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Full duality among graph algebras and flat graph algebras

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Abstract. We prove that among finite graph algebras and among finite flat graph algebras, dualizability, full dualizability, strong dualizability and entropicity are all equivalent. Any finite (flat) graph algebra which is not dualizable must be inherently non-κ-dualizable for every infinite cardinal κ. A new, general method for proving strong duality is presented.

1. Introduction

The familiar dualities of Stone [20] for Boolean algebras and of Priestley [16] for bounded distributive lattices exemplify the meaning of dualizability. These dualities fall under the broad umbrella of the theory of natural dualities expounded in the survey of Davey [3] and the monograph of Clark and Davey [1]. While the necessary definitions will be included below, these two references supply a detailed development of the theory of natural dualities.

Let **B** be a finite algebra and κ be a cardinal. By a **κ-alter ego** of **B** we mean a structured topological space $\mathbb{B}$ where the topology is the discrete topology on B and the additional structure consists of a system of (possibly infinitely many) operations, partial operations, and relations on B each of which must be a subuniverse of some direct power $\mathbf{B}^\lambda$ of the algebra **B**, where λ is a cardinal smaller than κ. (Such operations, partial operations and relations are called **κ-algebraic** for **B**.) Suppose now that **A** is isomorphic to a subalgebra of a direct power of **B** (which we denote $\mathbf{A} \in \boldsymbol{SP}\,\mathbf{B}$). Then $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ will be a nonempty topologically closed subuniverse of $\mathbb{B}^A$. We let $\mathbb{D}(\mathbf{A})$ denote the corresponding structured topological space and refer to it as the **dual** of **A** (with respect to $\mathbb{B}$). Likewise, suppose that $\mathbb{X}$ is a structured topological space which is isomorphic to a topologically closed substructure of some nontrivial power of $\mathbb{B}$ (which we denote $\mathbb{X} \in \mathbb{S}_c\mathbb{P}^+\mathbb{B}$). Then the set $\mathrm{Hom}(\mathbb{X}, \mathbb{B})$ of continuous structure preserving maps from $\mathbb{X}$ into $\mathbb{B}$ is a nonempty subuniverse of the algebra $\mathbf{B}^X$. We denote the corresponding subalgebra by $\mathbf{E}(\mathbb{X})$ and refer to it as the **dual** of $\mathbb{X}$ (with respect to **B**). Under the stipulations set out above, there is a natural embedding e

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of the algebra **A** into its double dual $\mathbf{E}(\mathbb{D}(\mathbf{A}))$. Indeed, e merely assigns to each $a \in A$ the evaluation map e_a defined via

$e_a(\alpha) = \alpha(a)$ for all $\alpha \in \mathbb{D}(\mathbf{A})$.

Generally, $\mathbf{E}(\mathbb{D}(\mathbf{A}))$ will have members that are not such evaluation maps, that is, e will fail to map onto $\mathbf{E}(\mathbb{D}(\mathbf{A}))$. In the event that e maps onto $\mathbf{E}(\mathbb{D}(\mathbf{A}))$ we say that $\mathbb{B}$ **yields a** κ**-duality on A**. We call the algebra **B** κ**-dualizable** provided it has a κ-alter ego $\mathbb{B}$ so that $\mathbb{B}$ yields a κ-duality on **A** for every algebra $\mathbf{A} \in \boldsymbol{SP}\,\mathbf{B}$. Thus if **B** is κ-dualizable via the κ-alter ego $\mathbb{B}$, then the category $\boldsymbol{SP}\,\mathbf{B}$ of algebras is dually equivalent to a certain subcategory of the category $\mathbb{S}_c\mathbb{P}^+\mathbb{B}$ of structured topological spaces.

When $\kappa \leq \omega$ we suppress it and simply say that **B** is dualizable, that an operation, partial operation or relation is algebraic, and that $\mathbb{B}$ is an alter ego of **B**, as the case may be. In seeking duality results we hope for ω-duality (and, even more, we hope the alter ego can be devised using only a handful of operations, relations, and partial operations which each have small finite rank). In seeking nonduality results, on the other hand, we hope for the failure of κ-duality for every infinite cardinal κ. The notion of κ-dualizability emerges in [2, 6].

Any two finite algebras which are term equivalent exhibit the same behavior with respect to dualizability, in the sense that one is κ-dualizable if and only if the other is as well, for any cardinal κ. Indeed, any κ-alter ego of an algebra **A** is also a κ-alter ego for each algebra term equivalent to **A**. Now up to term equivalence there are only 2^ω finite algebras. To each of these continuum-many finite algebras assign the smallest κ such that the algebra is κ-dualizable, if such a κ exists, and the value ∞ as a default otherwise. Call the resulting set of cardinals (and ∞) the **dualizability spectrum** S_d. Let μ be the smallest cardinal that is strictly larger than every cardinal in S_d. This cardinal μ is called the **Hanf number for dualizability**. See [9] where Hanf introduced this notion in a quite general context, specialized here to dualizability. It is an interesting open problem to find the Hanf number for dualizability, and to describe in detail the dualizability spectrum. Little seems to be known about these matters at present.

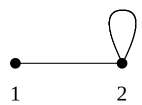
The notion of the dualizability spectrum and of the Hanf number can be relativized to any class of algebras. Some interesting possibilities are the similarity classes, the class of all n-element algebras for some fixed finite n, the class of all finite algebras generating congruence distributive (modular, meet-semidistributive, etc.) varieties, or other interesting classes of algebras. For example, according to results in [6] the dualizability spectrum for the class of graph algebras is contained in $\{0, 1, 2, 3, 4, \infty\}$, while by [6] and [1, Theorem 3.4], the dualizability spectrum of the class of algebras which generate congruence distributive varieties is contained in $\omega \cup \{\infty\}$. Among other things, in this paper we show that the spectrum for the class of flat graph algebras is contained in $\{0, 1, 2, 3, 4, \infty\}$. Results in [2] show that the set of (finite) commutative rings with (named) identity has spectrum contained in $\{0, 1, 2, 3, 4, 5, \infty\}$. It would be interesting to know the dualizability spectra for the class of all groups and for the class of all rings.

Graph algebras were introduced by C. Shallon in her dissertation [19] as a general framework for constructing finite algebras with unusual properties. Given a graph G, possibly with loops at some of its vertices, the algebra $\mathbf{A}(G)$, called the **graph algebra of** G, is the algebra with universe $V \cup \{0\}$, where V is the set of vertices of G and we insist that $0 \notin V$. The algebra $\mathbf{A}(G)$ has just one basic operation $\cdot$ which is binary and defined as follows:

$$u \cdot v = \begin{cases} u & \text{if } u, v \in V \text{ and an edge of } G \text{ joins } u \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$

An example of a graph M and the multiplication table for the algebra $\mathbf{A}(M)$ are displayed below.

M



$\mathbf{A}(M)$

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

Flat graph algebras, which were introduced by Willard in [22] as M-graph algebras and investigated by Dejan Delić [8] and Zoltan Szekely [21] in their respective Ph.D. dissertations, are obtained by expanding graph algebras by an additional basic binary operation $\wedge$ which provides the structure of a meet-semilattice of height one with least element 0. The flat graph algebra of the graph G is denoted by $\mathbf{F}(G)$.

In [6], the graph algebras that are dualizable were characterized. It turned out that for a finite graph algebra $\mathbf{A}(G)$, the following are equivalent ([6, Theorem 1]): (i) $\mathbf{A}(G)$ is dualizable; (ii) $\mathbf{A}(G)$ is finitely based; (iii) none of the graphs M, L_3, T, nor P_4 (see Figure 1) is an induced subgraph of G; (iv) $\mathbf{A}(G)$ is entropic. (An algebra is **entropic** if each of its fundamental operations is a homomorphism from the appropriate power of the algebra to the algebra.)

Using the methods of [6], we characterize which flat graph algebras are dualizable. We originally guessed that the dualizable flat graph algebras would coincide with the finitely based flat graph algebras (characterized by Delić), but this guess proved to be wrong. It does turn out, however, that this is the only one of the four conditions from [6] that must be completely eliminated. The list of forbidden subgraphs must be modified by replacing L_3 with L_2 and P_4 with P_3. The equivalence of these three conditions for flat graph algebras is proved in Sections 2 and 3.

The connection between the forbidden subgraphs and entropicity of the flat graph algebras rests on the following observations:

- The operation of the graph algebra must already be entropic with respect to itself (that is the equation $(xy)(uv) = (xu)(yv)$ must be true in the graph algebra). As pointed out in [6] this is equivalent to forbidding M, T, L_3, and P_4 as induced subgraphs.

