NEW TOOLS FOR PROVING DUALIZABILITY

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We address the problem of proving that a finite algebra $\underline{\mathbf{M}}$ is dualizable, or strongly dualizable, in the sense of [1]. There are generally two aspects to the problem: (1) handling the finite members of $\mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$, and (2) handling the infinite members of $\mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$. In principle, the problem rests entirely at the infinite level (that is, in the second aspect). In practice, however, the real work is typically done at the finite level; then the analysis is "lifted" to the infinitely level by a suitable combinatorial argument.

In this paper we give two "lifting theorems," one for dualizability and the other for strong dualizability, which may prove useful in this enterprise.

1. Dualizability

Throughout this paper $\underline{\mathbf{M}}$ will denote a fixed finite algebra. The theory of natural dualities attempts to produce a "nice" dual category for the quasivariety generated by $\underline{\mathbf{M}}$. See the book [1] for the full story. In this and the next section, we shall give a bare minimum of definitions needed to state our theorems.

By an *algebraic relation of* $\underline{\mathbf{M}}$ is meant any subuniverse of $\underline{\mathbf{M}}^n$ for some $n < \omega$, construed as an n-ary relation on the set M. Fix $\mathcal{R}$, a set of algebraic relations of $\underline{\mathbf{M}}$.

Definition 1.1. (1) $\underset{\sim}{\mathbf{M}}$ is the relational-topological structure $\langle M, \mathcal{R}, \mathcal{T}_d \rangle$, whose universe is M (the universe of $\underline{\mathbf{M}}$), whose relations are those in $\mathcal{R}$, and whose topology $\mathcal{T}_d$ is the discrete topology.

(2) If $\mathbf{A} \in \mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$, then $\mathbf{D}(\mathbf{A})$ denotes the induced relational-topological substructure of $\underset{\sim}{\mathbf{M}}^A$ with universe $\mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$. ($\mathbf{D}(\mathbf{A})$ is the *dual* of $\mathbf{A}$.)

Fix $\mathbf{A} \in \mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$ and $a \in A$. The "evaluation-at-a" map $\mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}}) \to M$, given by $h \mapsto h(a)$, is an example of a continuous $\mathcal{R}$-preserving map from $\mathbf{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$.

Definition 1.2. (1) $\mathcal{R}$ *dualizes* $\underline{\mathbf{M}}$ if for every $\mathbf{A} \in \mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$, the continuous $\mathcal{R}$-preserving maps from $\mathbf{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$ are precisely the evaluation maps.

(2) $\mathcal{R}$ *dualizes* $\underline{\mathbf{M}}$ *at the finite level* if the previous item holds for every $\mathbf{A} \in \mathbf{S}\,\mathbf{P}_{\mathrm{fin}}(\underline{\mathbf{M}})$.

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(3) $\underline{\mathbf{M}}$ is *dualizable* if there exists a set $\mathcal{R}$ of algebraic relations which dualizes $\underline{\mathbf{M}}$.

Our first "lifting theorem" is due to Zádori [4] and the author [2], and is dubbed the "Duality Compactness Theorem" in [1].

Theorem 1.3. *Let* $\underline{\mathbf{M}}$ *be a finite algebra and* $\mathcal{R}$ *a* finite *set of algebraic relations for* $\underline{\mathbf{M}}$*. If* $\mathcal{R}$ *dualizes* $\underline{\mathbf{M}}$ *at the finite level, then* $\mathcal{R}$ *dualizes* $\underline{\mathbf{M}}$*.*

Sketch of the proof. Fix $\mathbf{A} \in \mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$, and let $\varphi : \mathbf{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ be continuous and $\mathcal{R}$-preserving. Because φ is continuous, there exists a finite subalgebra $\mathbf{A}_0 \leq \mathbf{A}$ which "determines" φ in the following sense: if $h, h' \in \mathbf{D}(\mathbf{A})$ and $h|_{A_0} = h'|_{A_0}$, then $\varphi(h) = \varphi(h')$.

