x})$ and $\Psi(\bar{x})$ are n-ary predicates on U, where as usual $U = A \setminus \{0\}$.

Suppose furthermore that the following technical condition holds:

if

$$J = \{j : x_j \text{ occurs explicitly in some } s \in S\}$$

and

$$\Sigma(\bar{x}) \quad \text{is} \quad \mathop{\&}_{t=1}^{N} x_{j_t} = x_{k_t}$$

then

$$\{k_1, \ldots, k_N\} \subseteq J \text{ and } J \cup \{j_1, \ldots, j_N\} = \{1, \ldots, n\}.$$

The "$\mathcal{C}$-condition corresponding to S, Σ, Φ, Ψ," denoted by $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$, is the following condition attributed to $\mathbf{B} \leq \mathbf{A}^I$ and $p \in B(U)$:

> For all $f_1, \ldots, f_n \in B(U)$, if $s(\bar{f}) = t(\bar{f}) \in B_p$ for all $s, t \in S$ and if $\Phi(\bar{f})$ holds (i.e., is true at each coordinate in I), then $\Sigma(\bar{f})$ is true and $\Psi(\bar{f})$ holds (at each coordinate in I).

We say that $(\mathbf{B}, p)$ **satisfies** $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$ if the above condition holds. Note that this extends the notion of $\mathcal{C}$-condition defined earlier since $\mathcal{C}^{s=t}_{\Sigma} = \mathcal{C}^{\{s,t\},\mathsf{T}}_{\mathsf{T},\Sigma}$ where T is the universally true n-ary predicate on U.

We shall also say that $(\mathbf{B}, p)$ **strongly satisfies** $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$ if the displayed condition remains true when the hypothesis "$s(\bar{f}) = t(\bar{f}) \in B_p$" is replaced by "$s(\bar{f}) = t(\bar{f}) \in B(U)$." This will be the case if e.g. $(\mathbf{B}, p)$ satisfies $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$ and $B_p = B(U)$.

As before, to each condition $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$ there is a corresponding operation symbol $\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}$. If $\mathcal{X}$ is a set of $\mathcal{B}$- and $\mathcal{C}$-conditions then $\tau(\mathcal{X})$ is again the corresponding type expanding the type of $\mathbf{A}$. Now, however, we can define $\mathbf{D}^{\otimes}$ only when $\mathbf{D} = \mathbf{B}(p)$. The definition of the new operations in $\mathbf{B}(p)^{\otimes}$ is as before except for the symbols corresponding to the $\mathcal{C}$-conditions. These are now defined as follows:

$$\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{f}) = \begin{cases} \bigwedge_{s \in S} s(\bar{f}) & \text{if } \Phi(\bar{f}) \text{ holds (coordinatewise in } \mathbf{B}) \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2.7 *[McKenzie's Method]* *Suppose* $\mathbf{A}$ *is a finite M-algebra and that* $\mathcal{X}$ *is a collection of* $\mathcal{B}$*- and* $\mathcal{C}$*-conditions. Let* $\mathcal{K}_1$ *be the class of* *all algebras $\mathbf{B}(p)$ where $\mathbf{B} \leq \mathbf{A}^I$, $p \in B(U)$, and $(\mathbf{B}, p)$ satisfies all of the conditions in $\mathcal{X}$, and let $\mathcal{K}_2$ be the subclass consisting of those members $\mathbf{B}(p)$ of $\mathcal{K}_1$ for which $(\mathbf{B}, p)$ strongly satisfies the $\mathcal{C}$-conditions in $\mathcal{X}$.*

There exists a finite algebra $\mathbf{A}_1^+$, of type $\tau(\mathcal{X})$, having $\mathbf{A}$ as a subreduct, and such that

1. $\mathcal{K}_2^{\otimes} \subseteq \mathsf{V}_{\mathrm{si}}(\mathbf{A}_1^+)$;
2. *Conversely, every member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A}_1^+) \setminus \mathsf{HS}(\mathbf{A}_1^+)$ is isomorphic to a member of $\mathcal{K}_1^{\otimes}$.*

3 Proofs

Proof of Theorem 1.2. Let $\mathbf{A}$ be an M-algebra. Suppose $\mathbf{B} \leq \mathbf{A}^I$ $(I \neq \emptyset)$ and $p \in B(U)$. Let $\theta_p = 0_B \cup (B \setminus B_p)^2$. Then $\theta_p \in \operatorname{Con}\mathbf{B}$ and $\mathbf{B}/\theta_p \cong \mathbf{B}(p)$, which proves that $\mathbf{B}(p) \in \mathsf{V}(\mathbf{A})$. $\mathbf{B}(p)$ is subdirectly irreducible by Lemma 1.1.

