DETERMINING WHETHER $\mathsf{V}(\mathbf{A})$ HAS A MODEL COMPANION IS UNDECIDABLE

ROSS WILLARD

Abstract. Using techniques pioneered by R. McKenzie, we prove that there is no algorithm which, given a finite algebra in a finite language, determines whether the variety (equational class) generated by the algebra has a model companion. In particular, there exists a finite algebra such that the variety it generates has no model companion; this answers a question of Burris and Werner from 1979.

1. Introduction

The notions of a *model completion* and *model companion* of a first-order theory arose in the work of A. Robinson as a generalization of the relationship that the theory of algebraically closed fields has to the theory of fields (see [8, 3]). In 1979 S. Burris and H. Werner [2] proved that the universal Horn theory of a finite set of finite structures always has a model companion. In particular, the equational theories of many finite algebraic structures (including finite lattices, Heyting algebras, relative Stone algebras, abelian groups) have a model companion. Burris and Werner asked whether there exists any finite algebra whose equational theory does *not* have a model companion.

A *variety* is the class of models of an equational theory of algebras. It is *finitely generated* if it is the class of models of the equational theory of some finite algebra. In the fall of 1993, R. McKenzie [5] refuted some long-standing conjectures of universal algebra by showing how to construct finitely generated varieties in which one has remarkable control over the class of irreducible members. In particular [6], he showed how to effectively incorporate the computations of a Turing machine into the construction so that the resulting variety is residually finite if and only if the machine halts.

In this paper we adapt McKenzie's construction so that the equational theory of the constructed algebra has a model companion if and only if the Turing machine from which it was built halts. This proves the title of this paper, as well as settling the question of Burris and Werner in the negative.

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The idea of the construction is relatively simple. Let L^* be the 2-sorted language (with the sorts named G and X respectively) which contains a binary relation symbol E on the first sort, a binary operation symbol $\times$ on the second sort, and a right action $*$ of the second sort on the first sort (i.e., a binary operation symbol $* : G \times X \to G$). Let $\mathcal{K}$ be the class of L^*-structures $\mathbf{M} = \langle G, X; E, \times, * \rangle$ axiomatized by:

- $\mathbf{G} := \langle G; E \rangle$ is a disjoint union of directed chains. That is, $\mathbf{G}$ is an irreflexive acyclic directed graph and satisfies $\forall abcd[(aEc \ \&\ bEd) \to (a = b \leftrightarrow c = d)]$.
- $\mathbf{X} := \langle X; \times \rangle$ is a Boolean group (i.e., a group of exponent at most 2).
- $*$ is an action of $\mathbf{X}$ on $\mathbf{G}$. That is, $\mathbf{M}$ satisfies $\forall xya[a * (x \times y) = (a * x) * y]$, $\forall a[a * 1 = a]$, and $\forall abx[aEb \to (a * x)E(b * x)]$.
- $*$ is *regular* in the following sense: for all $a \in G$ and $x \in X$, if $a * x = a$ then $x = 1$.

Also let $\mathcal{K}_n$ denote the class of those $\mathbf{M} \in \mathcal{K}$ such that every chain in $\mathbf{G}$ has length at most n.

If $\mathbf{M} \in \mathcal{K}$ and $a, b \in G$, say that a, b are *conjugate in* $\mathbf{M}$ if there exists $x \in X$ such that $a * x = b$; and are *weakly connected in* $\mathbf{M}$ if a is conjugate to some element in the connected component of $\mathbf{G}$ containing b. Say that $\mathbf{M}$ is weakly connected if every pair of elements in $\mathbf{G}$ is weakly connected.

The existentially closed members of $\mathcal{K}$ are weakly connected. In fact, if $\mathbf{M} \in \mathcal{K}$ and a, b are not weakly connected in $\mathbf{M}$, then there exists $\mathbf{M}' \in \mathcal{K}$ such that $\mathbf{M} \leq \mathbf{M}'$ and a, b are conjugate in $\mathbf{M}'$. Since the relation of weak connectivity is not definable in $\mathcal{K}$ by a first-order formula, the existentially closed members of $\mathcal{K}$ do not form an axiomatizable class. This implies that the theory of $\mathcal{K}$ has no model companion.

On the other hand, weak connectivity is definable in each subclass $\mathcal{K}_n$ and the theory of each $\mathcal{K}_n$ does have a model companion.

Our strategy in this paper is to design, for each Turing machine $\mathcal{T}$, a finite algebra $\mathbf{M}(\mathcal{T})$ so that the irreducible members of the variety generated by $\mathbf{M}(\mathcal{T})$ encode all the members of $\mathcal{K}$ (actually, just the weakly connected ones) if $\mathcal{T}$ does not halt, but only the members of $\mathcal{K}_n$ for some $n < \omega$ if $\mathcal{T}$ does halt. Readers who are familiar with McKenzie's work will realize how this can be done. Two problems arise: (1) in either case, the variety will contain many more irreducible members than the intended ones, and (2) the theory of the variety is not the same as the theory of its irreducible members. In case $\mathcal{T}$ halts, neither problem is serious; we are able to show that there exist finitely many $\aleph_0$-categorical irreducible algebras $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in the variety such that every countable irreducible member embeds into some $\mathbf{S}_i$, and this suffices. On the other hand, if $\mathcal{T}$ does not halt then a brute force argument is needed to lift noncompanionability up to the full variety.

We assume the notation and basic results of universal algebra as given in [1] or [7]. We also assume the reader is familiar with the formalism of first-order logic. Some tedious arguments concerning the exact description of the irreducible members of the constructed varieties are merely sketched or even omitted. The reader may profit by first reading [6].

2. Model companions

A first-order theory T is *model-complete* if every embedding between models of T is an elementary embedding. T is a *model companion* of a theory T_0 if (i) T is model-complete and (ii) every model of T_0 can be embedded in a model of T and vice versa. See [4, Section 8.3].

A model $\mathbf{M}$ of a theory T is *existentially closed in* T if whenever $\mathbf{M} \leq \mathbf{N} \models T$, every existential sentence with parameters from M which is true in $\mathbf{N}$ is already true in $\mathbf{M}$. If $\mathcal{K}$ is an axiomatizable class, then we use $\mathcal{K}_{ec}$ to denote the class of structures which are existentially closed in the theory of $\mathcal{K}$. The following basic result may be found in several places, e.g. [4, Theorem 8.3.6].

Lemma 2.1. *If T is an $\forall\exists$-axiomatizable theory and $\mathcal{K}$ is its class of models, then T has a model companion if and only if $\mathcal{K}_{ec}$ is axiomatizable. If $\mathcal{K}_{ec}$ is axiomatizable, then the theory of $\mathcal{K}_{ec}$ is the unique model companion of T.*

When $\mathcal{T}$ halts, we shall establish that the equational theory of $\mathbf{M}(\mathcal{T})$ has a model companion by citing the following result of Burris and Werner.

Lemma 2.2. [2, Theorem 8.4] *Suppose T is a first-order theory such that for every $n < \omega$ there are only finitely many existential formulas in n free variables, up to equivalence in T. Then the theory of universal Horn consequences of T has a model companion.*

When $\mathcal{T}$ does not halt, we shall appeal to the following elementary result.

Lemma 2.3. *Suppose T is an $\forall\exists$-axiomatizable theory, N is an infinite set, $\phi(x,y)$, $\psi(x,y)$ and $\theta_n(x,y)$ ($n \in N$) are existential formulas, and $\{\delta_i(x,y) : i \in I\} = \Delta$ is a family of first-order formulas in the language of T. Suppose moreover that the following conditions hold:*

1. *For all $\mathbf{A} \models T$ and $a, b \in A$, if $\mathbf{A} \models \phi(a,b)$ & $\bigwedge_{\delta\in\Delta} \delta(a,b)$, then there exists $\mathbf{B} \models T$ such that $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \models \psi(a,b)$.*
2. *$T \vdash [\phi(x,y)$ & $\psi(x,y)] \to \bigwedge_{n\in N} \neg\theta_n(x,y)$.*
3. *There exists a function r with domain Δ such that*
   - *(a) $r(\delta)$ is a finite set for all $\delta \in \Delta$.*
   - *(b) For all $n \in N$,*
   $$T \vdash [\phi(x,y) \ \&\ \theta_n(x,y)] \to \bigwedge_{\{\delta\,:\,n\notin r(\delta)\}} \delta(x,y).$$
4. *For all $n \in N$ there exists $\mathbf{A} \models T$ satisfying $\exists xy[\phi(x,y)$ & $\theta_n(x,y)]$.*

Then T does not have a model companion.

Proof. Let $\mathcal{K}$ be the class of models of T. Using item 4, for each $n \in N$ choose $\mathbf{A}_n \in \mathcal{K}$ and $a_n, b_n \in A$ such that $\mathbf{A}_n \models \phi(a_n, b_n)$ & $\theta_n(a_n, b_n)$. Pick $\mathbf{B}_n \in \mathcal{K}_{ec}$ such that $\mathbf{A}_n \leq \mathbf{B}_n$; thus $\mathbf{B}_n \models \phi(a_n, b_n)$ & $\theta_n(a_n, b_n)$ as well. Moreover, $\mathbf{B}_n \models \neg\psi(a_n, b_n)$ by item 2, and $\mathbf{B}_n \models \delta(a_n, b_n)$ for all $\delta \in \Delta$ such that $n \notin r(\delta)$ by item 3.

Let U be a nonprincipal ultrafilter over N and put $\mathbf{C} = \prod \mathbf{B}_n/U$ and $a = (a_n)/U$ and $b = (b_n)/U$. Then $\mathbf{C} \models \neg\psi(a, b)$ & $\phi(a, b)$ & $\bigwedge_{\delta\in\Delta} \delta(a, b)$ by construction. By item 1, there exists $\mathbf{D} \in \mathcal{K}$ such that $\mathbf{C} \leq \mathbf{D}$ and $\mathbf{D} \models \psi(a, b)$. Thus $\mathbf{C} \notin \mathcal{K}_{ec}$, which proves that $\mathcal{K}_{ec}$ is not axiomatizable and therefore T does not have a model companion by Lemma 2.1. □

3. Turing machines

Let $\mathcal{T}$ be a Turing machine of the type considered in [6]. Thus the tape alphabet of $\mathcal{T}$ is $\{0, 1\}$, the set of internal states of $\mathcal{T}$ is $\{q_0, q_1, \ldots, q_{k-1}\}$ for some $k = k(\mathcal{T}) \geq 2$ and the transition function of $\mathcal{T}$ is a total function

$$\mathcal{T} : \{q_1, \ldots, q_{k-1}\} \times \{0, 1\} \to \{0, 1\} \times \{L, R\} \times \{q_0, \ldots, q_{k-1}\}.$$

0 is the *blank* symbol, q_1 is the *initial state*, and q_0 is the unique *halting state*. The instruction $\mathcal{T}(q, r) = (s, D, q')$ means "If the read/write head is viewing the symbol r while in state q, then replace the symbol by s, move one step in direction D ($D = L$ means *left*, $D = R$ means *right*), and change to state q'." We interpret $\mathcal{T}$ to operate on two-way infinite tapes, so that the machine never halts for lack of space.

Somewhat more precisely, *configurations* are triples (t, m, i) where $t \in \{0, 1\}^{\mathbb{Z}}$ (t is a tape), $m \in \mathbb{Z}$ (m is the current location of the read/write head), and $0 \leq i < k$ (q_i is the current state of the machine). $\mathrm{Config}_{\mathcal{T}}$ (or simply Config) is the set of all configurations for $\mathcal{T}$, $\vdash_{\mathcal{T}}$ (or simply $\vdash$) is the one-step transition relation on Config determined by the instructions of $\mathcal{T}$, and $\vdash^*$ is the reflexive transitive closure of $\vdash$. Thus $\mathcal{Q} \vdash^* \mathcal{Q}'$ means that the computation of $\mathcal{T}$ starting from the configuration $\mathcal{Q}$ will eventually reach the configuration $\mathcal{Q}'$.

A configuration (t, m, i) is *in the halting state* if $i = 0$. Letting t_0 denote the constant zero (i.e., blank) tape, we say that $\mathcal{T}$ *halts* if there exists a configuration $\mathcal{Q}$ in the halting state such that $(t_0, 0, 1) \vdash^* \mathcal{Q}$. The *halting problem* we use in this paper is the decision problem which asks, for each Turing machine of the kind described here, whether the machine halts in this sense.

More generally, suppose N is a nonempty interval (convex subset) of $\mathbb{Z}$. An *N-configuration* is a configuration (t, m, i) such that (1) $t(\ell) = 0$ for all $\ell \notin N$, and (2) $m \in N$. We let Config_N denote the set of all N-configurations, let $\vdash_N$ denote the restriction of $\vdash$ to Config_N, and let $\vdash_N^*$ be the reflexive transitive closure of $\vdash_N$. Thus $\mathcal{Q} \vdash_N^* \mathcal{Q}'$ if and only if the tape of $\mathcal{Q}$ is blank outside of N, the computation of $\mathcal{T}$ starting from $\mathcal{Q}$ eventually reaches $\mathcal{Q}'$, and the read/write head never moves outside of N during this computation.

We shall be especially interested in the directed graph $\langle \mathrm{Config}_N, \vdash_N \rangle$ and its connected components. Note that, for a fixed Turing machine $\mathcal{T}$, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each n, $f(n)$ is an upper bound to the sizes of the connected components of $\langle \mathrm{Config}_N, \vdash_N \rangle$ for any interval N with $|N| < n$. Also note that, because $\mathcal{T}$ has no defined action on configurations in the halting state, if $\mathcal{Q} \in \mathrm{Config}_N$ then the connected component containing $\mathcal{Q}$ contains a configuration $\mathcal{Q}'$ in the halting state if and only if $\mathcal{Q} \vdash_N^* \mathcal{Q}'$.

It will be helpful to simulate the computations of $\mathcal{T}$ by a nondeterministic machine $\mathcal{T}^\partial$. The instructions of $\mathcal{T}^\partial$ will be of the following kind. Given $i, i' \in \{0, \ldots, k-1\}$ and $r, s, r', s' \in \{0, 1\}$, an instruction $irsRi'r's'$ will have the following intended effect: if $\mathcal{Q} = (t, m, i)$ where $(t(m), t(m+1)) = (r, s)$, then the above instruction can be applied to $\mathcal{Q}$ and it will produce a new configuration $\mathcal{Q}' = (t', m+1, i')$ where $(t'(m), t'(m+1)) = (r', s')$, and $t'(\ell) = t(\ell)$ for all $\ell \in \mathbb{Z} \setminus \{m, m+1\}$. Informally, the instruction $irsRi'r's'$ tells the machine to check not only the current state (it should be q_i) and current symbol being read (it should be r), but also the symbol to the right of the head (it should be s); if this is the case, then the machine is instructed to change not only the symbol being read (to r') but also the symbol to the right of the head (to s') and then move right, changing the state to $q_{i'}$.