Since $\mathcal{R}$ is finite, we can define $n = \max(\text{arities of members of } \mathcal{R})$. A combinatorial argument can be given to prove the existence of a finite algebra $\mathbf{A}_1$ satisfying

- $\mathbf{A}_0 \leq \mathbf{A}_1 \leq \mathbf{A}$, and
- For all $k \leq n$ and $\mathbf{h} \in \mathrm{Hom}(\mathbf{A}_1, \underline{\mathbf{M}}^k)$, there exists $\mathbf{h}' \in \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}}^k)$ such that $\mathbf{h}'|_{A_0} = \mathbf{h}|_{A_0}$ and $\mathrm{range}(\mathbf{h}') \subseteq \mathrm{range}(\mathbf{h})$.

Now define $\varphi^* : \mathbf{D}(\mathbf{A}_1) \to \underset{\sim}{\mathbf{M}}$ by

$$\varphi^*(h) = \varphi(h'), \quad h' \in \mathbf{D}(\mathbf{A}), \quad h'|_{A_0} = h|_{A_0}.$$

By the choice of $\mathbf{A}_1$, φ^* preserves the same ($\leq n$)-ary relations as does φ. Hence φ^* is $\mathcal{R}$-preserving. Since $\mathbf{A}_1$ is finite, φ^* is automatically continuous. Since $\mathcal{R}$ dualizes $\underline{\mathbf{M}}$ at the finite level, φ^* is an evaluation map. It follows that φ is an evaluation map as well. □

2. Characterization of strong dualizability

By an *algebraic operation of* $\underline{\mathbf{M}}$ is meant any $h \in \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$, where $\mathbf{B} \leq \underline{\mathbf{M}}^n$ for some $n < \omega$, construed as an n-ary partial operation on the set M. Fix $\mathcal{F}$, a set of algebraic operations of $\underline{\mathbf{M}}$.

Definition 2.1. (1) $\widetilde{\mathbf{M}}$ is the topological algebraic structure $\langle M, \mathcal{F}, \mathcal{T}_d \rangle$ built on the universe M with the partial operations in $\mathcal{F}$ and the discrete topology.

(2) If $\mathcal{R}$ is fixed as in Section 1, then $\underset{\sim}{\widetilde{\mathbf{M}}} = \langle M, \mathcal{R}, \mathcal{F}, \mathcal{T}_d \rangle$.

Clark and Davey [1, Chapter 3] formulate the so-called *Term Closure* condition and the *Finite Term Closure* condition as conditions on $\underset{\sim}{\widetilde{\mathbf{M}}}$. In reality they are conditions on $\mathcal{F}$ only, so we shall write these conditions as $\mathrm{TC}(\mathcal{F})$ and $\mathrm{FTC}(\mathcal{F})$ respectively. We shall give alternate characterizations of them below, but first we state the connection between $\mathrm{TC}(\mathcal{F})$ and strong duality.

Definition 2.2. Let $\underline{\mathbf{M}}$ be a finite algebra, $\mathcal{F}$ a set of algebraic operations of $\underline{\mathbf{M}}$, and $\mathcal{R}$ a set of algebraic relations of $\underline{\mathbf{M}}$ such that $\mathcal{R}$ contains the graph of each member of $\mathcal{F}$.

(1) $\langle \mathcal{R}, \mathcal{F} \rangle$ *strongly dualizes* $\underline{\mathbf{M}}$ if $\mathcal{R}$ dualizes $\underline{\mathbf{M}}$ and $\mathrm{TC}(\mathcal{F})$ holds.

(2) $\underline{\mathbf{M}}$ is *strongly dualizable* if there exist $\mathcal{R}$ and $\mathcal{F}$ which strongly dualize $\underline{\mathbf{M}}$.

The next lemma can be found in [2].

Lemma 2.3. (1) $\mathrm{TC}(\mathcal{F})$ *holds if and only if for every* $\mathbf{A} \leq \underset{\sim}{\underline{\mathbf{M}}}^I$ *(I arbitrary), if* $X \subseteq \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ *is a topologically closed subuniverse of* $\mathbf{M}^A$ *which contains the projection homomorphisms* $\{\pi_i : i \in I\}$*, then* $X = \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$.

