AN ALGEBRA THAT IS DUALIZABLE BUT NOT FULLY DUALIZABLE

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Abstract. We give an example of a finite algebra which is dualizable but not fully dualizable in the sense of natural duality theory.

1. Introduction

Fix a finite algebra $\mathbf{M}$, in the sense of universal algebra, and let $\mathbb{M}$ be an infinitary topological structure whose universe is identical to that of $\mathbf{M}$, whose topology is discrete, and whose signature consists only of λ-ary relations (λ a nonzero ordinal) which are universes of subalgebras of $\mathbf{M}^\lambda$, and λ-ary operations or partial operations (λ an ordinal) whose graphs are universes of subalgebras of $\mathbf{M}^{\lambda+1}$. (Such a structure is called an *alter ego of* $\mathbf{M}$ in [1].) The quasivariety $\mathcal{A}$ generated by $\mathbf{M}$ and the topological quasivariety $\mathcal{X}$ generated by $\mathbb{M}$ are dually adjoint via the functors $\mathbb{D} = \mathrm{Hom}(-, \mathbf{M})$ and $\mathbf{E} = \mathrm{Hom}(-, \mathbb{M})$ and the "evaluation map" natural transformations $e : 1_\mathcal{A} \to \mathbf{E}\mathbb{D}$ and $\varepsilon : 1_\mathcal{X} \to \mathbb{D}\mathbf{E}$. The transformation e is defined as follows: given $\mathbf{A} \in \mathcal{A}$ and $a \in A$, define $e^\mathbf{A}(a) : \mathbb{D}(\mathbf{A}) \to A$ by $e^\mathbf{A}(a)(h) = h(a)$; then $e^\mathbf{A} : \mathbf{A} \hookrightarrow \mathbf{E}(\mathbb{D}(\mathbf{A}))$ is the map $a \mapsto e^\mathbf{A}(a)$. The definition of ε is similar.

$\mathbb{M}$ *yields a duality on* $\mathbf{M}$ if $e^\mathbf{A}$ is an isomorphism for every $\mathbf{A} \in \mathcal{A}$, and *yields a full duality on* $\mathbf{M}$ if in addition $\varepsilon^\mathbb{X}$ is an isomorphism for every $\mathbb{X} \in \mathcal{X}$. Thus if $\mathbb{M}$ yields a full duality on $\mathbf{M}$, then the above dual adjunction provides a "natural" dual equivalence between $\mathcal{A}$ and a finitely generated topological quasivariety.

If the signature of $\mathbb{M}$ consists of the proper class of all permissible relations, operations and partial operations, then $\mathbb{M}$ automatically yields a full duality on $\mathbf{M}$, but we consider this to be cheating. Following [3], we say that $\mathbf{M}$ is *dualizable* if there exists an alter ego which yields a duality on $\mathbf{M}$ and whose signature consists of *finitary* relations, operations and partial operations only. $\mathbf{M}$ is *fully dualizable* if there exists such an alter ego which yields a full duality on $\mathbf{M}$. Not all finite algebras are fully dualizable or even dualizable in this sense; however, in 1991 B. A. Davey noted that every algebra known to be dualizable had been shown to be fully dualizable, and asked [3, Problem 4] whether the two notions are equivalent. We give a

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strong negative answer to this question by displaying a 3-element algebra which is dualizable but is not fully dualized by any alter ego having only a *set* of (possibly infinitary) relations, operations and partial operations in its signature. This also solves the "Strong Upgrade Problem" in [1].

2. $\mathbf{M}$ is dualizable

Our algebra is $\mathbf{M} = \langle M, f, g\rangle$ where $M = \{0, 1, 2\}$ and f, g are the unary operations defined by the following table:

x	$f(x)$	$g(x)$
0	1	0
1	2	0
2	2	1

Consider $\langle M, \leq\rangle$ as an ordered chain with $0 < 1 < 2$. Let $\wedge$ and $\vee$ denote the lattice meet and join operations in this chain. Define $E \subseteq M^2$ and $R \subseteq M^4$ as follows:

$$\begin{aligned} E &= \{(x, y) : x \leq y \text{ and } (x, y) \neq (0, 2)\} \\ R &= \{(x, y, z, w) : x \leq y \leq z \leq w \text{ and } x = y \text{ or } z = w\}. \end{aligned}$$

Define $\mathbb{M} = \langle M, \wedge, \vee, E, R, \text{discrete topology}\rangle$. It can be easily checked that E is a subuniverse of $\mathbf{M}^2$, the graphs of $\wedge$ and $\vee$ are subuniverses of $\mathbf{M}^3$, and R is a subuniverse of $\mathbf{M}^4$; in other words, that $\mathbb{M}$ is a finitary alter ego for $\mathbf{M}$.

Theorem 2.1. $\mathbb{M}$ *dualizes* $\mathbf{M}$.

Proof. We will show that $\mathbb{M}$ satisfies the so-called *Interpolation Condition* relative to $\mathbf{M}$. The result then follows either by the Second Duality Theorem [1, Theorem 2.2.7], or by [1, Lemma 2.2.5] and the Duality Compactness Theorem of L. Zádori (see [6, Corollary 3.5] or [1, Theorem 2.2.11]).

To say that $\mathbb{M}$ satisfies the Interpolation Condition relative to $\mathbf{M}$ is just to say that if $1 \leq n < \omega$ and $\mathbb{X} \leq \mathbb{M}^n$ and $h \in \mathrm{Hom}(\mathbb{X}, \mathbb{M})$, then h is the restriction to X of an n-ary term operation of $\mathbf{M}$. The proof is by cases.

Case 1. $|\mathrm{range}(h)| = 1$.

That is, h is constant. Then h is the restriction to X of either $gg(x_1)$, $fgg(x_1)$ or $ff(x_1)$.

Case 2. $|\mathrm{range}(h)| = 2$.

Then $\langle X, \wedge, \vee\rangle$ is a finite distributive lattice and h is a homomorphism from this lattice onto a two-element lattice. Write $\mathrm{range}(h) = \{0', 1'\}$ with $0' < 1'$. Thus there exist $a, b \in X$ such that $h^{-1}(0') = \{x \in X : x \leq a\}$ and $h^{-1}(1') = \{x \in X : x \geq b\}$. Let $a^* = a \vee b$.

Suppose first that $\mathrm{range}(h) = \{0, 2\}$. Since $\langle h(a), h(a^*)\rangle \notin E$ and h preserves E, there must exist $i \in \{1, \ldots, n\}$ such that $(a_i, a_i^*) \notin E$. The only way this can happen is if $(a_i, a_i^*) = (0,2)$. Since $a_i^* = a_i \vee b_i$ and $\langle M, \wedge, \vee\rangle$ is a chain, we get $b_i = 2$. It follows that for all $x \in X$, $h(x) = 0$ implies $x_i = 0$ while $h(x) = 2$ implies $x_i = 2$. Thus h is the restriction to X of the coordinate projection x_i.