Note that the intended effect of an instruction of $\mathcal{T}$ of the form $\mathcal{T}(q_i, r) = (s, R, q_j)$ is simulated by the pair of generalized instructions $ir0Rjs0$ and $ir1Rjs1$.

The instruction $irsLi'r's'$ shall have the reverse effect; that is, if $\mathcal{Q}' = (t', m+1, i')$ is a configuration where $(t'(m), t'(m+1)) = (r', s')$, then this instruction can be applied to $\mathcal{Q}'$ to produce the configuration $\mathcal{Q} = (t, m, i)$ where $(t(m), t(m+1)) = (r, s)$ and $t(\ell) = t'(\ell)$ for all $\ell \in \mathbb{Z} \setminus \{m, m+1\}$. In other words, the instruction $irsLi'r's'$ tells the machine to check not only the current state (it should be $q_{i'}$) and the symbol being read (it should be s'), but also the symbol to the left of the head (it should be r'); if this is the case, then the machine changes not only the symbol being read (to s) but also the symbol to the left of the head (to r) and then moves left, changing the state to q_i. Again, an instruction of $\mathcal{T}$ of the form $\mathcal{T}(q_i, r) = (s, L, q_j)$ is simulated by the pair of generalized instructions $j0sLi0r$ and $j1sLi1r$.

Let $\mathcal{T}'$ be the (deterministic) machine whose instructions are obtained by replacing each instruction of $\mathcal{T}$ by the corresponding pair of generalized instructions. Then let $\mathcal{T}^\partial$ be the (nondeterministic) machine obtained from $\mathcal{T}'$ by adding both $irsRi'r's'$ and $irsLi'r's'$ as instructions whenever at least one of them is an instruction of $\mathcal{T}'$. It should be clear that $\mathcal{T}$ and $\mathcal{T}'$ give rise to identical directed graphs $\langle \mathrm{Config}_N, \vdash_N \rangle$ for any interval N, and that the graphs corresponding to $\mathcal{T}^\partial$ (suitably defined) are the symmetric closures of the graphs for $\mathcal{T}'$. Thus all three machines give the identical decompositions of Config_N into connected components, for any interval N.

4. The intended subdirectly irreducibles

Throughout this and the next two sections, fix a Turing machine $\mathcal{T}$ of the kind described in the previous section, let $\mathcal{T}^\partial$ be the corresponding nondeterministic machine described at the end of the previous section, and let *Instr* be the collection of all 6-tuples $irsjpq$ such that $irsRjpq$ and $irsLjpq$ are instructions of $\mathcal{T}^\partial$.

In Section 5 we shall effectively construct from $\mathcal{T}$ a finite algebra $\mathbf{M}(\mathcal{T})$ such that the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ reflect the strategy outlined in the introduction. $\mathbf{M}(\mathcal{T})$ will be built on a $(40k+25)$-element universe where $k = k(\mathcal{T})$ is the number of distinct internal states of $\mathcal{T}$, and will have $4c+16$ operations where $c = |\mathit{Instr}|$. The explicit definition of $\mathbf{M}(\mathcal{T})$ is quite tedious. In this section we shall be content to describe the language of $\mathbf{M}(\mathcal{T})$ and (most of) the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$.

The language consists of:

- Two nullary operations: 0 and e.
- Two unary operations: I and ν.
- Five binary operations: $\wedge, \cdot, \circ, *, \times$.
- $2c+2$ ternary operations: J, J', and a pair L_τ, R_τ for each $\tau \in \mathit{Instr}$.
- $2c+4$ quaternary operations: S_0, S_1, T_1, T_2, and a pair U^1_τ, U^2_τ for each $\tau \in \mathit{Instr}$.
- One 5-ary operation: S_2.

For psychological convenience we group the symbols as follows. $\cdot$ and $\circ$ are the *chain* operations. I and the operations L_τ, R_τ ($\tau \in \mathit{Instr}$) are the *machine* operations. $*, \times, e$ are the *group action* operations. The remaining operations — $0, \wedge, \nu, J, J', S_0, S_1, S_2, T_1, T_2$, and the pairs U^1_τ, U^2_τ ($\tau \in \mathit{Instr}$) — are the *enabling* operations. We also let L_0 denote the language consisting of the chain operations, machine operations, and 0; let L_1 denote the union of L_0 with the the group action operations and $\wedge$; and let L denote the full language consisting of all the listed symbols.

Now we define some algebras in these languages. Given a nonempty interval N (possibly infinite) in $\mathbb{Z}$, let $\mathbf{S}_N(\mathcal{T})$ be the algebra in the language L_0 defined as follows. The universe of $\mathbf{S}_N(\mathcal{T})$ is

$$\{0\} \cup \{a_n : n \in N\} \cup \{v_n : n \in N \text{ or } n-1 \in N\}$$

where the indicated elements are distinct. The operations are defined by interpreting 0 as itself, putting

$$\begin{aligned} a_n \cdot v_{n+1} &= v_n && \text{for } n \in N,\\ x \cdot y &= 0 && \text{in all other cases,}\\ a_n \circ v_n &= v_{n+1} && \text{for } n \in N,\\ x \circ y &= 0 && \text{in all other cases,} \end{aligned}$$

and setting each machine operation to be identically zero. $\mathbf{S}_N(\mathcal{T})$ will be called the **chain type building block** corresponding to the interval N. Note that the formula $x \neq 0$ & $y \neq 0$ & $\exists z(z \circ x = y$ & $z \cdot y = x)$ defines in $\mathbf{S}_N(\mathcal{T})$ a directed chain of edge order-type N; the elements $\{v_n : n \in N \text{ or } n-1 \in N\}$ are the vertices.

Again, let N be a nonempty interval in $\mathbb{Z}$ and let C be a nonempty connected component of $\langle \mathrm{Config}_N, \vdash_N \rangle$ in which no configuration is in the halting state. The algebra $\mathbf{P}_{(N,C)}(\mathcal{T})$ in the language L_0 is defined as follows. Its universe is

$$\{0\} \cup \{a_n : n \in N\} \cup C$$

where again the indicated elements are distinct. 0 is interpreted as itself, the chain operations $\cdot$ and $\circ$ are defined to be identically zero, and the machine operations are defined as follows. For each $n \in N$, if there exists $\mathcal{Q} = (t, n, 1) \in C$ such that $t(\ell) = 0$ for all $\ell \in N$ (such $\mathcal{Q}$, if it exists, is unique), then define $I(a_n) = \mathcal{Q}$. Put $I(x) = 0$ in all other cases. Next, suppose $\tau = irsjpq \in \mathit{Instr}$ and $\mathcal{Q}, \mathcal{Q}' \in C$ where $\mathcal{Q} = (t, n, i)$ and $\mathcal{Q}' = (t', n+1, j)$. If the generalized instruction $irsRjpq$ can be applied to $\mathcal{Q}$ to produce $\mathcal{Q}'$, then $R_\tau(a_n, a_{n+1}, \mathcal{Q})$ and $L_\tau(a_n, a_{n+1}, \mathcal{Q}')$ are defined to be $\mathcal{Q}'$ and $\mathcal{Q}$ respectively. All other values of $R_\tau(x, y, z)$ and $L_\tau(x, y, z)$ are declared to be 0. The algebra $\mathbf{P}_{(N,C)}(\mathcal{T})$ is called a **machine type building block** corresponding to the interval N.

Note that in both $\mathbf{S}_N(\mathcal{T})$ and $\mathbf{P}_{(N,C)}(\mathcal{T})$, the elements $\{a_n : n \in N\}$ are parameters which serve to coordinate the sequential structure of the remaining elements relative to the fundamental operations. In a chain type building block, a_n witnesses the existential parameter in the definition of the edge from v_n to v_{n+1}. In a machine type building block, the parameters are used in the fundamental operations to "point to" the two adjacent squares that participate in the generalized instruction of the Turing machine.

Definition 4.1. Suppose that $\mathbf{B}$ is an algebra in the language L_0 and X is a Boolean group. Define an algebra $\mathbf{B} \circledast X$ in the language L_1 as follows. The universe of $\mathbf{B} \circledast X$ is the disjoint union of X, $(B \setminus \{0^{\mathbf{B}}\}) \times X$, and $\{0\}$. 0 is interpreted as itself while each chain or machine operation H is defined as follows:

$$\begin{aligned} H((b_1, v), \ldots, (b_n, v)) &= (H^{\mathbf{B}}(\bar{b}), v) \text{ if } H^{\mathbf{B}}(\bar{b}) \neq 0^{\mathbf{B}} \\ H(x_1, \ldots, x_n) &= 0 \text{ in all other cases.} \end{aligned}$$

e is interpreted as the identity element of X, while $*$ and $\times$ are defined by

$$\begin{aligned} x \times y &= \begin{cases} xy & \text{if } x, y \in X \\ 0 & \text{otherwise} \end{cases} \\ (b, u) * v &= (b, uv) \text{ if } b \in B \setminus \{0^{\mathbf{B}}\} \text{ and } u, v \in X; \\ x * y &= 0 \text{ in all other cases.} \end{aligned}$$

Finally, $\wedge$ is defined by setting $x \wedge x = x$ and $x \wedge y = 0$ for $x \neq y$. This completes the definition.

Definition 4.2. For any algebra $\mathbf{C}$ in the language L_1, $\mathbf{C}^{\otimes}$ is the trivial expansion of $\mathbf{C}$ to the language L obtained by defining

$$\begin{array}{rcll} S_i(\bar{x}, u, z, w) & = & 0 & \text{for } i = 0, 1, 2 \\ T_1(x, y, z, w) & = & (x \wedge z) \cdot (y \wedge w) & \\ T_2(x, y, z, w) & = & (x \wedge z) \circ (y \wedge w) & \\ U^1_\tau(x, y, z, u) & = & R_\tau(x, y \wedge z, u) & \\ U^2_\tau(x, y, z, u) & = & R_\tau(x \wedge y, z, u) & \\ J(x, y, z) & = & x \wedge y & \\ J'(x, y, z) & = & x \wedge y \wedge z & \\ \nu(x) & = & x. & \end{array}$$

Definition 4.3. $\mathbf{T}$ is the algebra in the language L_1 built on the meet semilattice $\langle \{0, 1\}, \wedge \rangle$ by setting all other operations in L_1 equal to zero. $\mathbf{2}$ is the reduct of $\mathbf{T}$ to the language L_0. $\mathbf{1}$ is the unique one-element subalgebra of $\mathbf{2}$. $\mathbf{1}$ and $\mathbf{2}$ are the **trivial building blocks**.

The algebra $\mathbf{M}(\mathcal{T})$ to be defined in the next section will have the following property.

Theorem 4.4. *Let $\mathbf{M}(\mathcal{T})$ be the algebra defined in the next section.*

(1) *Suppose $\mathcal{T}$ does not halt. Then the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ are, up to isomorphism:*
  (a) *$\mathbf{T}^{\otimes}$ and all algebras of the form $(\mathbf{B} \circledast X)^{\otimes}$ where $\mathbf{B}$ is a chain type or machine type or trivial building block as described above, and X is a Boolean group.*
  (b) *A finite list of finite algebras, each of which is in $\mathsf{HS}(\mathbf{M}(\mathcal{T}))$ and in which the operation S_i is not identically equal to 0 for some $i = 0, 1, 2$.*

(2) *Conversely, suppose $\mathcal{T}$ does halt. Let M be the finite interval in $\mathbb{Z}$ representing the set of tape squares actually "visited" by $\mathcal{T}$ during the computation starting from the standard initial configuration $(t_0, 0, 1)$ in Config$_{\mathbb{Z}}$, and put $\pi(\mathcal{T}) = |M|$. Then the description of the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ is as in the previous item* except *that algebras of the form $(\mathbf{B} \circledast X)^{\otimes}$, where $\mathbf{B}$ is $\mathbf{S}_N(\mathcal{T})$ or $\mathbf{P}_{(N,C)}(\mathcal{T})$, are included only if $|N| < \pi(\mathcal{T})$.*

5. Definition of $\mathbf{M}(\mathcal{T})$

Recall that $\mathcal{T}$ is a fixed Turing machine having $k = k(\mathcal{T})$ distinct internal states. Before defining the algebra $\mathbf{M}(\mathcal{T})$, we briefly recall McKenzie's scheme for encoding the elements of the chain type and machine type building blocks. Let V be the $10k$-element set

$$\{C^s_{ir}, D^s_{ir}, M^r_i \, : \, 0 \leq i < k, \, \{r, s\} \subseteq \{0, 1\}\}$$

where the indicated elements are pairwise distinct and also distinct from $1, 2, C, D, H$ and 0. For future reference, define $V_0 = \{C^s_{0r}, D^s_{0r}, M^r_0 : \{r, s\} \subseteq \{0, 1\}\}$. Given a nonempty interval $N \subseteq \mathbb{Z}$, define sequences $[a_n] \in \{1, H, 2\}^N$ (if $n \in N$), $[v_n] \in \{C, D\}^N$ (if $n \in N$ or $n - 1 \in N$), and $[\mathcal{Q}] \in V^N$ (if $\mathcal{Q} = (t, n, i) \in \mathrm{Config}_N$) as follows:

$$\begin{aligned}
[a_n](\ell) &= \begin{cases} 1 & \text{if } \ell < n, \\ H & \text{if } \ell = n, \\ 2 & \text{if } \ell > n, \end{cases} \\
[v_n](\ell) &= \begin{cases} C & \text{if } \ell < n, \\ D & \text{if } \ell \geq n, \end{cases} \\
[(t, n, i)](\ell) &= \begin{cases} C^{t(\ell)}_{it(n)} & \text{if } \ell < n, \\ M^{t(n)}_i & \text{if } \ell = n, \\ D^{t(\ell)}_{it(n)} & \text{if } \ell > n. \end{cases}
\end{aligned}$$

Note in particular that each element in the sequence representing the configuration $\mathcal{Q}$ records not only the symbol at the corresponding position of the tape, but also (1) the direction from that position to the read/write head, (2) the symbol currently under the read/write head, and (3) the current state of the machine.

Comparing the intended implementations of the elements a_n, v_n and $\mathcal{Q}$ displayed above with the definitions of the fundamental operations in the chain type and machine type building blocks should help the reader make sense of the next definition.