Conversely, assume that $\mathbf{A}$ is finite and $\mathbf{S} \in \mathsf{V}_{\mathrm{si}}(\mathbf{A})$.

CASE 1: S is finite.

The proof is implicit in [3]. Choose a set I of least cardinality such that $\mathbf{S} \in \mathsf{HS}(\mathbf{A}^I)$, and choose $\mathbf{B} \leq \mathbf{A}^I$ and $\theta \in \operatorname{Con}\mathbf{B}$ such that $\mathbf{S} \cong \mathbf{B}/\theta$. I is finite since $\mathbf{A}$ and $\mathbf{S}$ are. Let $\bar{\theta}$ be the unique cover of θ in $\operatorname{Con}\mathbf{B}$. Because $\mathbf{B}$ is finite and has a semilattice reduct, there exists $(p, q) \in \bar{\theta} \setminus \theta$ such that p covers q in the semilattice ordering $\leq$ of $\langle B, \wedge \rangle$. Choose such (p, q) with p minimal relative to $\leq$.

The following six claims complete the proof by showing that $p \in B(U)$ (Claim 3), and that θ is the congruence θ_p defined above (Claims 5 and 6).

Claim 1. $(f, p) \in \theta$ implies $f \geq p$.

Claim 2. $(f, g) \notin \theta$ implies that there exists $\lambda \in \operatorname{Pol}_1 \mathbf{B}$ such that $\lambda(f) < p$ and $\lambda(g) = p$, or vice versa.

Claim 1 follows from the presence of $\wedge$ and the choice of p. Claim 2 follows from the presence of the polynomial $\lambda(x) := x \wedge p$ and the fact that $(f, g) \notin \theta$ implies $(p, q) \in \theta \vee \operatorname{Cg}^{\mathbf{B}}(f, g)$.

Claim 3. $0 \notin \operatorname{range}(p)$.

To prove this, pick any $i \in I$. By the minimality of $|I|$, there exist $f, g \in B$ such that $(f, g) \notin \theta$ and $f(j) = g(j)$ for all $j \in I \setminus \{i\}$. By Claims 1 and 2 we may assume that $f < p$ and $g = p$. Then we must have $f(i) < p(i)$, so $p(i) \neq 0$.

Recall that $B_p = \{f \in B(U) : f \ggg p\} = \{f \in B \setminus \{\hat{0}\} : f \ggg p\}$.

Claim 4. For $f \in B$ the following are equivalent: (i) there exists a nonconstant $\lambda \in \mathrm{Pol}_1 \mathbf{B}$ such that $\lambda(f) = p$; (ii) there exists $\lambda \in \mathrm{Pol}_1 \mathbf{B}$ such that $\lambda(f) = p$ and $\lambda(\hat{0}) = \hat{0}$; (iii) $f \in B_p$.

This follows from the characterization of B_p and the fact that $\hat{0}$ is an absorbing element for every fundamental operation of $\mathbf{B}$.

Claim 5. $(f, \hat{0}) \notin \theta$ implies $f \in B_p$.

This follows from Claims 2 and 4 and the fact that every fundamental operation of $\mathbf{B}$ is order-preserving.

Claim 6. $f \in B_p$ implies $f/\theta = \{f\}$.

Indeed, suppose $f \in B_p$. By Claim 4, there exists $\lambda \in \mathrm{Pol}_1 \mathbf{B}$ such that $\lambda(f) = p$ and $\lambda(\hat{0}) = \hat{0}$. Hence $(p, \hat{0}) \in \mathrm{Cg}^{\mathbf{B}}(f, \hat{0})$. Since $(p, \hat{0}) \notin \theta$ by Claim 1, it follows that $(f, \hat{0}) \notin \theta$. Now suppose in addition that $(f, g) \in \theta$. Then $(g, \hat{0}) \notin \theta$, hence $g \in B_p$ by Claim 5. This proves $f/\theta \subseteq B(U)$. But this is only possible if $f/\theta = \{f\}$, since $B(U)$ is an antichain in $\langle B, \leq \rangle$. This completes the proof of Case 1.

CASE 2: $\mathbf{S}$ is infinite.