Suppose on the other hand that $1 \in \mathrm{range}(h)$. Choose any $i \in \{1, \dots, n\}$ such that $a_i < a_i^*$. Then as in the previous paragraph we get $a_i < b_i$. Clearly if $x, y \in X$ and $x_i = y_i$, then it is impossible to have $x \leq a$ while $y \geq b$, or vice versa. Thus $x_i = y_i$ implies $h(x) = h(y)$. This means that if $X_i = \{x_i : x \in X\}$ then we can define $h_i : X_i \to \mathrm{range}(h)$ so that $h(x) = h_i(x_i)$ for all $x \in X$. Our goal is now to prove that h_i is the restriction to X_i of a unary term operation t of $\mathbf{M}$, for then h will be $t(x_i){\restriction}_X$. Since $1 \in \mathrm{range}(h_i)$, it suffices to prove that h_i is order-preserving. If $x, y \in X$ with $x_i \leq y_i$, then

$$h_i(x_i) = h_i((x \wedge y)_i) = h(x \wedge y) = h(x) \wedge h(y) = h_i(x_i) \wedge h_i(y_i),$$

so $h_i(x_i) \leq h_i(y_i)$ as required.

Case 3. $|\mathrm{range}(h)| = 3$.

That is, h is surjective. Then h is a lattice homomorphism from $\langle X, \wedge, \vee\rangle$ onto a three-element chain. Thus there exist $a, b, c, d \in X$ such that

$$\begin{aligned} h^{-1}(0) &= \{x \in X : x \leq a\} \\ h^{-1}(1) &= \{x \in X : b \leq x \leq c\} \\ h^{-1}(2) &= \{x \in X : d \leq x\}. \end{aligned}$$

Let $a^* = a \vee b$ and $c^* = c \vee d$, and note that $a^* \leq c$ (as $h(a^*) = h(a) \vee h(b) = 1$).

Since $\langle h(a), h(a^*), h(c), h(c^*)\rangle \notin R$, and since h preserves R, there must exist i such that $(a_i, a_i^*, c_i, c_i^*) \notin R$. This forces $(a_i, a_i^*, c_i, c_i^*) = (0,1,1,2)$, hence $b_i = 1$ and $d_i = 2$. It follows that h is the restriction to X of the coordinate projection x_i. □

3. Reduction to a simpler category

By a *looped dag* we mean a directed graph $\mathbf{G} = \langle G, \to\rangle$ in which the vertex set G is possibly empty, the edge relation $\to \subseteq G \times G$ possibly has loops $a \to a$, and there do not exist *distinct* vertices $a_0, a_1, \dots, a_n$, $n \geq 1$, such that $a_0 \to a_1 \to \cdots \to a_n \to a_0$. $\mathcal{D}$ denotes the class of all looped dags.

If $\mathbf{G} \in \mathcal{D}$, then we canonically define a larger looped dag $\langle G^+, \stackrel{+}{\to}\rangle$ as follows: let G^+ be the disjoint union of G and the set $\{0,1\}$; then for $x, y \in G^+$ define $x \stackrel{+}{\to} y$ iff $x = 0$ or $y = 1$ or $x \to y$. Also define $\mathbf{G}^+$ to be the structure $\langle G^+, \stackrel{+}{\to}, 0, 1\rangle$ and $\mathcal{D}^+ = \{\mathbf{G}^+ : \mathbf{G} \in \mathcal{D}\}$. We consider $\mathcal{D}^+$ as a category with the usual homomorphisms. It turns out that $\mathbf{ISP}(\mathbf{M})$ is categorically equivalent to $\mathcal{D}^+$; we shall prove most of this statement. We use $\mathcal{D}$ to efficiently name the members of $\mathcal{D}^+$.

If $\mathbf{G} \in \mathcal{D}$, we wish to define an algebra $\mathbf{B}(\mathbf{G}^+)$ in the language of $\mathbf{M}$. Its universe $B(\mathbf{G}^+)$ is the edge set $\xrightarrow{+}$ of $\mathbf{G}^+$. The operations f, g are defined on $B(\mathbf{G}^+)$ as follows:

$$\begin{aligned} f(x,y) &= (y,1) \\ g(x,y) &= (0,x). \end{aligned}$$

Note in particular that if $\varnothing$ is the empty looped dag, then $\varnothing^+ = \mathbf{2}_{01}$, the 2-element bounded poset, and $\mathbf{B}(\varnothing^+) \cong \mathbf{M}$ via the isomorphism ι sending $(0,0) \mapsto 0$, $(0,1) \mapsto 1$, and $(1,1) \mapsto 2$.

Given $\mathbf{G}, \mathbf{H} \in \mathcal{D}$ and $h \in \operatorname{Hom}(\mathbf{G}^+, \mathbf{H}^+)$, define the map $\beta_{\mathbf{G}^+}(h) : B(\mathbf{G}^+) \to B(\mathbf{H}^+)$ by applying h coordinatewise: $\beta_{\mathbf{G}^+}(h)(x,y) = (h(x), h(y))$.

Lemma 3.1.

(1) *The maps $\mathbf{G}^+ \mapsto \mathbf{B}(\mathbf{G}^+)$, $h \mapsto \beta_{\mathbf{G}^+}(h)$, define a functor from $\mathcal{D}^+$ to $\mathbf{ISP}(\mathbf{M})$.*
(2) *For all $\mathbf{G}, \mathbf{H} \in \mathcal{D}$, $\operatorname{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{H}^+)) = \{\beta_{\mathbf{G}^+}(h) : h \in \operatorname{Hom}(\mathbf{G}^+, \mathbf{H}^+)\}$.*
(3) *For any $\mathbf{G} \in \mathcal{D}$, the map $h \mapsto \iota\beta_{\mathbf{G}^+}(h)$ is an order isomorphism from $\langle \operatorname{Hom}(\mathbf{G}^+, \mathbf{2}_{01}), \leq \rangle$ to $\langle \operatorname{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{M}), \leq \rangle$. (Here $\leq$ denotes the order relations evaluated pointwise in 2 and M respectively.)*

Proof. The first item will be clear once it is seen that $\mathbf{B}(\mathbf{G}^+) \in \mathbf{ISP}(\mathbf{M})$ when $\mathbf{G} \in \mathcal{D}$, and $\beta_{\mathbf{G}^+}(h) \in \operatorname{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{H}^+))$ when $h \in \operatorname{Hom}(\mathbf{G}^+, \mathbf{H}^+)$. Let $\prec$ be the reflexive, transitive closure of $\xrightarrow{+}$ on $\mathbf{G}^+$. $\prec$ is a partial order so there is some set I and an embedding of $\langle G^+, \prec, 0, 1 \rangle$ into the bounded poset $(\mathbf{2}_{01})^I$. Using this embedding to relabel the vertices of $\mathbf{G}^+$, we may assume that $\langle G^+, \prec, 0, 1 \rangle \leq (\mathbf{2}_{01})^I$. Now define an embedding $\tau : \mathbf{B}(\mathbf{G}^+) \hookrightarrow \mathbf{M}^I$ as follows. For $(x,y) \in B(\mathbf{G}^+)$, we have $x \leq y$ in $(\mathbf{2}_{01})^I$ and we define $\tau(x,y)$ by

$$\tau(x,y)_i = \begin{cases} 0 & \text{if} \quad x_i = y_i = 0 \\ 1 & \text{if} \quad x_i = 0, y_i = 1 \\ 2 & \text{if} \quad x_i = y_i = 1. \end{cases}$$