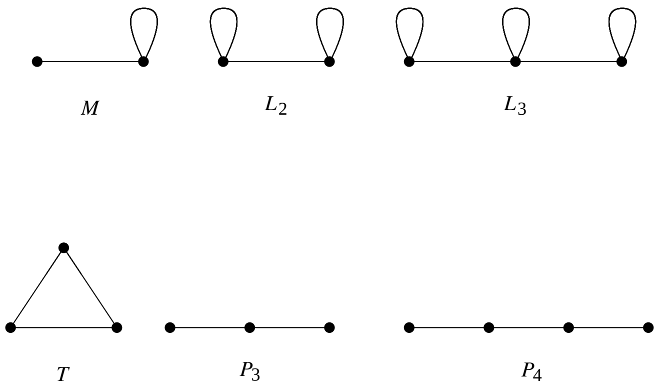


Figure 1 Forbidden Subgraphs

- A semilattice operation is always entropic with respect to itself.
- 0 is an *absorbing* element with respect to the graph algebra operation and with respect to the flat meet-semilattice operation. (This means that $x0 = 0y = 0 = x \wedge 0 = 0 \wedge x$ holds for such operations.)
- The graph operation $\cdot$ and the flat meet-semilattice operation $\wedge$ are entropic with respect to each other if and only if $xy = uv \neq 0 \Longrightarrow x = u$ and $y = v$ in the graph algebra. This last condition means that in the graph each vertex lies on at most one edge.

Apart from the blatantly graph theoretical portions, these observations can be made into a really quite general description of when adding a flat semilattice operation to an entropic algebra with an absorbing element results in a flat entropic algebra.

All the basic operations of a finite entropic algebra $\mathbf{B}$ can be included in the structure of any κ-alter ego $\mathbb{B}$ (this being another way to frame the definition of entropic algebra). This promises to be of considerable use in devising an alter ego which can be used successfully to establish a duality. While the alter egos used in [6] and in the present paper all have structure in addition to the basic operations, it does not seem unreasonable to speculate that every finite entropic algebra is dualizable. More information about entropic algebras can be found in [13, 14, 17, 18].

A strong notion of the failure of κ-dualizability was introduced in [6]. A finite algebra $\mathbf{H}$ is **inherently non-κ-dualizable** in case $\mathbf{B}$ is not κ-dualizable whenever $\mathbf{B}$ is a finite algebra with $\mathbf{H} \in \boldsymbol{SP}\,\mathbf{B}$. As was done in [6] for graph algebras, we show that if a finite flat graph algebra is not dualizable, then it is inherently non-κ-dualizable for every cardinal κ. This contrasts strongly with a recently announced result of Davey and Jane Pitkethly. They showed that every nondualizable unary algebra is embeddable in a dualizable one.

Suppose **B** is dualizable via the alter ego $\mathbb{B}$. For each $\mathbb{X} \in \mathbb{S}_c\mathbb{P}^+\mathbb{B}$ there is a natural embedding ε of $\mathbb{X}$ into its double dual $\mathbb{D}(\mathbf{E}(\mathbb{X}))$. Indeed, ε assigns to each $x \in X$ the evaluation map ε_x defined via

$$\varepsilon_x(\alpha) = \alpha(x) \text{ for all } \alpha \in \mathbf{E}(\mathbb{X}).$$

In the event that ε maps onto $\mathbb{D}(\mathbf{E}(\mathbb{X}))$ for every $\mathbb{X} \in \mathbb{S}_c\mathbb{P}^+\mathbb{B}$, we say that **B** is **fully dualizable** via the alter ego $\mathbb{B}$. Thus if **B** is fully dualizable via $\mathbb{B}$, then the category ***SP*** **B** of algebras is dually equivalent to the full category $\mathbb{S}_c\mathbb{P}^+\mathbb{B}$ of structured topological spaces. That full dualizability is strictly stronger than dualizability follows from a recent result of Hyndman and Willard [12].

There is yet another notion, **strong dualizability**, more technical but more tractable than full dualizability, which is explained in Section 4. Every strongly dualizable finite algebra is fully dualizable. The converse is an open problem.

In [6] there was no attempt to prove that every dualizable graph algebra is fully dualizable. In Section 5 we prove in fact that they must be strongly dualizable, and then do the same for dualizable flat graph algebras in Section 6. The proofs of both of these results rely on a general method for proving strong dualizability which is developed in Section 4. This method is sufficiently general that we know of no strongly dualizable algebra which cannot be proved to be strongly dualizable using the method.

The authors thank the referee for kindly providing the source file for the commutative diagrams which appear in Section 4.

2. Nondualizable flat graph algebras

THEOREM 2.1 *If* $\mathbf{F}(G)$ *is a flat graph algebra whose graph* G *contains one of* M, L_2, T *or* P_3 *as an induced subgraph, then* $\mathbf{F}(G)$ *is inherently non-*κ*-dualizable for any cardinal* κ.

Proof. It suffices to show that if G is one of the graphs M, L_2, T or P_3 then $\mathbf{F}(G)$ is inherently non-κ-dualizable for any cardinal κ. We start with the vertices of the four graphs labeled as shown below.

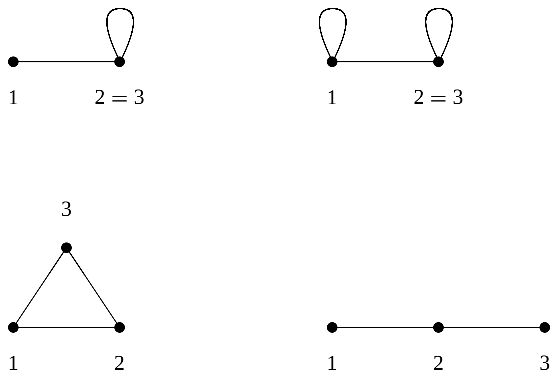
We note that, with these labelings, in each of the flat graph algebras we have

$$1 \cdot 2 = 1 \quad 3 \cdot 2 = 3 \quad \text{and} \quad 2 \cdot 1 = 2 = 2 \cdot 3.$$

Of course it is also the case that

$$1 \wedge 3 = 3 \wedge 1 = 0 \text{ and } x \wedge x = x \text{ for any } x.$$

It is these few simple equations that will allow us to show that each of these flat graph algebras is inherently non-κ-dualizable for any cardinal κ.



Some notation proves useful. Let κ be a cardinal Suppose x and $y_1, \ldots, y_k$ are objects and $\alpha_1, \ldots, \alpha_k$ are ordinals less than κ such that $y_i = y_j$ whenever $\alpha_i = \alpha_j$. Then we let the sequence

$$x^{y_1,\ldots,y_k}_{\alpha_1,\ldots,\alpha_k} = \langle z_\beta : \beta < \kappa \rangle$$

where $z_{\alpha_j} = y_j$, for $1 \leq j \leq k$, and $z_\beta = x$ otherwise. For example, $2^{3,1}_{7,4}$ denotes the sequence

$$2, 2, 2, 2, 1, 2, 2, 3, 2, 2, 2, \ldots$$

where the 1 sits at coordinate 4, and 3 sits at coordinate 7, and 2 sits at every other coordinate.

Suppose κ is an infinite cardinal. Let

$$T = \{1^3_\alpha : \alpha < \kappa\}.$$

We let $\mathbf{D}$ be the subalgebra of $\mathbf{F}(G)^\kappa$ generated by

$$T \cup \{\mathbf{2}\}$$

where $\mathbf{2}$ is the constant sequence in $F(G)^\kappa$ with value 2.

Let θ be any congruence of $\mathbf{D}$. Suppose $\alpha \neq \beta$ and $1^3_\alpha \equiv 1^3_\beta \pmod{\theta}$ and $\gamma \neq \delta$ and $1^3_\gamma \equiv 1^3_\delta \pmod{\theta}$. But then we have

$$1^3_\alpha = 1^3_\alpha \wedge 1^3_\alpha \equiv 1^3_\alpha \wedge 1^3_\beta = 1^{0,0}_{\alpha,\beta} \pmod{\theta}$$

Similarly $1^3_\gamma \equiv 1^{0,0}_{\gamma,\delta} \pmod{\theta}$. Thus

$$1^3_\alpha = 1^3_\alpha \cdot (\mathbf{2} \cdot 1^3_\gamma) \equiv 1^{0,0}_{\alpha,\beta} \cdot (\mathbf{2} \cdot 1^{0,0}_{\gamma,\delta}) = 1^{0,0,0,0}_{\alpha,\beta,\gamma,\delta} \pmod{\theta}.$$

Symmetrically, we have $1^3_\gamma \equiv 1^{0,0,0,0}_{\alpha,\beta,\gamma,\delta} \pmod{\theta}$. But then we obtain

$$1^3_\alpha \equiv 1^3_\gamma \pmod{\theta}.$$

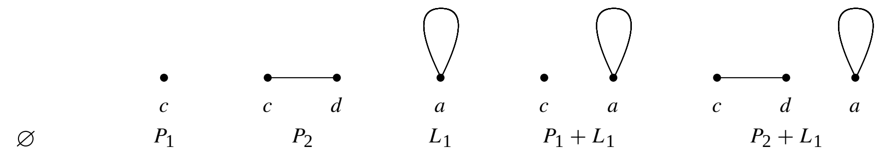
It follows that the restriction of θ to T has at most one nontrivial block. So if θ is of finite index, then the restriction of θ to T has exactly one nontrivial block, and that block is cofinite.