Let $\mathcal{H}_1$ be the class of all structures $\langle \mathbf{B}, B(U), \theta \rangle$ where

1. $\mathbf{B} \leq \mathbf{A}^I$ for some $I \neq \emptyset$
2. $\theta \in \mathrm{Con}\, \mathbf{B}$
3. $B \setminus B(U) \subseteq \hat{0}/\theta$

and where $B(U)$ and θ are construed as relations (unary and binary respectively) in the structure. Also let $\mathcal{H}_2 = \{\mathbf{B}/\theta : \langle \mathbf{B}, B(U), \theta \rangle \in \mathcal{H}_1\}$.

Claim 7. $\mathcal{H}_1$ is closed under substructures. $\mathcal{H}_2$ is closed under subalgebras.

Here is the proof for $\mathcal{H}_2$. Suppose $\mathbf{C} \leq \mathbf{B}/\theta$ where $\langle \mathbf{B}, B(U), \theta \rangle \in \mathcal{H}_1$. Then $\mathbf{C} = \mathbf{B}'/\theta|_{\mathbf{B}'}$ for some $\mathbf{B}' \leq \mathbf{B}$. Clearly $\langle \mathbf{B}', B'(U), \theta|_{\mathbf{B}'} \rangle$ is a substructure of $\langle \mathbf{B}, B(U), \theta \rangle$ and hence is in $\mathcal{H}_1$, so $\mathbf{C} \in \mathcal{H}_2$.

Claim 8. $\mathbf{I}(\mathcal{H}_1)$ is axiomatized by universal first-order sentences.

A suitable set of axioms is the union of the universal Horn theory of $\langle \mathbf{A}, U\rangle$, a (possibly infinite) set of universal Horn sentences asserting that $\theta \in \operatorname{Con} \mathbf{B}$, and a final universal sentence asserting item 3 above.

As a consequence, $\mathbf{I}(\mathcal{H}_1)$ is closed under ultraproducts.

Claim 9. $\mathbf{I}(\mathcal{H}_2)$ is closed under ultraproducts.

Suppose $(\langle \mathbf{B}_x, B_x(U), \theta_x\rangle)_{x\in X}$ is a family of members of $\mathcal{H}_1$ and $\mathcal{U}$ is an ultrafilter on X, and let

$$\begin{aligned}
\mathbf{C} &= \prod_{x\in X} (\mathbf{B}_x/\theta_x)/\mathcal{U} \\
\langle \mathbf{D}, V, \theta\rangle &= \prod_{x\in X} \langle \mathbf{B}_x, B_x(U), \theta_x\rangle/\mathcal{U}.
\end{aligned}$$

Then $\langle \mathbf{D}, V, \theta\rangle \in \mathbf{I}(\mathcal{H}_1)$ by Claim 8 and $\mathbf{D}/\theta \cong \mathbf{C}$ canonically, so $\mathbf{C} \in \mathbf{I}(\mathcal{H}_2)$, which proves the claim.

Now consider the infinite member $\mathbf{S}$ of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$. By a result of Dziobiak [1], $\mathbf{S}$ can be embedded in an ultraproduct of finite members of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$. We showed in Case 1 that every finite member of $\mathsf{V}_{\mathrm{si}}(\mathbf{A})$ is isomorphic to a member of $\mathcal{H}_2$. So $\mathbf{S} \in \mathbf{I}(\mathcal{H}_2)$ by Claims 7 and 9, say $\mathbf{S} \cong \mathbf{B}/\theta$ where $\langle \mathbf{B}, B(U), \theta\rangle \in \mathcal{H}_1$. Clearly $B(U) \neq \emptyset$ since $|S| > 1$, so $\mathbf{B}/\theta$ is an M-algebra. By Lemma 1.1(1) there must exist $p \in B\setminus(\hat{0}/\theta)$ such that $B\setminus(\hat{0}/\theta) \subseteq B_p$. On the other hand, if $f \in B_p$ then $(f, \hat{0}) \notin \theta$ by the first half of the proof of Claim 5 in Case 1. This shows that $B \setminus (\hat{0}/\theta) = B_p$ and hence θ is the congruence θ_p defined at the start of the proof of this Theorem. So $\mathbf{S} \cong \mathbf{B}/\theta_p \cong \mathbf{B}(p)$ as desired. ∎

Proof of Theorem 2.7. The fiendishly clever ideas are all due to McKenzie. Let $\mathbf{A}$ and $\mathcal{X}$ be given, where $A = U \cup \{0\}$. Let U' be a set disjoint from A and satisfying $|U'| = |U|$, let $U_1 = U \cup U'$, let ∂ be an involution of U_1 sending U to U' and vice versa, and let $A_1 = \{0\} \cup U_1$. Extend ∂ to A_1 by defining $\partial(0) = 0$; then extend the fundamental operations F of $\mathbf{A}$ other than $\wedge$ to A_1 so that

$$F(x_1, \ldots, \partial(x_i), \ldots, x_n) = \partial F(x_1, \ldots, x_i, \ldots, x_n) \quad \text{for each } i,$$

and let $\mathbf{A}_1$ be the resulting M-algebra.