Definition 5.1. $\mathbf{M}_0(\mathcal{T})$ is the algebra in the language L_0 whose universe is $V \cup \{0, 1, 2, C, D, H\}$ and whose operations are defined as follows. 0 is interpreted as itself. The chain operations are defined by:

$$\begin{aligned}
1 \cdot C &= C, \\
H \cdot C &= 2 \cdot D = D, \\
x \cdot y &= 0 \ \text{ for all other pairs } (x, y),
\end{aligned}$$

$$\begin{aligned}
1 \circ C &= H \circ D = C, \\
2 \circ D &= D, \\
x \circ y &= 0 \ \text{ for all other pairs } (x, y).
\end{aligned}$$

The unary machine operation I is defined by:

$$I(x) = \begin{cases} C^0_{10} & \text{if } x = 1, \\ M^0_1 & \text{if } x = H, \\ D^0_{10} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For each $\tau = irsjpq \in \mathit{Instr}$ define R_τ and L_τ as follows:

$$R_\tau(x,y,z) \;=\; \begin{cases} C^\sigma_{jq} & \text{if } (x,y,z) = (1,1,C^\sigma_{ir}) \text{ for some } \sigma, \\ C^p_{jq} & \text{if } (x,y,z) = (H,1,M^r_i), \\ M^q_j & \text{if } (x,y,z) = (2,H,D^s_{ir}), \\ D^\sigma_{jq} & \text{if } (x,y,z) = (2,2,D^\sigma_{ir}) \text{ for some } \sigma, \\ 0 & \text{in all other cases,} \end{cases}$$

$$L_\tau(x,y,z) \;=\; \begin{cases} C^\sigma_{ir} & \text{if } (x,y,z) = (1,1,C^\sigma_{jq}) \text{ for some } \sigma, \\ M^r_i & \text{if } (x,y,z) = (H,1,C^p_{jq}), \\ D^s_{ir} & \text{if } (x,y,z) = (2,H,M^q_j), \\ D^\sigma_{ir} & \text{if } (x,y,z) = (2,2,D^\sigma_{jq}) \text{ for some } \sigma, \\ 0 & \text{in all other cases.} \end{cases}$$

This completes the definition of $\mathbf{M}_0(\mathcal{T})$. Note that $|M_0(\mathcal{T})| = 10k+6$. The algebra just defined is clearly stolen from McKenzie's $\mathbf{A}(\mathcal{T})$ in [6], the only changes being that both the chain operations and machine operations have been made "reversible" as in [9].

Definition 5.2. $\mathbf{M}_1(\mathcal{T}) := \mathbf{M}_0(\mathcal{T}) \circledast K_2$ where K_2 denotes a two-element cyclic group with identity element e.

Thus $\mathbf{M}_1(\mathcal{T})$ is a $(20k+13)$-element algebra in the language L_1 essentially consisting of K_2 together with two copies of $\mathbf{M}_0(\mathcal{T})$ amalgamated over 0 and being acted upon by K_2. This doubling of $\mathbf{M}_0(\mathcal{T})$ is the only step at which our construction differs in an essential way from McKenzie's $\mathbf{A}(\mathcal{T})$. Our version of his binary relation $\preccurlyeq$ is defined as follows: for $x, y \in M_1(\mathcal{T})$ we write $x \preccurlyeq y$ if and only if $x = (x_1, c)$ and $y = (y_1, c)$ with $c \in K_2$ and $(x_1, y_1) \in \{(1,1), (H,1), (2,H), (2,2)\}$.

Finally, we implement McKenzie's enabling operations almost exactly as he did for his $\mathbf{A}(\mathcal{T})$ in [6]. A general context for the implementation is explained in [10]. First let U denote the set of nonzero elements of $\mathbf{M}_1(\mathcal{T})$ and put $U_1 = U \times \mathbb{Z}_2$. Also put $U_0 = U \times \{0\} \subseteq U_1$ and note that $U_0 \cup \{0\}$ is in obvious bijective correspondence with $M_1(\mathcal{T})$. We interpret $\preccurlyeq$ defined in the previous paragraph as a binary relation on U_0 via this correspondence. Also let $\partial : U_1 \cup \{0\} \to U_1 \cup \{0\}$ be the involution given by $\partial((u,c)) = (u, c{+}1)$ and $\partial(0) = 0$.

Definition 5.3. The algebra $\mathbf{M}(\mathcal{T})$ has universe $U_1 \cup \{0\}$ and operations defined as follows. 0 is interpreted by itself, e is interpreted by $(e^{\mathbf{M}_1(\mathcal{T})}, 0)$, and $\wedge$ is defined in the usual way: $x \wedge x = x$ while $x \wedge y = 0$ when $x \neq y$. Each operation $H \in L_1 \setminus \{0, e, \wedge\}$

is defined (with $n =$ arity of H) by

$$
\begin{aligned}
H((w_1,c_1),\ldots,(w_n,c_n)) &= \begin{cases} (H^{\mathbf{M}_1(\mathcal{T})}(\mathbf{w}), \sum_{i=1}^n c_i) & \text{if } H^{\mathbf{M}_1(\mathcal{T})}(\mathbf{w}) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \\
H(x_1,\ldots,x_n) &= 0 \ \text{ if } 0 \in \{x_1,\ldots,x_n\}.
\end{aligned}
$$

The remaining operations in $L \setminus L_1$ are defined as follows.

$$
\begin{aligned}
J(x,y,z) &= \begin{cases} x & \text{if } x = y \text{ or } z = x = \partial(y), \\ 0 & \text{otherwise,} \end{cases} \\
J'(x,y,z) &= J(x,\partial(y),z), \\
\nu(x) &= \begin{cases} (x_1,0) & \text{if } x = (x_1,c) \in W \times K_2, \\ 0 & \text{if } x = 0, \end{cases} \\
S_0(x,u,z,w) &= \begin{cases} u & \text{if } x \in V_0 \times K_2 \times \{0\} \text{ and } (u = z \text{ or } u = w), \\ 0 & \text{otherwise,} \end{cases} \\
S_1(x,u,z,w) &= \begin{cases} u & \text{if } x \in \{1,2\} \times K_2 \times \{0\} \text{ and } (u = z \text{ or } u = w), \\ 0 & \text{otherwise,} \end{cases} \\
S_2(x,y,u,z,w) &= \begin{cases} u & \text{if } x = \partial(y) \neq 0 \text{ and } (u = z \text{ or } u = w), \\ 0 & \text{otherwise,} \end{cases} \\
T_1(x,y,z,w) &= \begin{cases} x \cdot y & \text{if } x,y,z,w \in U_0 \text{ and } x \cdot y = z \cdot w \neq 0 \\ & \qquad \text{and } (x = z \text{ and } y = w), \\ \partial(x \cdot y) & \text{if } x,y,z,w \in U_0 \text{ and } x \cdot y = z \cdot w \neq 0 \\ & \qquad \text{and } (x \neq z \text{ or } y \neq w), \\ 0 & \text{otherwise,} \end{cases} \\
T_2(x,y,z,w) &= \begin{cases} x \circ y & \text{if } x,y,z,w \in U_0 \text{ and } x \circ y = z \circ w \neq 0 \\ & \qquad \text{and } (x = z \text{ and } y = w), \\ \partial(x \circ y) & \text{if } x,y,z,w \in U_0 \text{ and } x \circ y = z \circ w \neq 0 \\ & \qquad \text{and } (x \neq z \text{ or } y \neq w), \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

and, for each $\tau \in \mathit{Instr}$,

$$U^1_\tau(x,y,z,u) = \begin{cases} R_\tau(x,y,u) & \text{if } x,y,z,u \in U_0 \text{ and } R_\tau(x,y,u) \neq 0 \text{ and } x \preccurlyeq z \\ & \quad \text{and } y = z, \\ \partial(R_\tau(x,y,u)) & \text{if } x,y,z,u \in U_0 \text{ and } R_\tau(x,y,u) \neq 0 \text{ and } x \preccurlyeq z \\ & \quad \text{and } y \neq z, \\ 0 & \text{otherwise,} \end{cases}$$

$$U^2_\tau(x,y,z,u) = \begin{cases} R_\tau(x,z,u) & \text{if } x,y,z,u \in U_0 \text{ and } R_\tau(x,z,u) \neq 0 \text{ and } y \preccurlyeq z \\ & \quad \text{and } x = y, \\ \partial(R_\tau(x,z,u)) & \text{if } x,y,z,u \in U_0 \text{ and } R_\tau(x,z,u) \neq 0 \text{ and } y \preccurlyeq z \\ & \quad \text{and } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

6. Proof of Theorem 4.4

In this section we sketch the proof of Theorem 4.4.

Recall that $\mathcal{T}$ is a fixed Turing machine and $\mathbf{M}_1(\mathcal{T})$ is the $(20k+13)$-element algebra in the language L_1 defined in the previous section. Observe that (i) the reduct of $\mathbf{M}_1(\mathcal{T})$ to $\{\wedge\}$ is a height-1 semilattice with absorbing element 0, and (ii) each nonnullary fundamental operation $H \in L_1$ has the property that for any $a_1, \ldots, a_n \in M_1(\mathcal{T})$ ($n =$ arity of H), if $0 \in \{a_1, \ldots, a_n\}$ then $H(\mathbf{a}) = 0$.

Our analysis of the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ will be accomplished by studying certain members of $\mathsf{V}(\mathbf{M}_1(\mathcal{T}))$ satisfying "intended conditions."

Definition 6.1. ([10]) Suppose $I \neq \emptyset$ and $\mathbf{B} \leq \mathbf{M}_1(\mathcal{T})^I$.

1. $B_{\neq 0}$ denotes the set $B \cap U^I$.
2. $\gg$ is the binary relation on $B_{\neq 0}$ defined as follows: for $f, g \in B_{\neq 0}$, $f \gg g$ iff $H(f_1, \ldots, f_n) = g$ for some $H \in L_1 \setminus \{0, e\}$ and some $f_1, \ldots, f_n \in B_{\neq 0}$ ($n =$ arity of H) such that $f \in \{f_1, \ldots, f_n\}$.
3. $\ggg$ is the transitive closure of $\gg$.
4. If $p \in B_{\neq 0}$ then $B_p := \{f \in B_{\neq 0} : f \ggg p\}$ and $\theta_p := (B \setminus B_p)^2 \cup 0_B$.
5. If $p \in B_{\neq 0}$ then $\mathbf{B}(p) := \mathbf{B}/\theta_p$. We call $\mathbf{B}(p)$ a **neighborhood algebra**.

Note that neighborhood algebras belong to $\mathsf{V}(\mathbf{M}_1(\mathcal{T}))$. Suppose $\mathbf{B}(p)$ is a neighborhood algebra with $\mathbf{B} \leq \mathbf{M}_1(\mathcal{T})^I$. Here are some desirable restrictions on $\mathbf{B}$ and p that may or may not be satisfied.

1. $B \cap (V_0 \times K_2)^I = \emptyset$.
2. $B \cap (\{1,2\} \times K_2)^I = \emptyset$.
3. If $f, g, h, k \in B_{\neq 0}$ and $f \cdot g = h \cdot k \in B_p$, then $f = h$ and $g = k$.
4. If $f, g, h, k \in B_{\neq 0}$ and $f \circ g = h \circ k \in B_p$, then $f = h$ and $g = k$.
5. For each $\tau \in \mathit{Instr}$, if $f, g, h, k \in B_{\neq 0}$ and $R_\tau(f, g, k) \in B_p$ and $f(i) \preccurlyeq h(i)$ for all $i \in I$, then $g = h$.

(6) For each $\tau \in \mathit{Instr}$, if $f, g, h, k \in B_{\neq 0}$ and $R_\tau(f, h, k) \in B_p$ and $g(i) \preccurlyeq h(i)$ for all $i \in I$, then $f = g$.

Also let $3'$–$6'$ denote the versions of 3–6 obtained by replacing " $\in B_p$" with " $\in B_{\neq 0}$."

Definition 6.2. Let $\mathbf{B}(p)$ be a neighborhood algebra.

(1) $\mathbf{B}(p)$ is **good** if 1–6 hold.
(2) $\mathbf{B}(p)$ is **very good** if 1–2 and $3'$–$6'$ hold.

Lemma 6.3.

(1) If $\mathbf{S}$ is a subdirectly irreducible member of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$, then either $\mathbf{S} \in \mathsf{HS}(\mathbf{M}(\mathcal{T}))$ and some operation S_i $(i = 0, 1, 2)$ is not identically equal to 0 in $\mathbf{S}$, or $\mathbf{S} \cong \mathbf{B}(p)^{\otimes}$ for some good neighborhood algebra $\mathbf{B}(p)$.
(2) Conversely, if $\mathbf{B}(p)$ is a very good neighborhood algebra, then $\mathbf{B}(p)^{\otimes}$ is a subdirectly irreducible member of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$.

Proof. This lemma essentially follows from Theorem 2.7 of [10]. In the notation used there, $\mathbf{A}$ is $\mathbf{M}_1(\mathcal{T})$, restrictions 1–2 on neighborhood algebras are "$\mathcal{B}$-conditions," 3–6 are "$\mathcal{C}$-conditions, $3'$–$6'$ are the corresponding "strong $\mathcal{C}$-conditions," and $\mathbf{M}(\mathcal{T})$ as defined in the previous section is isomorphic to the algebra $\mathbf{A}_1^+$ built from $\mathbf{A} := \mathbf{M}_1(\mathcal{T})$ and this list of $\mathcal{B}$- and $\mathcal{C}$-conditions according to the recipe in the proof of [10, Theorem 2.7].

There are just two niggling details that must be addressed: $\mathbf{M}_1(\mathcal{T})$ fails to satisfy the hypotheses placed on $\mathbf{A}$ in the statement of [10, Theorem 2.7]; and the stated conclusions of [10, Theorem 2.7] do not imply the conclusions of the above lemma. We explain briefly what the problems are and why they are not serious.

The first problem is simply that the language of $\mathbf{M}_1(\mathcal{T})$ contains a nullary constant $e \neq 0$, yet this is disallowed in the definitions preceding the statement of [10, Theorem 2.7]. In fact, the reader of the proof of [10, Theorem 2.7] will find no point at which this assumption plays a role. The only effect of dropping the assumption is the extra care needed in defining $\mathbf{B}(p)$ (cf. [10, p. 150]).

