ALGEBRAS FROM FINITE GROUP ACTIONS AND A QUESTION OF EILENBERG AND SCHÜTZENBERGER

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Abstract. In 1976 S. Eilenberg and M.-P. Schützenberger posed the following diabolical question: if **A** is a finite algebraic structure, Σ is the set of all identities true in **A**, and there exists a finite subset F of Σ such that F and Σ have exactly the same *finite* models, must there also exist a finite subset F' of Σ such that F' and Σ have exactly the same *finite and infinite* models? (That is, must the identities of **A** be "finitely based"?) It is known that any counter-example to their question (if one exists) must fail to be finitely based in a particularly strange way. In this paper we show that the "inherently nonfinitely based" algebras constructed by Lawrence and Willard from group actions do *not* fail to be finitely based in this particularly strange way, and so do not provide a counter-example to the question of Eilenberg and Schützenberger. As a corollary, we give the first known examples of inherently nonfinitely based "automatic algebras" constructed from group actions.

1. Introduction

The problem motivating the work presented here concerns finite algebras, as understood in universal algebra, and the identities satisfied in them. A finite algebra **A** (we will always assume our algebras have just finitely many basic operations) is said to be *finitely based* (FB) if the set of identities valid in **A** can be axiomatized by some finite subset; if this is not the case, then **A** is *nonfinitely based* (NFB). While many finite algebras of general interest, including all finite groups [25] and finite rings [14, 16], are FB, the situation for general finite algebras is more delicate [18]. Even determining which finite semigroups are FB is an enormous, ongoing problem [33]. In this paper we consider two open questions about finite algebras which are NFB in a particularly strong way, and solve them in two special classes of algebras that arise from finite group actions.

To explain the questions, let us consider a finite algebra **A**. For each $n \geq 1$ we let $\mathcal{V}(\mathbf{A})^{(n)}$ denote the class of all models of the set of all n-variable identities valid in

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$\mathbf{A}$, so $\mathcal{V}(\mathbf{A})^{(1)} \supseteq \mathcal{V}(\mathbf{A})^{(2)} \supseteq \cdots$, and we let $\mathcal{V}(\mathbf{A}) = \bigcap_{n=1}^{\infty} \mathcal{V}(\mathbf{A})^{(n)}$. Thus $\mathcal{V}(\mathbf{A})$ is the class of all models of the set of all identities valid in $\mathbf{A}$; $\mathcal{V}(\mathbf{A})$ is called the *variety generated by* $\mathbf{A}$. By a theorem of Birkhoff, $\mathbf{A}$ is FB if and only if $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{A})^{(n)}$ for some $n \geq 1$. Furthermore, $\mathcal{V}(\mathbf{A})$ is *locally finite*: every finitely generated member of $\mathcal{V}(\mathbf{A})$ is finite. So if it happens that $\mathcal{V}(\mathbf{A})^{(n)}$ is *not* locally finite, for every $n \geq 1$, then $\mathbf{A}$ must be NFB. We say that $\mathbf{A}$ is *inherently* NFB (INFB) if it satisfies the hypothesis of the previous sentence. Many known NFB finite algebras are actually INFB [1, 10, 11, 13, 15, 18, 20, 22, 26, 27, 29].

A notion nominally stronger even than INFB arises if we require finite witnesses to nonlocal finiteness. Let $\mathcal{K}$ be a class of algebras closed under forming subalgebras and arbitrary direct products. If $\mathcal{K}$ is not locally finite, then for some $d \geq 1$ there must exist in $\mathcal{K}$ an infinite, d-generated algebra. A stronger property, which may or may not hold, is that for some $d \geq 1$ there exist in $\mathcal{K}$ arbitrarily large *finite* d-generated algebras. Guided by this observation, we say that a finite algebra $\mathbf{A}$ is INFB *with finite witnesses*, or *in the finite sense* (INFB_{fin}), if for every $n \geq 1$ there exists $d \geq 1$ such that $\mathcal{V}(\mathbf{A})^{(n)}$ contains arbitrarily large finite d-generated algebras.

This brings us to the work of Eilenberg and Schützenberger from 1976. Motivated by applications to the theory of automata, they considered finite monoids and asked, for a finite monoid $\mathbf{M}$, whether the existence of a finite set of identities whose class of *finite* models coincides with the class of *finite* members of $\mathcal{V}(\mathbf{M})$ is sufficient to imply that $\mathbf{M}$ is FB. They also noted that their question can be posed for arbitrary finite algebras, not just monoids:

Eilenberg-Schützenberger Problem [7]:

If $\mathbf{A}$ is a finite algebra for which there exists a finite set of identities whose class of finite models coincides with the class of finite members of $\mathcal{V}(\mathbf{A})$, does it follow that $\mathbf{A}$ is FB?

As Sapir noted in his positive solution to the Eilenberg-Schützenberger Problem in the case of semigroups [29], if a finite semigroup is a counter-example to the Eilenberg-Schützenberger Problem, then it must be INFB but not INFB_{fin}. His reasoning works generally: any counter-example to the Eilenberg-Schützenberger Problem must be INFB but not INFB_{fin}. This motivates the following problem, first stated by McNulty et al in 2008:

INFB_{fin} Problem [21]:

Does there exist a finite algebra which is INFB but not INFB_{fin}?

Both the Eilenberg-Schützenberger Problem and the INFB_{fin} Problem are open. By the foregoing remarks, a negative answer to the Eilenberg-Schützenberger Problem provides a positive answer to the INFB_{fin} Problem. (It is not known whether the converse also holds, i.e., whether a positive answer to the INFB_{fin} Problem supplies a counter-example to the Eilenberg-Schützenberger Problem.) A reasonable strategy for approaching both problems is to identify known classes of INFB algebras and prove that they are all $\mathrm{INFB}_{\mathit{fin}}$. This is how Sapir answered the Eilenberg-Schützenberger Problem affirmatively for semigroups, using his classification of INFB semigroups [28]. Another source of known INFB algebras is provided by the so-called *shift automorphism method* [1]; McNulty et al [21] proved that every algebra which can be proved to be INFB by this method is $\mathrm{INFB}_{\mathit{fin}}$. A third sporadic source of INFB algebras is the family of 5-dimensional nonassociative bilinear K-algebras constructed by Isaev [10]. These have been studied in a recent preprint by Carlson et al [6] and shown there to be $\mathrm{INFB}_{\mathit{fin}}$.