$\mathbf{A}_1^+$ is the expansion of $\mathbf{A}_1$ to the type $\tau(\mathcal{X})$ defined as follows. First define

$$\mathrm{S}_2(x,y,u,z,w) = \begin{cases} u & \text{if } x = \partial(y) \neq 0 \text{ and } (u = z \text{ or } u = w) \\ 0 & \text{otherwise} \end{cases}$$

$$\nu(x) = \begin{cases} \partial(x) & \text{if } x \in U' \\ x & \text{otherwise} \end{cases}$$

$$\mathrm{J}(x,y,z) = \begin{cases} x & \text{if } x = y \text{ or } z = x = \partial(y) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{J}'(x,y,z) = \mathrm{J}(x,\partial(y),z)$$

Next, note that any n-ary predicate $\Phi(\bar{x})$ on U can also be construed as a predicate on A_1, since $U \subseteq A_1$. With this understanding define

(for each $\mathcal{B}_\Phi \in \mathcal{X}$):

$$\mathrm{B}_\Phi(\bar{x},u,z,w) = \begin{cases} u & \text{if } \Phi(\bar{x}) \text{ and } (u = z \text{ or } u = w) \\ 0 & \text{otherwise} \end{cases}$$

(for each $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma} \in \mathcal{X}$):

$$\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{x}) = \begin{cases} \bigwedge_{s \in S} s(\bar{x}) & \text{if } \Phi(\bar{x}) \text{ and } \Psi(\bar{x}) \text{ and } \Sigma(\bar{x}) \\ \partial\left(\bigwedge_{s\in S} s(\bar{x})\right) & \text{if } \Phi(\bar{x}) \text{ and } \neg\left(\Psi(\bar{x}) \text{ and } \Sigma(\bar{x})\right) \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that $\mathbf{A}_1^+$ satisfies the requirements of Theorem 2.7. The proof roughly follows that of Theorem 1.2, though more work is required. Throughout the proof we shall refer to the fundamental operations of $\mathbf{A}_1$ as the **original operations**, and the remaining fundamental operations of $\mathbf{A}_1^+$ as the **new operations**.

PART I. Suppose first that $\mathbf{B} \leq \mathbf{A}^I$, $p \in B(U)$, and $(\mathbf{B},p)$ satisfies the $\mathcal{B}$-conditions and strongly satisfies the $\mathcal{C}$-conditions in $\mathcal{X}$. Arguing as in [4, Lemma 4.1], define

$$B' = B(U) \cup \{f \in (A_1)^I \,:\, 0 \in \mathrm{range}(f)\}.$$

CLAIM 1. B' is a subuniverse of $(\mathbf{A}_1^+)^I$.

To prove it, one must show that B' is closed under the fundamental operations of $\mathbf{A}_1^+$. This is clear for the original operations and J, J$'$, ν and S_2. It is closed under each B_Φ ($\mathcal{B}_\Phi \in \mathcal{X}$) since $(\mathbf{B}, p)$ satisfies $\mathcal{B}_\Phi$, and is closed under each $\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}$ ($\mathcal{C}^{S,\Phi}_{\Psi,\Sigma} \in \mathcal{X}$) since $(\mathbf{B}, p)$ strongly satisfies $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma}$.

$\mathbf{B}'$ shall denote the subalgebra of $(\mathbf{A}_1^+)^I$ whose universe is B'. Next define $\theta = 0_{B'} \cup (B' \setminus B_p)^2$. The next two claims will show that $\theta \in \operatorname{Con} \mathbf{B}'$ and $\mathbf{B}'/\theta \cong \mathbf{B}(p)^\otimes$. The first claim can be deduced from $(\mathbf{B}, p)$ satisfying (strongly satisfying) the $\mathcal{B}$- ($\mathcal{C}$-) conditions in $\mathcal{X}$, and the definitions of the new operations in $\mathbf{A}_1^+$.

Claim 2. In $\mathbf{B}'$ the following are true.