The following shows that τ preserves f:

$$\begin{aligned} f(\tau(x,y))_i &= \begin{cases} 1 & \text{if} \quad y_i = 0 \\ 2 & \text{if} \quad y_i = 1 \end{cases} \\ &= \tau(y,1)_i = \tau(f(x,y))_i. \end{aligned}$$

Similarly, τ preserves g. Clearly τ is injective. Thus $\mathbf{B}(\mathbf{G}^+) \in \mathbf{ISP}(\mathbf{M})$.

Next, assume $h \in \operatorname{Hom}(\mathbf{G}^+, \mathbf{H}^+)$. If $(x,y) \in B(\mathbf{G}^+)$ then (x,y) is an edge of $\mathbf{G}^+$, hence $(h(x), h(y))$ is an edge of $\mathbf{H}^+$ and is an element of $B(\mathbf{H}^+)$. Let $(x,y) \in B(\mathbf{G}^+)$. Since $h(0) = 0$ and $h(1) = 1$ we have

$$\begin{aligned} \beta_{\mathbf{G}^+}(h)(g(x,y)) &= \beta_{\mathbf{G}^+}(h)(0,x) = (h(0), h(x)) = (0, h(x)) \\ &= g(h(x), h(y)) = g(\beta_{\mathbf{G}^+}(h)(x,y)). \end{aligned}$$

Similarly, $\beta_{\mathbf{G}^+}(h)(f(x,y)) = f(\beta_{\mathbf{G}^+}(h)(x,y))$. Thus $\beta_{\mathbf{G}^+}(h) \in \operatorname{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{H}^+))$.

To prove the second item, let $\alpha \in \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{H}^+))$. Note that $\alpha(0,0) = (0,0)$ and $\alpha(0,1) = (0,1)$, since in both $\mathbf{B}(\mathbf{G}^+)$ and $\mathbf{B}(\mathbf{H}^+)$, $(0,0)$ is the unique element in the range of gg and $(0,1) = f(0,0)$. Furthermore, in either $\mathbf{B}(\mathbf{G}^+)$ or $\mathbf{B}(\mathbf{H}^+)$ we have $g(x,y) = (0,0)$ iff $x = 0$. Thus there exists a function $h : G^+ \to H^+$ such that $\alpha(0,y) = (0, h(y))$ for all $y \in H^+$. Note in particular that $h(0) = 0$ and $h(1) = 1$.

Assume $x \xrightarrow{+} y$ in $\mathbf{G}^+$; then $(x,y) \in B(\mathbf{G}^+)$ with, say, $\alpha(x,y) = s \in B(\mathbf{H}^+)$. Then $g(s) = \alpha(0,x) = (0, h(x))$ and $gf(s) = g(\alpha(y,1)) = \alpha(0,y) = (0,h(y))$. By checking the definition of $\mathbf{B}(\mathbf{H}^+)$ we see that this forces $s = (h(x), h(y))$, hence $(h(x), h(y)) \in B(\mathbf{H}^+)$ and therefore $h(x) \xrightarrow{+} h(y)$ in $\mathbf{H}^+$. This proves $h \in \mathrm{Hom}(\mathbf{G}^+, \mathbf{H}^+)$.

We claim that whenever $\alpha, \alpha' \in \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{H}^+))$ and for all $y \in G^+$, $\alpha(0,y) = \alpha'(0,y)$, then $\alpha = \alpha'$. Indeed, let $(x,y) \in B(\mathbf{G}^+)$.

$$\begin{aligned} f(\alpha(x,y)) &= \alpha(y,1) = \alpha(f(0,y)) = f(\alpha(0,y)) = f(\alpha'(0,y)) = \alpha'(f(0,y)) \\ &= \alpha'(y,1) = \alpha'(f(x,y)) = f(\alpha'(x,y)). \end{aligned}$$

That $g(\alpha(x,y)) = g(\alpha'(x,y))$ is a similar but simpler calculation. In $\mathbf{B}(\mathbf{H}^+)$, $f(u) = f(v)$ and $g(u) = g(v)$ imply $u = v$ so we have $\alpha = \alpha'$, proving the claim. Since $\beta_{\mathbf{G}^+}(h)(0,y) = (h(0), h(y)) = (0, h(y)) = \alpha(0,y)$, the claim yields $\beta_{\mathbf{G}^+}(h) = \alpha$.

To prove the third item, let $\rho\colon \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01}) \to \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{B}(\mathbf{2}_{01}))$ be given by $h \mapsto \beta_{\mathbf{G}^+}(h)$. ρ is surjective by item 2. It will suffice to show that ρ is an order embedding where the order in $\mathbf{B}(\mathbf{2}_{01})$ is the one inherited from $\mathbf{M}$ via ι. Assume $\beta_{\mathbf{G}^+}(h_1) = \beta_{\mathbf{G}^+}(h_2)$. For all $y \in G^+$, $(0,y) \in B(\mathbf{G}^+)$ so $\beta_{\mathbf{G}^+}(h_1)(0,y) = \beta_{\mathbf{G}^+}(h_2)(0,y)$ or $h_1(y) = h_2(y)$. Hence ρ is injective.

If $h_1, h_2 \in \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01})$ with $h_1 \le h_2$ then for all $(x,y) \in B(\mathbf{G}^+)$, $(h_1(x), h_1(y)) \le (h_2(x), h_2(y))$ co-ordinatewise and hence in $\mathbf{B}(\mathbf{2}_{01})$. That is, $\beta_{\mathbf{G}^+}(h_1) \le \beta_{\mathbf{G}^+}(h_2)$. Conversely, assume $h_1 \not\le h_2$; choose $y \in G^+$ such that $h_1(y) = 1$ while $h_2(y) = 0$. Then $\beta_{\mathbf{G}^+}(h_1)(0,y) \not\le \beta_{\mathbf{G}^+}(h_2)(0,y)$, proving $\beta_{\mathbf{G}^+}(h_1) \not\le \beta_{\mathbf{G}^+}(h_2)$. $\square$

Given $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$, we shall define a directed graph $\mathbf{G}'_{\mathbf{A}}$ as follows. The vertex set is $g(A)$, the range of g in $\mathbf{A}$. The edge set consists of a directed edge $e_a = (g(a), gf(a))$ for each $a \in A$. Note that if 0 denotes the unique element in the range of gg in $\mathbf{A}$ and 1 denotes $f(0)$, then $0, 1 \in g(A)$. Define $\mathbf{G}^+_{\mathbf{A}} = \langle \mathbf{G}'_{\mathbf{A}}, 0, 1\rangle$. Note that $\mathbf{G}^+_{\mathbf{M}} = \mathbf{2}_{01}$ and, more generally, $\mathbf{G}^+_{\mathbf{M}^X} = (\mathbf{2}_{01})^X$ for any set X.