In this paper we study another sporadic class of known INFB algebras for which these questions are not already resolved: the algebras constructed from group actions by Lawrence and Willard in [15]. These algebras are constructed as follows. Let α be a faithful action of a finite group G on a finite set S, written $(g, s) \mapsto gs$. Define an algebra $\mathbf{Alg}(G, S; \alpha)^*$ with universe $G \times S$ and two basic operations:

(1) A unary operation f given by $f((g, s)) = (g, gs)$.
(2) A binary operation d given by $d((g_1, s_1), (g_2, s_2)) = (g_1, s_2)$.

In essence, the binary operation allows one to view $\mathbf{Alg}(G, S; \alpha)^*$ as a 2-sorted algebra with sorts G and S respectively, and whose only operation is the action $G \times S \to S$. In particular, although *elements* of G are present and can act on elements of S, the group *operation* of G is not available for use in identities. Lawrence and Willard [15] proved that (i) if G is nilpotent, then $\mathbf{Alg}(G, S; \alpha)^*$ is FB, while (ii) if G is not nilpotent, then $\mathbf{Alg}(G, S; \alpha)^*$ is INFB. The main result of our paper is

Theorem 1.1. *If G is not nilpotent, then $\mathbf{Alg}(G, S; \alpha)^*$ is $\mathrm{INFB}_{\mathit{fin}}$.*

Hence no algebra $\mathbf{Alg}(G, S; \alpha)^*$ arising from a group action in this way is a counterexample to the Eilenberg-Schützenberger Problem.

Another natural way to model a group action without the group operation is as an "automatic algebra." Given nonempty sets G, S and a function σ assigning to each $a \in G$ a partial self-map σ_a on S (that is, a function $\sigma_a : U_a \to S$ for some $U_a \subseteq S$), the *automatic algebra* determined by (G, S, σ) is an algebra $\mathbf{Auto}(G, S; \sigma)$ whose universe is the disjoint union of G, S and $\{0\}$ where 0 plays the role of a default element, and which has one binary operation $\cdot$ given by

$$x \cdot y = \begin{cases} \sigma_x(y) & \text{if } x \in G \text{ and } y \in \mathrm{dom}(\sigma_x) \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if α is an action of a group G on a set S, we let $\mathbf{Auto}(G, S; \alpha)$ denote the automatic algebra $\mathbf{Auto}(G, S; \sigma)$ where each σ_a is the total map $S \to S$ given by $\sigma_a(s) = as$.

A few small automatic algebras are known to be INFB [5, 13, 21], either explicitly or implicitly via the shift automorphism method. A limitation of the shift automorphism method is that it doesn't play well with elements (of an algebra) which "act" as permutations (on other elements of the algebra). In particular, the shift automorphism method cannot be applied to any automatic algebra arising from a group action. Using our methods presented here, we can overcome this limitation:

Theorem 1.2. *Let α be a faithful action of a finite group G on a finite set S. If G is not nilpotent, then $\mathbf{Auto}(G, S; \alpha)$ is INFB_{fin}.*

As in [15], our proofs ultimately rest on the existence of a family of finitely generated groups with special properties. The innovation here is the fact that the finitely generated groups used in [15], which are infinite, have arbitrarily large finite homomorphic images.

Here is an overview of the rest of the paper. In Section 2 we construct the finitely presented groups which underpin our theorems, and prove their needed properties. In Section 3 we construct and analyze the 2-sorted versions of the algebras considered in Theorem 1.1. In Section 4 we convert the 2-sorted algebras to their 1-sorted avatar described above and prove Theorem 1.1. In Section 5 we show how Theorem 1.1 can be extended to actions of certain semigroups with zero. Then in Section 6 we prove Theorem 1.2. The paper concludes with some open problems.

2. Improving the group construction from [15]

Recall [23] that $\mathcal{A}_n\mathcal{A}_m$ is the variety of all extensions of abelian groups of exponent dividing n by abelian groups of exponent dividing m. Given $n \geq 2$ and primes p, q, let $\mathcal{F}_{n,p,q}$ denote the set of all groups G such that

($*$) There exists $X \subseteq G$ such that $G = \langle X \rangle$, $|X| = n + 1$, and every subgroup of G generated by a proper subset of X is in $\mathcal{A}_p\mathcal{A}_q$.

A key result from [15] is the construction, for any $n \geq 2$ and any primes p, q, of an infinite group in $\mathcal{F}_{n,p,q}$. In this section we improve the construction in order to show that $\mathcal{F}_{n,p,q}$ contains arbitrarily large finite groups. For this we need the following two facts about finite nilpotent groups.

Lemma 2.1. *If G is a nilpotent group generated by finitely many elements of finite order, then G is finite.*

Lemma 2.2. *For each prime p there exist arbitrarily large finite p-groups generated by two elements of order p.*

A proof of Lemma 2.1 can be found in [30, Theorem 3.9(iii)]; see also [12] for an elementary proof. We are indebted to Eamonn O'Brien, who pointed us to the following proof of Lemma 2.2.

Proof of Lemma 2.2. Let $G = C_p * C_p$ be the free product of two cyclic groups of order p. By a theorem of Nielsen [24], a proof of which can be found in [17], the derived subgroup G' is free of rank $r = (p-1)^2$. Clearly $G/G' \cong C_p \times C_p$. Now fix $n \geq 1$.