1. $B'(U_1) = B(U)$.
2. The ranges of the operations B_Φ ($\mathcal{B}_\Phi \in \mathcal{X}$) and S_2 are disjoint from $B(U)$.
3. Let $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma} \in \mathcal{X}$ and $s \in S$. $\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{f}) \in B(U)$ if and only if $f_1, \ldots, f_n \in B(U)$ and $s(\bar{f}) = t(\bar{f}) \in B(U)$ for all $t \in S$ and $\Phi(\bar{f})$ holds; in which case $\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{f}) = s(\bar{f})$.
4. $\mathrm{J}(f, g, h) \in B(U)$ if and only if $f = g \in B(U)$, in which case $\mathrm{J}(f, g, h) = f$.
5. $\mathrm{J}'(f, g, h) \in B(U)$ if and only if $f = g = h \in B(U)$, in which case $\mathrm{J}'(f, g, h) = f$.
6. $\nu(f) \in B(U)$ if and only if $f \in B(U)$, in which case $\nu(f) = f$.

Recall the definition of B_p (from $\mathbf{B}$ and p) following Lemma 1.1.

Claim 3. (For $\mathcal{C}^{S,\Phi}_{\Psi,\Sigma} \in \mathcal{X}$): If $\mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{f}) \in B_p$ then $f_1, \ldots, f_n \in B_p$.

For the proof, let $r := \mathrm{C}^{S,\Phi}_{\Psi,\Sigma}(\bar{f}) \in B_p$. By Claim 2(3) and the proof of Claim 1, we already know that $(*)$ $f_1, \ldots, f_n \in B$ and $r = s(\bar{f})$ for each $s \in S$, and that $\Sigma(\bar{f})$ holds. Let $J = \{j : x_j \text{ occurs explicitly in some } s \in S\}$. Since $r \in B_p$ and each $s \in S$ is a term built from original operations, it follows from $(*)$ that $f_j \in B_p$ for each $j \in J$. On the other hand, the technical condition in the definition of $\mathcal{C}$-conditions plus the truth of $\Sigma(\bar{f})$ imply $\{f_1, \ldots, f_n\} = \{f_j : j \in J\}$.

Claims 2 and 3 imply $\theta \in \operatorname{Con} \mathbf{B}'$ and $\mathbf{B}(p)^{\otimes} \cong \mathbf{B}'/\theta \in \mathsf{V}(\mathbf{A}_1^{+})$. $\mathbf{B}(p)^{\otimes}$ is subdirectly irreducible since its reduct $\mathbf{B}(p)$ is. This proves the easy half of the theorem.

PART II. Conversely, assume that $\mathbf{S} \in \mathsf{V}_{\mathrm{si}}(\mathbf{A}_1^{+}) \setminus \mathsf{HS}(\mathbf{A}_1^{+})$. We must find $\mathbf{B} \leq \mathbf{A}^I$ and $p \in B(U)$ such that $(\mathbf{B}, p)$ satisfies the conditions in $\mathcal{X}$ and $\mathbf{S} \cong \mathbf{B}(p)^{\otimes}$.

CASE 1: S is finite.

The proof is essentially McKenzie's [3, Lemmas 6.4–6.9]. As in the proof of Theorem 1.2, choose a set I of least cardinality such that $\mathbf{S} \in \mathsf{HS}((\mathbf{A}_1^{+})^I)$, and choose $\mathbf{B}' \leq (\mathbf{A}_1^{+})^I$ and $\theta \in \operatorname{Con} \mathbf{B}'$ such that $\mathbf{S} \cong \mathbf{B}'/\theta$. Let $\bar{\theta}$ be the unique cover of θ in $\operatorname{Con} \mathbf{B}'$. Let $\leq$ denote the partial ordering on B' derived from the semilattice operation of $\mathbf{B}'$, and choose $(p, q) \in \bar{\theta} \setminus \theta$ as in the proof of Theorem 1.2, Case 1. Also let $\mathbf{B}^*$ be the reduct of $\mathbf{B}'$ to the original type of $\mathbf{A}$. Thus $\mathbf{B}^* \leq (\mathbf{A}_1)^I$.

We shall complete Case 1 by showing the following: (i) $p \in U^I$; (ii) if B_p^* is defined in $\mathbf{B}^*$ relative to the M-algebra $\mathbf{A}_1$ (as explained in Section 1), then $B_p^* \subseteq U^I$; (iii) $\theta = 0_{B'} \cup (B' \setminus B_p^*)^2$. Then letting $\mathbf{B}$ be the subalgebra of $\mathbf{A}^I$ generated by B_p^* it will follow that $\mathbf{B}(p) = \mathbf{B}^*(p) \cong \mathbf{B}^*/\theta$. The analysis of $\mathbf{B}'$ will also reveal that $\mathbf{B}(p)^{\otimes} \cong \mathbf{B}'/\theta$ and that $(\mathbf{B}, p)$ satisfies the conditions in $\mathcal{X}$. Now here are the details.