(2) $\mathrm{FTC}(\mathcal{F})$ *holds if and only if the above condition is true for all* $\mathbf{A} \leq \underline{\mathbf{M}}^n$, $n < \omega$.

Now the question is whether there are general situations in which it can be guaranteed that $\mathrm{FTC}(\mathcal{F})$ lifts to $\mathrm{TC}(\mathcal{F})$. For example, if $\mathcal{F}$ is finite and $\mathrm{FTC}(\mathcal{F})$ holds, does it follow that $\mathrm{TC}(\mathcal{F})$ holds? We do not know, but we can proving a lifting theorem from a different hypothesis. The hypothesis is explained in the next section, and the theorem is given in Section 4.

3. Ranks

In this section we assign to each algebraic operation of $\underline{\mathbf{M}}$ an ordinal number (or ∞), which we call its *rank*. First we define some auxiliary notions.

Definition 3.1. (1) If $\mathbf{B} \leq \underline{\mathbf{M}}^n$ and $\mathbf{B}' \leq \underline{\mathbf{M}}^{n+k}$, then we write $\mathbf{B} \Rrightarrow \mathbf{B}'$ if there exists $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$ such that

$$B' = \{(b_1, \ldots, b_n, b_{\sigma 1}, \ldots, b_{\sigma k}) : (b_1, \ldots, b_n) \in B\}.$$

If this holds, then there is an obvious isomorphism from $\mathbf{B}$ to $\mathbf{B}'$ induced by σ, which we also denote by σ and write $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$.

(2) If $\mathbf{B} \leq \mathbf{C}$ and $Y \subseteq \mathrm{Hom}(\mathbf{C}, \underline{\mathbf{M}})$, then $\mathbf{B}|_Y$ denotes $\mathbf{B}/\bigcap\{\ker(h|_B) : h \in Y\}$.

(3) If $\mathbf{B} \leq \mathbf{C} \leq \mathbf{D}$ and $Y \subseteq \mathrm{Hom}(\mathbf{D}, \underline{\mathbf{M}})$, then the natural map $\mathbf{B} \to \mathbf{C}|_Y$ shall be denoted by ν.

(4) If $\alpha : \mathbf{B} \to \mathbf{C}$ and $h \in \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$, then we say that h *lifts to* $\mathbf{C}$ (through α) if there exists $h' \in \mathrm{Hom}(\mathbf{C}, \underline{\mathbf{M}})$ such that $h'\alpha = h$.

Now we define the notion "rank$(h) \leq \alpha$" for h an algebraic operation of $\underline{\mathbf{M}}$ and α an ordinal. The definition is inductive.

Definition 3.2. Let $\mathbf{B} \leq \underline{\mathbf{M}}^n$ $(n < \omega)$ and $h \in \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$.

(1) $\mathrm{rank}(h) \leq 0$ if and only if h is a coordinate projection map π_i for some $i = 1, \ldots, n$.

(2) For $\alpha \in On \setminus \{0\}$, $\mathrm{rank}(h) \leq \alpha$ if and only if there exists $N < \omega$ such that the following condition holds: whenever $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \underline{\mathbf{M}}^{n+k}$ and h lifts to $\mathbf{D}$ through σ, then there exists $Y \subseteq \mathrm{Hom}(\mathbf{D}, \underline{\mathbf{M}})$ such that

(a) $|Y| \leq N$;

(b) h lifts to $\mathbf{C}|_Y$ through $\nu\sigma$; and

(c) $\mathrm{rank}(f|_C) < \alpha$ (i.e., $\leq \beta$ for some $\beta < \alpha$) for all $f \in Y$.

(3) Write $\operatorname{rank}(h) = \alpha$ if α is the least ordinal for which $\operatorname{rank}(h) \leq \alpha$ is true. If there is no such ordinal, then write $\operatorname{rank}(h) = \infty$.