There exists a characteristic subgroup N of G' such that $G'/N \cong C_{p^n} \times \cdots \times C_{p^n}$ (r factors). Then $N \lhd G$ and G/N is a finite p-group of order p^{nr+2}. Finally, G/N inherits from G the property that it is generated by two elements of order p. □

Recall that the *left-normed higher commutators* in a group are defined by $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ for $n > 2$, and that a group G is nilpotent of class $\le c$ if and only if $[x_1, \ldots, x_{c+1}]$ is identically equal to 1 in G. For each prime p and integer $c \ge 1$ let $H_{p,c}$ be the group presented using generators a_1, a_2 and the following relations:

$$a_i^p : i = 1, 2$$
$$[a_{i_0}, a_{i_1}, \ldots, a_{i_c}] : (i_0, i_1, \ldots, i_c) \in \{1, 2\}^{c+1}.$$

Then $H_{p,c}$ is nilpotent of class $\le c$ (see [15, Claim 1]), and since $H_{p,c}$ is generated by two elements of finite order, we get that $H_{p.c}$ is finite by Lemma 2.1. On the other hand, every (finite) nilpotent group generated by two elements of order p is nilpotent of some class and thus is a quotient of some $H_{p,c}$; thus Lemma 2.2 implies $\lim_{c\to\infty} |H_{p,c}| = \infty$ for each prime p.

Proposition 2.3. *For each $n \ge 2$ and all primes p, q, $\mathcal{F}_{n,p,q}$ contains arbitrarily large finite groups.*

Proof. Fix n, p, q as in the statement of the Proposition, and fix $\ell > 0$. We will construct a finite group in $\mathcal{F}_{n,p,q}$ of order at least ℓ.

Let $(V, +, 0)$ denote a vector space of dimension n over the q-element field, and let $B = \{e_1, \ldots, e_n\}$ be a basis. For each $i \in \{1, \ldots, n\}$ let $V_i = \operatorname{span}(B \setminus \{e_i\})$.

Choose $c > 0$ large enough so that $|H_{p,c}| \ge \ell$. Define $G_{V,p,c}$ to be the group presented using V as the set of generators, and using the following relations:

$$v^p : v \in V,$$
$$[v_0, v_1, \ldots, v_c] : v_0, \ldots, v_c \in V,$$
$$[v, w] : v, w \in V \text{ and } v - w \in V_1 \cup \cdots \cup V_n.$$

It follows from [15, Claim 1] that $G_{V,p,c}$ is nilpotent of class $\le c$, and since it is generated by finitely many elements of order p, $G_{V,p,c}$ is finite by Lemma 2.1. Because there exist $v, w \in V$ with $v - w \notin V_1 \cup \cdots \cup V_n$, we have that $H_{p,c}$ is a retract of $G_{V,p,c}$ and hence $|G_{V,p,c}| \ge \ell$.

Now we define an action of V on $G_{V,p,c}$. For each $x \in V$ define $\phi_x : V \to V$ by $\phi_x(v) = v + x$. Observe that for any $x, v, w \in V$ we have $v - w \in V_1 \cup \cdots \cup V_n$ if and only if $\phi_x(v) - \phi_x(w) \in V_1 \cup \cdots \cup V_n$. Hence ϕ_x extends to an automorphism ϕ_x^* of $G_{V,p,c}$. Moreover, the map $\phi^* : x \mapsto \phi_x^*$ is a group homomorphism from V to $\operatorname{Aut} G_{V,p,c}$.

Let $G = G_{V,p,c} \rtimes_{\phi^*} V$ be the semidirect product of $G_{V,p,c}$ by V with respect to ϕ^*. Clearly G is finite and $|G| \ge |G_{V,p,c}| \ge \ell$, and it remains to show $G \in \mathcal{F}_{n,p,q}$. To distinguish an element $v \in V$ from its image as a generator of $G_{V,p,c}$, we shall denote the latter by $[v]$. Then the set $X = \{e_1, \ldots, e_n, [0]\}$ generates G. Suppose Y is a subset of X of size n. If $Y = \{e_1, \ldots, e_n\}$ then $\langle Y \rangle = V \in \mathcal{A}_q \subseteq \mathcal{A}_p\mathcal{A}_q$. Otherwise, there exists i such that $e_i \notin Y$. Then $\langle Y \rangle = H_i \rtimes V_i$ where H_i is the subgroup of $G_{V,p,c}$ generated by $\{[v] : v \in V_i\}$. By design, H_i is abelian and so $H_i \in \mathcal{A}_p$. Since $V_i \in \mathcal{A}_q$, we have $\langle Y \rangle \in \mathcal{A}_p\mathcal{A}_q$ as required. $\square$

3. Two-sorted algebras

Recall that a *signature for 2-sorted algebras* is a function τ whose domain is a set of operation symbols and which, for each symbol in its domain, assigns an expression of the form $i_1 \times \cdots \times i_n \to j$ where $n \geq 0$ and $i_1, \ldots, i_n, j \in \{1, 2\}$. This expression is called the *type* of the symbol. A *2-sorted algebra* in the signature τ is a structure $\mathbf{A} = (A_1, A_2; \mathcal{F})$ where A_1 and A_2 are sets (the *universes*) and $\mathcal{F}$ is a set of finitary operations indexed by the symbols in the domain of τ, subject to the requirement that if the type of a symbol is $i_1 \times \cdots \times i_n \to j$ then the corresponding operation must be a function from $A_{i_1} \times \cdots \times A_{i_n}$ to A_j.

The standard notions of subalgebras, products, homomorphisms, congruences, quotient algebras, terms, and identities are easily extended from ordinary (1-sorted) algebras to 2-sorted algebras (see e.g. [4,9,19,31,32]). The only subtlety arises around the question of whether or not to admit algebras with one or more empty universe. In this paper we will only have need to consider 2-sorted algebras in which both universes are nonempty; these are called *everywhere nonempty* in [31]. If K is a class of everywhere nonempty 2-sorted algebras in the same signature τ, then we let $\mathcal{V}(K)$ denote the closure of K under products, everywhere nonempty subalgebras, and homomorphic images. If $\mathbf{A} = (A_1, A_2; \mathcal{F})$ is a 2-sorted algebra and $\varnothing \neq U_i \subseteq A_i$ for $i = 1, 2$, then we say that (U_1, U_2) *generates* $\mathbf{A}$ if the only subalgebra $\mathbf{B} = (B_1, B_2; \mathcal{F})$ of $\mathbf{A}$ with $U_i \subseteq B_i$ for $i = 1, 2$ is $\mathbf{A}$ itself. If $n_1, n_2 \geq 1$ then we say that $\mathbf{A}$ is (n_1, n_2)*-generated* if $\mathbf{A}$ is generated by some (U_1, U_2) with $0 < |U_i| \leq n_i$ for $i = 1, 2$.