Because of the way the operations work in $\mathbf{F}(G)$, if $\mathbf{x}$ is in D and no entry in $\mathbf{x}$ is 0, then $\mathbf{x} \in T \cup \{\mathbf{2}\}$. So $\mathbf{1}$, the constant sequence with value 1, is not in D since it is not in $T \cup \{\mathbf{2}\}$. Now Lemma 2 of [6] yields the fact that $\mathbf{F}(G)$ is inherently non-κ-dualizable. $\square$

3. Dualizable flat graph algebras

THEOREM 3.1 *If* $\mathbf{F}(G)$ *is a flat graph algebra whose graph* G *omits each of* M, L_2, T, *and* P_3 *as induced subgraphs, then* $\mathbf{F}(G)$ *is dualizable.*

Proof. Suppose G is a graph not having any of M, L_2, T, and P_3 as induced subgraphs. Then each connected component consists of a single point, with or without a loop, or a single edge without loops. (A connected component consisting of a single point without a loop is called a *loose vertex*.) There are infinitely many such algebras but we only need to consider six of them. The graphs of these six essential algebras are displayed below:



We call an algebra $\mathbf{B}$ a **point separating retract** of an algebra $\mathbf{A}$ provided $\mathbf{B}$ is a retract of $\mathbf{A}$ and $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ separates the points of $\mathbf{A}$. If $\mathbf{B}$ is a point separating retract of $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ generate the same quasivarieties. In [7] Davey and Willard prove that if $\mathbf{A}$ and $\mathbf{B}$ are finite algebras which generate the same quasivariety, and one of them is dualizable, then so is the other. Another route to the same conclusion (with a more convenient alter ego) is via a result in [4].

Given any finite flat graph algebra whose components are copied from among P_1, P_2 or L_1, we can find a point separating retract among the six flat graph algebras associated with the graphs displayed above. Indeed, this follows using maps similar to the maps k_D below. So it only remains to prove that each of these is dualizable.

Each of our six algebras has either no components, one component with one or two vertices, or two components one of which is a single vertex with a loop and the other is completely loopless. We prepare to describe the alter ego $\mathbb{F}(G)$ of the flat graph algebra $\mathbf{F}(G)$ of these kinds.

If one takes such a flat graph algebra and forgets the meet then the resulting graph algebra $\mathbf{A}(G)$ is dualizable (but not conversely). So we will borrow freely from the proof in [6] of the dualizability of the dualizable graph algebras. First we borrow the definition of some operations.

Unary Operations:

$$s(x) = \begin{cases} c & \text{if } x = d, \\ d & \text{if } x = c, \\ x & \text{otherwise.} \end{cases} \qquad k_D(x) = \begin{cases} x & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

where D is a connected component.

A Binary Operation:

$$x * y = \begin{cases} x & \text{if } x \text{ is a loopless vertex and } y = a, \\ 0 & \text{otherwise.} \end{cases}$$

We take $\mathbb{F}(G)$, the alter ego of $\mathbf{F}(G)$, to be the topological structure having universe $F(G)$ and endowed with the discrete topology, 0 as a nullary operation (or constant), a as a nullary operation (if $a \in G$), the following unary operations: s (if $d \in G$), and k_D (if D is a component of G), the following binary operations: $\wedge$ and $*$, and the unary relation $\{0, d\}$ (if $d \in G$). Proving that these are algebraic is a straightforward but tedious task, which we omit. It may seem surprising that, though the graph algebra operation $\cdot$ is algebraic, yet we do not include it among the operations of $\mathbb{F}(G)$. In fact, in every case $x \cdot y$ is either constantly 0 or is given $x \wedge s(y)$ or by $k_{L_1}(x \wedge y)$.

Suppose that n is a natural number and $\mathbb{X}$ is a substructure of $\mathbb{F}(G)^n$ with universe X and $f : X \longrightarrow F(G)$ is a structure preserving mapping. To complete the proof, it suffices to show that f is the restriction to X of some term function, according to the IC Duality Theorem, [1, Corollary 2.2.12].

We will eventually show that f is the restriction to X of the function associated with one of the following terms.

1. $\bigwedge_{i \in I} x_i$ for some nonempty subset I of $\{0, \ldots, n-1\} = n$
2. $\bigwedge_{i \in I} (x_i \cdot x_i)$ for some nonempty subset I of n
3. $(\bigwedge_{i \in I} x_i) \cdot (\bigwedge_{j \in J} x_j)$ for some nonempty subsets I and J of n

CLAIM 3.2 *$\langle 0, \ldots, 0 \rangle \in X$ and $f(\langle 0, \ldots, 0 \rangle) = 0$, and so $0 \in \mathrm{Rng(f)}$. If $a \in G$, then $\langle a, \ldots, a \rangle \in X$ and $f(\langle a, \ldots, a \rangle) = a$, and so $a \in \mathrm{Rng(f)}$.*

This follows easily because a (if $a \in G$) and 0 are nullary operations of $\mathbb{F}(G)$.

CLAIM 3.3 *If some nonlooped vertex is in $\mathrm{Rng(f)}$, then $\mathrm{Rng(f)} = \mathrm{F(G)}$.*

Proof. If $a \in G$, then $a \in \mathrm{Rng(f)}$ by Claim 3.2. Suppose, without loss of generality, that $c \in \mathrm{Rng(f)}$ and that $d \in G$. To see that $d \in \mathrm{Rng(f)}$, pick $\mathbf{x} \in X$ so that $f(\mathbf{x}) = c$. Now apply s. We get $d = s(c) = s(f(\mathbf{x})) = f(s(\mathbf{x}))$. So $d \in \mathrm{Rng(f)}$. $\square$

Suppose v is a vertex in the range of f. We set

$$\mathbf{c}_v = \bigwedge_{f(\mathbf{x})=v} \mathbf{x}.$$

We call $\mathbf{c}_v$ the **canonical tuple** of v. Observe that

$$f(\mathbf{c}_v) = f\left(\bigwedge_{f(\mathbf{x})=v} \mathbf{x}\right) = \bigwedge_{f(\mathbf{x})=v} f(\mathbf{x}) = \bigwedge_{f(\mathbf{x})=v} v = v.$$

Hence, $\mathbf{c}_v$ is the smallest $\mathbf{x}$ such that $f(\mathbf{x}) = v$.

CLAIM 3.4 *Suppose v is a vertex in the range of f. Then $f(\mathbf{x}) = v$ iff $\mathbf{c}_v \leq \mathbf{x}$.*

Proof. We already know that if $f(\mathbf{x}) = v$ then $\mathbf{c}_v \leq \mathbf{x}$. So we suppose $\mathbf{c}_v \leq \mathbf{x}$. Now $v = f(\mathbf{c}_v) = f(\mathbf{c}_v \wedge \mathbf{x}) = f(\mathbf{c}_v) \wedge f(\mathbf{x}) = v \wedge f(\mathbf{x})$. That is, $v = v \wedge f(\mathbf{x})$. Since the semilattice reduct of $\mathbf{F}(G)$ is a flat semilattice and $v \neq 0$, we have $v = f(\mathbf{x})$ completing the proof. $\square$

CLAIM 3.5 *Let v be a vertex in* $\mathrm{Rng(f)}$. *Every nonzero entry in $\mathbf{c}_v$ belongs to the same connected component as v.*

Proof. Let D be the connected component containing v. Then

$$v = k_D(v) = k_D(f(\mathbf{c}_v)) = f(k_D(\mathbf{c}_v)).$$

So $\mathbf{c}_v \leq k_D(\mathbf{c}_v)$. Thus every nonzero entry of $\mathbf{c}_v$ belongs to D. $\square$

CLAIM 3.6 *If v is a vertex in* $\mathrm{Rng(f)}$, *then v is one of the entries of $\mathbf{c}_v$.*

Proof. Since $f(\langle 0, \ldots, 0\rangle) = 0$ while $f(\mathbf{c}_v) = v \neq 0$, we see that $\mathbf{c}_v$ must have a nonzero entry, and by Claim 3.5 all such entries belong to the connected component of v. Since P_2 is the only possible component with more than one element, we need only consider the case when $v \in \{c, d\} \subseteq G$. Suppose $v = c$. Since $\{0, d\}$ is a unary relation of $\mathbb{F}(G)$ preserved by f and since $f(\mathbf{c}_c) = c \notin \{0, d\}$, at least one entry of $\mathbf{c}_c$ is c. The automorphism s that transposes c and d and fixes every other elements settles the question when $v = d$. $\square$

We shall say that i is a *null coordinate* of v iff the ith entry in $\mathbf{c}_v$ is 0. Suppose $c, d \in$ Rng(f) and $\{v, w\} = \{c, d\}$. We call a coordinate i *an opposite coordinate for* v iff the ith entry in $\mathbf{c}_v$ is w.

CLAIM 3.7 *Suppose* $d \in$ Rng(f). *The null coordinates for* c *and* d *are identical and the opposite coordinates for* c *and* d *are identical.*

Proof. We employ the automorphism s of $\mathbf{F}(G)$. We have $c = s(d) = s(f(\mathbf{c}_d)) = f(s(\mathbf{c}_d))$ and $d = s(c) = s(f(\mathbf{c}_c)) = f(s(\mathbf{c}_c))$. Hence $\mathbf{c}_c \leq s(\mathbf{c}_d)$ and $\mathbf{c}_d \leq s(\mathbf{c}_c)$. □

REMARK. We might like to prove that the null coordinates for any two vertices in the range are identical. If $X = F(G)^n$, then we could prove that, but in general we cannot. But then we do not really need such a strong result.

CLAIM 3.8 *Suppose* v *is a nonlooped vertex belonging to* Rng(f) *and* $a \in G$ *and* $\mathbf{y} \in X$. *Then* $f(\mathbf{y}) = a$ *iff* $y_i = a$ *for every* i *which is* not *a null coordinate for* v.