The second problem is that, while [10, Theorem 2.7] establishes item 2 of the above lemma, it falls short of implying item 1. What it fails to yield is the following: if $\mathbf{S}$ is a subdirectly irreducible member of $\mathsf{HS}(\mathbf{M}(\mathcal{T}))$ and each operation S_i $(i = 0, 1, 2)$ is identically equal to 0 in $\mathbf{S}$, then $\mathbf{S} \cong \mathbf{B}(p)^{\otimes}$ for some good neighborhood algebra. However, the proof of [10, Theorem 2.7], Part II, Case 1, can be easily modified to establish this last fact. The only change needed occurs near Claim B. Just before this claim the hypothesis that $\mathbf{S} \notin \mathsf{HS}(\mathbf{M}(\mathcal{T}))$ was used to deduce $m > 1$, but in the current context neither the hypothesis nor the deduction are true. Hence the proof of Claim B is invalid in the current context. However, Claim B was used only in proving Claims C and D, which assert that S_0, S_1, S_2 are identically equal to 0 in $\mathbf{S}$. Fortunately, in the current context this is already assumed to be true. □

Let $\mathcal{K}_0$ be the set of all chain type building blocks $\mathbf{S}_N(\mathcal{T})$ and machine type building blocks $\mathbf{P}_{(N,C)}(\mathcal{T})$ such that $|N|$ satisfies the restriction in Theorem 4.4(2) if $\mathcal{T}$ halts. Let $\mathcal{K}$ be the class consisting of $\mathbf{T}$ and all algebras $\mathbf{B} \circledast X$ where $\mathbf{B}$ ranges over the members of $\mathcal{K}_0 \cup \{\mathbf{1}, \mathbf{2}\}$ and X ranges over all Boolean groups. In light of Lemma 6.3, the proof of Theorem 4.4 will be completed by showing the following.

Lemma 6.4.

(1) Every member of $\mathcal{K}$ is isomorphic to a very good neighborhood algebra.

(2) Conversely, every good neighborhood algebra is isomorphic to a member of $\mathcal{K}$.

Sketch of proof. To prove item 1, we give the definitions of the appropriate neighborhood algebras and leave the tedious verification of all details to the reader. The reader might want to first study [6, Section 4].

To represent $\mathbf{T}$, let $\mathbf{B}$ be the subalgebra of $\mathbf{M}_1(\mathcal{T})^2$ generated by the element $p = \langle e, (1, e)\rangle$. Then $B_{\neq 0} = \{p, \langle e, e\rangle\}$, $B_p = \{p\}$, $\mathbf{B}(p) \cong \mathbf{T}$, and $\mathbf{B}(p)$ is very good.

Given a chain type building block $\mathbf{S}_N(\mathcal{T})$ (respectively, a machine type building block $\mathbf{P}_{(N,C)}(\mathcal{T})$) in $\mathcal{K}_0$ and a Boolean group X, choose a linearly ordered set I containing N as an initial segment and sufficiently large so that X can be embedded in $(K_2)^I$; in fact, assume that X is a subgroup of $(K_2)^I$. For each $n \in N$ let C_n be the connected component in $\langle \mathrm{Config}_N, \vdash_N\rangle$ containing $(t_0, n, 1)$ where, recall, t_0 denotes the blank tape. Put $D = \bigcup_{n\in N} C_n$ (respectively $D = C \cup \bigcup_{n\in N} C_n$) and

$$E = X \;\cup\; (D \cup \{a_n : n \in N\} \cup \{v_n : n \in N \text{ or } n-1 \in N\}) \times X,$$

and note that the nonzero elements of $\mathbf{S}_N(\mathcal{T}) \circledast X$ (respectively, of $\mathbf{P}_{(N,C)}(\mathcal{T}) \circledast X$) are included in E. For each element $\alpha \in E$ we define a corresponding element $[\alpha] \in \mathbf{M}_1(\mathcal{T})^I$ as follows:

$$\begin{aligned}
[x] &= x \quad \text{for } x \in X,\\
[(a_n, x)](\ell) &= \begin{cases} (1, x(\ell)) & \text{if } \ell < n,\\ (H, x(\ell)) & \text{if } \ell = n,\\ (2, x(\ell)) & \text{if } \ell > n,\end{cases}\\
[(v_n, x)](\ell) &= \begin{cases} (C, x(\ell)) & \text{if } \ell < n,\\ (D, x(\ell)) & \text{if } \ell \geq n,\end{cases}\\
[((t, n, i), x)](\ell) &= \begin{cases} (C^{t(\ell)}_{it(n)}, x(\ell)) & \text{if } \ell < n,\\ (M^{t(n)}_i, x(\ell)) & \text{if } \ell = n,\\ (D^{t(\ell)}_{it(n)}, x(\ell)) & \text{if } \ell > n.\end{cases}
\end{aligned}$$

Let $[E] = \{[\alpha] : \alpha \in E\}$ and $F_0 = \{f \in M_1(\mathcal{T})^I : 0 \in \mathrm{range}(f)\}$, and put $B = [E] \cup F_0$. Then $\mathbf{B} \leq \mathbf{M}_1(\mathcal{T})^I$ and $B_{\neq 0} = [E]$, and for every $p \in B_{\neq 0}$, $\mathbf{B}(p)$ is very good.

Choose $n \in N$. If the building block with which we began the construction is $\mathbf{S}_N(\mathcal{T})$, put $p = [(v_n, e)]$. Then $\mathbf{B}(p) \cong \mathbf{S}_N(\mathcal{T}) \circledast X$. On the other hand, if the building block is $\mathbf{P}_{(N,C)}(\mathcal{T})$, choose $\mathcal{Q} \in C$ and put $p = ([\mathcal{Q}, e])$. Then $\mathbf{B}(p) \cong \mathbf{P}_{(N,C)}(\mathcal{T}) \circledast X$. $\mathbf{1} \circledast X$ and $\mathbf{2} \circledast X$ can also be realized by choosing $p = [e]$ and $p = [(a_n, e)]$ respectively. This proves the first item.

The proof of item 2 is very similar to McKenzie's proofs of Lemmas 5.4, 5.5, 5.6, 5.7(iv), 5.8, and 5.9 in [6] for $\mathbf{A}(\mathcal{T})$, the added complication being the group action. Given $\mathbf{B} \leq \mathbf{M}_1(\mathcal{T})^I$ and $p \in B_{\neq 0}$ such that $\mathbf{B}(p)$ is good, let $X = B \cap (K_2)^I$ and note that $\langle X, \times \rangle$ is a Boolean group. Also define $V_B = B \cap (V \times K_2)^I$, $G_B = B \cap (\{C, D\} \times K_2)^I$, and $E_B = B \cap (\{1, H, 2\} \times K_2)^I$.

Now consider cases. By examining the definition of $\mathbf{M}_1(\mathcal{T})$, the definition of $\mathbf{B}(p)$, and the assumption that $\mathbf{B}(p)$ is good, it is possible to prove the following.

(1) If $p \in X$, then $B_p = X$ and $\mathbf{B}(p) \cong \mathbf{1} \circledast X$.
(2) If $p \in G_B$, then $X \cup \{p\} \subseteq B_p \subseteq X \cup G_B \cup E_B$ and $\mathbf{B}(p) \cong \mathbf{S}_N(\mathcal{T}) \circledast X$ for some chain type building block $\mathbf{S}_N(\mathcal{T})$.
(3) If $p \in V_B$, then $X \cup \{p\} \subseteq B_p \subseteq X \cup V_B \cup E_B$ and $\mathbf{B}(p) \cong \mathbf{B}_{(N,C)}(\mathcal{T}) \circledast X$ for some machine type building block $\mathbf{P}_{(N,C)}(\mathcal{T})$.
(4) If $p \in ((V \cup \{C, D, 1, H, 2\}) \times K_2)^I$ but $p \notin G_B \cup V_B$, then $\mathbf{B}(p) \cong \mathbf{2} \circledast X$.
(5) If none of the above hold, then $B_p = \{p\}$ and $\mathbf{B}(p) \cong \mathbf{T}$.

Furthermore, the building blocks arising in cases 2 and 3 are such that $|N|$ satisfies the restriction in Theorem 4.4(2) if $\mathcal{T}$ halts. The tedious details are left to the reader. $\square$

7. When $\mathcal{T}$ halts

Let $\mathcal{T}$ be a fixed Turing machine, let $\mathcal{H}$ denote the class of subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$, and let $\mathcal{A}_2$ denote the class of all Boolean groups.

Suppose $\mathcal{T}$ halts. Then by Theorem 4.4 there exist finitely many finite building blocks $\mathbf{C}_1, \ldots, \mathbf{C}_k$ and finitely many finite algebras $\mathbf{S}_1, \ldots, \mathbf{S}_l$ in $\mathsf{HS}(\mathbf{M}(\mathcal{T})) \cup \{\mathbf{T}^{\otimes}\}$ such that $\mathcal{H}$ is, up to isomorphism,

$$\left(\bigcup_{i=1}^{k} \{(\mathbf{C}_i \circledast X)^{\otimes} \,:\, X \in \mathcal{A}_2\}\right) \cup \{\mathbf{S}_1, \ldots, \mathbf{S}_l\}.$$

For $i = 1, \ldots, k$ define

$$\mathcal{K}_i = \{(\mathbf{C}_i \circledast X)^{\otimes} \,:\, X \in \mathcal{A}_2,\ X \text{ infinite}\}$$

and put $\mathcal{K} = \left(\bigcup_{i=1}^{k} \mathcal{K}_i\right) \cup \{\mathbf{S}_1, \ldots, \mathbf{S}_l\}$. Clearly $\mathcal{H} \subseteq \mathsf{S}(\mathcal{K})$ and therefore $\mathsf{V}(\mathbf{M}(\mathcal{T})) = \mathsf{SP}(\mathcal{K})$. On the other hand, it is not hard to see that each class $\mathcal{K}_i$ is (up to isomorphism) the class of models of an $\aleph_0$-categorical theory, and therefore for every $n < \omega$ there are only finitely many pairwise $\mathcal{K}$-inequivalent existential formulas in the free variables $x_1, \ldots, x_n$. Hence by Lemma 2.2 we have:

Theorem 7.1. *If $\mathcal{T}$ halts, then the theory of* $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ *has a model companion.*

8. Chain type algebras

We need to look more closely at chain type algebras in the language L_1, i.e., algebras of the form $\mathbf{S}_N(\mathcal{T}) \circledast X$. Recall that the universe of $\mathbf{S}_N(\mathcal{T}) \circledast X$ is the disjoint union of $\{0\}$, X, $\{a_n : n \in N\} \times X$ and $\{v_n : n \in N \text{ or } n-1 \in N\} \times X$. We shall use the notation

$$\begin{aligned} U &:= \{a_n : n \in N\} \times X \\ W &:= \{v_n : n \in N \text{ or } n-1 \in N\} \times X \end{aligned}$$

(superseding the notation of Section 5). Define a binary relation E on W by the rule $(y,z) \in E$ iff $\exists x[x \cdot y = z]$. (Note: this E is the converse of the edge relation discussed in the paragraphs preceding Definition 4.1.) E is naturally in bijective correspondence with U by $(y,z) \rightsquigarrow x$ iff $x \cdot y = z$. Thus we can consider W to be the set of vertices and U the set of edges of a directed graph. Up to isomorphism, this graph consists of the disjoint union of $|X|$-many isomorphic chains. X acts on this graph as a group of automorphisms via the operation $*$; furthermore, this action is **regular** in the sense that for all $u \in X$, if there exists a vertex $x \in W$ such that $x * u = x$, then $u = e$.

Conversely, suppose that $\langle G; E\rangle$ is a directed graph, $X = \langle X, \times, e\rangle$ is a Boolean group, and $*$ is a regular action of X on $\langle G; E\rangle$. We shall construct an algebra in the language L_1. The universe is the disjoint union of G, E, X and $\{0\}$. $\times$ and $*$ are extended to this set by putting $x \times y = 0$ if $\{x, y\} \not\subseteq X$ and $x * u = 0$ if $x \notin G \cup E$ or $u \notin X$. Then define

$$x \cdot y = \begin{cases} z & \text{if } x = (y,z) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$x \circ y = \begin{cases} z & \text{if } x = (z,y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

0 and e are interpreted as themselves, $\wedge$ is defined by $x \wedge x = x$, $x \wedge y = 0$ if $x \neq y$, and the machine operations are defined to be identically 0. The algebra just defined shall be denoted $\mathbf{S}(G, E, X)$. Observe that if U, W and E are obtained from $\mathbf{S}_N(\mathcal{T}) \circledast X$ as in the previous paragraph, then $\mathbf{S}_N(\mathcal{T}) \circledast X \cong \mathbf{S}(W, E, X)$.

Say that two vertices $x, y \in G$ are **weakly connected** if there exist $z \in G$ and $u \in X$ such that x, z are connected in the graph $\langle G; E\rangle$ and $z * u = y$. Weak connectedness is an equivalence relation on G; we shall call the equivalence classes the **weak components** of $\mathbf{S}(G, E, X)$.

Suppose $x \in G$. Let us temporarily use $[x]$ to denote the subgraph induced on the connected component of x in $\langle G; E\rangle$, and $[[x]]$ to denote the weak component of $\mathbf{S}(G, E, X)$ containing x. If $y \in [[x]]$, then $[x]$ and $[y]$ are isomorphic graphs. Hence the isomorphism type of $[x]$ is an invariant for $[[x]]$, which we shall call the **type** of $[[x]]$.

Obviously if $\mathbf{A} \cong \mathbf{S}(G,E,X)$ then we may also speak of the vertices, edges, connected components, weak components and their types, etc. in $\mathbf{A}$. For example, $\mathbf{S}_N(\mathcal{T}) \circledast X$ has one weak component; its type is a chain determined by N.

Lemma 8.1. *Suppose $\langle G;E\rangle$ is a directed graph, $X \in \mathcal{A}_2$, and $*$ is a regular action of X on $\langle G;E\rangle$. If the type of each weak component of $\mathbf{S}(G,E,X)$ is a chain, then $\mathbf{S}(G,E,X)$ can be embedded in $\mathbf{S}_{\mathbb{Z}}(\mathcal{T}) \circledast Y$ for sufficiently large $Y \in \mathcal{A}_2$.*

Proof. First, for each interval N (possibly empty) in $\mathbb{Z}$ we define a chain $\mathcal{C}(N) = \langle C(N); E(N)\rangle$ as follows: if $N \neq \emptyset$ then

$$\begin{aligned} C(N) &= \{n : n \in N \text{ or } n-1 \in N\} \\ E(N) &= \{e_n : n \in N\} \end{aligned}$$

where $e_n = (n{+}1, n)$ for $n \in N$, while $\langle C(\emptyset); E(\emptyset)\rangle = \langle \{0\}; \emptyset\rangle$. Next, let $([[x_\alpha]] : \alpha < \eta)$ be a one-to-one enumeration of the weak components of $\mathbf{S}(G,E,X)$. For each $\alpha < \eta$ pick an interval N_α in $\mathbb{Z}$ such that $0 \in C(N_\alpha)$ and $[x_\alpha] \cong \mathcal{C}(N_\alpha)$ via an isomorphism sending x_α to 0. Define a new graph $\langle \bar{G}; \bar{E}\rangle$ as follows:

$$\begin{aligned} \bar{G} &= \bigcup_{\alpha<\eta} C(N_\alpha) \times \{\alpha\} \times X \\ \bar{E} &= \bigcup_{\alpha<\eta} E(N_\alpha) \times \{\alpha\} \times X \end{aligned}$$

where (e_n, α, u) now represents the edge from $(n{+}1, \alpha, u)$ to (n, α, u). Define an action of X on $\langle \bar{G}; \bar{E}\rangle$ by $(x,\alpha,u) * v = (x, \alpha, u \times v)$. Then there exists an isomorphism $f : \mathbf{S}(G,E,X) \cong \mathbf{S}(\bar{G},\bar{E},X)$ which sends x_α to $(0,\alpha,e)$ for all $\alpha < \eta$. (Here we use the regularity of the action of X on $\langle G;E\rangle$, plus the fact that chains have no automorphisms of order 2.)

