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Homogeneous locally finite varieties

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Let $\mathscr{V}$ be a locally finite variety of finite type. $\mathscr{V}$ is *homogeneous* iff every isomorphism between subalgebras of a finite algebra in the variety extends to an automorphism of the algebra. In [5] it was shown that if $\mathscr{V}$ is homogeneous and finite axiomatizable, then it is the varietal product of some generalized G-sets and some generalized modules over a finite semisimple ring. The immediate goal of this note is to show that the second hypothesis of this claim is not needed, since it follows from the first.

As in [5], our proof hinges on recent results (in particular, the spectacular theorem of McKenzie and Valeriote [4]) on the structure of locally finite varieties whose first-order theory is decidable. The larger goal of this note is to advertise the potential of these recent results.

All terminology and notation occurring in the following theorem is defined in [3, 4, 5].

THEOREM 1. *Let $\mathscr{V}$ be a locally finite variety of finite type. Then $\mathscr{V}$ is homogeneous iff $\mathscr{V} = \mathscr{S} \otimes \mathscr{A}$ where:*

(i) *$\mathscr{S}$ is a (strongly abelian) subvariety of $\mathscr{V}$ which is term-equivalent to a variety $\mathscr{W}[k]$, where $\mathscr{W}$ is a k-sorted unary variety such that every term of $\mathscr{W}$ is either constant or left-invertible (i.e., for every term $t(x)$, either $\mathscr{W} \models t(x) \approx t(y)$ or there is some term $h(y)$ of the appropriate sort such that $\mathscr{W} \models h(t(x)) \approx x$).*

(ii) *$\mathscr{A}$ is an affine subvariety of $\mathscr{V}$ whose associated (finite) ring is semisimple (i.e., Jacobson radical is (0)).*

Proof. ($\Leftarrow$). This is straightforward, and is proved in [5].

($\Rightarrow$). Though not explicitly stated in the book of McKenzie and Valeriote [4], it follows from their proof (and proofs in [2]), or in any event can be deduced from

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their main result, that if $\mathscr{V}$ is a locally finite variety of finite type, all of whose finitely generated subvarieties are decidable, then $\mathscr{V} = \mathscr{S} \otimes \mathscr{A} \otimes \mathscr{D}$ where $\mathscr{S}$ is strongly abelian, $\mathscr{A}$ is affine, and $\mathscr{D}$ is a discriminator variety. We use their result in the following way. Suppose $\mathscr{V}$ is a locally finite homogeneous variety of finite type; choose an arbitrary finite algebra $\mathbf{A} \in \mathscr{V}$, and let $\mathscr{V}_1 = \mathrm{V}(\mathbf{A})$. $\mathscr{V}_1$ is still homogeneous, and its universal theory is decidable, so it follows from the proof of [3, Theorem 3.2] that $\mathrm{V}(\mathscr{V}_1^t)$, the discriminator variety generated by $\mathscr{V}_1$ augmented by a ternary discriminator term, is decidable. Hence $\mathscr{V}_1$ itself is decidable, so because $\mathbf{A}$ was arbitary and by the theorem of McKenzie and Valeriote, $\mathscr{V} = \mathscr{S} \otimes \mathscr{A} \otimes \mathscr{D}$.

We want to show that $\mathscr{D}$ is trivial. This is an easy consequence of homogeneity, but in keeping with the spirit of this note we instead argue as follows. Suppose $\mathscr{D}$ were nontrivial; pick a finite nontrivial $\mathbf{D} \in \mathscr{D}$ and let $\mathscr{V}_2 = \mathrm{V}(\mathbf{D})$. Then $\mathrm{V}(\mathscr{V}_2^t)$ is decidable by the argument in the previous paragraph, while $\mathscr{V}_2$ is undecidable by a theorem of Burris on iterated discriminator varieties [1].

Thus $\mathscr{V} = \mathscr{S} \otimes \mathscr{A}$. It now follows that $\mathscr{V}$ is finitely axiomatizable ([4, Theorem 0.17] and [2, page 54]), so we can appeal to the main result of [5]. □

COROLLARY 2. *Suppose $\mathscr{V}$ is a locally finite homogeneous variety of finite type. Then $\mathscr{V}$ is abelian, finitely generated, and finitely axiomatizable.*

It would be interesting to know whether more elementary arguments establishing this corollary can be found.

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