For the remainder of this paper, we fix τ to consist of one binary operation symbol s of type $1 \times 2 \to 2$. Let Ω_τ denote the class of all everywhere-nonempty algebras in this signature.

Definition 3.1. Suppose G is a group, S is a set, and $\alpha : G \times S \to S$ is a faithful left action of G on S. We define $\mathbf{Alg}(G, S; \alpha)$ to be the two-sorted algebra $(G, S; \alpha)$ in Ω_τ.

In [15, §4] the authors worked with the 2-sorted algebra $(S, G; \alpha)$ where the type of α is now $2 \times 1 \to 1$; the choice of how to order the universes of $\mathbf{Alg}(G, S; \alpha)$ is a matter of taste and makes no material difference to the results in this paper.

We need the following definition and lemma guaranteeing that certain algebras are in $\mathcal{V}(\mathbf{Alg}(G, S; \alpha))$.

Definition 3.2. If H is a group and $r \geq 1$, then we let $H^{\otimes r}$ denote the disjoint union of r copies of H, and let $\mathbf{L}(H,r)$ denote $\mathbf{Alg}(H, H^{\otimes r}; \lambda)$ where λ is the action of H on $H^{\otimes r}$ by left multiplication.

Lemma 3.3. *Suppose G is a group, $\alpha : G \times S \to S$ is a faithful left action, and $H \in \mathcal{V}(G)$. Then $\mathbf{L}(H,r) \in \mathcal{V}(\mathbf{Alg}(G,S;\alpha))$ for all $r \geq 1$.*

Proof. For each $g \in G$ define $\Delta(g)$ to be the constant map in G^S with value g, and define $R(g)$ to be the map in S^S given by $R(g)(s) = \alpha(g,s)$. It can be easily checked that (Δ, R) is an embedding of $\mathbf{L}(G,1)$ into $\mathbf{Alg}(G,S;\alpha)^S$, so $\mathbf{L}(G,1) \in \mathcal{V}(\mathbf{Alg}(G,S;\alpha))$.

Next note that for any groups H, K we have:

(1) $\mathbf{L}(H^X,1) = \mathbf{L}(H,1)^X$ for any set X.
(2) If $K \leq H$ then $\mathbf{L}(K,1) \leq \mathbf{L}(H,1)$.
(3) If $N \lhd K$ and θ_K is the corresponding congruence of K, then $(\theta_N, \theta_N) \in \mathrm{Con}(\mathbf{L}(K,1))$ and $\mathbf{L}(K,1)/(\theta_N,\theta_N) \cong \mathbf{L}(K/N,1)$.

It follows from these facts that if $H \in \mathcal{V}(G)$ then $\mathbf{L}(H,1) \in \mathcal{V}(\mathbf{L}(G,1))$ and hence $\mathbf{L}(H,1) \in \mathcal{V}(\mathbf{Alg}(G,S;\alpha))$.

Finally, given $r \geq 1$, note that $H \in \mathcal{V}(G)$ implies $H^r \in \mathcal{V}(G)$ so $\mathbf{L}(H^r,1) \in \mathcal{V}(\mathbf{Alg}(G,S;\alpha))$. One can show that $\mathbf{L}(H,r)$ embeds into $\mathbf{L}(H^r,1)$. (Hint: H^r has a subgroup isomorphic to H; identify $H^{\otimes r}$ with the union of r distinct right cosets of this subgroup.) So $\mathbf{L}(H,r) \in \mathcal{V}(\mathbf{Alg}(G,S;\alpha))$ as well. □

Theorem 3.4. *Suppose G is a finite group, $\alpha : G \times S \to S$ is a faithful left action, and G is not nilpotent. Then for every $n \geq 2$ and $\ell > 0$ there exists a 2-sorted algebra $\mathbf{B}(n,\ell) \in \Omega_\tau$ satisfying:*

(1) *$\mathbf{B}(n,\ell)$ is $(n+1,1)$-generated.*
(2) *Both universes of $\mathbf{B}(n,\ell)$ are finite and nonempty.*
(3) *The second universe of $\mathbf{B}(n,\ell)$ has size $\geq \ell$.*
(4) *Each everywhere-nonempty (n,n)-generated subalgebra of $\mathbf{B}(n,\ell)$ belongs to $\mathcal{V}(\mathbf{Alg}(G,S;\alpha))$.*

Proof. By [15, Theorem 2.4], we can choose and fix primes p, q with $p \neq q$ and such that $\mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{V}(G)$. Given $n \geq 2$ and $\ell > 0$, choose $P \in \mathfrak{F}_{n,p,q}$ so that P is finite and $|P| \geq \ell$. Choose $X = \{x_0, x_1, \ldots, x_n\} \subseteq P$ witnessing the fact that $P \in \mathfrak{F}_{n,p,q}$; that is, $P = \langle X \rangle$ and every subgroup of P generated by a proper subset of X is in $\mathcal{A}_p\mathcal{A}_q$. Recall that $\mathbf{L}(P,1)$ is the algebra $(P,P;\lambda)$ where λ is left multiplication. Now let $\mathbf{B}(n,\ell)$ be the subalgebra of $\mathbf{L}(P,1)$ with universes X and P respectively.