Claim 1. $(f, p) \in \theta$ implies $f \geq p$.

Claim 2. $(f, g) \notin \theta$ implies that there exists $\lambda \in \operatorname{Pol}_1 \mathbf{B}'$ such that $\lambda(f) < p$ and $\lambda(g) = p$, or vice versa.

Claim 3. $0 \notin \operatorname{range}(p)$; and for every $i \in I$, if $p_i \in (A_1)^I$ is defined by $p_i(j) = p(j)$ for $j \neq i$ and $p_i(i) = 0$, then $p_i \in B'$.

These claims are proved just as in the proof of Theorem 1.2. (The second half of Claim 3 follows from the proof given there.)

Claims 4–6 from the corresponding part of the proof of Theorem 1.2 will also eventually appear here, but first we need to establish several intermediate claims.

Claim A. If $f < p$, $\lambda \in \operatorname{Pol}_1 \mathbf{B}'$, and $\lambda(f) = p$, then $\lambda(q) = p$.

For the proof, first note that each fundamental operation of $\mathbf{A}_1^{+}$ is order-preserving, hence the same is true of every polynomial in $\mathbf{B}'$. Secondly, note that $f < p$ implies $(f, f \wedge q) \in \theta$ by the choice of (p, q). Observe that p is maximal in $\langle B', \leq \rangle$ by Claim 3. Now use Claim 2 and the monotonicity of λ (for help, see [3, Lemma 6.5]).

Let $L = \{f \in B' : f \leq p\}$ and $m = |I|$. The hypothesis $\mathbf{S} \notin \mathsf{HS}(\mathbf{A}_1^+)$ implies $m > 1$. By Claim 3, $\langle L, \wedge \rangle$ is the meet semilattice reduct of a 2^m-element Boolean lattice $\langle L, \wedge, \vee \rangle$.

Claim B. $\mathbf{B}'$ has no binary polynomial whose restriction to L is $\vee$.

Otherwise, $\theta|_L$ and $\bar{\theta}|_L$ would be congruences of $\langle L, \wedge, \vee \rangle$, which contradicts what we already know about θ, $\bar{\theta}$, p, q, and the fact that $m > 1$.

Claim C. Let $\mathcal{B}_\Phi \in \mathcal{X}$. There do not exist $\bar{f}$ in $B'(U_1)$ satisfying $\Phi(\bar{f})$.

Claim D. There do not exist $f, g \in B'(U_1)$ satisfying $\partial(f) = g$.

Otherwise $\mathrm{B}_\Phi(\bar{f}, p, x, y)$ or $\mathrm{S}_2(f, g, p, x, y)$ would yield $x \vee y$ on L, contradicting Claim B.

Claim E. If $f \in B'$ and $\nu(f) = \nu(p)$, then $f = p$.

(See [3, Lemma 6.6(ii)].) Assume that $\nu(f) = \nu(p)$ but $f \neq p$. Also $\partial(f) \neq p$ by Claim D. Hence there exist $i, j \in I$ such that $f(i) = p(i)$ and $f(j) = \partial(p(j))$. Choose $k \in I$ such that $q = p_k$ (see the discussion before Claim B). The definition of J in $\mathbf{A}_1^+$ and $f(i) = p(i)$ imply $\mathrm{J}(p, f, p_i) = p$, hence $\mathrm{J}(p, f, q) = p$ by Claim A, which forces $f(k) = p(k)$. But a similar argument using j and J' in place of i and J forces $f(k) = \partial(p(k))$.

It is necessary to make some technical remarks concerning B_p^*. Let Π_0 be the set of all unary polynomials of $\mathbf{B}'$ of the form

$$F(g_1, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_n)$$

where F is an original n-ary operation other than $\wedge$, $1 \leq i \leq n$, and the g_j $(j \neq i)$ are in $B'(U_1)$. Let Π be the closure of $\Pi_0 \cup \{\mathrm{id}(x)\}$ under composition. Then:

1. $B_p^* = \{f \in B' : \lambda(f) = p \text{ for some } \lambda \in \Pi\}$.
2. Suppose $\lambda \in \Pi$, $f, g \in B'(U_1)$, $i \in I$, and $f(i) = \partial(g(i))$. Then $\lambda(f)(i) = \partial(\lambda(g)(i))$.