Lemma 3.2.

(1) *The maps* $\mathbf{A} \mapsto \mathbf{G}^+_{\mathbf{A}}$ *(for* $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$*) and* $h \mapsto h|_{g(A)}$ *(for* $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$*) define a functor from* $\mathbf{ISP}(\mathbf{M})$ *to* $\mathcal{D}^+$.

(2) *For each* $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$*, the map* $\eta_{\mathbf{A}} : A \to B(\mathbf{G}^+_{\mathbf{A}})$ *defined by* $\eta_{\mathbf{A}}(a) = e_a$ *is an isomorphism from* $\mathbf{A}$ *to* $\mathbf{B}(\mathbf{G}^+_{\mathbf{A}})$.

(3) $\mathbf{ISP}(\mathbf{M}) = \mathbf{I}(\{\mathbf{B}(\mathbf{G}^+) : \mathbf{G} \in \mathcal{D}\})$.

(4) *If* $\mathbf{G} \in \mathcal{D}$, $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$, $h \in \mathrm{Hom}(\mathbf{G}^+, \mathbf{G}^+_{\mathbf{A}})$*, and* $y \in G^+$*, then* $\beta_{\mathbf{G}^+}(h)(0,y) = \eta_{\mathbf{A}}(h(y))$.

Proof. First note that if $\mathbf{A}, \mathbf{B} \in \mathbf{ISP}(\mathbf{M})$ and $h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$, then $h|_{g(A)}$ is easily shown to be in $\mathrm{Hom}(\mathbf{G}^+_{\mathbf{A}}, \mathbf{G}^+_{\mathbf{B}})$. In particular, an embedding of $\mathbf{A}$ into $\mathbf{M}^X$ induces an injective homomorphism from $\mathbf{G}^+_{\mathbf{A}}$ to $(\mathbf{2}_{01})^X$. It follows that $\mathbf{G}'_{\mathbf{A}}$ is a looped dag. Since $gg(x) = 0$ and $gfg = g$ in $\mathbf{A}$, it follows that if $a \in g(A)$ then $e_a = (0, a)$ while $e_{f(a)} = (a, 1)$. Thus in $\mathbf{G}'_{\mathbf{A}}$ there is an edge from 0 to every vertex, and from every vertex to 1. Hence $\mathbf{G}^+_{\mathbf{A}} \in \mathcal{D}^+$. This proves the nontrivial parts of item 1.

The quasi-identity $[g(x) = g(y) \;\&\; gf(x) = gf(y)] \Rightarrow x = y$ is true in $\mathbf{M}$ and hence holds throughout $\mathbf{ISP}(\mathbf{M})$. Thus the map $\eta_{\mathbf{A}} : A \to B(\mathbf{G}^+_{\mathbf{A}})$ is injective and hence is a bijection. To prove it is an isomorphism, observe that

$$\begin{aligned} e_{g(a)} &= (gg(a), gfg(a)) = (0, g(a)) = g(e_a) \\ e_{f(a)} &= (gf(a), gff(a)) = (gf(a), 1) = f(e_a). \end{aligned}$$

This proves item 2. The third item follows from item 2 and Lemma 3.1. The fourth item follows from comments in the first paragraph and the fact that $h(y) \in g(A)$. $\square$

An (infinitary) *algebraic operation* of $\mathbf{M}$ is any function α such that for some nonzero ordinal λ and some $\mathbf{D} \leq \mathbf{M}^\lambda$ we have $\alpha \in \mathrm{Hom}(\mathbf{D}, \mathbf{M})$. The ordinal λ is the *arity* of α. Fix an algebraic operation α of $\mathbf{M}$ (say $\alpha \in \mathrm{Hom}(\mathbf{D}, \mathbf{M})$ with $\mathbf{D} \leq \mathbf{M}^\lambda$), and for each $i < \lambda$ let $\rho_i : \mathbf{D} \to \mathbf{M}$ denote the ith projection. If $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$ and $X \subseteq \mathrm{Hom}(\mathbf{A}, \mathbf{M})$, then X is said to be *closed under* α if for all $\delta \in \mathrm{Hom}(\mathbf{A}, \mathbf{D})$ the following implication holds: if $\rho_i \circ \delta \in X$ for every $i < \lambda$, then $\alpha \circ \delta \in X$.

We wish to define the appropriate analogous notions for $\mathbf{2}_{01}$.

Definition 3.3. An *algebraic operation* of $\mathbf{2}_{01}$ is a pair $(\mathbf{H}^+, h)$ where $\mathbf{H}^+ \in \mathcal{D}^+$, $H^+ \subseteq 2^\lambda$ for some nonzero ordinal λ, and the inclusion map $H^+ \hookrightarrow 2^\lambda$ is a homomorphism from $\mathbf{H}^+$ to $(\mathbf{2}_{01})^\lambda$, and where $h \in \mathrm{Hom}(\mathbf{H}^+, \mathbf{2}_{01})$. The ordinal λ is the *arity* of $(\mathbf{H}^+, h)$.

Note that in general, the edge set of $\mathbf{H}^+$ cannot be recovered from the graph of h, hence the explicit inclusion of $\mathbf{H}^+$ in the definition.

Definition 3.4. Suppose $\mathbf{G} \in \mathcal{D}$, $Y \subseteq \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01})$, and $(\mathbf{H}^+, h)$ is a λ-ary algebraic operation of $\mathbf{2}_{01}$. For each $i < \lambda$ let $r_i : \mathbf{H}^+ \to \mathbf{2}_{01}$ denote the ith projection. We say that Y is *closed under* $(\mathbf{H}^+, h)$ *relative to* $\mathbf{G}^+$ if for all $d \in \mathrm{Hom}(\mathbf{G}^+, \mathbf{H}^+)$ the following implication holds: if $r_i \circ d \in Y$ for all $i < \lambda$, then $h \circ d \in Y$.