Clearly $\mathbf{B}(n,\ell)$ is generated by $(X,\{1\})$, proving (1). Items (2) and (3) are obviously true. So it remains to prove (4). Suppose $\mathbf{C} = (C_1, C_2; \lambda|_{C_1 \times C_2})$ is an (n,n)-generated subalgebra of $\mathbf{B}(n,\ell)$ with $C_1, C_2 \neq \varnothing$. Note that C_1 is a proper subset of X. Let H be the subgroup of P generated by C_1. By hypothesis, $H \in \mathcal{A}_p\mathcal{A}_q$ so

$H \in \mathcal{V}(G)$. C_2 is closed under the action of H by left multiplication, so C_2 is the union of some number, r, of left cosets of H in P. Thus $\mathbf{C}$ embeds into $\mathbf{L}(H, r)$. As $\mathbf{L}(H, r) \in \mathcal{V}(\mathbf{Alg}(G, S; \alpha))$ by Lemma 3.3, we get $\mathbf{C} \in \mathcal{V}(\mathbf{Alg}(G, S; \alpha))$. □

4. Conversion to 1-sorted algebras

In this section we exploit a general categorical conversion of the variety of all everywhere-nonempty k-sorted algebras in a fixed signature to a variety of 1-sorted algebras, due ultimately to Barr [2, Theorem 5] (see [9, §5] for a fuller account, and [19, chapter 11] for explicit details). The resulting 1-sorted variety is defined only up to term equivalence, meaning that the conversion is really between multi-sorted and 1-sorted clones (for universal algebraists) or algebraic theories (for category theorists). To avoid having to explain the machinery of clones or algebraic theories, we will keep this presentation concrete by explaining the conversion in the special case of the signature considered in the previous section, choosing one specific presentation of the clone of the 1-sorted variety which is produced.

So again let τ denote the 2-sorted signature consisting of exactly one binary operation s of sort $1 \times 2 \to 2$, and recall that Ω_τ is the class of all everywhere-nonempty 2-sorted algebras in the signature τ. Let τ^* be the signature of 1-sorted algebras consisting of a binary operation d and a unary operation f. Let Ω_τ^* denote the variety of 1-sorted algebras in the signature τ^* axiomatized by the following identities:

$$\begin{aligned} d(x, x) &\approx x \\ d(d(x, y), d(z, w)) &\approx d(x, w) \\ d(f(x), y) &\approx d(x, y). \end{aligned}$$

For every $\mathbf{A} = (A_1, A_2; s) \in \Omega_\tau$, define a 1-sorted algebra $\mathbf{A}^*$ in the signature τ^* as follows: the universe is $A_1 \times A_2$, and the operations d, f are given by

$$\begin{aligned} d((a_1, a_2), (b_1, b_2)) &= (a_1, b_2) \\ f((a_1, a_2)) &= (a_1, s(a_1, a_2)). \end{aligned}$$

Proposition 4.1 (essentially Barr [2]). $\mathbf{A} \mapsto \mathbf{A}^*$ *is (the object map of) a category equivalence from* Ω_τ *to* Ω_τ^*.

Indeed, given an algebra $\mathbf{C} \in \Omega_\tau^*$, define binary relations E_1, E_2 on C by aE_1b if and only if there exists $c \in C$ with $d(a, c) = d(b, c)$, and aE_2b if and only if there exists $c \in C$ with $d(c, a) = d(c, b)$. One can use the defining identities of Ω_τ^* to show that E_1 and E_2 are equivalence relations and the map $C \to C/E_1 \times C/E_2$ given by $c \mapsto (c/E_1, c/E_2)$ is a bijection with inverse given by $(a/E_1, b/E_2) \mapsto d(a, b)$. In particular, if aE_1a' and bE_2b' then $d(a, b) = d(a', b')$. Now define the 2-sorted algebra $\mathbf{C}^\square \in \Omega_\tau$ to have universes C/E_1 and C/E_2 and the operation $s : C/E_1 \times C/E_2 \to C/E_2$ given by $s(a/E_1, b/E_2) = f(d(a, b))/E_2$. One can verify $(\mathbf{A}^*)^\square \cong \mathbf{A}$ naturally for all $\mathbf{A} \in \Omega_\tau$, and $(\mathbf{C}^\square)^* \cong \mathbf{C}$ naturally for all $\mathbf{C} \in \Omega_\tau^*$.

Additionally, the following facts are easily verified.

(F1) For all $\mathbf{A} \in \Omega_\tau$ and $n > 0$, $\mathbf{A}$ is (n,n)-generated if and only if $\mathbf{A}^*$ is n-generated.
(F2) For all $\mathbf{A} \in \Omega_\tau$, if $\mathbf{C} \leq \mathbf{A}^*$ then $\mathbf{C} = \mathbf{B}^*$ for some $\mathbf{B} \leq \mathbf{A}$.
(F3) For all $\mathbf{A}, \mathbf{B} \in \Omega_\tau$, $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ if and only if $\mathbf{B}^* \in \mathcal{V}(\mathbf{A}^*)$.

Now we can prove the main result of this paper.

Theorem 1.1. *Suppose G is a finite group, $\alpha : G \times S \to S$ is a faithful left action, and G is not nilpotent. Then $\mathbf{Alg}(G, S; \alpha)^*$ is* $\mathrm{INFB}_{\mathit{fin}}$.

Proof. Let $V = \mathcal{V}(\mathbf{Alg}(G, S; \alpha)^*)$. Fix $n \geq 2$. We will show that $V^{(n)}$ contains arbitrarily large finite d-generated algebras where $d = n + 1$. Fix $\ell > 0$. Let $\mathbf{B} = \mathbf{B}(n, \ell) = (B_1, B_2; s)$ be the 2-sorted algebra from Theorem 3.4. Then $\mathbf{B} \in \Omega_\tau$, both B_1 and B_2 are finite, $|B_1| \geq 1$, and $|B_2| \geq \ell$. Consider the 1-sorted image $\mathbf{B}^* \in \Omega^*_\tau$ of $\mathbf{B}$. We have $\mathbf{B}^*$ is finite and $|B^*| = |B_1| \cdot |B_2| \geq \ell$. Moreover, (F1) implies $\mathbf{B}^*$ is d-generated because $\mathbf{B}$ is $(d, 1)$-generated.