Now choose $Y \in \mathcal{A}_2$ and $\{y_\alpha : \alpha < \eta\} \subseteq Y$ such that X is a subgroup of Y and $y_\alpha \times y_\beta \notin X$ whenever $\alpha < \beta < \eta$. Define a map g from $\mathbf{S}(\bar{G},\bar{E},X)$ to $\mathbf{S}_{\mathbb{Z}}(\mathcal{T}) \circledast Y$ by $x \mapsto x$ for $x \in X \cup \{0\}$ and

$$\begin{aligned} (n,\alpha,u) &\mapsto (v_n, u \times y_\alpha) \quad \text{for } (n,\alpha,u) \in \bar{G} \\ (e_n,\alpha,u) &\mapsto (a_n, u \times y_\alpha) \quad \text{for } (e_n,\alpha,u) \in \bar{E}. \end{aligned}$$

gf is the desired embedding. □

In the preceding proof, let $h = gf$. Then $h(x_\alpha) * (y_\alpha \times y_\beta) = h(x_\beta)$ for all $\alpha, \beta < \eta$. Since each x_α can be chosen arbitrarily from $[[x_\alpha]]$, and since $(\mathbf{S}_{\mathbb{Z}}(\mathcal{T}) \circledast Y)^{\otimes} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ when $\mathcal{T}$ does not halt, we have proved

Corollary 8.2. *Assume $\mathcal{T}$ does not halt. Suppose $\mathbf{B} \cong \mathbf{S}(G,E,X)$ where $\langle G;E\rangle$ and X satisfy the hypotheses of Lemma 8.1. If a, b are vertices of $\mathbf{B}$ belonging to* *different weak components, then there exists* $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ *such that* $\mathbf{B}^{\otimes} \leq \mathbf{D}$ *and* $\mathbf{D} \models \exists u[a * u = b \ \& \ b * u = a]$.

The next lemma provides useful criteria which imply that an algebra $\mathbf{B}$ is isomorphic to some $\mathbf{S}(G, E, X)$ satisfying the hypotheses of Lemma 8.1.

Lemma 8.3. *Let* $\mathbf{B}$ *be an algebra in the language* L_1. *Suppose* $\mathbf{B}$ *satisfies the following properties:*

(1) $\mathbf{B}^{\otimes} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$.
(2) *The machine operations are identically equal to 0 (* $:= 0^{\mathbf{B}}$*).*
(3) B *is the disjoint union of sets* X, W, U *and* $\{0\}$; W *is nonempty.*
(4) $e := e^{\mathbf{B}} \in \mathsf{X}$. X *is closed under* $\times$. *Conversely, if* $x \times y \neq 0$ *then* $x \in \mathsf{X}$.
(5) *If* $x \cdot y \neq 0$ *then* $x \in \mathsf{U}$ *and* $y \in \mathsf{W}$. *For all* $x \in \mathsf{U}$ *there exists* $y \in \mathsf{W}$ *such that* $x \cdot y \neq 0$.
(6) *If* $x \cdot y \neq 0$ *and* $x \cdot z \neq 0$ *then* $y = z$.
(7) *If* $x \in \mathsf{W}$ *and* $u \in \mathsf{X}$, *then* $x * u \in \mathsf{W}$.
(8) *If* $x * u = x \in \mathsf{W}$ *then* $u = e$.
(9) *If* $x \neq y$ *then* $x \wedge y = 0$.

Then $\mathbf{B}$ *is isomorphic to some* $\mathbf{S}(G, E, X)$ *satisfying the hypotheses of Lemma 8.1.* W *and* U *are the sets of vertices and edges, respectively, of* $\mathbf{B}$.

Proof. Since $\mathbf{B}^{\otimes} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ by item 1, $\wedge$ is a meet semilattice operation with least element 0. By items 3 and 9, $\langle B; \wedge \rangle$ has height 1. Since $\mathbf{M}(\mathcal{T}) \models x \times x \leq e$, $\mathbf{B}$ also satisfies this identity (again using item 1). Items 4 and 9 then imply $\mathsf{X} = \{x \in B : x \times x = e\}$. Using the fact that $\times$ is commutative and associative and satisfies $x \times e \leq x$ in $\mathbf{M}(\mathcal{T})$ (and hence in $\mathbf{B}$), one can deduce that $\langle \mathsf{X}; \times, e \rangle \in \mathcal{A}_2$.

We claim that $x \cdot y \neq 0$ implies $x \cdot y \in \mathsf{W}$. For if $x \cdot y \in \mathsf{X}$, then $(x \cdot y) \times e \in \mathsf{X}$ by item 4, which contradicts $\mathbf{M}(\mathcal{T}) \models (x \cdot y) \times z = 0$. And if $x \cdot y \in \mathsf{U}$, then by item 5 there exists z such that $(x \cdot y) \cdot z \neq 0$, contradicting $\mathbf{M}(\mathcal{T}) \models (x \cdot y) \cdot z = 0$.

By items 5 and 6 it is now apparent that U is in bijective correspondence with the directed edge relation E defined on W by $E = \{(y, z) \in \mathsf{W}^2 : \exists x[x \cdot y = z]\}$, where the correspondence is $(y, z) \rightsquigarrow$ the unique $x \in B$ satisfying $x \cdot y = z$.

Next, we claim that $x \cdot y = z \neq 0$ iff $x \circ z = y \neq 0$. Indeed, using the identities (i) $x \circ (x \cdot y) \leq y$, (ii) $x \cdot (x \circ (x \cdot y)) = x \cdot y$, and (iii) $x \cdot 0 = 0$ (which are true in $\mathbf{M}(\mathcal{T})$, thus in $\mathbf{B}$), if $x \cdot y = z \neq 0$ then $0 \neq x \circ z \leq y$, which implies $x \circ z = y$ by item 9. The opposite implication is proved similarly.

Assume $x \in \mathsf{U}$ and $u \in \mathsf{X}$. By item 5 there exists y such that $x \cdot y \neq 0$; in fact, we have shown $x \cdot y \in \mathsf{W}$. Thus by item 7, $(x \cdot y) * u \neq 0$. Using the identity (iv) $(x \cdot y) * u = (x * u) \cdot (y * u)$ of $\mathbf{M}(\mathcal{T})$, we obtain $x * u \in \mathsf{U}$ by item 5.

Conversely, assume that $x, u \in B$ and $x * u \neq 0$. Using the identities $x * u = x * (u \times e)$ and $x * 0 = 0$, we can deduce $u \times e \neq 0$ and therefore $u \in \mathsf{X}$. Using the identity

$0 * u = 0$ we also get $x \neq 0$. Suppose $x \notin \mathsf{W} \cup \mathsf{U}$. Then $x \in \mathsf{X}$, so $x \times e = x$. But these facts contradict the identity $(x \times y) * z = 0$.

Thus far we have proved that $\times$ is nonzero only when restricted to X, where it is a Boolean group operation; that U represents a directed edge relation E on W and that $\cdot$ and $\circ$ are definable from E in the usual way; that $x * u \neq 0$ iff $x \in \mathsf{W} \cup \mathsf{U}$ and $u \in \mathsf{X}$; and that for each $u \in \mathsf{X}$, the unary polynomial $\lambda_u(x) := x * u$ maps W into W and U into U. Using the identities $(x * u) * v = x * (u \times v)$ and $x * e \leq x$, we see that $*$ is a group action of $\langle \mathsf{X}; \times, e\rangle$ on the set $\mathsf{W} \cup \mathsf{U}$. Using the identity (iv) above, we see in fact that each λ_u is an automorphism of the graph $\langle \mathsf{W}; E\rangle$. The action is regular by item 8. Thus we have proved $\mathbf{B} \cong \mathbf{S}(\mathsf{W}, E, \mathsf{X})$.

It remains to prove that every connected component of $\langle \mathsf{W}; E\rangle$ is a chain. As $\mathbf{B}^{\otimes} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$, we can represent $\mathbf{B}^{\otimes}$ as a subdirect product of subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$. Since S_i is identically equal to 0 in $\mathbf{B}^{\otimes}$ $(i = 0, 1, 2)$ by definition, and the machine operations are identically equal to 0 in $\mathbf{B}$ by item 2, the subdirectly irreducible algebras in this representation have the same property. So by Theorem 4.4, $\mathbf{B}$ is a subdirect product of chain type algebras, algebras of the form $\mathbf{2} \circledast X$ and $\mathbf{1} \circledast X$, and $\mathbf{T}$. Such algebras satisfy the following identities:

$$\begin{aligned}
(x \cdot y) \wedge (z \cdot w) &= (x \wedge z) \cdot (y \wedge w) \\
(x \circ y) \wedge (z \circ w) &= (x \wedge z) \circ (y \wedge w) \\
y_1 \cdot (y_2 \cdot (\cdots (y_{n-1} \cdot (y_n \cdot x)) \cdots)) \wedge x &= 0 \text{ for all } n \geq 1 \\
y_1 \circ (y_2 \circ (\cdots (y_{n-1} \circ (y_n \circ x)) \cdots)) \wedge x &= 0 \text{ for all } n \geq 1
\end{aligned}$$

Hence $\mathbf{B}$ satisfies these identities as well. The reader can check that they imply that each connected component of $\langle \mathsf{W}; E\rangle$ is a chain. □

We end this section with some syntactic notions. Let $\mathbf{B}$ be an algebra in either L_1 or L. A **basic translation** of $\mathbf{B}$ is any unary polynomial $\lambda(x)$ of $\mathbf{B}$ of one of the following forms (where $r \in B$ is arbitrary):

$$\begin{array}{ccccc}
x \cdot r & x \circ r & x * r & x \times r & x \wedge r \\
r \cdot x & r \circ x & r * x & r \times x & r \wedge x.
\end{array}$$

$\mathrm{Pol}_1^* \mathbf{B}$ shall denote the subset of $\mathrm{Pol}_1 \mathbf{B}$ consisting of the identity map plus all compositions of basic translations. Observe that if $\mathbf{B}$ is (a reduct of) a member of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$, then $\lambda(0) = 0$ for all $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$.

Let $x, v_1, v_2, \ldots$ be a fixed infinite sequence of variables. Let T^* denote the smallest set of terms $t(x; \bar{v})$ which contains x and is closed under the following rule: If $t(x; v_1, \ldots, v_n) \in \mathsf{T}^*$, then the following also belong to T^*:

$$\begin{array}{ccccc}
t \cdot v_{n+1} & t \circ v_{n+1} & t * v_{n+1} & t \times v_{n+1} & t \wedge v_{n+1} \\
v_{n+1} \cdot t & v_{n+1} \circ t & v_{n+1} * t. & &
\end{array}$$

Thus $\mathrm{Pol}_1^* \mathbf{B} = \{t^{\mathbf{B}}(x;\bar{r}) : t \in \mathsf{T}^*, \bar{r} \in B^n\}$ provided only that $\times$ and $\wedge$ are commutative in $\mathbf{B}$. Next we define some subsets of T^*.

Definition 8.4.

(1) T_X is the smallest set of terms $t(x;\bar{v})$ which contains x and is closed under the following rule: if $t(x;v_1,\ldots,v_n) \in \mathsf{T}_X$, then $t \wedge v_{n+1}$ and $t \times v_{n+1}$ are in T_X.

(2) T_U is the smallest set of terms $t(x,\bar{v})$ which contains x and is closed under the following rule: if $t(x;v_1,\ldots,v_n) \in \mathsf{T}_U$, then $t \wedge v_{n+1}$ and $t * v_{n+1}$ are in T_U.

(3) T_W is the smallest set of terms $t(x,\bar{v})$ which contains x and is closed under the following rule: if $t(x;v_1,\ldots,v_n) \in \mathsf{T}_W$, then $t \wedge v_{n+1}$, $t * v_{n+1}$, $v_{n+1} \cdot t$ and $v_{n+1} \circ t$ are in T_W.

(4) T_{UW} is the set of all terms of the form

$$\begin{array}{ll} & t(s(x;v_1,\ldots,v_n)\cdot v_{n+1};v_{n+2},\ldots,v_{n+m+1}) \\ \text{or} & t(s(x;v_1,\ldots,v_n)\circ v_{n+1};v_{n+2},\ldots,v_{n+m+1}) \end{array}$$

where $s(x;v_1,\ldots,v_n) \in \mathsf{T}_U$ and $t(x;v_1,\ldots,v_m) \in \mathsf{T}_W$.

(5) T_{XW} is the set of all terms of the form

$$t(v_{n+1} * s(x;v_1,\ldots,v_n);v_{n+2},\ldots,v_{n+m+1})$$

where $s(x;v_1,\ldots,v_n) \in \mathsf{T}_X$ and $t(x;v_1,\ldots,v_m) \in \mathsf{T}_W \cup \mathsf{T}_{UW}$.

Observe that the sets T_W, T_{UW} and T_{XW} are pairwise disjoint.

For $t \in \mathsf{T}_W \cup \mathsf{T}_{UW}$ we define the integer $\mathrm{displ}(t)$ inductively as follows:

$$\begin{array}{rcl} \mathrm{displ}(x) & = & 0 \\ \left.\begin{array}{rcl} \mathrm{displ}(t\cdot v_i) & = & 0 \\ \mathrm{displ}(t\circ v_i) & = & -1 \end{array}\right\} & & \text{for } t \in \mathsf{T}_U. \\ \left.\begin{array}{rcl} \mathrm{displ}(t\wedge v_i) & = & \mathrm{displ}(t) \\ \mathrm{displ}(t * v_i) & = & \mathrm{displ}(t) \\ \mathrm{displ}(v_i \cdot t) & = & \mathrm{displ}(t)+1 \\ \mathrm{displ}(v_i \circ t) & = & \mathrm{displ}(t)-1 \end{array}\right\} & & \text{for } t \in \mathsf{T}_W \cup \mathsf{T}_{UW} \end{array}$$

Now let $\mathbf{C}$ be a chain type algebra, say $\mathbf{C} = \mathbf{S}_N(\mathcal{T}) \circledast X$. Recall that we use the notation $W = \{v_n : n \in N \text{ or } n-1 \in N\} \times X$ for the vertex set of $\mathbf{C}$ and $U = \{a_n : n \in N\} \times X$ for the edge set of $\mathbf{C}$. The following Lemma can be proved by examining the definition of $\mathbf{S}_N(\mathcal{T}) \circledast X$.