Proof.

$$\begin{aligned} f(\mathbf{y}) = a \quad & \text{iff} \quad v = v * f(\mathbf{y}) \\ & \text{iff} \quad v = f(\mathbf{c}_v) * f(\mathbf{y}) = f(\mathbf{c}_v * \mathbf{y}) \\ & \text{iff} \quad \mathbf{c}_v \leq \mathbf{c}_v * \mathbf{y} \\ & \text{iff} \quad y_i = \text{ a for every } i \text{ not null for } \mathbf{c}_v. \end{aligned}$$

□

We are now in a position to show that f is the restriction of a term function.

CASE 1. f is the constant 0 function.

This means by Claim 3.2 that $a \notin G$. Then f is the restriction of the term function associated with the term $t = x_0 \cdot x_0$.

CASE 2. a is the only vertex in the range of f.

Let I be the set of coordinates nonnull for a. (Notice $I \neq \varnothing$ since $f(\langle 0, \ldots, 0\rangle) = 0$.) Let

$$t = \bigwedge_{i \in I} (x_i \cdot x_i).$$

In this case Rng(f) $= \{0, \text{a}\}$. Let $\mathbf{x} \in X$. Suppose first that $f(\mathbf{x}) = a$. Then by Claims 3.4 and 3.5 we know that $x_i = a$ for all $i \in I$. This means that $t^{\mathbf{F}(G)}(\mathbf{x}) = a$ and so $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. Now suppose that $f(\mathbf{x}) = 0$. Then again by Claims 3.4 and 3.5 we know that $x_i \neq a$ for some $i \in I$. This means that $x_i \cdot x_i = 0$ for some $i \in I$, and so $t^{\mathbf{F}(G)}(\mathbf{x}) = 0$. Hence, again $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. Consequently, f is the restriction of $t^{\mathbf{F}(G)}$ to X.

CASE 3. $c, d \in \mathrm{Rng(f)}$, and there are opposite coordinates for c.

Let J be the set of opposite coordinates for c and let I be the set of the remaining nonnull coordinates for c. (Notice $I \neq \varnothing$ by Claim 3.6.) Set

$$t = \left(\bigwedge_{i \in I} x_i\right) \cdot \left(\bigwedge_{j \in J} x_j\right).$$

In this case $\mathrm{Rng(f)} = \mathrm{G} \cup \{0\}$, with $a \in \mathrm{Rng(f)}$ if $a \in G$. This follows by Claim 3.3. Let $\mathbf{x} \in X$. First, suppose that $f(\mathbf{x}) = a$. Then $x_\ell = a$ for all $\ell \in I \cup J$, by Claim 3.8. Hence $t^{\mathbf{F}(G)}(\mathbf{x}) = a$ and $t^{\mathbf{F}(G)}(\mathbf{x}) = f(\mathbf{x})$. Second, suppose that $f(\mathbf{x}) = c$. Then $x_i = c$ for all $i \in I$ and $x_j = d$ for all $j \in J$, by Claims 3.4 and 3.5. Hence $t^{\mathbf{F}(G)}(\mathbf{x}) = c$ and $t^{\mathbf{F}(G)}(\mathbf{x}) = f(\mathbf{x})$. Third, suppose that $f(\mathbf{x}) = d$. Reasoning as we just did we find again that $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. Last, suppose that $f(\mathbf{x}) = 0$. We may assume that $0 \neq x_i = x_{i'}$ for all $i, i' \in I$ and that $0 \neq x_j = x_{j'}$ for all $j, j' \in J$, for otherwise we have the desired conclusion that $t^{\mathbf{F}(G)}(\mathbf{x}) = 0$. There remain three possibilities to exclude under which $t^{\mathbf{F}(G)}(\mathbf{x}) \neq 0$:

> *Possibility 1:* $x_i = a$ for all $i \in I$ and $x_j = a$ for all $j \in J$. But by Claim 3.8, this means that $f(\mathbf{x}) = a$, contradicting our supposition that $f(\mathbf{x}) = 0$.
>
> *Possibility 2:* $x_i = c$ for all $i \in I$ and $x_j = d$ for all $j \in J$. Hence $\mathbf{c}_c \leq \mathbf{x}$. So by Claim 3.4, $f(\mathbf{x}) = c$, contradicting our supposition that $f(\mathbf{x}) = 0$.
>
> *Possibility 3:* $x_i = d$ for all $i \in I$ and $x_j = c$ for all $j \in J$. Hence $\mathbf{c}_d \leq \mathbf{x}$, by Claim 3.7. So by Claim 3.4, $f(\mathbf{x}) = d$, contradicting our supposition that $f(\mathbf{x}) = 0$.

We conclude that f is the restriction to X of the term function $t^{\mathbf{F}(G)}$.

CASE 4. $c \in \mathrm{Rng(f)}$ and no vertex in the range has opposite coordinates.

Let I be the set of nonnull coordinates for c. Let

$$t = \bigwedge_{i \in I} x_i.$$

In this case, $\mathrm{Rng(f)} = \mathrm{G} \cup \{0\}$, by Claim 3.3. Let $\mathbf{x} \in X$. In the event that $f(\mathbf{x}) = a$, we use Claim 3.8 to conclude that $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. In the event that $f(\mathbf{x}) \in \{c, d\}$ we use Claims 3.4, 3.5, and 3.7 to conclude that $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. So last consider the situation when $f(\mathbf{x}) = 0$. By Claim 3.8 pick $i \in I$ with $x_i \neq a$. By Claim 3.4 pick $j \in I$ with $x_j \neq c$. By Claims 3.4 and 3.7 pick $k \in I$ with $x_k \neq d$. Then it must be that $x_i \wedge x_j \wedge x_k = 0$. Therefore, $t^{\mathbf{F}(G)}(\mathbf{x}) = 0$, and so $f(\mathbf{x}) = t^{\mathbf{F}(G)}(\mathbf{x})$. We conclude that f is the restriction to X of the term function $t^{\mathbf{F}(G)}$. $\square$

4. A method for proving strong duality

In this section we describe a method for proving that a finite algebra **A** is strongly dualizable. Recall that if **A** is a finite algebra and $X \subseteq A^I$, then X is said to be **term closed relative to A** if for every $\mathbf{a} \in A^I \setminus X$ there exist n-ary term operations s, t of **A** and coordinates $i_1, \ldots, i_n \in I$ such that (i) $s(x_{i_1}, \ldots, x_{i_n}) = t(x_{i_1}, \ldots, x_{i_n})$ for all $\mathbf{x} \in X$, but (ii) $s(a_{i_1}, \ldots, a_{i_n}) \neq t(a_{i_1}, \ldots, a_{i_n})$. If $\mathcal{F}$ is a set of algebraic partial operations of **A**, then we say that **A** satisfies the **term closure condition relative to** $\mathcal{F}$ provided every topologically closed subuniverse of a power of $\langle A, \mathcal{F}, \text{discrete topology}\rangle$ is term closed relative to **A**. The algebra **A** is **strongly dualizable** if and only if it is dualizable and **A** satisfies the term closure condition relative to some set of algebraic partial operations. Strong dualizability is discussed in detail in [1], where a proof can also be found that every strongly dualizable algebra is fully dualizable. In fact ([1], Theorem 3.1.7 and the first paragraph of Section 3.2), if **A** is dualized by an alter ego $\mathbb{A}$ and satisfies the term closure condition relative to the set $\mathcal{F}$ of algebraic partial operations, and if $\mathbb{A}^+$ is the alter ego obtained by adding the partial operations in $\mathcal{F}$ to the signature of $\mathbb{A}$, then $\mathbb{A}^+$ fully dualizes **A**.

Our method for proving that a finite algebra **A** is strongly dualizable involves a simplification of the notion of *rank* as defined in [23]. The simplification was implicitly used by J. Hyndman in [10].

DEFINITION 4.1

1. If $\mathbf{C} \leq \mathbf{D}$ and $Y \subseteq \mathrm{Hom}(\mathbf{D}, \mathbf{A})$, then $\mathbf{C}|_Y$ denotes $\mathbf{C}/\bigcap\{\ker(h|_C) : h \in Y\}$.
2. If $\mathbf{B} \leq \mathbf{C} \leq \mathbf{D}$ and $Y \subseteq \mathrm{Hom}(\mathbf{D}, \mathbf{A})$, then the natural map $\mathbf{B} \to \mathbf{C}|_Y$ shall be denoted by ν.
3. If $\alpha : \mathbf{B} \to \mathbf{D}$ and $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$, then we say that h **lifts to D** (through α) if there exists $h' \in \mathrm{Hom}(\mathbf{D}, \mathbf{A})$ such that $h'\alpha = h$.

DEFINITION 4.2 Let **A** be a finite algebra. We shall say that **A has enough total algebraic operations** if the following condition holds: there exists $\phi : \omega \to \omega$ such that for all $\mathbf{B} \leq \mathbf{C} \leq \mathbf{A}^n$ and every $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$, if h lifts to **C** then there exists $X \subseteq \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$ such that

1. $|X| \leq \phi(|B|)$, and
2. h lifts to $\mathbf{C}|_X$ through the natural map ν.