Definition 3.3. The rank of $\underline{\mathbf{M}}$ is the supremum of the ranks of all the algebraic operations of $\underline{\mathbf{M}}$.

This rank function seems to measure the degree to which $\underline{\mathbf{M}}$ fails to be injective in the quasivariety it generates. Indeed, if $\underline{\mathbf{M}}$ is injective in its quasivariety, then its rank is 0 or 1. The converse, however, is not true (e.g. when $\underline{\mathbf{M}}$ is the ring $\mathbb{Z}_4$), and in general we do not fully understand what this rank function is measuring. Its definition is motivated by the proof of the theorem in the next section.

4. A lifting theorem for strong dualizability

Theorem 4.1. *Let $\underline{\mathbf{M}}$ be a finite algebra, and suppose that $\underline{\mathbf{M}}$ has rank $< \infty$. Let $\mathcal{F}$ be a set of algebraic operations of $\underline{\mathbf{M}}$.*

(1) *If* $\mathrm{FTC}(\mathcal{F})$ *holds, then* $\mathrm{TC}(\mathcal{F})$ *holds.*
(2) *Thus if $\underline{\mathbf{M}}$ is dualizable, then $\underline{\mathbf{M}}$ is strongly dualizable.*

The second item is explained by the fact that $\mathrm{FTC}(\mathcal{F})$ always holds when $\mathcal{F}$ is chosen to be the set of *all* algebraic operations of $\underline{\mathbf{M}}$. The rest of this section is devoted to proving the first item.

It will be convenient to represent algebraic operations abstractly rather than concretely. If $\mathbf{B} \leq \underline{\mathbf{M}}^n$, then $\mathbf{B}$ *as a subalgebra of* $\underline{\mathbf{M}}^n$ is determined by $\mathbf{B}$ as an abstract algebra (that is, known up to isomorphism only) together with the list $\langle \pi_1, \dots, \pi_n \rangle$ of the coordinate projection homomorphisms from $\mathbf{B}$ to $\underline{\mathbf{M}}$. For our purposes, we need only the information carried by $\mathbf{B}$ and the set $\{\pi_1, \dots, \pi_n\}$.

Definition 4.2. (1) If $\mathbf{B} \in \mathbf{I}\,\mathbf{S}\,\mathbf{P}(\underline{\mathbf{M}})$ and $X \subseteq \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$, then we say that X *separates the points of* $\mathbf{B}$ if $\bigcap\{\ker(h) : h \in X\} = 0_B$.

(2) *Rel* is the collection of all pairs $(\mathbf{B}, X)$ where $\mathbf{B} \in \mathbf{I}\,\mathbf{S}\,\mathbf{P}_{\mathrm{fin}}(\underline{\mathbf{M}})$, $X \subseteq \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$, and X separates the points of $\mathbf{B}$.
(3) *Op* is the collection of all triples $(h, \mathbf{B}, X)$ where $(\mathbf{B}, X) \in \mathit{Rel}$ and $h \in \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$.

For example, if $\mathbf{B} \leq \underline{\mathbf{M}}^n$, $h \in \mathrm{Hom}(\mathbf{B}, \underline{\mathbf{M}})$ and $X = \{\pi_1, \dots, \pi_n\}$, then $(h, \mathbf{B}, X) \in \mathit{Op}$. We shall call this example the *collapse* of h. Conversely, given $(h, \mathbf{B}, X) \in \mathit{Op}$ we can list $X = \{p_1, \dots, p_n\}$ with $n = |X|$, define $\mathbf{p} : \mathbf{B} \to \underline{\mathbf{M}}^n$ by $\mathbf{p}(b) = (p_1(b), \dots, p_n(b))$, and define $\mathbf{B}' = \mathrm{image}(\mathbf{p}) \leq \underline{\mathbf{M}}^n$ and $h' \in \mathrm{Hom}(\mathbf{B}', \underline{\mathbf{M}})$ so that $h'\mathbf{p} = h$. We will call the $n!$ algebraic operations h' obtained in this way the *associates* of $(h, \mathbf{B}, X)$.