It remains to prove $\mathbf{B}^* \in V^{(n)}$. Because $V^{(n)}$ is defined by the n-variable identities valid in V, it suffices to show that every n-generated subalgebra of $\mathbf{B}^*$ is in V (see [8, Lemma 7.15]). Suppose $\mathbf{C}$ is an n-generated subalgebra of $\mathbf{B}^*$. Then $\mathbf{C} = \mathbf{D}^*$ for some (n, n)-generated subalgebra $\mathbf{D}$ of $\mathbf{B}$, by (F2) and (F1). By Theorem 3.4, $\mathbf{D} \in \mathcal{V}(\mathbf{Alg}(G, S; \alpha))$, so $\mathbf{C} \in \mathcal{V}(\mathbf{Alg}(G, S; \alpha)^*) = V$ by (F3). This proves $\mathbf{B}^* \in V^{(n)}$. □

5. Adjoining zero

The construction of $\mathbf{Alg}(G, S; \alpha)^*$ does not require a group action; any action of one finite set on another will do. In this section we extend our main theorem to actions by finite semigroups consisting of a nonnilpotent group with a zero element adjoined. More precisely, let G be a group and let α be an action of G on a set S. Let G^0 denote the semigroup obtained from G by adding a zero element 0, let S^0 denote the set obtained from S by adding a new element 0, and let α^0 denote the extension of α to a semigroup action of G^0 on S^0 satisfying $\alpha^0(0, y) = \alpha^0(x, 0) = 0$ for all $x \in G^0$ and all $y \in S^0$. We will show that if G and S are finite, G is nonnilpotent, and α is faithful, then $\mathbf{Alg}(G^0, S^0; \alpha^0)$ is $\mathrm{INFB}_{\mathit{fin}}$.

Given any 2-sorted algebra $\mathbf{B} = (B_1, B_2; s) \in \Omega_\tau$, let $\mathbf{B}^0$ denote the 2-sorted algebra $(B^0_1, B^0_2; s^0)$, where B^0_i is the disjoint union of B_i with $\{0\}$ and s^0 is the extension of s given by

$$s^0(b, b') = \begin{cases} s(b, b') & \text{if } b \in B_1 \text{ and } b' \in B_2 \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular that if G, S, α are as before, then $\mathbf{Alg}(G^0, S^0; \alpha^0) = \mathbf{Alg}(G, S; \alpha)^0$. For readability, we will also denote $\mathbf{Alg}(G, S; \alpha)^0$ by $\mathbf{Alg}^0(G, S; \alpha)$.

The following facts are easily verified.

(F_1') For $\mathbf{A}, \mathbf{B} \in \Omega_\tau$, if $\mathbf{B} \leq \mathbf{A}$ then $\mathbf{B}^0 \leq \mathbf{A}^0$.
(F_2') For $\mathbf{A}_i \in \Omega_\tau$, $(\prod_i \mathbf{A}_i)^0$ embeds into $\prod_i \mathbf{A}_i^0$.
(F_3') For $\mathbf{A}, \mathbf{B} \in \Omega_\tau$, any homomorphism $h : \mathbf{A} \to \mathbf{B}$ has an extension $h^0 : \mathbf{A}^0 \to \mathbf{B}^0$ such that $h^0(\mathbf{A}^0) = (h(\mathbf{A}))^0$. Furthermore, if $\mathbf{C} \leq \mathbf{A}$, then $h^0(\mathbf{C}^0) \cong (h(\mathbf{C}))^0$.
(F_4') For all $\mathbf{A} \in \Omega_\tau$ and $\mathbf{B} \leq \mathbf{A}^0$, $\mathbf{B}$ is (X_1, X_2)-generated if and only if $\mathbf{B}^- = (B_1^-, B_2^-; s)$ is (X_1^-, X_2^-)-generated, where $B_i^- = B_i \setminus \{0\}$ and $X_i^- = X_i \setminus \{0\}$.

Lemma 5.1. *Fix* $\mathbf{A} \in \Omega_\tau$. *If* $\mathbf{B} \in \mathcal{V}(\mathbf{A})$, *then* $\mathbf{B}^0 \in \mathcal{V}(\mathbf{A}^0)$.

Proof. By $(F_1'), (F_2')$ and (F_3'), the class $\{\mathbf{B} \in \Omega_\tau : \mathbf{B}^0 \in \mathcal{V}(\mathbf{A}^0)\}$ contains $\mathbf{A}$ and is closed under H, S and P. □

Theorem 5.2. *Let G be a finite nonnilpotent group and let α be a faithful action of G on a finite set S. Then* $\mathbf{Alg}^0(G, S; \alpha)^*$ *is* $\mathrm{INFB}_{\mathit{fin}}$.

Proof. Let $V = \mathcal{V}(\mathbf{Alg}(G, S; \alpha))$, $V_0 = \mathcal{V}(\mathbf{Alg}^0(G, S; \alpha))$, and $V_0^* = \{\mathbf{A}^* : \mathbf{A} \in V_0\}$, and observe that $\mathcal{V}(\mathbf{Alg}^0(G, S; \alpha)^*)$ is the closure of V_0^* under isomorphisms by Proposition 4.1 and Fact (F3) from the previous section. Fix $n \geq 2$. We will show that $(V_0^*)^{(n)}$ contains arbitrarily large finite d-generated algebras where $d = n + 2$. Now for fixed $n \geq 2$ and $\ell > 0$ Theorem 3.4 gives an algebra $\mathbf{B}(n, \ell) \in \Omega_\tau$ satisfying four properties. Let $\mathbf{B} = \mathbf{B}(n, \ell)$ and consider $\mathbf{B}^0$. Clearly

(1)′ $\mathbf{B}^0$ is $(n + 2, 2)$-generated.
(2)′ Both universes of $\mathbf{B}^0$ are finite and nonempty.
(3)′ The second universe of $\mathbf{B}^0$ has size $> \ell$.

Moreover we can show:

(4)′ Each everywhere-nonempty (n, n)-generated subalgebra of $\mathbf{B}^0$ is in V_0.

Indeed, suppose $\mathbf{D}$ is an everywhere-nonempty (n, n)-generated subalgebra of $\mathbf{B}^0$. If $D_1 = \{0\}$ or $D_2 = \{0\}$ then $\mathbf{D} \in V_0$ is easily verified. So assume that $D_1 \neq \{0\}$ and $D_2 \neq \{0\}$. Let $D_i^- = D_i \setminus \{0\}$. Then (D_1^-, D_2^-) are universes of an everywhere-nonempty (n, n)-generated subalgebra $\mathbf{D}^-$ of $\mathbf{B}$. Thus $\mathbf{D}^- \in V$ by one of the properties of $\mathbf{B}$ and so $(\mathbf{D}^-)^0 \in V_0$ by Lemma 5.1. Further, $\mathbf{D} \leq (\mathbf{D}^-)^0$, which implies $\mathbf{D} \in V_0$, proving (4)′.