Lemma 8.5. *Let $\mathbf{C}, W, U$ and X be as described above. Suppose $t \in \mathsf{T}^*$, $\bar{r} \in C^n$, and $\lambda(x) = t^{\mathbf{C}}(x;\bar{r})$.*

(1) $\lambda(W) \subseteq W \cup \{0\}$.

(2) *If* $\mathrm{range}(\lambda) \cap W \neq \emptyset$*, then* $t \in \mathsf{T}_W \cup \mathsf{T}_{UW} \cup \mathsf{T}_{XW}$.
(3) *If* $t \in \mathsf{T}_{UW}$*, then* $\mathrm{range}(\lambda) \subseteq W \cup \{0\}$ *and* $\lambda^{-1}(W) \subseteq U$.
(4) *If* $t \in \mathsf{T}_{XW}$*, then* $\mathrm{range}(\lambda) \subseteq W \cup \{0\}$ *and* $\lambda^{-1}(W) \subseteq X$.
(5) *If* $t \in \mathsf{T}_W$*, then* $\lambda^{-1}(W) \subseteq W$.
(6) *Suppose* $(a_n, u) \in U$ *and* $\lambda((a_n, u)) = (v_m, v)$ *(so* $t \in \mathsf{T}_{UW}$*). Then* $n - m = \mathrm{displ}(t)$.
(7) *Suppose* $(v_n, u) \in W$ *and* $\lambda((v_n, u)) = (v_m, v)$ *(so* $t \in \mathsf{T}_W$*). Then* $n - m = \mathrm{displ}(t)$.
(8) $\mathbf{C} \models \lambda(x) = \lambda(y) \neq 0 \to x = y$.

Observe that one can also show the following. Suppose $\mathbf{B} \cong \mathbf{S}(G, E, X)$ for some graph $\langle G; E\rangle$, $X \in \mathcal{A}_2$, and a regular action of X on $\langle G; E\rangle$, and suppose a, b are vertices of $\mathbf{B}$. Then the following are equivalent: (i) there exists $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ satisfying $\lambda(a) = b$; (ii) a and b belong to the same weak component of $\mathbf{B}$.

9. The brute-force argument when $\mathcal{T}$ does not halt

Define the following first-order formulas.

$$
\begin{aligned}
\phi_0(x) &: \exists z[z \circ (z \cdot x) = x \ \&\ S_1(z, x, x, x) = 0]\\
\phi_1(x) &: \exists w[w \cdot (w \circ x) = x \ \&\ S_1(w, x, x, x) = 0]\\
\phi_2(x) &: \phi_0(x) \ \&\ \phi_1(x)\\
\phi_3(x, y) &: \phi_2(x) \ \&\ \phi_2(y)\\
\phi(x, y) &: \phi_3(x, y) \ \&\ x \neq 0 \ \&\ y \neq 0\\
\psi(x, y) &: \exists u[x * u = y \ \&\ y * u = x]\\
\theta_n^+(x, y) &: \exists z_1 \cdots z_n[z_1 \cdot (z_2 \cdot (\cdots \cdot (z_{n-1} \cdot (z_n \cdot x)) \cdots)) = y] \ (n \geq 1)\\
\theta_n^-(x, y) &: \exists z_1 \cdots z_n[z_1 \circ (z_2 \circ (\cdots \circ (z_{n-1} \circ (z_n \circ y)) \cdots)) = x] \ (n \geq 1)\\
\theta_n(x, y) &: \theta_n^+(x, y) \ \&\ \theta_n^-(x, y) \ (n \geq 1)\\
\delta_0(x, y) &: \forall uv[x * u = x * v \leftrightarrow y * u = y * v]\\
\delta_{p,s,t}(x, y) &: \forall z u \bar{v}_1 \bar{v}_2 \bar{v}_3[x * u \leq p(z; \bar{v}_1) \to (s(z; \bar{v}_2) \wedge t(y; \bar{v}_3)) * u = 0]\\
\delta'_{p,s,t}(x, y) &: \forall z u \bar{v}_1 \bar{v}_2 \bar{v}_3[y * u \leq p(z; \bar{v}_1) \to (s(z; \bar{v}_2) \wedge t(x; \bar{v}_3)) * u = 0]
\end{aligned}
$$

where p, s vary over $\mathsf{T}_W \cup \mathsf{T}_{UW}$ and t varies over T_W in the last two lines.

Assuming $\mathcal{T}$ does not halt, we shall verify the hypotheses of Lemma 2.3 for $T = \mathit{Th}(\mathsf{V}(\mathbf{M}(\mathcal{T})))$, using the existential formulas ϕ, ψ, θ_n $(0 < n < \omega)$, the family of formulas $\{\delta_0\} \cup \{\delta_{p,s,t},\ \delta'_{p,s,t} : p, s \in \mathsf{T}_W \cup \mathsf{T}_{UW} \text{ and } t \in \mathsf{T}_W\} =: \Delta$, and the function r defined by

$$
\begin{aligned}
r(\delta_0) &= \emptyset\\
r(\delta_{p,s,t}) &= \{\mathrm{displ}(s) - \mathrm{displ}(p) - \mathrm{displ}(t)\}\\
r(\delta'_{p,s,t}) &= \{\mathrm{displ}(p) + \mathrm{displ}(t) - \mathrm{displ}(s)\}.
\end{aligned}
$$

Lemma 9.1. *$\mathsf{V}(\mathbf{M}(\mathcal{T})) \models \delta(0,0)$ for all $\delta \in \Delta$.*

Proof. This follows from the fact that $\mathsf{V}(\mathbf{M}(\mathcal{T})) \models 0 * u = 0$. □

Lemma 9.2. *Suppose $\mathbf{S}$ is a subdirectly irreducible member of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$. If $\mathbf{S}$ is* not *of the form $(\mathbf{S}_N(\mathcal{T}) \circledast X)^{\otimes}$, then $\mathbf{S} \models \phi_2(x) \to x = 0$. If $\mathbf{S}$* is *of the form $\mathbf{C}^{\otimes}$ for some chain type algebra $\mathbf{C}$, and if W is the vertex set of $\mathbf{C}$ and $a \in S$ and $\mathbf{S} \models \phi_2(a)$, then $a \in W \cup \{0\}$.*

Proof. If $\mathbf{S} \cong \mathbf{T}^{\otimes}$ or $\mathbf{S} \cong (\mathbf{B} \circledast X)^{\otimes}$ for some building block $\mathbf{B}$ and Boolean space X, then the result is obvious. If $\mathbf{S} \in \mathsf{HS}(\mathbf{M}(\mathcal{T}))$, then use the fact that the positive sentence

$$\forall xzw[z \circ (z \cdot x) = 0 \ \vee \ w \cdot (w \circ x) = 0 \ \vee \ S_1(z,x,x,x) = x \ \vee \ S_1(w,x,x,x) = x]$$

is true in $\mathbf{M}(\mathcal{T})$ and therefore also in $\mathbf{S}$. (The sentence and $\phi_2(x)$ logically imply $x = 0$.) By Theorem 4.4, there are no other possibilities. □

Lemma 9.3. *If $\mathbf{C}$ is a chain type algebra, then for all $n \geq 1$,*

(1) $\mathbf{C} \models [\psi(x,y) \ \& \ \theta_n(x,y)] \to x = 0$.

(2) $\mathbf{C} \models \theta_n(x,y) \to \delta(x,y)$ for all $\delta \in \Delta$ such that $n \notin r(\delta)$.

Proof. Let $\mathbf{C} = \mathbf{S}_N(\mathcal{T}) \circledast X$ with edge set $U = \{a_n : n \in N\}$ and vertex set $W = \{v_n : n \in N \text{ or } n-1 \in N\}$ and suppose $\mathbf{C} \models \theta_n(a,b)$. We shall prove $\mathbf{C} \models [\psi(a,b) \to a = 0] \ \& \ \delta(a,b)$. There are two cases:

Case 1. $a = b = 0$.

Then in particular $a = 0$, and also $\mathbf{C} \models \delta(a,b)$ by Lemma 9.1.

Case 2. $a, b \in W$.

Say $a = (v_l, x)$ and $b = (v_m, y)$. Since $\mathbf{C} \models \theta_n(a,b)$ we have $l - m = n$, so in particular, $l \neq m$. This implies $\mathbf{C} \models \neg\psi(a,b)$ by the definition of $*$ in $\mathbf{C}$. Next we consider $\delta(a,b)$.

Case (a): δ is δ_0. Suppose $u, v \in C$ and $b * u \neq b * v$. This can only happen if (i) $u \in X$ but $v \notin X$, or (ii) $v \in X$ but $u \notin X$, or (iii) $u, v \in X$ but $u \neq v$. Each case implies $a * u \neq a * v$. The argument is symmetric in a and b, so $\mathbf{C} \models \delta_0(a,b)$.

Case (b): δ is $\delta_{p,s,t}$ with $n \notin r(\delta_{p,s,t})$. Suppose $c, u, \bar{v}_1, \bar{v}_2, \bar{v}_3$ are elements or tuples of elements from C such that $(s^{\mathbf{C}}(c;\bar{v}_2) \wedge t^{\mathbf{C}}(b;\bar{v}_3)) * u \neq 0$. Then $u \in X$ and $s^{\mathbf{C}}(c;\bar{v}_2) = t^{\mathbf{C}}(b;\bar{v}_3) = (v_j, z) \in W$ where $m - j = \mathrm{displ}(t)$, by Lemma 8.5. Hence $c = (v_k, z) \in W$ or $c = (a_k, z) \in U$ where $k - j = \mathrm{displ}(s)$, again by Lemma 8.5. Suppose for the sake of contradiction that $a * u \leq p^{\mathbf{C}}(c;\bar{v}_1)$. Then $p^{\mathbf{C}}(c;\bar{v}_1) = a * u = (v_l, x \times u)$, hence $k - l = \mathrm{displ}(p)$ by Lemma 8.5. These facts contradict $l - m = n \notin r(\delta_{p,s,t})$, so $\mathbf{C} \models \delta_{p,s,t}(a,b)$.

Case (c): δ is $\delta'_{p,s,t}$ with $n \notin r(\delta'_{p,s,t})$. The argument is similar to the previous one. Thus in all cases, $\mathbf{C} \models \delta(a,b)$. □

Lemma 9.4. *For each $n \geq 1$ and $\delta \in \Delta$ satisfying $n \notin r(\delta)$,*

(1) $\mathsf{V}(\mathbf{M}(\mathcal{T})) \models [\phi(x,y) \ \&\ \psi(x,y)] \to \neg\theta_n(x,y)$
(2) $\mathsf{V}(\mathbf{M}(\mathcal{T})) \models [\phi(x,y) \ \&\ \theta_n(x,y)] \to \delta(x,y)$.

Proof. It suffices to prove the stronger claims

(3) $\mathsf{V}(\mathbf{M}(\mathcal{T})) \models [\phi_3(x,y) \ \&\ \psi(x,y) \ \&\ \theta_n(x,y)] \to x = 0$
(4) $\mathsf{V}(\mathbf{M}(\mathcal{T})) \models [\phi_3(x,y) \ \&\ \theta_n(x,y)] \to \delta(x,y)$.

The sentences in items 3 and 4 are equivalent to (conjunctions of) quasi-identities, hence it suffices to check their truth in the subdirectly irreducible members of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$. This was done in Lemmas 9.1 through 9.3. □

Lemma 9.4 establishes the second and third hypotheses of Lemma 2.3. To establish the remaining two hypotheses, we need to assume that $\mathcal{T}$ does not halt.

Lemma 9.5. *Assume that $\mathcal{T}$ does not halt. For every $n \geq 1$ there exists $\mathbf{A} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ such that $\mathbf{A} \models \exists xy[\phi(x,y) \ \&\ \theta_n(x,y)]$.*

Proof. By Theorem 4.4, we can choose $\mathbf{A} = (\mathbf{S}_{\mathbb{Z}}(\mathcal{T}) \circledast \{e\})^{\otimes}$. □

Lemma 9.6. *Assume that $\mathcal{T}$ does not halt. Suppose $\mathbf{A} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ and $a, b \in A$ and $\mathbf{A} \models \phi(a,b) \ \&\ \bigwedge_{\delta\in\Delta} \delta(a,b)$. Then there exists $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ such that $\mathbf{A} \leq \mathbf{D}$ and $\mathbf{D} \models \psi(a,b)$.*

Proof. Since ψ is a Horn formula, it suffices to show the following:

(∗) For all $c, d \in A$ with $c \neq d$, there exist $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ and a homomorphism $h : \mathbf{A} \to \mathbf{D}$ such that $\mathbf{D} \models \psi(h(a), h(b))$ and $h(c) \neq h(d)$.

We can assume that $\mathbf{A} \leq_{sd} \prod_{j\in J} \mathbf{S}_j$ where each $\mathbf{S}_j$ is subdirectly irreducible. Since ϕ_3 is a positive formula, $\mathbf{S}_j \models \phi_3(a_j, b_j)$ for each $j \in J$. Hence Lemma 9.2 implies $\mathbf{A} \models a * e = a \ \&\ b * e = b$. Thus if $(c,d) \notin \mathrm{Cg}^{\mathbf{A}}(a,b)$ then we can prove (∗) by defining $\mathbf{D} = \mathbf{A}/\theta$ and $h(x) = x/\theta$, where $\theta = \mathrm{Cg}^{\mathbf{A}}(a,b)$.

Therefore, for the remainder of this proof we shall assume that $(c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b)$. Define

$$K = \{j \in J : \mathbf{S}_j = \mathbf{C}_j^{\otimes} \text{ where } \mathbf{C}_j \text{ is a chain type algebra}\}.$$

Since $c \neq d$ we must have either $c \not\leq d$ or $d \not\leq c$. We shall assume with no loss of generality that $d \not\leq c$. Observe that

$$\begin{array}{rcll} [\![c \neq d]\!] & \subseteq & [\![a \neq b]\!] & \text{since } (c,d) \in \mathrm{Cg}^{\mathbf{A}}(a,b) \\ & \subseteq & [\![a \neq 0]\!] \cup [\![b \neq 0]\!] & \\ & \subseteq & K & \text{by Lemma 9.2.} \end{array}$$

Hence $d|_K \not\leq c|_K$. Let $\mathbf{A}|_K$ be the image of the homomorphism from $\mathbf{A}$ to $\prod_{j\in K} \mathbf{S}_j$ given by $x \mapsto x|_K$, and note that $(c|_K, d|_K) \in \mathrm{Cg}^{\mathbf{A}|_K}(a|_K, b|_K)$.