Lemma 3.5. *For every algebraic operation α of $\mathbf{M}$ there is an algebraic operation $(\mathbf{H}^+, h)$ of $\mathbf{2}_{01}$ of the same arity as α making the following true: for any $\mathbf{G} \in \mathcal{D}$ and any $Y \subseteq \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01})$, if Y is closed under $(\mathbf{H}^+, h)$ relative to $\mathbf{G}^+$, then the set $X := \{\iota\beta_{\mathbf{G}^+}(y) : y \in Y\} \subseteq \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{M})$ is closed under α.*

Proof. Let $\mathbf{D} \leq \mathbf{M}^\lambda$ and $\alpha \in \mathrm{Hom}(\mathbf{D}, \mathbf{M})$ be given, and put $\mathbf{H}^+ = \mathbf{G}^+_{\mathbf{D}}$. Then the inclusion map $\mathbf{H}^+ \hookrightarrow (\mathbf{2}_{01})^\lambda$ is a homomorphism. Let $\alpha^* = \iota^{-1}\alpha\eta^{-1}_{\mathbf{D}} \in \mathrm{Hom}(\mathbf{B}(\mathbf{H}^+), \mathbf{B}(\mathbf{2}_{01}))$. By Lemma 3.1(2) there exists $h \in \mathrm{Hom}(\mathbf{H}^+, \mathbf{2}_{01})$ such that $\alpha^* = \beta_{\mathbf{H}^+}(h)$. Thus

$(\mathbf{H}^+, h)$ is an algebraic operation of $\mathbf{2}_{01}$ of the same arity as α; we shall show that it witnesses the claim of Lemma 3.5.

Let $\mathbf{G} \in \mathcal{D}$ and $Y \subseteq \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01})$, and assume that Y is closed under $(\mathbf{H}^+, h)$ relative to $\mathbf{G}^+$. Define $X = \{\iota\beta_{\mathbf{G}^+}(y) : y \in Y\}$, and let $\delta \in \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{D})$ be such that $\rho_i \circ \delta \in X$ for all $i < \lambda$; we must show $\alpha \circ \delta \in X$.

By Lemma 3.1(2) there exists $d \in \mathrm{Hom}(\mathbf{G}^+, \mathbf{H}^+)$ such that $\eta_{\mathbf{D}}\delta = \beta_{\mathbf{G}^+}(d)$. For each $i < \lambda$ choose $y_i \in Y$ such that $\rho_i \circ \delta = \iota\beta_{\mathbf{G}^+}(y_i)$. We first verify that $r_i \circ d = y_i$ for all $i < \lambda$.

Fix $a \in G^+$ and define $s = \delta(0, a) \in D$ and $t = \delta(a, 1) \in D$. As $g(t) = s$ in $\mathbf{D}$, i.e., $s \in g(D)$, we have that s is simultaneously an element of $\mathbf{D}$ and of $\mathbf{H}^+$, and hence $\rho_i(s) = r_i(s)$ for all $i < \lambda$. Moreover, it follows from remarks in the proof of Lemma 3.2(1) that $\eta_{\mathbf{D}}(s) = (0, s)$, as $s \in g(D)$. On the other hand, recall from the proof of Lemma 3.1(2) that $\eta_{\mathbf{D}}(s) = \beta_{\mathbf{G}^+}(d)(0, a) = (0, d(a))$; hence $s = d(a)$. Finally, note that since $\mathbf{2}_{01} = \mathbf{G}^+_{\mathbf{M}}$ and using Lemma 3.2(4), for each $i < \lambda$ we have $\beta_{\mathbf{G}^+}(y_i)(0, a) = \eta_{\mathbf{M}}(y_i(a))$; hence $\iota\beta_{\mathbf{G}^+}(y_i)(0, a) = y_i(a)$ (note that $\iota = \eta_{\mathbf{M}}^{-1}$). Thus $(r_i \circ d)(a) = r_i(s) = \rho_i(s) = (\rho_i \circ \delta)(0, a) = \iota\beta_{\mathbf{G}^+}(y_i)(0, a) = y_i(a)$, proving $r_i \circ d = y_i$ as desired.

In particular, $r_i \circ d \in Y$ for all $i < \lambda$. As Y is closed under $(\mathbf{H}^+, h)$ relative to $\mathbf{G}^+$, the map $h \circ d$ is in Y and hence $\iota\beta_{\mathbf{G}^+}(h \circ d) \in X$. But $\iota\beta_{\mathbf{G}^+}(h \circ d) = \iota\beta_{\mathbf{H}^+}(h)\beta_{\mathbf{G}^+}(d) = \iota\alpha^*\eta_{\mathbf{D}}\delta = \alpha\delta$, which proves $\alpha \circ \delta \in X$ as required. □

4. $\mathbf{M}$ is not fully dualizable

Lemma 4.1. *For every infinite cardinal κ there exists a structure $\mathbb{G}_\kappa = \langle G_\kappa, \rightarrow, \twoheadrightarrow\rangle$ satisfying:*

1. *$\twoheadrightarrow$ is a linear ordering of G_κ; $\rightarrow$ is a partial ordering of G_κ.*
2. *$\rightarrow$ is a proper subset of $\twoheadrightarrow$.*
3. *For all $x, y \in G_\kappa$ with $x \twoheadrightarrow y$ but $x \not\twoheadrightarrow y$, there exists $\{a_i : i < \kappa\} \cup \{b_i : i < \kappa\} \subseteq G_\kappa$ such that for all $i < \kappa$, $x \rightarrow a_i \twoheadrightarrow b_i \rightarrow y$ and $b_i \twoheadrightarrow a_{i+1}$.*

Proof. It suffices to note that if $\mathbb{G} = \langle G, \rightarrow, \twoheadrightarrow\rangle$ is a model of items 1 and 2 and has cardinality κ, and if $(x, y) \in G^2$ is a single failure of item 3, then $\mathbb{G}$ can be embedded in a larger model $\mathbb{H}$ of items 1 and 2, still of cardinality κ and in which the designated instance of item 3 is now true. Then a chain of models of 1 and 2 can be arranged so that its union is a model of all three items. □

For each infinite cardinal κ fix a structure $\mathbb{G}_\kappa = \langle G_\kappa, \rightarrow, \twoheadrightarrow\rangle$ as in the previous lemma. Put $\mathbf{G}_\kappa = \langle G_\kappa, \rightarrow\rangle$ and $\mathbf{L}_\kappa = \langle G_\kappa, \twoheadrightarrow\rangle$. Define $Y_\kappa = \mathrm{Hom}(\mathbf{L}^+_\kappa, \mathbf{2}_{01})$ and note that $Y_\kappa \subsetneq \mathrm{Hom}(\mathbf{G}^+_\kappa, \mathbf{2}_{01})$.

Lemma 4.2. *Y_κ is closed relative to $\mathbf{G}^+_\kappa$ under all algebraic operations of $\mathbf{2}_{01}$ of arity less than κ.*

Proof. Let $(\mathbf{H}^+, h)$ be an algebraic operation of $\mathbf{2}_{01}$ where $H^+ \subseteq 2^\lambda$ with $\lambda < \kappa$. Suppose $d \in \mathrm{Hom}(\mathbf{G}_\kappa^+, \mathbf{H}^+)$ is such that $r_i \circ d \in Y_\kappa$ for all $i < \lambda$. Let $\nu : H^+ \hookrightarrow 2^\lambda$ denote the inclusion map. Then $\nu \circ d \in \mathrm{Hom}(\mathbf{L}_\kappa^+, (\mathbf{2}_{01})^\lambda)$. It must be shown that $h' := h \circ d$ is in Y_κ.