Consider the 1-sorted image $(\mathbf{B}^0)^* \in \Omega_\tau^*$ of $\mathbf{B}^0$. We have $(\mathbf{B}^0)^*$ is finite, $|(B^0)^*| \geq \ell$ and $(\mathbf{B}^0)^*$ is $n + 2$-generated. Suppose $\mathbf{C}$ is an n-generated subalgebra of $(\mathbf{B}^0)^*$. By fact (F1) from the previous section, $\mathbf{C} = \mathbf{D}^*$ for some (n, n)-generated subalgebra $\mathbf{D}$ of $\mathbf{B}^0$. Then $\mathbf{D} \in V_0$ by (4)′, so $\mathbf{C} \in V_0^*$. This proves that $(\mathbf{B}^0)^* \in (V_0^*)^{(n)}$. □

6. Application to automatic algebras

Recall that an automatic algebra is determined by nonempty sets G, S and a function σ assigning to each $a \in G$ a partial self-map σ_a on S, and is denoted by

Auto$(G, S; \sigma)$. We also use the notation **Auto**$(G, S; \alpha)$ when α is an action of a group G on a set S.

In this section we show how the proof of Theorem 5.2 can be modified to prove the second main result of this paper.

Theorem 1.2. *Suppose G is a finite nonnilpotent group and α is a faithful action of G on a finite set S. Then* **Auto**$(G, S; \alpha)$ *is* $\mathrm{INFB}_{\mathit{fin}}$.

Before starting the proof, we introduce some notation and one fact which will be useful. Let X be an infinite set of variables. Given a word w over X and a variable y we define a term $[w]y$ in the signature of automatic algebras as follows:

- If ε is the empty word then $[\varepsilon]y = y$.
- If $w = xw'$ with $x \in X$, then $[w]y = x \cdot [w']y$.

For example, $[x_1x_2x_3]y = x_1 \cdot (x_2 \cdot (x_3 \cdot y))$.

Now consider the 2-sorted signature τ with its single operation s of type $1 \times 2 \to 2$. If w is a word over X and y is a variable not in X, then we can reinterpret $[w]y$ as a term of sort 2 in the signature τ by:

- reinterpreting the variables in X as having sort 1, and the variable y as having sort 2;
- replacing each occurrence of $\cdot$ by s.

Denote this reinterpretation by $[w]y^\sharp$. For example, $[x_1x_2x_3]y^\sharp = s(x_1, s(x_2, s(x_3, y)))$ where now x_1, x_2, x_3 are variables of sort 1 and y is a variable of sort 2.

Let **Auto**$(G, S; \sigma)$ be an arbitrary automatic algebra. We use $\mathbf{0}$ as an abbreviation for the term $z \cdot z$. Note that $\mathbf{0}$ is identically equal to 0 in **Auto**$(G, S; \sigma)$. Let **Alg**$(G^0, S^0; \sigma)$ denote the 2-sorted algebra $(G^0, S^0; s)$ in Ω_τ whose operation s is the restriction of the operation $\cdot$ of **Auto**$(G, S; \sigma)$ to $G^0 \times S^0$.

Now define the following sets of identities in the signature of automatic algebras:

$$\begin{aligned}
\Delta = \{&\mathbf{0} \cdot x \approx \mathbf{0},\ x \cdot \mathbf{0} \approx \mathbf{0},\ x \cdot x \approx \mathbf{0},\ (x \cdot y) \cdot z \approx \mathbf{0}\} \\
&\cup\ \{[x_0x_1 \cdots x_n]x_0 \approx \mathbf{0} \,:\, n \geq 0\},
\end{aligned}$$

$$\Psi(G, S, \sigma) = \{[w]y \approx [w']y \,:\, [w]y^\sharp \approx [w']y^\sharp \text{ is identically true in } \mathbf{Alg}(G^0, S^0; \sigma)\},$$

$$\Psi_0(G, S, \sigma) = \{[w]y \approx \mathbf{0} \,:\, [w]y^\sharp \text{ is identically equal to 0 in } \mathbf{Alg}(G^0, S^0; \sigma)\}.$$

We need the following fact, which is a consequence of the "Core Theorem" in John Boozer's PhD thesis.

Fact 6.1 (Boozer [5])**.** *Let* **Auto**$(G, S; \sigma)$ *be an automatic algebra. Then*

$$\Delta \cup \Psi(G, S, \sigma) \cup \Psi_0(G, S, \sigma)$$

is a basis for $\mathcal{V}(\mathbf{Auto}(G, S; \sigma))$.

We also need one more well-known fact, due to Birkhoff.

Fact 6.2 (Birkhoff [3])**.** *If* $\mathbf{A}$ *is a finite algebra in a finite signature and* $n > 0$*, then* $\mathcal{V}(\mathbf{A})^{(n)}$ *is finitely based.*

Proof of Theorem 1.2. Recall that G is a finite nonnilpotent group, α is a faithful action of G on the finite set S, and we wish to show that the automatic algebra $\mathbf{Auto}(G, S; \alpha)$ is $\mathrm{INFB}_{\mathit{fin}}$.

Fix $n \geq 2$. By Fact 6.2, there exists a finite basis Σ_n for $\mathcal{V}(\mathbf{Auto}(G, S; \alpha))^{(n)}$. By Fact 6.1 and the compactness theorem, there exists a finite subset $\Sigma'_n \subseteq \Psi(G, S, \sigma) \cup \Psi_0(G, S, \sigma)$ such that every identity in Σ_n is a logical consequence of $\Delta \cup \Sigma'_n$. In fact, because $\alpha(x, y) \neq 0$ for all $x \in G$ and $y \in S$, it follows that $\Psi_0(G, S, \sigma) = \varnothing$, so we have $\Sigma'_n \subseteq \Psi(G, S, \sigma)$. Let k be the maximum number of variables occurring in an identity in Σ'_n. Let $d = k + 4$. We will show that $\mathcal{V}(\mathbf{Auto}(G, S; \alpha))^{(n)}$ has arbitrarily large finite d-generated members.