Now we define the rank of each $(h, \mathbf{B}, X) \in \mathit{Op}$ in the same way as we defined ranks of algebraic operations. Here is the key part of the definition.

Definition 4.3. Let $(h, \mathbf{B}, X) \in \mathit{Op}$.

(1) $\text{rank}^*(h, \mathbf{B}, X) \le 0$ if and only if $h \in X$.
(2) For $\alpha \in On \setminus \{0\}$, $\text{rank}^*(h, \mathbf{B}, X) \le \alpha$ if and only if there exists $N < \omega$ such that the following condition holds: whenever $\mathbf{C}, \mathbf{D} \in \mathbf{I}\,\mathbf{S}\,\mathbf{P}_{\text{fin}}(\underline{\mathbf{M}})$ and $X' \subseteq \text{Hom}(\mathbf{D}, \underline{\mathbf{M}})$ satisfy
   - $\mathbf{B} \le \mathbf{C} \le \mathbf{D}$;
   - X' separates the points of $\mathbf{D}$ and $X'|_B = X$; and
   - h can be lifted to $\mathbf{D}$ (through the inclusion map);

   then there exists $Y \subseteq \text{Hom}(\mathbf{D}, \underline{\mathbf{M}})$ such that
   (a) $|Y| \le N$;
   (b) h lifts to $\mathbf{C}|_Y$ through ν; and
   (c) $\text{rank}^*(f|_C, \mathbf{C}, X'|_C) < \alpha$ for all $f \in Y$.

It can be shown that if h is an algebraic operation and $(h, \mathbf{B}, X)$ is its collapse, then $\text{rank}(h) = \text{rank}^*(h, \mathbf{B}, X)$; and conversely, if $(h, \mathbf{B}, X) \in Op$ and h' is an associate, then $\text{rank}^*(h, \mathbf{B}, X) = \text{rank}(h')$.

The next lemma may be found e.g. in [3] (Theorem 1, p. 132).

Lemma 4.4. *Suppose that P is a poset in which any two elements have an upper bound, and F is a function with domain P such that for each $x \in P$, $F(x)$ is a finite nonempty set. Suppose moreover that for all pairs $(x, y) \in P^2$ with $x \le y$ we have a specified function $f_{x,y} : F(y) \to F(x)$ and that these functions satisfy (i) $f_{x,y} \circ f_{y,z} = f_{x,z}$ whenever $x \le y \le z$ in P, and (ii) $f_{x,x} = \text{id}_{F(x)}$ for all $x \in P$. Then there is a function φ with domain P such that $\varphi(x) \in F(x)$ for all $x \in P$, and $f_{x,y}(\varphi(y)) = \varphi(x)$ for all $x \le y$.*

The heart of the proof of Theorem 4.1 is contained in the next proposition.

Proposition 4.5. *Let $\underline{\mathbf{M}}$ be a finite algebra and $\mathcal{F}$ a set of algebraic operations of $\underline{\mathbf{M}}$ such that* FTC($\mathcal{F}$) *holds. Suppose $\mathbf{A} \in \mathbf{I}\,\underset{\sim}{\mathbf{S}}\,\mathbf{P}(\underline{\mathbf{M}})$ and $X \subseteq \text{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ where X is a topologically closed subuniverse of $\mathbf{M}^A$ and X separates the points of $\mathbf{A}$. Then for every finite subalgebra $\mathbf{B} \le \mathbf{A}$ and every $h \in \text{Hom}(\mathbf{A}, \underline{\mathbf{M}})$, either $\text{rank}^*(h|_B, \mathbf{B}, X|_B) = 0$ or $\text{rank}^*(h|_B, \mathbf{B}, X|_B) = \infty$.*