That is, **A** has enough total operations if there exists a map $\varphi : \omega \to \omega$ such that for all n, all **B** and **C** with $\mathbf{B} \leq \mathbf{C} \leq \mathbf{A}^n$ and $\iota : \mathbf{B} \to \mathbf{C}$ denoting the inclusion map, and all homomorphisms $h : \mathbf{B} \to \mathbf{A}$, if there exists a homomorphism $h' : \mathbf{C} \to \mathbf{A}$ such that the diagram below commutes (i.e. h lifts to **C** through ι),

$$\begin{tikzcd}
\mathbf{B} \arrow[r, hook, "\iota"] \arrow[d, "h"'] & \mathbf{C} \arrow[r, hook] \arrow[dl, dotted, "h'"] & \mathbf{A}^n \\
\mathbf{A} & &
\end{tikzcd}$$

then there exist $\ell \leq \varphi(|B|)$ and homomorphisms $g : \mathbf{A}^n \to \mathbf{A}^\ell$ and $h'' : g(\mathrm{C}) \to \mathbf{A}$ such that the diagram below commutes (i.e. h lifts to $g(\mathbf{C})$ through $g{\restriction}_{\mathbf{C}} \circ \iota$).

$$\begin{tikzcd}
\mathbf{B} \arrow[r, hook, "\iota"] \arrow[d, "h"'] & \mathbf{C} \arrow[r, hook] \arrow[d, "g{\restriction}_C"] & \mathbf{A}^n \arrow[d, "g"] \\
\mathbf{A} & g(\mathbf{C}) \arrow[l, dotted, "h''"] \arrow[r, hook] & \mathbf{A}^\ell
\end{tikzcd}$$

THEOREM 4.3 *Suppose* **A** *is a finite algebra and* **A** *has enough total algebraic operations. Then* **A** *satisfies the term closure condition relative to the set of all of its algebraic partial operations. Hence if* **A** *is dualizable, then it is strongly dualizable.*

Proof. We shall first show that, in general, an algebraic partial operation that is a restriction of a total algebraic operation always has rank at most 1 in the sense of [23]. Then we show that having enough total algebraic operations implies that every algebraic partial operation has rank at most 2. This will suffice, since according to [23, Theorem 4.1], if every algebraic partial operation of a finite algebra has rank $< \infty$, then the algebra satisfies the term closure condition relative to the set of all of its algebraic partial operations.

Suppose h is an algebraic partial operation of **A**, say $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ where $\mathbf{B} \leq \mathbf{A}^n$. We state the definition of "rank$(h) \leq 1$" from [23]. First some notation: if $\mathbf{B} \leq \mathbf{A}^n$ and $\mathbf{B}' \leq \mathbf{A}^{n+k}$, then we write $\mathbf{B} \Rrightarrow \mathbf{B}'$ if there exists $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$ such that $B' = \{(b_1, \ldots, b_n, b_{\sigma 1}, \ldots, b_{\sigma k}) : (b_1, \ldots, b_n) \in B\}$. If this holds, then there is an obvious isomorphism from **B** to $\mathbf{B}'$ induced by σ, which we also denote by σ and write $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$. Now if $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ where $\mathbf{B} \leq \mathbf{A}^n$, then rank$(h) \leq 1$ iff

> there exists $N < \omega$ such that whenever $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{A}^{n+k}$ and h lifts to $\mathbf{C}$ through σ, then there exists $Y \subseteq \{\rho_i : i < n+k\}$ (where $\rho_i : A^{n+k} \to A$ is the ith coordinate projection map) such that
>
> 1. $|Y| \le N$, and
> 2. h lifts to $\mathbf{C}|_Y$ through $\nu\sigma$.

If h is the restriction to B of the total algebraic operation H, we shall show that the above condition is true with $N = n$. Suppose $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{A}^{n+k}$. Define $h' \in \mathrm{Hom}(\mathbf{B}', \mathbf{A})$ by $h'\sigma = h$. Clearly h' is the restriction to B' of the $(n+k)$-ary total algebraic operation $H'(\mathbf{x}, \mathbf{y}) = H(\mathbf{x})$. Define $Y = \{\rho_i : i < n\}$. It should be clear that we can define $h^+ : \mathbf{C}|_Y \to \mathbf{A}$ by $h^+(c|_Y) = H'(c)$, for each $c \in C$. Then $h = h^+\nu\sigma$, verifying item 2.

Next we assume that $\mathbf{A}$ has enough total algebraic operations, and prove that every algebraic partial operation of $\mathbf{A}$ has rank at most 2. Here is the definition from [23]: If $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ where $\mathbf{B} \le \mathbf{A}^n$, then $\mathrm{rank}(h) \le 2$ iff

> there exists $N < \omega$ such that whenever $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{D} \le \mathbf{A}^{n+k}$ and h lifts to $\mathbf{D}$ through σ, then there exists $Y \subseteq \mathrm{Hom}(\mathbf{D}, \mathbf{A})$ such that
>
> 1. $|Y| \le N$,
> 2. h lifts to $\mathbf{C}|_Y$ through $\nu\sigma$, and
> 3. $\mathrm{rank}(f|_C) \le 1$ for every $f \in Y$.

Let $\mathbf{B} \le \mathbf{A}^n$ and $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A})$ be given; let ϕ witness the definition of $\mathbf{A}$ having enough total algebraic operations. We shall show that definition of "$\mathrm{rank}(h) \le 2$" is true with $N = \phi(|B|)$. Suppose $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{D} \le \mathbf{A}^{n+k}$ and h lifts to $\mathbf{D}$ through σ. Define $h' = h\sigma^{-1}$. Then h' certainly lifts to $\mathbf{C}$. Because $\mathbf{A}$ has enough total algebraic operations, there exists $X \subseteq \mathrm{Hom}(\mathbf{A}^{n+k}, \mathbf{A})$ such that $|X| \le \phi(|B'|) = N$ and h' lifts to $\mathbf{C}|_X$ through ν. Hence h lifts to $\mathbf{C}|_X$ through $\nu\sigma$. Define $Y = \{f|_D : f \in X\}$. Then $\mathbf{C}|_Y = \mathbf{C}|_X$ and for each $f \in Y$, $f|_C$ is the restriction to C of a total algebraic operation, hence $\mathrm{rank}(f|_C) \le 1$ by the discussion in the previous paragraph. This proves $\mathrm{rank}(h) \le 2$. □

5. Strong duality for dualizable graph algebras

The goal of this section is to argue that every dualizable graph algebra is strongly dualizable and hence fully dualizable.

In [6] there is a list of twelve small graph algebras, each with no more than two components and no component with more than three vertices, which are essential in the sense that (i) each is dualizable, and (ii) given any dualizable graph algebra one of these twelve is a point separating retract. Now Hyndman [11] and Davey and Haviar [5] independently have obtained general arguments which in this case show that for the task of this section it will suffice to prove that each of these twelve essential graph algebras is strongly dualizable.

Thus, one approach would be to establish that each of these small concrete algebras has enough total algebraic operations. Our argument below, which does not follow this strategy, seems to be simpler if more abstract.

Throughout this section, $\mathbf{A}(G)$ shall be a fixed (finite) dualizable graph algebra. We use $x \sqsubseteq y$ to mean $xy = x$. For $n \geq 1$, the $(n+1)$-ary terms $c_n(\mathbf{x}, y)$ and $d_n(\mathbf{x}, y)$ are defined as follows:

$$\begin{aligned} c_n(\mathbf{x}, y) &= y(x_1(y(x_2(\cdots(yx_n)\cdots)))) \\ d_n(\mathbf{x}, y) &= x_1 c_n(\mathbf{x}, y). \end{aligned}$$

The following identities are true in any graph algebra and therefore hold throughout $\mathbf{V}(\mathbf{A}(G))$:

$$xy \sqsubseteq y \tag{5.1}$$

$$xy \sqsubseteq yx \tag{5.2}$$

$$x(yz) \sqsubseteq y \tag{5.3}$$

$$x(yx) = xy \tag{5.4}$$

$$x(y(z(yx))) = x(yz) \tag{5.5}$$

$$(x(y(zw)))w = x(y((zy)(wx))) \tag{5.6}$$

$$c_n(\mathbf{x}, y) \sqsubseteq x_i \quad \text{for all } i = 1, \ldots, n. \tag{5.7}$$

(For those readers for whom the above identities are a mystery, we suggest the discussion on p. 209 of [15] and the following remark: if θ, ϕ are terms and $T(\theta), T(\phi)$ resemble each other, then the identity $\theta = \phi$ is true in every graph algebra.) Consequences of the above identities are:

$$x \sqsubseteq yz \Rightarrow x \sqsubseteq y \qquad \text{(by 5.3)} \tag{5.8}$$

$$x \sqsubseteq y \Rightarrow x \sqsubseteq yx \qquad \text{(by 5.4)} \tag{5.9}$$

$$d_n(\mathbf{x}, y) \sqsubseteq c_n(\mathbf{x}, y) \qquad \text{(by 5.1)} \tag{5.10}$$

$$c_n(\mathbf{x}, y) \sqsubseteq d_n(\mathbf{x}, y) \qquad \text{(by 5.7 and 5.9)} \tag{5.11}$$