Assume $h' \notin Y_\kappa$; then there exist $x, y \in G_\kappa^+$ such that $x \overset{+}{\twoheadrightarrow} y$, $h'(x) = 1$, and $h'(y) = 0$. Because $h' \in \mathrm{Hom}(\mathbf{G}_\kappa^+, \mathbf{2}_{01})$ we get $x, y \in G_\kappa$, $x \twoheadrightarrow y$ and $x \not\twoheadrightarrow y$. Choose $\{a_i : i < \kappa\} \cup \{b_i : i < \kappa\}$ for x, y as in the statement of Lemma 4.1(3).

$\mathbf{L}_\kappa^+$ is a chain, and $\nu \circ d \in \mathrm{Hom}(\mathbf{L}_\kappa^+, (\mathbf{2}_{01})^\lambda)$; hence the image of $\nu \circ d$ in $(\mathbf{2}_{01})^\lambda$ is also a chain. It follows that $\ker(\nu \circ d) = \ker(d)$ partitions $\mathbf{L}_\kappa^+$ into fewer than κ many intervals. In particular, there must exist $i < \kappa$ such that $d(a_i) = d(b_i)$. But $x \overset{+}{\to} a_i$ and $b_i \overset{+}{\to} y$ then imply $h'(x) \le h'(a_i) = h'(b_i) \le h'(y)$, a contradiction. □

Fix an infinite cardinal κ and let $\mathbb{M}$ be an alter ego for $\mathbf{M}$ whose signature consists of relations, operations and partial operations of arities less than κ. Let $\mathbf{G}_\kappa$, $\mathbf{L}_\kappa$ and Y_κ be as above, and define $X_\kappa = \{\iota\beta_{\mathbf{G}_\kappa^+}(y) : y \in Y_\kappa\} \subseteq M^{B(\mathbf{G}_\kappa^+)}$. Note that $\mathbf{B}(\mathbf{G}_\kappa^+) \le \mathbf{B}(\mathbf{L}_\kappa^+)$ and $X_\kappa = \{\alpha\restriction_{B(\mathbf{G}_\kappa^+)} : \alpha \in \mathrm{Hom}(\mathbf{B}(\mathbf{L}_\kappa^+), \mathbf{M})\}$. These facts automatically imply that X_κ is topologically closed in $M^{B(\mathbf{G}_\kappa^+)}$. By Lemmas 3.5 and 4.2, X_κ is closed under all algebraic operations of $\mathbf{M}$ of arities less than κ. Hence X_κ is a subuniverse of $\mathbb{M}^{B(\mathbf{G}_\kappa^+)}$ and the corresponding substructure $\mathbb{X}_\kappa$ belongs to the topological quasivariety generated by $\mathbb{M}$.

Clearly $X_\kappa \subseteq \mathrm{Hom}(\mathbf{B}(\mathbf{G}_\kappa^+), \mathbf{M})$, and the elements of X_κ separate the points of $\mathbf{B}(\mathbf{G}_\kappa^+)$; that is, for any $a, b \in B(\mathbf{G}_\kappa^+)$ with $a \ne b$ there exists $\alpha \in X_\kappa$ such that $\alpha(a) \ne \alpha(b)$. It should also be clear that $X_\kappa \ne \mathrm{Hom}(\mathbf{B}(\mathbf{G}_\kappa^+), \mathbf{M})$, since $Y_\kappa \ne \mathrm{Hom}(\mathbf{G}_\kappa^+, \mathbf{2}_{01})$. When $\kappa = \omega$ these facts plus Lemma 3.8 from [2] imply that $\mathbf{M}$ is not strongly dualizable. To prove that $\mathbf{M}$ is not fully dualizable, indeed is not fully dualized by $\mathbb{M}$, further argument is needed.

Definition 4.3. Suppose $I \ne \varnothing$ and $\mathbb{X}$ is a substructure of $\langle M, \le, E, R, \ldots\rangle^I$, where $\le, E, R$ are as defined in section 2 and $\langle M, \le, E, R, \ldots\rangle$ is any alter ego for $\mathbf{M}$ whose signature includes $\le, E, R$. The *bi-graph associated with* $\mathbb{X}$, denoted $\mathrm{bg}(\mathbb{X})$, is defined as follows. Let

$$G_{\mathbb{X}} = \{\alpha \in X : \mathbb{X} \models \exists y \forall z([z \le \alpha \ \&\ \neg(z = \alpha)] \leftrightarrow z \le y)\}.$$

For $\alpha \in G_{\mathbb{X}}$ define α_* to be the unique lower cover of α in $\langle X, \le\rangle$. Define relations $\to_{\mathbb{X}}$ and $\vdash_{\mathbb{X}}$ on $G_{\mathbb{X}}$ as follows: for $\alpha, \beta \in G_{\mathbb{X}}$

$$\alpha \to_{\mathbb{X}} \beta \quad \text{iff} \quad \beta \le \alpha \text{ and } \begin{cases} \beta = \alpha \text{ and } \mathbb{X} \models \neg E(\alpha_*, \alpha), \text{ or} \\ \beta \ne \alpha \text{ and } \mathbb{X} \models \neg R(\beta_*, \beta, \alpha_*, \alpha), \end{cases}$$

$$\alpha \vdash_{\mathbb{X}} \beta \quad \text{iff} \quad \beta \le \alpha.$$

Then $\mathrm{bg}(\mathbb{X}) = \langle G_{\mathbb{X}}, \to_{\mathbb{X}}, \vdash_{\mathbb{X}}\rangle$.

Lemma 4.4. *Suppose* $\mathbf{G} = \langle G, \to \rangle \in \mathcal{D}$ *and* $\mathbf{G}_1 = \langle G, \twoheadrightarrow \rangle \in \mathcal{D}$ *have the same universe and* $\to$ *is included in* $\twoheadrightarrow$*. Define*

$$\begin{aligned} \vdash &= \text{the reflexive transitive closure of } \twoheadrightarrow \text{ in } G \\ Y &= \mathrm{Hom}(\mathbf{G}_1^+, \mathbf{2}_{01}) \subseteq \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01}) \\ X &= \{\iota\beta_{\mathbf{G}^+}(h) : h \in Y\} \subseteq \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{M}). \end{aligned}$$

If X *is a subuniverse of* $\langle M, \leq, E, R, \ldots\rangle^{B(\mathbf{G}^+)}$ *and* $\mathbb{X}$ *is the corresponding substructure, then*

(1) $\mathrm{bg}(\mathbb{X}) \cong \langle G, \to, \vdash\rangle$.
(2) *In particular, if* $\mathbf{G}_1 = \mathbf{G}$ *so that* $X = \mathrm{Hom}(\mathbf{B}(\mathbf{G}^+), \mathbf{M})$*, then* $\vdash_{\mathbb{X}}$ *is the reflexive transitive closure of* $\to_{\mathbb{X}}$ *in* $\mathrm{bg}(\mathbb{X})$.