Sketch of the proof. We shall show, by induction on α, that if $\mathbf{B} \le \mathbf{A}$ is finite, $h \in \text{Hom}(\mathbf{A}, \underline{\mathbf{M}})$, and $\text{rank}^*(h|_B, \mathbf{B}, X|_B) \le \alpha$, then $\text{rank}^*(h|_B, \mathbf{B}, X|_B) = 0$. We may assume $\alpha > 0$ and that the claim has been proved for all $\alpha' < \alpha$. Let $\mathbf{B}$ and h be given with $\text{rank}^*(h|_B, \mathbf{B}, X|_B) \le \alpha$. Choose $N \in \omega$ to witness the definition of the rank being $\le \alpha$. For the time being, fix a finite algebra $\mathbf{C}$ satisfying $\mathbf{B} \le \mathbf{C} \le \mathbf{A}$, and let $X_1 = X|_C$. Let $\mathcal{C}$ be the set of all pairs $(m, \mathbf{g})$ where $1 \le m \le N$, $\mathbf{g} = (g_1, \ldots, g_m) \in \text{Hom}(\mathbf{C}, \underline{\mathbf{M}})^m$, $\text{rank}^*(g_i, \mathbf{C}, X_1) < \alpha$ for all $i = 1, \ldots, m$, and $h|_B$ lifts to $\mathbf{C}|_Y$ through the natural map, where $Y = \{g_1, \ldots, g_m\}$.

Let P be the poset of all finite subalgebras of $\mathbf{A}$ which contain $\mathbf{C}$, ordered by inclusion. For any $\mathbf{D} \in P$, let $X_2 = X|_D$; as $h|_B$ obviously lifts to $\mathbf{D}$, the definition of '$\text{rank}^*(h|_B, \mathbf{B}, X|_B) \le \alpha$' says there must exist $Y \subseteq \text{Hom}(\mathbf{D}, \underline{\mathbf{M}})$ satisfying $|Y| \le N$,

$\mathrm{rank}^*(f|_C, \mathbf{C}, X_1) < \alpha$ for all $f \in Y$, and $h|_B$ lifts to $\mathbf{C}|_Y$ through the natural map. Let $F(\mathbf{D})$ be the set of all pairs $(m, \mathbf{f})$ such that $1 \le m \le N$, $\mathbf{f} \in \mathrm{Hom}(\mathbf{D}, \underline{\mathbf{M}})^m$, and $(m, \mathbf{f}|_C) \in \mathcal{C}$. $F(\mathbf{D})$ is finite and, by the above discussion, is nonempty for each $\mathbf{D} \in P$. Moreover, if $\mathbf{D}, \mathbf{E} \in P$ with $\mathbf{D} \le \mathbf{E}$, then $(m, \mathbf{f}) \in F(\mathbf{E})$ implies $(m, \mathbf{f}|_D) \in F(\mathbf{D})$. Thus Lemma 4.4 yields $m \le N$ and $\mathbf{h} = (h_1, \ldots, h_m) \in (M^A)^m$ such that $(m, \mathbf{h}|_D) \in F(\mathbf{D})$ for all $\mathbf{D} \in P$. It follows that $h_1, \ldots, h_m \in \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ and $(m, \mathbf{h}|_C) \in \mathcal{C}$. By the inductive hypothesis we must have $\mathrm{rank}^*(h_i|_C, \mathbf{C}, X|_C) = 0$ for all i, which means there exist $g_1, \ldots, g_m \in X$ with $\mathbf{g}|_C = \mathbf{h}|_C$.

Thus we have proved the following: there exists $Y \subseteq X|_C$ such that $|Y| \le N$ and $h|_B$ lifts to $\mathbf{C}|_Y$ through the natural map ν. Enumerate $Y = \{y_1, \ldots, y_m\}$ with $m \le N$. The last condition (of $h|_B$ lifting) can be restated as follows: there is an m-ary algebraic operation h^* of $\underline{\mathbf{M}}$ such that

(1) $\langle y_1(c), \ldots, y_m(c)\rangle$ is in the domain of h^* for every $c \in C$.
(2) If $h' \in \mathrm{Hom}(\mathbf{C}, \underline{\mathbf{M}})$ is defined by $h'(c) = h^*(y_1(c), \ldots, y_m(c))$, then $h'|_B = h|_B$.