Since $\mathbf{A}(G)$ is dualizable, the following identities hold in $\mathbf{A}(G)$ and hence throughout $\mathbf{V}(\mathbf{A}(G))$:

$$(xy)(zw) = (xz)(yw) \qquad \text{(by [6, Theorem 1])} \tag{5.12}$$

$$x(y(zw)) \sqsubseteq w \qquad \text{(by 5.6, 5.12 and 5.5)} \tag{5.13}$$

$$(xy)(zw) \sqsubseteq yw \qquad \text{(by 5.12 and 5.1).} \tag{5.14}$$

As a consequence, the following quasi-identities hold throughout $\mathbf{V}(\mathbf{A}(G))$:

$$x \sqsubseteq y \sqsubseteq z \sqsubseteq w \Rightarrow x \sqsubseteq w \qquad \text{(by 5.13)} \tag{5.15}$$

$$x \sqsubseteq y \;\&\; x \sqsubseteq z \sqsubseteq w \Rightarrow x \sqsubseteq yw \qquad \text{(by 5.14).} \tag{5.16}$$

According to [6, Theorem 1], each connected component of G is either complete looped, bipartite complete, or a loose vertex. List the complete looped components of $\mathbf{A}(G)$ as $\mathbb{X} = \{X_i : i \in I\}$, the complete bipartite components of $\mathbf{A}(G)$ as $\mathbb{Q} = \{Q_j : j \in J\}$, and let T be the set of loose vertices of $\mathbf{A}(G)$. For each $Q \in \mathbb{Q}$ choose a *left side* L_Q and *right side* R_Q. Let $\mathbb{S} = \mathbb{X} \cup \{L_Q, R_Q : Q \in \mathbb{Q}\}$. This ends the initial setup for the arguments to follow.

Until further notice (i.e., through Corollary 5.5), $\mathbf{B}, \mathbf{C}$ are fixed finite structures in $\boldsymbol{SP}(\mathbf{A}(G))$ with $\mathbf{B} \leq \mathbf{C}$, and h is a homomorphism from $\mathbf{B}$ to $\mathbf{A}(G)$. For each $S \in \mathbb{S} \cup \{T\}$ define $B_S = h^{-1}(S) \subseteq B$.

LEMMA 5.1

1. *For each $S \in \mathbb{X}$:*
   (a) *If $a, b \in B$ with $a \in B_S$ and $a \sqsubseteq b$, then $b \in B_S$.*
   (b) *If $B_S \neq \varnothing$, then there exists $g \in B_S$ such that $B_S = \{x \in B : g \sqsubseteq x\}$.*
2. *For each $Q \in \mathbb{Q}$ with sides S and S':*
   (a) *If $a, b \in B$ with $a \in B_S$ and $a \sqsubseteq b$, then $b \in B_{S'}$.*
   (b) *If $B_S \neq \varnothing$ and $B_{S'} \neq \varnothing$, then there exists $g \in B_S$ so that $B_{S'} = \{x \in B : g \sqsubseteq x\}$.*

Proof. 1a and 2a are proved by noting that $a \sqsubseteq b$ implies $h(a) \sqsubseteq h(b)$ and then arguing in $\mathbf{A}(G)$. To prove 2b, choose $b \in B_S$, enumerate $B_{S'} = \{a_1, \ldots, a_n\}$, and put $g = c_n(\mathbf{a}, b)$. One proves $g \in B_S$ by arguing in $\mathbf{A}(G)$. That $B_{S'} = \{x \in B : g \sqsubseteq x\}$ follows from 2a and equation 5.7. Conclusion 1b is proved similarly, putting $g = c_n(\mathbf{a}, a_1)$ where $B_S = \{a_1, \ldots, a_n\}$. $\square$

LEMMA 5.2 *Let $\{C_S : S \in \mathbb{S} \cup \{T\}\}$ be a family of subsets of C. The following are equivalent:*

- *There exists $h_1 \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}(G))$ extending h and such that $h_1^{-1}(S) = C_S$ for all $S \in \mathbb{S} \cup \{T\}$.*
- 1. *$C_S \cap C_{S'} = \varnothing$ whenever $S \neq S'$;*
  2. *$C_S \cap B = B_S$ for all $S \in \mathbb{S} \cup \{T\}$.*
  3. *For all $a, b \in C$:*
     (a) *Suppose $S \in \mathbb{X}$. Then $ab \in C_S$ iff $a \in C_S$ and $b \in C_S$. (In particular, if $a \in C_S$ and $a \sqsubseteq b$, then $b \in C_S$.)*
     (b) *Suppose $Q \in \mathbb{Q}$ with sides S and S'. Then $ab \in C_S$ iff $a \in C_S$ and $b \in C_{S'}$. (In particular, if $a \in C_S$ and $a \sqsubseteq b$, then $b \in C_{S'}$.)*
     (c) *$ab \notin C_T$. (In particular, if $a \in C_T$, then $a \not\sqsubseteq b$.)*

Proof. The necessity of the conditions is easy to verify. Conversely, if conditions 1–3 hold, then any map h_1 satisfying

- $h_1(x) = x$ if $x \in B$,
- $h_1(x) \in S$ if $S \in \mathbb{S} \cup \{T\}$ and $x \in C_S \setminus B$,
- $h_1(x) = 0$ for all other x,

will do. □

DEFINITION 5.3 Let $S \mapsto G_S$ be a function which assigns to each $S \in \mathbb{S}$ a subset G_S of C. We say that this assignment is *adequate relative to* $\langle \mathbf{B}, \mathbf{C}, h\rangle$ provided it meets the following requirements:

R1. If $S \in \mathbb{X}$ with $B_S \neq \varnothing$, then G_S is a singleton set $\{g\}$ for some g as in Lemma 5.1(1b).
R2. If $S \in \mathbb{X}$ with $B_S = \varnothing$, then $G_S = \varnothing$.
R3. If $Q \in \mathbb{Q}$ with sides S and S' and if $B_S \neq \varnothing$ and $B_{S'} \neq \varnothing$, then G_S is a singleton set $\{g\}$ for some g as in Lemma 5.1(2b).
R4. Suppose $Q \in \mathbb{Q}$ with sides S_1, S_2 such that $B_{S_2} = \varnothing$ and there exist $a \in B_{S_1}$ and $b \in C$ so that $a \sqsubseteq b$. Then G_{S_1} and G_{S_2} are singleton sets of the form $\{d_n(\mathbf{a}, b)\}$ and $\{c_n(\mathbf{a}, b)\}$ respectively, for some enumeration $B_{S_1} = \{a_1, \ldots, a_n\}$ and some $b \in C$ with $a_1 \sqsubseteq b$.
R5. Suppose $Q \in \mathbb{Q}$ with sides S_1, S_2 such that $B_{S_2} = \varnothing$ and there do not exist $a \in B_{S_1}$ and $b \in C$ so that $a \sqsubseteq b$. Then $G_{S_1} = B_{S_1}$ and $G_{S_2} = \varnothing$.

PROPOSITION 5.4 *Suppose* $\mathbf{B}$, $\mathbf{C}$ *and* h *are fixed as before. Let* $S \mapsto G_S$ *be an adequate assignment relative to* $\langle \mathbf{B}, \mathbf{C}, h\rangle$*. Then* h *lifts to* $\mathbf{C}$ *if and only if the following conditions are met:*

1. *For all* $S, S' \in \mathbb{S}$ *with* $S \neq S'$*, there do not exist* $x \in G_S$*,* $y \in G_{S'}$*, and* $z \in C$ *with* $x \sqsubseteq z$ *and* $y \sqsubseteq z$.
2. *Suppose* $Q \in \mathbb{Q}$ *with sides* S *and* S' *such that* $B_{S'} = \varnothing$ *and there exist* $a \in B_S$ *and* $b \in C$ *such that* $a \sqsubseteq b$*; let* $G_S = \{g\}$ *and* $G_{S'} = \{g'\}$*. Then*
   (a) *there does not exist* $x \in B$ *with* $g \sqsubseteq x$*, and*
   (b) *there does not exist* $x \in B \setminus B_S$ *with* $g' \sqsubseteq x$.
3. *There do not exist* $x \in B_T$ *and* $y \in C$ *with* $x \sqsubseteq y$.

Proof. Assume first that h lifts to $\mathbf{C}$; pick $h_1 \in \mathrm{Hom}(\mathbf{C}, \mathbf{A}(G))$ such that $h_1|_B = h$ and define the sets $C_S = h_1^{-1}(S)$ for each $S \in \mathbb{S} \cup \{T\}$. We claim that $G_S \subseteq C_S$ for every $S \in \mathbb{S}$. Indeed, referring to Definition 5.3, $G_S \subseteq B_S$ in every case except R4. If $Q \in \mathbb{Q}$ with sides S_1, S_2 and $S \in \{S_1, S_2\}$, such that $B_{S_1} = \{a_1, \ldots, a_n\}$, $B_{S_2} = \varnothing$, $b \in C$ and $a_1 \sqsubseteq b$, then argue as follows. Use Lemma 5.2(3b) to get $b \in C_{S_2}$, then argue in $\mathbf{A}(G)$ to get $d_n(\mathbf{a}, b) \in C_{S_1}$ (if $S = S_1$) or $c_n(\mathbf{a}, b) \in C_{S_2}$ (if $S = S_2$). These remarks and Lemma 5.2 are sufficient to prove Conditions 1–3 of the proposition.