Proof. Let $\mathbf{P}$ be the poset $\langle G, \vdash\rangle$ and note that $Y = \mathrm{Hom}(\mathbf{P}^+, \mathbf{2}_{01})$. For each $a \in G$ define $h_a, h'_a \in Y$ by

$$\begin{aligned} h_a(x) &= \begin{cases} 1 & \text{if } a \vdash x \\ 0 & \text{otherwise} \end{cases} \\ h'_a(x) &= \begin{cases} 1 & \text{if } a \vdash x \text{ and } x \neq a \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and let $J = \{h_a : a \in G\}$. As is well-known from the theory of posets,

$$\begin{aligned} J &= \text{the set of completely join-irreducible elements of } \langle Y, \leq\rangle \\ a \vdash b &\text{ iff } h_b \leq h_a, \quad \text{for } a, b \in G \\ h'_a &= \text{the unique lower cover of } h_a \text{ in } \langle Y, \leq\rangle, \quad \text{for } a \in G. \end{aligned}$$

On the other hand, note that $\langle Y, \leq\rangle$ is a subposet of $\langle \mathrm{Hom}(\mathbf{G}^+, \mathbf{2}_{01}), \leq\rangle$. Thus by Lemma 3.1(3), the map $h \mapsto \iota\beta_{\mathbf{G}^+}(h)$ is an order isomorphism $\varphi : \langle Y, \leq\rangle \to \langle X, \leq\rangle$ and so

$$\begin{aligned} G_{\mathbb{X}} &= \{\varphi(h_a) : a \in G\} \\ a \vdash b &\text{ iff } \varphi(h_a) \vdash_{\mathbb{X}} \varphi(h_b) \\ \varphi(h'_a) &= \text{the unique lower cover of } \varphi(h_a) \text{ in } \mathbb{X}. \end{aligned}$$

Our isomorphism $\langle G, \to, \vdash\rangle \cong \mathrm{bg}(\mathbb{X})$ will be the map $a \mapsto \varphi(h_a)$. It remains to prove that if $a, b \in G$ with $a \vdash b$ and $a \neq b$, then

$$\begin{aligned} a \to a &\text{ iff } \mathbb{X} \models \neg E(\varphi(h'_a), \varphi(h_a)), \quad \text{while} \\ a \to b &\text{ iff } \mathbb{X} \models \neg R(\varphi(h'_b), \varphi(h_b), \varphi(h'_a), \varphi(h_a)). \end{aligned}$$

We shall prove the second equivalence, the first being similar. Let $\langle \beta_*, \beta, \alpha_*, \alpha\rangle$ be the 4-tuple $\langle \varphi(h'_b), \varphi(h_b), \varphi(h'_a), \varphi(h_a)\rangle$. Since $a \vdash b$ and $a \neq b$ we have $h'_b \leq h_b \leq h'_a \leq h_a$ and thus $\beta_* \leq \beta \leq \alpha_* \leq \alpha$. Thus the only way that $R(\beta_*, \beta, \alpha_*, \alpha)$ can fail to be true is if at some coordinate $(x, y) \in B(\mathbf{G}^+)$ we have $\beta_*(x, y) = 0$, $\beta(x, y) = \alpha_*(x, y) = 1$, and $\alpha(x, y) = 2$. Note that for all $(x, y) \in B(\mathbf{G}^+)$ we have $x \to y$ and therefore $x \vdash y$; thus for $c \in \{a, b\}$ we have

$$\begin{aligned}
\varphi(h_c)(x,y) &= \begin{cases} 2 & \text{if } c \vdash x \\ 1 & \text{if } c \vdash y \text{ and } c \nvdash x \\ 0 & \text{if } c \nvdash y \end{cases} \\
\varphi(h_c')(x,y) &= \begin{cases} 2 & \text{if } c \vdash x \text{ and } c \neq x \\ 1 & \text{if } c \vdash y \text{ and } c \neq y \text{ and either } c \nvdash x \text{ or } c = x \\ 0 & \text{if } c \nvdash y \text{ or } c = y. \end{cases}
\end{aligned}$$

It follows that $\langle \beta_*(x, y), \beta(x, y), \alpha_*(x, y), \alpha(x, y)\rangle = \langle 0, 1, 1, 2\rangle$ iff $(x, y) = (a, b)$; so $\mathbb{X} \models \neg R(\beta_*, \beta, \alpha_*, \alpha)$ iff $(a, b) \in B(\mathbf{G}^+)$, which is equivalent to $a \to b$. $\square$

Definition 4.5. Suppose $\mathbf{A}$ is a finite algebra, $\mathbf{S} \leq \mathbf{A}^n$, and S is the corresponding n-ary relation on A. S is *balanced* if $|\mathrm{Hom}(\mathbf{S}, \mathbf{A})| = n$ and S has no repeated coordinates; i.e., $\rho_i{\restriction}_S \neq \rho_j{\restriction}_S$ whenever $i \neq j$, where $\rho_1, \dots, \rho_n$ are the projections $A^n \to A$.

Note that $\mathrm{Hom}(\mathbf{S}, \mathbf{A}) = \{\rho_i{\restriction}_S : 1 \leq i \leq n\}$ if S is balanced.

Lemma 4.6. *$\leq$, E, and R are balanced for $\mathbf{M}$.*

Proof. Here is a way to compute the sizes of the relevant hom-sets "by inspection." Let $\mathbf{L}$ denote the subalgebra of $\mathbf{M}^2$ whose universe is $\leq$. Then $\mathbf{L}, \mathbf{E}, \mathbf{R}$ are isomorphic to $\mathbf{B}(\mathbf{G}_1^+), \mathbf{B}(\mathbf{G}_2^+), \mathbf{B}(\mathbf{G}_3^+)$ respectively, where $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ are pictured below:

$\mathbf{G}_1$ $\mathbf{G}_2$ $\mathbf{G}_3$

Now use Lemma 3.1(3). $\square$

Lemma 4.7. *Suppose $\mathbf{A}$ is a finite algebra, $\mathbb{A}$ is an alter ego (whose relations and operations may be infinitary) which dualizes $\mathbf{A}$, and $\mathbb{A}^*$ is an alter ego obtained from $\mathbb{A}$ by adding one or more balanced relations of $\mathbf{A}$ to the signature.*

(1) *If S is a balanced n-ary relation of $\mathbf{A}$, then S is defined in $\mathbb{A}$ by a finite conjunction $\Phi(\mathsf{x}_1, \dots, \mathsf{x}_n)$ of atomic formulas in the signature of $\mathbb{A}$.*
(2) *$\mathbb{A}$ fully dualizes $\mathbf{A}$ iff $\mathbb{A}^*$ fully dualizes $\mathbf{A}$.*