In the foregoing discussion, $\mathbf{C}$ was fixed. Now unfix it. Let P' be the poset of all finite subalgebras of $\mathbf{A}$ which contain $\mathbf{B}$. For each $\mathbf{C} \in P'$ let $F'(\mathbf{C})$ be the set of all triples $(m, \mathbf{y}, h^*)$ where $1 \le m \le N$, $\mathbf{y} \in (X|_C)^m$, and h^* is an m-ary algebraic operation over $\underline{\mathbf{M}}$ satisfying items (1,2) above with respect to $\mathbf{y}$. $F'(\mathbf{C})$ is finite and nonempty, by the above discussion. Moreover, if $\mathbf{C}, \mathbf{D} \in P'$ with $\mathbf{C} \le \mathbf{D}$, then $(m, \mathbf{y}|_C, h^*) \in F'(\mathbf{C})$ whenever $(m, \mathbf{y}, h^*) \in F'(\mathbf{D})$. Thus by Lemma 4.4 there exist $m \le N$, $\mathbf{y} \in (M^A)^m$, and an m-ary algebraic operation h^* of $\underline{\mathbf{M}}$ such that $(m, \mathbf{y}|_C, h^*) \in F'(\mathbf{C})$ for all $\mathbf{C} \in P'$. Define $h' \in M^A$ by $h'(a) = h^*(y_1(a), \ldots, y_m(a))$. Since X is topologically closed we get $y_1, \ldots, y_m \in X$. Since X is closed under the pointwise application of the operations of $\mathcal{F}$, and since $\mathrm{FTC}(\mathcal{F})$ holds, it follows that X must be closed under h^* and therefore $h' \in X$. As $h'|_B = h|_B$ by construction, this proves $\mathrm{rank}^*(h|_B, \mathbf{B}, X|_B) \le 0$. $\square$

Now we can finish the proof of Theorem 4.1.

Proof of Theorem 4.1(1). Assume $\mathrm{FTC}(\mathcal{F})$ holds and $\underline{\mathbf{M}}$ has rank $< \infty$. To prove that $\mathrm{TC}(\mathcal{F})$ holds, we will verify the condition given in Lemma 2.3(1). Let $\mathbf{A} \le \underset{\sim}{\mathbf{M}}^I$ and $X \subseteq \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ be given so that X is a topologically closed subuniverse of $\mathbf{M}^A$ which contains the projection homomorphisms. Clearly X separates the points of $\mathbf{A}$, so the previous proposition applies. Since the rank of $\underline{\mathbf{M}}$ is $< \infty$, it follows that for every $h \in \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ and every finite subalgebra $\mathbf{B} \le \mathbf{A}$, $\mathrm{rank}^*(h|_B, \mathbf{B}, X|_B) = 0$, i.e., $h|_B \in X|_B$. As X is topologically closed, this implies $h \in X$ and therefore $X = \mathrm{Hom}(\mathbf{A}, \underline{\mathbf{M}})$ as required. $\square$

5. Discussion

Our discovery of Theorems 1.3 and 4.1 was motivated by our attempt (with P. Idziak and D. Clark) to prove that the ring $\mathbb{Z}_4$ is strongly dualizable. With the additional help of L. Sabourin and Cs. Szabó, we are able to prove that if R is any finite commutative ring with identity whose Jacobson radical J satisfies $J^2 = 0$, then R is strongly dualizable (see [2]). In particular, we show (implicitly) that if R is such a ring, then $\text{rank}(R) \leq 1$.

We have an example of a finite (dualizable) algebra whose rank is 2. Beyond that, we know nothing.

Problem 5.1. Is there a finite algebra whose rank is ∞? (If not, then every dualizable algebra is strongly dualizable.)

Problem 5.2. How big can $\text{rank}(\underline{\mathbf{M}})$ be if $\text{rank}(\underline{\mathbf{M}}) < \infty$? (Certainly $\text{rank}(\underline{\mathbf{M}})$ must be a countable ordinal or ∞. We know of no other restrictions, nor do we know of examples with rank > 2.)

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