Conversely, assume that Conditions 1–3 of the proposition hold. For $S \in \mathbb{S} \cup \{T\}$ define $C_S \subseteq C$ as follows:

- If $S = T$, then $C_S = B_T$.
- If $S \in \mathbb{X}$ and $G_S = \{g\}$, then $C_S = \{x \in C : g \sqsubseteq x\}$.
- If $S \in \mathbb{X}$ and $G_S = \varnothing$, then $C_S = \varnothing$.
- If $Q \in \mathbb{Q}$ with sides S and S' and if $G_S = \{g\}$ and $G_{S'} = \{g'\}$, then $C_S = \{x \in C : g' \sqsubseteq x\}$.
- If $Q \in \mathbb{Q}$ with sides S and S' and $G_S = \varnothing$ or $G_{S'} = \varnothing$, then $C_S = B_S$.

We claim that the assignment $S \mapsto C_S$ satisfies Conditions 1–3 of Lemma 5.2. The proof is tedious, but can be organized as follows. First show $B_S \subseteq C_S$ for all $S \in \mathbb{S} \cup \{T\}$. All cases follow immediately from the definitions and Lemma 5.1, except the case when $Q \in \mathbb{Q}$ with sides S and S' and $B_S = \{a_1, \ldots, a_n\}$, $B_{S'} = \varnothing$, $a_1 \sqsubseteq b \in C$, and $G_{S'} = \{c_n(\mathbf{a}, b)\}$. In this case, use equation 5.7.

Next show $C_S \cap B \subseteq B_S$ for each $S \in \mathbb{S} \cup \{T\}$. Lemma 5.1 is sufficient in most cases. Condition 2 of the proposition yields the remaining cases. This establishes item 2 of Lemma 5.2.

Suppose $S_1, S_2 \in \mathbb{S} \cup \{T\}$ with $S_1 \neq S_2$. If $C_{S_i} \subseteq B$ for some $i = 1, 2$, then $C_{S_1} \cap C_{S_2} \subseteq B_{S_1} \cap B_{S_2} = \varnothing$ by the previous paragraph. In the remaining cases there exist $S_1', S_2' \in \mathbb{S}$ with $S_1' \neq S_2'$ such that $G_{S_i'} = \{g_i'\}$ and $C_{S_i} = \{x \in C : g_i' \sqsubseteq x\}$. Then Condition 1 of the proposition yields $C_{S_1} \cap C_{S_2} = \varnothing$. This proves item 1 of Lemma 5.2.

Suppose $S \in \mathbb{X}$ with $C_S \neq \varnothing$, so $G_S = \{g\}$, and let $x, y \in C$. If $xy \in C_S$, then $g \sqsubseteq x$ and $g \sqsubseteq g \sqsubseteq y$, so $g \sqsubseteq xy$ by equation 5.16, proving $xy \in C_S$. Conversely, assume $xy \in C_S$, so $g \sqsubseteq g \sqsubseteq xy \sqsubseteq y$, the last holding by equation 5.1. Then $g \sqsubseteq x$ and $g \sqsubseteq y$ by 5.8 and 5.15 respectively, proving $x, y \in C_S$. This proves item 3(a) of Lemma 5.2.

Suppose $Q \in \mathbb{Q}$ with sides S and S' so that and $G_S = \{g\}$ and $G_{S'} = \{g'\}$. Then $g \sqsubseteq g'$ and $g' \sqsubseteq g$, either by Lemma 5.1 or equations 5.10 and 5.11. Then by arguments similar to those in the previous paragraph, $g' \sqsubseteq xy$ iff $g' \sqsubseteq x$ and $g \sqsubseteq y$, for all $x, y \in C$. Next suppose that $Q \in \mathbb{Q}$ with sides S and S' so that $B_{S'} = \varnothing$ and there do not exist $a \in B_S$ and $b \in C$ with $a \sqsubseteq b$. If there exist $x, y \in C$ with $xy \in C_S$, then $xy \in B_S$ and $xy \sqsubseteq yx \in C$ by equation 5.2, contradicting the assumption. Hence in all cases, $xy \in C_S$ iff $x \in C_S$ and $y \in C_{S'}$, proving item 3(b) of Lemma 5.2.

Finally, if $xy \in C_T$, then $xy \in B_T$ and $xy \sqsubseteq yx \in C$, contradicting Condition 3 of the proposition. So all items of Lemma 5.2 hold; hence h lifts to $\mathbf{C}$. $\square$

COROLLARY 5.5 *Suppose* $\mathbf{B}$, $\mathbf{C}$ *and* h *are fixed as before. If* h *lifts to* $\mathbf{C}$, *then there exists* $\Sigma \subseteq C$ *with* $|\Sigma| \leq |B| + |A(G)|$ *such that for all* $\theta \in \operatorname{Con} \mathbf{C}$, *if*

1. $\mathbf{C}/\theta \in \boldsymbol{SP}(\mathbf{A}(G))$,
2. $\theta|_B = 0_B$,
3. *for all* $a \in \Sigma$ *and* $y \in C$, *if* $a \not\sqsubseteq y$, *then* $a/\theta \not\sqsubseteq y/\theta$,

then h *lifts to* $\mathbf{C}/\theta$ *through the natural map.*

Proof. Choose an adequate assignment $S \mapsto G_S$ (for each $S \in \mathbb{S}$) relative to $\langle \mathbf{B}, \mathbf{C}, h \rangle$, and define

$$\Sigma = \left(\bigcup_{S \in \mathbb{S}} G_S \right) \cup B_T.$$

Clearly $|\Sigma| \leq |B| + |\mathbf{A}(G)|$. Suppose that $\theta \in \operatorname{Con} \mathbf{C}$ and θ satisfies the three items above. Let ν denote the natural map $\mathbf{B} \hookrightarrow \mathbf{C}/\theta$, and define $\mathbf{B}' = \nu(\mathbf{B})$, $\mathbf{C}' = \mathbf{C}/\theta$, and $h' = h\nu^{-1} \in \operatorname{Hom}(\mathbf{B}', \mathbf{A}(G))$. For each $S \in \mathbb{S}$ define $B'_S = B_S/\theta \subseteq B'$ and $G'_S = G_S/\theta \subseteq C'$. We claim that for each $S \in \mathbb{S}$, the item in Definition 5.3 whose hypothesis is satisfied by B_S relative to $\langle \mathbf{B}, \mathbf{C}, h \rangle$ is the same item whose hypothesis is satisfied by B'_S relative to $\langle \mathbf{B}', \mathbf{C}', h' \rangle$. For example, suppose B_S satisfies the hypothesis of R5 with $S = S_1$; so $Q \in \mathbb{Q}$ has sides S and S' such that $B_{S'} = \varnothing$ and there do not exist $a \in B_S$ and $b \in C$ so that $a \sqsubseteq b$. Then $B_S = G_S \subseteq \Sigma$ and so, by item 3 above, if $a \in B_S$ and $b \in C$, then $a/\theta \not\sqsubseteq b/\theta$; hence B'_S satisfies the hypothesis of R5 with $S = S_1$.

As $S \mapsto G_S$ is adequate relative to $\langle \mathbf{B}, \mathbf{C}, h \rangle$, it follows that $S \mapsto G'_S$ is adequate relative to $\langle \mathbf{B}', \mathbf{C}', h' \rangle$. Since h lifts to $\mathbf{C}$, and using Proposition 5.2 twice and item 3 above, it follows that h' lifts to $\mathbf{C}'$; hence h lifts to $\mathbf{C}'$ through ν. ☐

We are almost in a position to prove that $\mathbf{A}(G)$ has enough total algebraic operations. The following lemma supplies the last ingredient.

LEMMA 5.6 *Let $n \geq 1$ and $\mathbf{a} \in A(G)^n$ be fixed. There exists $f_{\mathbf{a}} : \mathbf{A}(G)^n \to \mathbf{A}(G)$ so that $\mathbf{a} \not\sqsubseteq \mathbf{y}$ implies $f_{\mathbf{a}}(\mathbf{a}) \not\sqsubseteq f_{\mathbf{a}}(\mathbf{y})$, for all $\mathbf{y} \in A(G)^n$.*

Proof. Look at the entries of $\mathbf{a}$ and consider cases.

CASE 1. Every entry of $\mathbf{a}$ is 0. Then $\mathbf{a} \sqsubseteq \mathbf{y}$ for all $\mathbf{y} \in \mathbf{A}(G)^n$, so $f_{\mathbf{a}}$ can be chosen to be a projection.

CASE 2. The entry of $\mathbf{a}$ at coordinate i is a loose vertex. Then $f_{\mathbf{a}}$ can be chosen to be the projection at coordinate i.