Claim. $\mathbf{A}|_K \models \phi(a|_K, b|_K)$ & $\bigwedge_{\delta\in\Delta} \delta(a|_K, b|_K)$.

Indeed, $a = a|_K \cup 0|_{K^c}$ and $b = b|_K \cup 0|_{K^c}$ by Lemma 9.2. Since $a \neq 0$ and $b \neq 0$ we get $a|_K \neq 0$ and $b|_K \neq 0$. As $\mathbf{A} \models \phi_3(a,b)$ and ϕ_3 is a positive formula, we get $\mathbf{A}|_K \models \phi_3(a|_K, b|_K)$ and hence $\mathbf{A}|_K \models \phi(a|_K, b|_K)$. Suppose $u, v \in A$ and $a|_K * u|_K = a|_K * v|_K$. Then $a * u = a * v$ (as $\mathbf{M}(\mathcal{T}) \models 0 * x = 0$), so $b * u = b * v$ (using $\mathbf{A} \models \delta_0(a,b)$) and hence $b|_K * u|_K = b|_K * v|_K$. The converse is also true, which proves $\mathbf{A}|_K \models \delta_0(a|_K, b|_K)$. A similar argument works for the other members of Δ, proving the claim.

Thus in completing the proof of $(*)$ for the given pair (c,d) we can assume with no loss of generality that $K = J$. Hence $\mathbf{A} = \mathbf{B}^{\otimes}$ where $\mathbf{B}$ is the reduct of $\mathbf{A}$ to the language L_1. For the remainder of this proof we shall work with $\mathbf{B}$ rather than $\mathbf{A}$. So far we know that $\mathbf{B} \leq_{sd} \prod_{j\in J} \mathbf{C}_j$ where each $\mathbf{C}_j$ is a chain type algebra, that $\mathbf{B} \models \phi(a,b)$ & $\bigwedge_{\delta\in\Delta} \delta(a,b)$, that $d \not\leq c$, and that $(c,d) \in \mathrm{Cg}^{\mathbf{B}}(a,b)$. What remains to be shown is:

$(**)$ There exist $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ and a homomorphism $h : \mathbf{B}^{\otimes} \to \mathbf{D}$ such that $\mathbf{D} \models \psi(h(a), h(b))$ and $h(c) \neq h(d)$.

For each $j \in J$ write $\mathbf{C}_j = \mathbf{S}_{N_j}(\mathcal{T}) \circledast X_j$ and let W_j and U_j denote the sets of vertices and edges, respectively, of $\mathbf{C}_j$. For $x \in B$ we shall use the notation $[\![x \in W]\!]$, $[\![x \in X]\!]$ etc. to denote the sets $\{j \in J : x_j \in W_j\}$, $\{j \in J : x_j \in X_j\}$, etc. As was already noted, $[\![a \neq 0]\!] = [\![a \in W]\!]$ and $[\![b \neq 0]\!] = [\![b \in W]\!]$.

Because all operations of $\mathbf{B}$ except $\wedge$, $\cdot$, $\circ$, $\times$ and $*$ are constant, the congruences of $\mathbf{B}$ are precisely the equivalence relations which are compatible with the basic translations defined in Section 8. Since $(c,d) \in \mathrm{Cg}^{\mathbf{B}}(a,b)$, there exists $\lambda_0 \in \mathrm{Pol}_1^* \mathbf{B}$ such that $d \in \{\lambda_0(a), \lambda_0(b)\}$. We shall assume with no loss of generality that $d = \lambda_0(b)$. Pick $t \in \mathsf{T}^*$ and $\bar{r} \in B^n$ such that $\lambda_0(x) = t^{\mathbf{B}}(x; \bar{r})$. By examining the equation $t^{\mathbf{B}}(b; \bar{r}) = d$ at each coordinate and using Lemma 8.5, we get $[\![d \neq 0]\!] = [\![d \in W]\!]$. A similar argument shows $[\![c \neq 0]\!] = [\![c \in W]\!]$.

Our strategy for proving $(**)$ is as follows. We shall define a certain congruence θ of $\mathbf{B}$ having the property that $(c,d) \notin \theta$. Then we shall show that $\mathbf{B}/\theta$ is isomorphic to some $\mathbf{S}(G,E,X)$ satisfying the hypotheses of Lemma 8.1. By construction, $\mathbf{B}/\theta$ will have exactly two weak components, one of which will contain a/θ as a vertex, the other containing b/θ and d/θ as vertices. Corollary 8.2 will then provide $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ such that $(\mathbf{B}/\theta)^{\otimes} \leq \mathbf{D}$ and $\mathbf{D} \models \psi(a/\theta, b/\theta)$, which will prove $(**)$.

Define

$$\begin{aligned}
Z &= \{u \in B : d * u = d\} \\
H &= \{f \in B : \exists u \in Z \text{ such that } f \geq a * u\} \\
\theta &= \{(f,g) \in B^2 \ : \ \forall \lambda \in \operatorname{Pol}_1^* \mathbf{B},\ \lambda(f) \in H \leftrightarrow \lambda(g) \in H \\
&\qquad \text{and } \lambda(f) \geq d \leftrightarrow \lambda(g) \geq d\} \\
W_a &= \{f \in B : \exists t \in \mathsf{T}_W \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \in H\} \\
W_d &= \{f \in B : \exists t \in \mathsf{T}_W \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \geq d\}
\end{aligned}$$

$$\begin{aligned}
U_a &= \{f \in B : \exists t \in \mathsf{T}_{UW} \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \in H\} \\
U_d &= \{f \in B : \exists t \in \mathsf{T}_{UW} \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \geq d\} \\
X_a &= \{f \in B : \exists t \in \mathsf{T}_{XW} \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \in H\} \\
X_d &= \{f \in B : \exists t \in \mathsf{T}_{XW} \text{ and } \bar{r} \in B^n \text{ such that } t^{\mathbf{B}}(f;\bar{r}) \geq d\}.
\end{aligned}$$

Clearly $\theta \in \operatorname{Con} \mathbf{B}$ and $(c,d) \notin \theta$.

Recall that $\lambda_0 \in \operatorname{Pol}_1^* \mathbf{B}$ and $\lambda_0(b) = d$. Hence $[\![d \neq 0]\!] \subseteq [\![b \neq 0]\!]$. Now suppose that $u, v \in B$ and $a * u = a * v$. Then $b * u = b * v$, as $\mathbf{C} \models \delta_0(a,b)$. As $[\![b \neq 0]\!] = [\![b \in W]\!]$, this implies

$$\begin{aligned}
&[\![b \neq 0]\!] \subseteq [\![u = v \text{ or } \{u,v\} \cap X = \emptyset]\!] \\
\text{hence} \qquad &[\![d \neq 0]\!] \subseteq [\![u = v \text{ or } \{u,v\} \cap X = \emptyset]\!]
\end{aligned}$$

which implies $d * u = d * v$. Thus we have proved:

Claim 1. For all $u, v \in B$, if $a * u = a * v$ then $d * u = d * v$.

If we set $v = 0$ in the above claim we get the following: if $a * u = 0$ then $d * u = 0$. Since $d * u = d \neq 0$ for all $u \in Z$, we then get:

Claim 2. For all $u \in Z$, $a * u \neq 0$. Hence $0 \notin H$.

Claim 3. For all $f \in B$, the following are equivalent: (i) $(0,f) \notin \theta$; (ii) there exists $\lambda \in \operatorname{Pol}_1^* \mathbf{B}$ such that $\lambda(f) \geq d$ or $\lambda(f) \in H$; (iii) $f \in W_a \cup W_d \cup U_a \cup U_d \cup X_a \cup X_d$.

Indeed, (iii) $\Rightarrow$ (ii) is obvious, while (i) $\Leftrightarrow$ (ii) follows from Claim 2 and the fact that $\lambda(0) = 0$ for all $\lambda \in \operatorname{Pol}_1^* \mathbf{B}$. Now assume that $\lambda \in \operatorname{Pol}_1^* \mathbf{B}$ and either $\lambda(f) \geq d$ or $\lambda(f) \in H$. In either case, $[\![\lambda(f) \in W]\!] \neq \emptyset$. Pick $t \in \mathsf{T}^*$ and $\bar{r} \in B^n$ such that $\lambda(x) = t^{\mathbf{B}}(x;\bar{r})$. Lemma 8.5 implies $t \in \mathsf{T}_W \cup \mathsf{T}_{UW} \cup \mathsf{T}_{XW}$, which proves (ii) $\Rightarrow$ (iii).

Claim 4. Each of the sets W_a, W_d, U_a, U_d, X_a, X_d is a union of θ-classes.

For example, suppose $f \in W_a$ and $(f,g) \in \theta$. Choose $t \in \mathsf{T}_W$ and $\bar{r} \in B^n$ such that $t^{\mathbf{B}}(f;\bar{r}) \in H$. Then $t^{\mathbf{B}}(g;\bar{r}) \in H$ as well, by the definition of θ, proving $g \in W_a$. The proof is identical for the other sets.

Claim 5. For any $f \in B$, the following are equivalent:

(1) $f \in X_d$
(2) $[\![d \neq 0]\!] \subseteq [\![f \in X]\!]$
(3) $f \in X_a$
(4) There exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f \in X]\!]$.

Here is the proof. $1 \Rightarrow 2$ follows from Lemma 8.5 and the fact that $[\![d \neq 0]\!] = [\![d \in W]\!]$. Assume item 2. Then $d*(f \times f) = d$, proving $f \times f \in Z$. Hence $a*(f \times f) \in H$, which proves item 3 using the term $v_2*(x \times v_1) \in \mathsf{T}_{XW}$. Now assume $f \in X_a$. Choose $t \in \mathsf{T}_{XW}$ and $\bar{r} \in B^n$ and $u \in Z$ such that $t^{\mathbf{B}}(f;\bar{r}) \geq a * u$. By Lemma 8.5 we get $[\![a * u \neq 0]\!] = [\![a * u \in W]\!] \subseteq [\![f \in X]\!]$, which proves $3 \Rightarrow 4$. Finally, assume $u \in Z$ and $[\![a * u \neq 0]\!] \subseteq [\![f \in X]\!]$. Then $a * (f \times f \times u) = a * u$, so

$$\begin{aligned} d * (f \times f \times u) &= d * u \text{ by Claim 1} \\ &= d \text{ as } u \in Z. \end{aligned}$$

Thus $f \in X_d$ via the term $v_2 * (x \times v_1)$, which proves $4 \Rightarrow 1$.

It follows that $X_a = X_d$. We shall denote this common set by X. The next claim follows from Lemma 8.5.

Claim 6. 1. If $f \in W_d$ then $[\![d \neq 0]\!] \subseteq [\![f \in W]\!]$.

(2) If $f \in W_a$ then there exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f \in W]\!]$.
(3) If $f \in U_d$ then $[\![d \neq 0]\!] \subseteq [\![f \in U]\!]$.
(4) If $f \in U_a$ then there exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f \in U]\!]$.

Claim 7. If $u, v \in Z$, then $u \wedge v \in Z$. (Obvious.)

Claim 8. B is the disjoint union of $0/\theta$, X, W_a, W_d, U_a and U_d.

In light of Claims 3 and 5, to prove Claim 8 it will suffice to establish the following:

(1) X_d, W_d and U_d are pairwise disjoint.
(2) X_a, W_a and U_a are pairwise disjoint.
(3) $(W_a \cup U_a) \cap (W_d \cup U_d) = \emptyset$.

To prove the first item, suppose for example that $f \in X_d \cap W_d$. Then $[\![d \neq 0]\!] \subseteq [\![f \in X]\!]$ by Claim 5 ($1 \Rightarrow 2$), while $[\![d \neq 0]\!] \subseteq [\![f \in W]\!]$ by Claim 6 (1). This would imply $d = 0$, a contradiction.

To prove the second item, suppose for example that $f \in X_a \cap W_a$. Then there exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f \in X]\!]$ by Claim 5 ($3 \Rightarrow 4$), and there exists $v \in Z$ such that $[\![a * v \neq 0]\!] \subseteq [\![f \in W]\!]$ by Claim 6 (2). By Claim 7 there exists $w \in Z$ such that $[\![w \in X]\!] \subseteq [\![u \in X]\!] \cap [\![v \in X]\!]$. This would imply $[\![a * w \neq 0]\!] = \emptyset$, which contradicts Claim 2.

To prove the third item, suppose $f \in (W_a \cup U_a) \cap (W_d \cup U_d)$. Pick $p(x;\bar{v}_1) \in \mathsf{T}_W \cup \mathsf{T}_{UW}$ and $\bar{r}_1 \in B^{n_1}$ and $u \in Z$ such that $p^{\mathbf{B}}(f;\bar{r}_1) \geq a * u$, and pick $s(x;\bar{v}_2) \in \mathsf{T}_W \cup \mathsf{T}_{UW}$ and $\bar{r}_2 \in B^{n_2}$ such that $s^{\mathbf{B}}(f;\bar{r}_2) \geq d$. Finally, recall that $\lambda_0 \in \mathrm{Pol}_1^* \mathbf{B}$

and $\lambda_0(b) = d$. Choose $t(x;\bar{v}_3) \in \mathsf{T}^*$ and $\bar{r}_3 \in B^{n_3}$ such that $t^{\mathbf{B}}(b;\bar{r}_3) = d$. Since $\emptyset \neq [\![d \in W]\!] \subseteq [\![b \in W]\!]$, Lemma 8.5 implies $t \in \mathsf{T}_W$. Now observe that

$$(s^{\mathbf{B}}(f;\bar{r}_2) \wedge t^{\mathbf{B}}(b;\bar{r}_3)) * u \;=\; d * u \;=\; d \;\neq\; 0.$$

But this contradicts the fact that $\mathbf{B} \models \delta_{p,s,t}(a,b)$. Hence Claim 8 is proved.

Claim 9. θ has the following characterization.

(1) $\theta \subseteq (0/\theta)^2 \cup \mathsf{X}^2 \cup (W_a)^2 \cup (W_d)^2 \cup (U_a)^2 \cup (U_d)^2$.
(2) For all $(f,g) \in \mathsf{X}^2$, the following are equivalent:
  (a) $(f,g) \in \theta$.
  (b) $[\![d \neq 0]\!] \subseteq [\![f = g]\!]$.
  (c) There exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f = g]\!]$.
(3) For all $(f,g) \in (W_d)^2 \cup (U_d)^2$, the following are equivalent:
  (a) $(f,g) \in \theta$.
  (b) $[\![d \neq 0]\!] \subseteq [\![f = g]\!]$.
(4) For all $(f,g) \in (W_a)^2 \cup (U_a)^2$, the following are equivalent:
  (a) $(f,g) \in \theta$.
  (b) There exists $u \in Z$ such that $[\![a * u \neq 0]\!] \subseteq [\![f = g]\!]$.