Recall the 2-sorted algebra $\mathbf{Alg}^0(G, S; \alpha)$ defined in the previous section, and let $V_0 = \mathcal{V}(\mathbf{Alg}^0(G, S; \alpha))$. Note that the 2-sorted algebra $\mathbf{Alg}(G^0, S^0; \sigma)$ defined before Fact 6.1 is in our current context just $\mathbf{Alg}^0(G, S; \alpha)$.

Fix $\ell > 0$. Recall that in the proof of Theorem 5.2 we constructed a 2-sorted algebra $\mathbf{B}^0 = \mathbf{B}(k, \ell)^0$ satisfying these properties:

(1)′ $\mathbf{B}^0$ is $(k + 2, 2)$-generated.
(2)′ Both universes of $\mathbf{B}^0$ are finite and nonempty.
(3)′ The second universe of $\mathbf{B}^0$ has size $> \ell$.
(4)′ Each everywhere-nonempty (k, k)-generated subalgebra of $\mathbf{B}^0$ is in V_0.
(5)′ Let the universes of $\mathbf{B}^0$ be (B_1, B_2), and let its operation be s. Then each B_i contains an element denoted 0 such that $s(0, y) = s(x, 0) = 0$ for all $x \in B_1$ and all $y \in B_2$.

Define an algebra $\mathbf{C}$ in the signature of automatic algebras as follows: arrange that $B_1 \cap B_2 = \{0\}$, let the universe of $\mathbf{C}$ be $B_1 \cup B_2$, and define the operation $\cdot$ on $B_1 \cup B_2$ by

$$x \cdot y = \begin{cases} s(x, y) & \text{if } x \in B_1 \text{ and } y \in B_2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that if (X_1, X_2) generates $\mathbf{B}^0$ then $X_1 \cup X_2$ generates $\mathbf{C}$, so $\mathbf{C}$ is $k + 4$-generated. Clearly $\mathbf{C}$ is finite and $|C| \geq |B_2| > \ell$. It remains to show that $\mathbf{C}$ is in $\mathcal{V}(\mathbf{Auto}(G, S; \alpha))^{(n)}$, or equivalently, that $\mathbf{C} \models \Sigma_n$. It suffices to show $\mathbf{C} \models \Delta \cup \Sigma'_n$. Clearly $\mathbf{C}$ satisfies every identity in Δ, so we need only check the identities in Σ'_n. Suppose $[w]y \approx [w']y$ is an identity in Σ'_n. By construction, $[w]y \approx [w']y$ is in $\Psi(G, S, \sigma)$, which means y does not occur in w or w' and $\mathbf{Alg}(G^0, S^0; \sigma) \models [w]y^\sharp \approx [w']y^\sharp$, or equivalently, $\mathbf{Alg}^0(G, S; \alpha) \models [w]y^\sharp \approx [w']y^\sharp$. Observe that this last fact implies that w and w' contain exactly the same variables; otherwise if a variable x were to occur in w, say, but not in w', then we could falsify $[w]y^\sharp \approx [w']y^\sharp$ in $\mathbf{Alg}^0(G, S; \alpha)$ by an assignment sending $x \mapsto 0$ and all other variables to nonzero elements of the appropriate sorts.

We now prove that $\mathbf{C} \models [w]y \approx [w']y$. Let $x_1, \dots, x_r$ be a list of the distinct variables that occur in w (equivalently in w'). Note that $r < k$ by our choice of k. Let $x_i \mapsto a_i$, $y \mapsto b$ be an assignment of values in C to the variables in $\{x_1, \dots, x_n\} \cup \{y\}$. Assume that this assignment falsifies the identity $[w]y \approx [w']y$. This can only happen if $a_1, \dots, a_r \in B_1 \setminus \{0\}$, $b \in B_2 \setminus \{0\}$, and the same assignment falsifies $[w]y^\sharp \approx [w']y^\sharp$ in $\mathbf{B}^0$. This falsification is then witnessed in the subalgebra of $\mathbf{B}^0$ generated by $(\{a_1, \dots, a_r\}, \{b\})$. But every (k,k)-generated subalgebra of $\mathbf{B}^0$ is in V_0 by (4)$'$, so cannot falsify $[w]y^\sharp \approx [w']y^\sharp$ as $V_0 \models [w]y^\sharp \approx [w']y^\sharp$. □

7. Summary and questions

Given a finite nonnilpotent group acting faithfully on a finite set, we have described three finite algebras that capture the group action: $\mathbf{Alg}(G, S; \alpha)^*$, $\mathbf{Alg}^0(G, S; \alpha)^*$, and $\mathbf{Auto}(G, S; \alpha)$. The first of these was invented in [15] and shown there to be inherently nonfinitely based (INFB). The second is a simple variation of the first. The third is an example of an automatic algebra to which the shift automorphism method does not apply, and was not previously known to be INFB. In this paper we showed that all three algebras are inherently nonfinitely based in the finite sense (INFB$_{\mathit{fin}}$). Thus none of these algebras can be a counterexample to the Eilenberg-Schützenberger Problem.

In particular, if $G = S_3$ and α is the faithful representation of S_3 as the set of permutations on $\{1, 2, 3\}$, then the algebras $\mathbf{Alg}(G, S; \alpha)^*$, $\mathbf{Alg}^0(G, S; \alpha)^*$ and $\mathbf{Auto}(G, S; \alpha)$ have 18, 28, and 10 elements respectively, and all are inherently nonfinitely based in the finite sense.

We end by posing three problems.

Problem 1. For which finite semigroups G does there exist a finite G-action α such that $\mathbf{Alg}(G, S; \alpha)^*$ is inherently nonfinitely based (INFB)?

Problem 2. Does the Eilenberg-Schützenberger Problem have a positive answer for algebras of the form $\mathbf{Alg}(G, S; \alpha)^*$ where α is an action of a finite semigroup G on a finite set S?

Problem 3. Does the Eilenberg-Schützenberger Problem have a positive answer for automatic algebras?

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