Claim F. If $f \in B'$ and $g \in B_p^*$ and $\nu(f) = \nu(g)$, then $f = g$.

Indeed, choose $\lambda \in \Pi$ such that $\lambda(g) = p$. If $\nu(f) = \nu(g)$ but $f \neq g$, then it follows from the second technical remark that $\nu(\lambda(f)) = \nu(p)$ but $\lambda(f) \neq p$. This contradicts Claim E.

An immediate consequence of Claim F is:

Claim G. The following are true in $\mathbf{B}'$:

1. (For $\mathcal{C}_{\Psi,\Sigma}^{S,\Phi} \in \mathcal{X}$): If $s(\bar{f}) = t(\bar{f}) \in B_p^*$ for all $s, t \in S$ and if $\Phi(\bar{f})$ holds, then $\Psi(\bar{f})$ and $\Sigma(\bar{f})$ also hold.
2. $\mathrm{J}(f, g, h) \in B_p^*$ if and only if $f = g \in B_p^*$, in which case $\mathrm{J}(f, g, h) = f$.
3. $\mathrm{J}'(f, g, h) \in B_p^*$ if and only if $f = g = h \in B_p^*$, in which case $\mathrm{J}'(f, g, h) = f$.

For example, assume $f_1 := \mathrm{J}(f, g, h) \in B_p^*$. By examining the definition of J in $\mathbf{A}_1^+$, one sees that $\nu(f_1) = \nu(f) = \nu(g)$ and hence $f_1 = f = g$ by Claim F. The converse is clear, and the other items are proved similarly.

Claim H. $B_p^* \subseteq U^I$.

For if $g \in B_p^*$, then $f := \nu(g) \in B'$ (because ν is a fundamental operation of $\mathbf{B}'$) and $\nu(f) = \nu(g)$, hence $f = g$ by Claim F, i.e., $g \in U^I$.

Claim J. (For $\mathcal{C}_{\Psi,\Sigma}^{S,\Phi} \in \mathcal{X}$): If $\mathrm{C}_{\Psi,\Sigma}^{S,\Phi}(\bar{f}) \in B_p^*$ then $f_1, \ldots, f_n \in B_p^*$.

The proof is similar to the proof of Claim 3 in Part I.

We are finally in a position to prove the analogues of Claims 4–6 from the proof of Theorem 1.2.

Claim 4. For $f \in B'$ the following are equivalent: (i) there exists a nonconstant $\lambda \in \mathrm{Pol}_1 \mathbf{B}'$ such that $\lambda(f) = p$; (ii) there exists $\lambda \in \mathrm{Pol}_1 \mathbf{B}'$ such that $\lambda(f) = p$ and $\lambda(\hat{0}) = \hat{0}$; (iii) $f \in B_p^*$.

(ii) $\Rightarrow$ (i) is trivial. (iii) $\Rightarrow$ (ii) follows from the first technical remark concerning B_p^* and the fact that every $\lambda \in \Pi$ satisfies $\lambda(\hat{0}) = \hat{0}$. (i) $\Rightarrow$ (iii) is proved by establishing the following stronger claim:

> Assume that $f \in B'$, that $\lambda(x)$ is a nonconstant unary polynomial of $\mathbf{B}'$, and that $\lambda(f) \in B_p^*$. Then $f \in B_p^*$.

The proof is by induction on the complexity of λ, using Claims C, D, F, G, or J if the outermost fundamental operation in the construction of λ is a new operation (see [3, Lemma 6.8]).

Claim 5. $(f, \hat{0}) \notin \theta$ implies $f \in B_p^*$.

Claim 6. $f \in B_p^*$ implies $f/\theta = \{f\}$.

These follow exactly as in the proof of Theorem 1.2.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^I$ generated by B_p^*. Then $\mathbf{B}(p) = \mathbf{B}^*(p)$ and, by Claims 5 and 6, the map $\iota : f \mapsto f/\theta$ is an isomorphism from $\mathbf{B}(p)$ to $\mathbf{B}^*/\theta$. $(\mathbf{B}, p)$ satisfies the $\mathcal{B}$-conditions in $\mathcal{X}$ by Claim C, and the $\mathcal{C}$-conditions in $\mathcal{X}$ by Claim G. Finally, that ι is an isomorphism from $\mathbf{B}(p)^{\otimes}$ to $\mathbf{B}'/\theta$ follows from Claims C, D, G and H.

CASE 2: $\mathbf{S}$ is infinite.