The first item follows from Claims 3 and 4. Here is the proof of the second item. (a $\Rightarrow$ b) As $f \in X_d$ there exists $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\lambda(f) \geq d$. Then $\lambda(g) \geq d$ as well, since $(f,g) \in \theta$. Thus $[\![d \neq 0]\!] \subseteq [\![\lambda(f) = \lambda(g) \neq 0]\!]$, which implies $[\![d \neq 0]\!] \subseteq [\![f = g]\!]$ by Lemma 8.5 (8).

(a $\Rightarrow$ c) As $f \in X_a$ there exists $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ and $u \in Z$ such that $\lambda(f) \geq a * u$. Then $\lambda(g) \in H$, say $\lambda(g) \geq a * v$ with $v \in Z$, since $(f,g) \in \theta$. Let $w = u \wedge v$. Then $w \in Z$ by Claim 7, and $\lambda(f) \wedge \lambda(g) \geq a * w$. Hence $[\![a * w \neq 0]\!] \subseteq [\![\lambda(f) = \lambda(g) \neq 0]\!]$, which implies $[\![a * w \neq 0]\!] \subseteq [\![f = g]\!]$ by Lemma 8.5 (8).

(b $\Rightarrow$ c) Since $[\![d \neq 0]\!] \subseteq [\![f \in X]\!] \cap [\![f = g]\!]$ by hypothesis and Claim 5, we see that $d * (f \times g) = d$ and therefore $f \times g \in Z$. Define $u := e \wedge (f \times g) \in Z$ by Claim 7. Then $[\![a * u \neq 0]\!] \subseteq [\![f = g]\!]$.

(c $\Rightarrow$ b) Suppose $u \in Z$ and $[\![a * u \neq 0]\!] \subseteq [\![f = g]\!]$. Then $a * (u \times f) = a * (u \times g)$, hence

$$\begin{aligned} d * f \;=\; (d * u) * f \;&=\; d * (u \times f) \\ &=\; d * (u \times g) \text{ by Claim 1} \\ &=\; \cdots \;=\; d * g. \end{aligned}$$

Since $[\![d \neq 0]\!] \subseteq [\![f \in X]\!] \cap [\![g \in X]\!]$ by Claim 5, we get $[\![d \neq 0]\!] \subseteq [\![f = g]\!]$.

(b and c $\Rightarrow$ a) Suppose $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$. Clearly $\lambda(f) \geq d$ iff $\lambda(g) \geq d$, by item (b). Suppose $\lambda(f) \in H$, say $\lambda(f) \geq a * u$ with $u \in Z$. By item (c), there exists $v \in Z$ such that $[\![a * v \neq 0]\!] \subseteq [\![f = g]\!]$. Let $w = u \wedge v \in Z$. Then $\lambda(f) \geq a * w$ and $[\![a * w \neq 0]\!] \subseteq [\![f = g]\!]$, implying $\lambda(g) \geq a * w$ and hence $\lambda(g) \in H$. The converse is proved similarly. This proves $(f,g) \in \theta$.

The proofs of items 3 and 4 are similar. The proofs of (b $\Rightarrow$ a) are simplified by the following observation: If $f \in W_d \cup U_d$ then there does not exist $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\lambda(f) \in H$. Similarly, if $f \in W_a \cup U_a$ then there does not exist $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\lambda(f) \geq d$. (This observation follows from Claim 8 and the proof of Claim 3.)

Now we can finish the proof of Lemma 9.6. Define $\mathsf{W} = W_a \cup W_d$ and $\mathsf{U} = U_a \cup U_d$. We wish to show that the algebra $\overline{\mathbf{B}} := \mathbf{B}/\theta$ and the sets $\overline{\mathsf{W}} := \mathsf{W}/\theta$, $\overline{\mathsf{U}} := \mathsf{U}/\theta$ and $\overline{\mathsf{X}} := \mathsf{X}/\theta$ satisfy the hypotheses of Lemma 8.3. Hypotheses 1 and 2 are true since they already hold in $\mathbf{B}$. Hypothesis 3 follows from Claim 8 and the obvious fact that $a, d \in \mathsf{W}$. Clearly $e \in \mathsf{X}$ and X is closed under $\times$ by Claim 5. Thus to verify the remaining hypotheses of Lemma 8.3 it suffices to show the following (for all $f, g, h, u \in B$):

(4) If $f \times g \stackrel{\theta}{\not\equiv} 0$ then $f \in \mathsf{X}$.

(5) (a) If $f \cdot g \stackrel{\theta}{\not\equiv} 0$ then $f \in \mathsf{U}$ and $g \in \mathsf{W}$.

(b) For all $f \in \mathsf{U}$ there exists $g \in \mathsf{W}$ such that $f \cdot g \stackrel{\theta}{\not\equiv} 0$.

(6) If $f \cdot g \stackrel{\theta}{\not\equiv} 0$ and $f \cdot h \stackrel{\theta}{\not\equiv} 0$, then $g \stackrel{\theta}{\equiv} h$.

(7) If $f \in \mathsf{W}$ and $u \in \mathsf{X}$, then $f * u \in \mathsf{W}$.

(8) If $f * u \stackrel{\theta}{\equiv} f \in \mathsf{W}$ then $u \stackrel{\theta}{\equiv} e$.

(9) If $f \stackrel{\theta}{\not\equiv} g$ then $f \wedge g \stackrel{\theta}{\equiv} 0$.

(Proof of 4) Assume $f \times g \stackrel{\theta}{\not\equiv} 0$. Since $[\![f \times g \neq 0]\!] = [\![f \times g \in X]\!]$ we must have $f \times g \in \mathsf{X}$ by Claim 6 and therefore $[\![d \neq 0]\!] \subseteq [\![f \times g \in X]\!]$ by Claim 5. Since $[\![f \times g \in X]\!] \subseteq [\![f \in X]\!]$ we get $[\![d \neq 0]\!] \subseteq [\![f \in X]\!]$ and hence $f \in \mathsf{X}$ by Claim 5.

(Proof of 5a) Assume $f \cdot g \stackrel{\theta}{\not\equiv} 0$. Since $[\![f \cdot g \neq 0]\!] = [\![f \cdot g \in W]\!]$ we must have $f \cdot g \in \mathsf{W}$ by Claims 5 and 6. Choose $t(x; v_1, \dots, v_n) \in \mathsf{T}_W$ and $\bar{r} \in B^n$ such that either $t^{\mathbf{B}}(f \cdot g; \bar{r}) \geq d$ or $t^{\mathbf{B}}(f \cdot g; \bar{r}) \in H$. Define $t_1 \in \mathsf{T}_W$ and $t_2 \in \mathsf{T}_{UW}$ by

$$\begin{aligned} t_1(x; v_1, \dots, v_{n+1}) &= t(v_1 \cdot x; v_2, \dots, v_{n+1}) \\ t_2(x; v_1, \dots, v_{n+1}) &= t(x \cdot v_1; v_2, \dots, v_{n+1}). \end{aligned}$$

Clearly $t_1^{\mathbf{B}}(g; f, \bar{r}) = t_2^{\mathbf{B}}(f; g, \bar{r}) = t^{\mathbf{B}}(f \cdot g; \bar{r})$, which proves $f \in \mathsf{U}$ and $g \in \mathsf{W}$.

Remark: In fact, this proof shows that $f \cdot g \in W_d$ implies $f \in U_d$ and $g \in W_d$, while $f \cdot g \in W_a$ implies $f \in U_a$ and $g \in W_a$.

(Proof of 5b) Let $f \in \mathsf{U}$. Choose $t \in \mathsf{T}_{UW}$ and $\bar{r} \in B^n$ such that either $t^{\mathbf{B}}(f; \bar{r}) \geq d$ or $t^{\mathbf{B}}(f; \bar{r}) \in H$. By definition, t must have the form

$$\begin{aligned} & p(s(x; v_1, \dots, v_m) \cdot v_{m+1}; v_{m+2}, \dots, v_n) \\ \text{or} \quad & p(s(x; v_1, \dots, v_m) \circ v_{m+1}; v_{m+2}, \dots, v_n) \end{aligned}$$

for some $s \in \mathsf{T}_U$ and $p \in \mathsf{T}_W$. In fact, using the identity $x \circ y = x \circ (x \cdot (x \circ y))$ of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$, we can assume that t has the first form. Using the identities

$$\begin{aligned} (x \wedge u) \cdot v &= (x \cdot v) \wedge (u \cdot v) \\ (x * u) \cdot v &= (x \cdot (v * u)) * u \end{aligned}$$

which are true in $\mathbf{M}(\mathcal{T})$ and hence in $\mathbf{B}$, we can furthermore assume that $s = x$. Now let $g = r_1$. As $p^{\mathbf{B}}(f \cdot g; r_2, \dots, r_n) = t^{\mathbf{B}}(f; \bar{r})$ and $p \in \mathsf{T}_W$, we get $f \cdot g \in W$.

(Proof of 6) Assume $f \cdot g \overset{\theta}{\not\equiv} 0$ and $f \cdot h \overset{\theta}{\not\equiv} 0$. Then $f \cdot g \in \mathsf{W}$ and $f \cdot h \in \mathsf{W}$ as shown in the proof of (5a). Suppose first that $f \cdot g \in W_d$. Then $f \in U_d$ and $g \in W_d$ by the remark following the proof of (5a). Hence it must follow that $h \in W_d$ as well, since otherwise $f \cdot h \in W_a$ and therefore $f \in W_a$ by the same reasoning. From $f \cdot g \in W_d$ and $f \cdot h \in W_d$ we get $[\![d \neq 0]\!] \subseteq [\![f \cdot g \neq 0 \ \& \ f \cdot h \neq 0]\!]$ by Claim 6. Since

$$(x \cdot y \neq 0 \ \& \ x \cdot z \neq 0) \rightarrow y = z$$

is valid in all chain type algebras, we get $[\![d \neq 0]\!] \subseteq [\![g = h]\!]$ and therefore $(f, g) \in \theta$ by Claim 9. In case $f \cdot g \in W_a$, one argues in a similar manner, using Claim 7 to obtain $w \in Z$ satisfing $[\![a * w \neq 0]\!] \subseteq [\![f \cdot g \neq 0 \ \& \ f \cdot h \neq 0]\!]$.

(Proof of 7) Assume first that $f \in W_d$. Choose $t \in \mathsf{T}_W$ and $\bar{r} \in B^n$ such that $t^{\mathbf{B}}(f; \bar{r}) \geq d$. Since $u \in X$ we have $[\![d \neq 0]\!] \subseteq [\![u \times u = e]\!]$ by Claim 5 and hence $t^{\mathbf{B}}(f * (u \times u); \bar{r}) \geq d$. Since $f * (u \times u)$ can also be written as $(f * u) * u$, we get $f * u \in W_d$. A similar argument, using Claim 7, works in case $f \in W_a$.

(Proof of 8) Assume first that $f \in W_d$. Since $(f * u, f) \in \theta$, we get $f * u \in W_d$ by Claim 4 and $[\![d \neq 0]\!] \subseteq [\![f * u = f]\!]$ by Claim 9. Moreover, $[\![d \neq 0]\!] \subseteq [\![f \in W]\!]$ by Claim 6. This forces $[\![d \neq 0]\!] \subseteq [\![u = e]\!]$ and hence $u \in \mathsf{X}$ and $(u, e) \in \theta$ by Claims 5 and 9. A similar argument, using Claim 7, works in case $f \in W_a$.

(Proof of 9) Assume $f \wedge g \overset{\theta}{\not\equiv} 0$. Then $f \wedge g$ must belong to one of X, W_a, W_d, U_a or U_d. We argue by cases. For example, suppose $f \wedge g \in W_d$. Choose $t \in \mathsf{T}_W$ and $\bar{r} \in B^n$ such that $t^{\mathbf{B}}(f \wedge g; \bar{r}) \geq d$. Then by the monotonicity of t we also get $t^{\mathbf{B}}(f; \bar{r}) \geq d$ and $t^{\mathbf{B}}(g; \bar{r}) \geq d$, hence $f, g \in W_d$. Furthermore, $t^{\mathbf{B}}(f \wedge g; \bar{r}) \geq d$ implies

$$\begin{aligned} [\![d \neq 0]\!] &\subseteq [\![f \wedge g \neq 0]\!] \\ &\subseteq [\![f = g]\!] \end{aligned}$$

so $(f, g) \in \theta$ by Claim 9. Similar arguments work in the other cases.

Thus the hypotheses of Lemma 8.3 are met, so by that lemma $\mathbf{B}/\theta$ is isomorphic to some $\mathbf{S}(G, E, X)$ satisfying the hypotheses of Lemma 8.1. Obviously $a \in W_a$ and $d \in W_d$. Recall that $\lambda_0(b) = d$. Choose $t \in \mathsf{T}^*$ and $\bar{r} \in B^n$ such that $\lambda_0(x) = t^{\mathbf{B}}(x; \bar{r})$. As was shown in the proof of item 3 of Claim 8, $t \in \mathsf{T}_W$. It follows that $b \in W_d$. Hence a/θ and b/θ are vertices of $\mathbf{B}/\theta$. Suppose a/θ and b/θ belonged to the same weak component. Then there would exist $\lambda \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\lambda(a) \overset{\theta}{\equiv} b$ and therefore $\lambda(a) \in W_d$. Hence there exists $\mu \in \mathrm{Pol}_1^* \mathbf{B}$ such that $\mu(a) \geq d$. But this implies

$a \in W_d \cup U_d \cup \mathsf{X}$, which is false. So a/θ and b/θ belong to distinct weak components of $\mathbf{B}/\theta$.

By Corollary 8.2 there exists $\mathbf{D} \in \mathsf{V}(\mathbf{M}(\mathcal{T}))$ such that $(\mathbf{B}/\theta)^{\otimes} \leq \mathbf{D}$ and $\mathbf{D} \models \psi(a/\theta, b/\theta)$. Thus defining $h : \mathbf{B}^{\otimes} \to \mathbf{D}$ by $h(x) = x/\theta$ completes the proof of $(**)$ and therefore of $(*)$, proving the lemma. □

Theorem 9.7. *If $\mathcal{T}$ does not halt, then the theory of $\mathsf{V}(\mathbf{M}(\mathcal{T}))$ does not have a model companion.*

Proof. By Lemmas 2.3 and 9.4–9.6. □

Corollary 9.8. *The problem of deciding, given a description of a finite algebra $\mathbf{A}$ in a finite language, whether $\mathsf{V}(\mathbf{A})$ has a model companion is recursively unsolvable.*

Proof. Theorems 7.1 and 9.7, the recursive unsolvability of the halting problem, and the effectiveness of the construction of $\mathbf{M}(\mathcal{T})$ from $\mathcal{T}$. □

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

E-mail address: rdwillar@uwaterloo.ca