Proof. (1). The argument is essentially due to Zádori [6, Corollary 3.2 and Theorem 3.3] and, independently, Davey, Haviar and Priestley [4, Theorem 3.6]. Write $\mathbb{A} = \langle A, \mathcal{R}, \mathcal{F}, \mathcal{C}, \text{topology}\rangle$ where $\mathcal{R}, \mathcal{F}, \mathcal{C}$ are the sets of relations, functions of positive arity, and constants respectively that are included in the signature of $\mathbb{A}$. Define

$$\begin{aligned} X &= \operatorname{Hom}(\mathbf{S}, \mathbf{A}) \subseteq A^S \\ S^* &= \{e^{\mathbf{S}}(a) : a \in S\} \subseteq A^X. \end{aligned}$$

Since $\mathbb{A}$ dualizes $\mathbf{A}$,

$$S^* = \{\varphi \in A^X : \varphi \text{ preserves } \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}\}.$$

Now for $x = (x_1, \ldots, x_n) \in A^n$ define $\varphi_x \in A^X$ by $\varphi_x(\rho_i{\restriction}_S) = x_i$. Note that the map $x \mapsto \varphi_x$ is well-defined (as S is balanced) and injective, and that $\varphi_a = e^{\mathbf{S}}(a)$ for each $a \in S$. Thus

$$\begin{aligned} S &= \{x \in A^n : \varphi_x \in S^*\} \\ &= \{x \in A^n : \varphi_x \text{ preserves } \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}\}. \end{aligned}$$

This last equation will provide the desired first-order formula. To see how, suppose $c \in \mathcal{C}$. Then $\{c\}$ is a one-element subalgebra of $\mathbf{A}$. Hence among the homomorphisms from $\mathbf{S}$ to $\mathbf{A}$ is one which is the constant map with range $\{c\}$. Choose i so that $\rho_i{\restriction}_S$ is this homomorphism (equivalently, so that $a_i = c$ for all $a \in S$). Then for any $x \in A^n$, φ_x preserves c iff $x_i = c$. Define $\Phi_c(\mathsf{x}_1, \ldots \mathsf{x}_n)$ to be the atomic formula $\mathsf{x}_i = c$. Then φ_x preserves c iff $\mathbb{A} \models \Phi_c(x)$.

We give a similar argument for each member of $\mathcal{R} \cup \mathcal{F}$. If R is a λ-ary relation in $\mathcal{R}$, define

$$\Omega_R = \{\sigma \in \{1, \ldots, n\}^\lambda : (a_{\sigma(i)})_{i<\lambda} \in R \text{ for all } a \in S\}.$$

Then define $\Phi_R(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ to be $\bigwedge\{R((\mathsf{x}_{\sigma(i)})_{i<\lambda}) : \sigma \in \Omega_R\}$. As before, φ_x preserves R iff $\mathbb{A} \models \Phi_R(x)$. Finally, if F is a k-ary operation in $\mathcal{F}$, define

$$\Omega_F = \{(\sigma, j) \in \{1, \ldots, n\}^{\lambda+1} : F((a_{\sigma(i)})_{i<\lambda}) = a_j \text{ for all } a \in S\}.$$

Then define $\Phi_F(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ to be $\bigwedge\{F((\mathsf{x}_{\sigma(i)})_{i<\lambda}) = \mathsf{x}_j : (\sigma, j) \in \Omega_F\}$. As before, φ_x preserves F iff $\mathbb{A} \models \Phi_F(x)$. Thus a suitable formula which defines S in $\mathbb{A}$ is

$$\bigwedge_{R \in \mathcal{R}} \Phi_R(\mathsf{x}_1, \ldots, \mathsf{x}_n) \;\&\; \bigwedge_{F \in \mathcal{F}} \Phi_F(\mathsf{x}_1, \ldots, \mathsf{x}_n) \;\&\; \bigwedge_{c \in \mathcal{C}} \Phi_c(\mathsf{x}_1, \ldots, \mathsf{x}_n).$$

Since $S \subseteq A^n$ and A^n is finite, only finitely many of the atomic conjuncts in the above formula are needed to define S in $\mathbb{A}$.

(2). Clearly $\mathbb{A}^*$ continues to dualize $\mathbf{A}$ (see [1, Lemma 2.4.2 and Theorem 2.4.3(i)]), and $\mathbb{A}^I$ and $(\mathbb{A}^*)^I$ have the same topologically closed subuniverses for any set I. What remains to be shown is the following: if X, Y are topologically closed subuniverses of $\mathbb{A}^I, \mathbb{A}^J$ respectively, $\mathbb{X}, \mathbb{Y}$ are the corresponding topological substructures, and $\mathbb{X}^*, \mathbb{Y}^*$ are the corresponding topological substructures of $(\mathbb{A}^*)^I, (\mathbb{A}^*)^J$ respectively, then $\mathbb{X} \cong \mathbb{Y}$ iff $\mathbb{X}^* \cong \mathbb{Y}^*$. That this is true follows from item 1. $\square$

We now have all the ingredients needed to complete our argument.

Theorem 4.8. $\mathbf{M}$ *is dualizable but is not fully dualized by any alter ego having only a set of relations, operations and partial operations in its signature.*

Proof. Suppose $\mathbf{M}$ is fully dualized by an alter ego $\mathbb{M}$ whose signature is a set. By Lemma 4.7(2), we may assume that $\leq$, E and R are included in the signature of $\mathbb{M}$. Let κ be an infinite cardinal greater than all the arities of the relations, operations and partial operations in this signature. Let $\mathbb{G}_\kappa = \langle G_\kappa, \to, \twoheadrightarrow \rangle$ be as in Lemma 4.1, and let $\mathbb{X}_\kappa$ be as in the discussion preceding Definition 4.3. Since $\mathbb{M}$ fully dualizes $\mathbf{M}$ there exists $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$ so that, with $\mathbb{X}'$ denoting $\mathrm{Hom}(\mathbf{A}, \mathbf{M})$ as a topological substructure of $(\mathbb{M})^A$, we have $\mathbb{X}_\kappa \cong \mathbb{X}'$. It follows that $\mathrm{bg}(\mathbb{X}_\kappa) \cong \mathrm{bg}(\mathbb{X}')$. Using Lemma 4.4 twice and Lemma 3.2(3) we find that $\vdash_{\mathbb{X}'}$ is the reflexive transitive closure of $\to_{\mathbb{X}'}$ in $\mathrm{bg}(\mathbb{X}')$, while $\mathrm{bg}(\mathbb{X}_\kappa) \cong \mathbb{G}_\kappa$. Since $\twoheadrightarrow$ is not the reflexive transitive closure of $\to$ in $\mathbb{G}_\kappa$, we have our desired contradiction. $\square$

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