As in the proof of Theorem 1.2, this case reduces to the previous case by an ultraproduct argument. First note that if $\mathbf{B} \leq \mathbf{A}^I$ and $\theta \in \operatorname{Con}\mathbf{B}$ and $B \setminus B(U) \subseteq \hat{0}/\theta$, then one can define the notion of $(\mathbf{B}, \theta)$ "satisfying the conditions in $\mathcal{X}$" in the obvious way. The only item requiring modification is the definition of $\mathcal{C}_{\Psi,\Sigma}^{S,\Phi}$ before the statement of Theorem 2.7: simply replace the one occurrence of B_p in the original definition by $B \setminus (\hat{0}/\theta)$. This definition is compatible with the original one: if there exists $p \in B(U)$ such that $\theta = 0_B \cup (B \setminus B_p)^2$, then $(\mathbf{B}, \theta)$ satisfies the conditions in $\mathcal{X}$ in this new sense if and only if $(\mathbf{B}, p)$ satisfies the conditions in $\mathcal{X}$ in the original sense.

Again suppose that $\mathbf{B} \leq \mathbf{A}^I$. One can define an expansion $\mathbf{B}^{\otimes}$ of $\mathbf{B}$ to the type $\tau(\mathcal{X})$ using the same rules defining $\mathbf{B}(p)^{\otimes}$ (preceding the statements of Theorems 2.1 and 2.7). If $\theta \in \operatorname{Con}\mathbf{B}$ satisfies $B \setminus B(U) \subseteq \hat{0}/\theta$, and if $(\mathbf{B}, \theta)$ satisfies the $\mathcal{C}$-conditions in $\mathcal{X}$ in the sense of the previous paragraph, then this definition of $\mathbf{B}^{\otimes}$ is compatible with the original one in that (i) $\theta \in \operatorname{Con}\mathbf{B}^{\otimes}$, and (ii) if there exists $p \in B(U)$ such that $\theta = 0_B \cup (B \setminus B_p)^2$, then $\mathbf{B}^{\otimes}/\theta \cong \mathbf{B}(p)^{\otimes}$.

Now suppose $\mathbf{B}$ and θ are as above, with $B \setminus B(U) \subseteq \hat{0}/\theta$. $(\mathbf{B}, \theta)^{\odot}$ shall be the expansion of $\mathbf{B}^{\otimes}$ to include the relations $B(U)$, θ, and all Φ and Ψ (defined coordinatewise) for $\mathcal{B}_\Phi$ or $\mathcal{C}_{\Psi,\Sigma}^{S,\Phi} \in \mathcal{X}$. Let $\mathcal{H}_1$ be the class of all such $(\mathbf{B}, \theta)^{\odot}$ such that $(\mathbf{B}, \theta)$ satisfies the conditions in $\mathcal{X}$, and let $\mathcal{H}_2 = \{\mathbf{B}^{\otimes}/\theta : (\mathbf{B}, \theta)^{\odot} \in \mathcal{H}_1\}$. Then as in the proof of Theorem 1.2 it can be shown that $\mathbf{I}(\mathcal{H}_1)$, and hence also $\mathbf{I}(\mathcal{H}_2)$, is axiomatized by universal first-order sentences. The rest of the proof is like the proof of Theorem 1.2. ∎

Acknowledgment: I wish to thank Ralph McKenzie and Matthew Valeriote for permitting me to include details from their manuscripts in this paper; and the referee for several valuable suggestions.

References

[1] W. Dziobiak, On infinite subdirectly irreducible algebras in locally finite equational classes, ALGEBRA UNIVERSALIS **13** (1981), 393-394.

[2] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Amer. Math. Soc. (Providence, R.I.), 1988.

[3] R. McKenzie, The residual bounds of finite algebras, INT. J. ALGEBRA AND COMPUTATION, to appear.

[4] R. McKenzie, The residual bound of a finite algebra is not computable, INT. J. ALGEBRA AND COMPUTATION, to appear.

[5] G. McNulty and C. Shallon, Inherently nonfinitely based finite algebras, in *Universal Algebra and Lattice Theory*, eds. R. Freese and O. Garcia, Springer Lecture Notes in Mathematics no. 1004, 1983, pp. 206-231.

[6] R. Quackenbush, Equational classes generated by finite algebras, ALGEBRA UNIVERSALIS **1** (1971), 265-266.

[7] M. Valeriote, A residually small, finitely generated, semi-simple variety which is not residually finite, manuscript, 1994.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO N2L 3G1
CANADA

email: `rdwillar@flynn.uwaterloo.ca`