CASE 3. No entry of $\mathbf{a}$ is a loose vertex at any coordinate, and some entry is in a bipartite component. Choose vertices v, v' in opposite sides of some bipartite component of G and define

$$f_{\mathbf{a}}(\mathbf{x}) = \begin{cases} v & \text{if } \mathbf{a} \sqsubseteq x \\ v' & \text{if } \mathbf{a} \sqsubseteq \mathbf{y} \sqsubseteq \mathbf{x} \text{ for some } \mathbf{y} \in A(G)^n \\ 0 & \text{otherwise.} \end{cases}$$

CASE 4. No entry of $\mathbf{a}$ is a loose vertex or in a bipartite component, and some entry is in a complete component. Choose a vertex v in some complete component of G and define

$$f_{\mathbf{a}}(\mathbf{x}) = \begin{cases} v & \text{if } \mathbf{a} \sqsubseteq \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

□

THEOREM 5.7 *If* $\mathbf{A}(G)$ *is a finite dualizable graph algebra, then* $\mathbf{A}(G)$ *has enough total algebraic operations.*

Proof. Define $\phi : \omega \to \omega$ by $\phi(k) = k(k+1)/2 + |A(G)|$. Suppose $\mathbf{B} \leq \mathbf{C} \leq \mathbf{A}(G)^n$ and $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{A}(G))$ such that $|B| = k$ and h lifts to $\mathbf{C}$. Choose $\Sigma \subseteq C$ as in Corollary 5.5. For each $\mathbf{a} \in \Sigma$ choose $f_{\mathbf{a}}$ as in Lemma 5.6. Finally, let P be a minimal family of projections $\rho_i : A(G)^n \to A(G)$ large enough to separate the points of B, and put $X = \{f_{\mathbf{a}} : \mathbf{a} \in \Sigma\} \cup P$. Then $|X| \leq \phi(k)$ and items 1–3 of Corollary 5.5 are met with $\theta = \bigcap\{\ker(f) : f \in X\}$. Hence h lifts to $\mathbf{C}|_X$ through the natural map, proving that $\mathbf{A}(G)$ has enough total algebraic operations. □

COROLLARY 5.8 *Every dualizable graph algebra is strongly dualizable.*

6. Strong duality for dualizable flat graph algebras

Throughout this section, $\mathbf{F}(G)$ is a fixed dualizable flat graph algebra. By Theorem 2.1 and the first paragraph of the proof of Theorem 3.1, each connected component of G is either a one-element looped complete graph, a two-element complete bipartite graph, or a loose vertex. Define $\mathbb{X}$, $\mathbb{Q}$, $\mathbb{S}$ and T as in the preceding section; thus each element of $\mathbb{S}$ is a one-element set. Rather than dealing with $\mathbb{S}$, we shall work with $\mathcal{S} = \{v \in G : \{v\} \in \mathbb{S}\}$.

We shall outline a proof that $\mathbf{F}(G)$ has enough total algebraic operations. The proof follows the same rough outline as the proof of the corresponding result in the previous section. Therefore we shall indicate the essential changes without giving all the details of the arguments.

Let $\mathbf{B}$, $\mathbf{C}$ be fixed finite structures in $\boldsymbol{SP}(\mathbf{F}(G))$ with $\mathbf{B} \leq \mathbf{C}$, and let h be a homomorphism from $\mathbf{B}$ to $\mathbf{F}(G)$. The first change is that we need to know not only $h^{-1}(T)$ but also $h^{-1}(v)$ for each $v \in T$. Thus for each $v \in \mathcal{S} \cup T$ define $B_v = h^{-1}(v)$; also define $B_T = \bigcup_{v \in T} B_v$. By the discussion in Section 3 preceding Claim 3.4, if $v \in \mathcal{S} \cup T$ and $B_v \neq \varnothing$, then there exists a unique $g \in B_v$ such that $B_v = \{x \in B : g \leq x\}$. We shall call g the *canonical generator* for v; it is essentially the canonical tuple for v as this was defined in Section 3, except that in general it is not a tuple.

Lemma 5.2, Definition 5.3 and Proposition 5.4 are modified as follows.

LEMMA 6.2 *Let* $\{C_v : v \in \mathcal{S} \cup T\}$ *be a family of subsets of* C. *Put* $C_T = \bigcup_{v \in T} C_v$. *The following are equivalent:*

- *There exists* $h_1 \in \mathrm{Hom}(\mathbf{C}, \mathbf{F}(G))$ *extending* h *and such that* $h_1^{-1}(v) = C_v$ *for all* $v \in \mathcal{S} \cup T$.
- 1. $C_v \cap C_{v'} = \varnothing$ *whenever* $v \neq v'$;
  2. $C_v \cap B = B_v$ *for all* $v \in \mathcal{S} \cup T$.
  3. *For all* $a, b \in C$:
     (a) *Suppose* $\{v\} \in \mathbb{X}$. *Then* $ab \in C_v$ *iff* $a \in C_v$ *and* $b \in C_v$.
     (b) *Suppose* $\{v, v'\} \in \mathbb{Q}$. *Then* $ab \in C_v$ *iff* $a \in C_v$ *and* $b \in C_{v'}$.
     (c) $ab \notin C_T$.
     (d) *For all* $v \in \mathcal{S} \cup T$, *if* $C_v \neq \varnothing$ *then there exists* $g \in C_v$ *such that* $C_v = \{x \in C : g \leq x\}$.

DEFINITION 6.3 Let $v \mapsto G_v$ be a function which assigns to each $v \in \mathcal{S} \cup T$ a subset G_v of C. We say that this assignment is *adequate relative to* $\langle \mathbf{B}, \mathbf{C}, h \rangle$ provided it meets the following requirements:

R1′. If $v \in \mathcal{S} \cup T$ with $B_v \neq \varnothing$, then G_v is the singleton set $\{g\}$ where g is the canonical generator for v.
R2′. If $\{v, v'\} \in \mathbb{Q}$, $B_v \neq \varnothing$, $B_{v'} = \varnothing$, $G_v = \{g\}$, and there exists $b \in C$ such that $g \sqsubseteq b$, then $G_{v'}$ is a singleton set $\{bg\}$ for some b such that $g \sqsubseteq b$.
R3′. If $v' \in \mathcal{S} \cup T$ and $B_{v'} = \varnothing$ and the previous case does not apply, then $G_{v'} = \varnothing$.

PROPOSITION 6.4 *Suppose* $\mathbf{B}$, $\mathbf{C}$ *and* h *are fixed as before. Let* $v \mapsto G_v$ *be an adequate assignment relative to* $\langle \mathbf{B}, \mathbf{C}, h \rangle$. *Then* h *lifts to* $\mathbf{C}$ *if and only if the following conditions are met:*

1. *For all* $v, v' \in \mathcal{S} \cup T$ *with* $v \neq v'$, *there do not exist* $x \in G_v$, $y \in G_{v'}$, *and* $z \in C$ *with* $x \leq z$ *and* $y \leq z$.
2. *There do not exist* $x \in B_T$ *and* $y \in C$ *with* $x \sqsubseteq y$.

For the proof of the converse direction of Proposition 6.3, assume that Conditions 1–2 hold. For $v \in \mathcal{S} \cup T$ define $C_v \subseteq C$ as follows:

- If $G_v = \{g\}$, then $C_v = \{x \in C : g \leq x\}$.
- If $G_v = \varnothing$, then $C_v = \varnothing$.

We claim that the assignment $v \mapsto C_v$ satisfies the Conditions 1–3 of Lemma 6.2. Condition 1 follows from item 1 above. Obviously $B_v \subseteq C_v$ for all v. To prove $C_v \cap B \subseteq B_v$ in the one nontrivial case, we simply need to know that

$$x \sqsubseteq y \ \& \ yx \leq z \ \Rightarrow \ x \sqsubseteq z$$

holds in **C**. This proves Condition 2. Condition 3(d) obviously holds. Conditions 3(a–c) can be proved using

$$\begin{aligned} xy &\leq x \\ x \leq y \sqsubseteq z &\Rightarrow x \sqsubseteq z \\ x \sqsubseteq y \leq zw &\Rightarrow x \leq w \\ x \leq y \,\&\, x \sqsubseteq z \leq w &\Rightarrow x \leq yw. \end{aligned}$$

The statement of Corollary 5.5 must be changed as follows (the proof remains the same).

COROLLARY 6.5 *Suppose* $\mathbf{B}$, $\mathbf{C}$ *and* h *are fixed as before. If* h *lifts to* $\mathbf{C}$, *then there exists* $\Sigma \subseteq C$ *with* $|\Sigma| \leq |B| + |F(G)|$ *such that for all* $\theta \in \operatorname{Con} \mathbf{C}$, *if*

1. $\mathbf{C}/\theta \in \boldsymbol{SP}(\mathbf{F}(G))$,
2. $\theta|_B = 0_B$,
3. *for all* $a \in \Sigma$ *and* $y \in C$, *if* $a \not\sqsubseteq y$, *then* $a/\theta \not\sqsubseteq y/\theta$,
4. *for all* $a \in \Sigma$ *and* $y \in C$, *if* $a \not\leq y$, *then* $a/\theta \not\leq y/\theta$,

then h *lifts to* $\mathbf{C}/\theta$ *through the natural map.*

Lemma 5.6 remains true with $\mathbf{A}(G)$ replaced by $\mathbf{F}(G)$, since the operations f_a defined in the proof also preserve $\wedge$ in this context. We also need the following lemma.

LEMMA 6.6 *Let* $n \geq 1$ *and* $\mathbf{a} \in F(G)^n$ *be fixed. There exists* $g_{\mathbf{a}} : \mathbf{F}(G)^n \to \mathbf{F}(G)$ *so that for all* $\mathbf{y} \in F(G)^n$, $\mathbf{a} \not\leq \mathbf{y}$ *implies* $g_{\mathbf{a}}(\mathbf{a}) \not\leq g_{\mathbf{a}}(\mathbf{y})$.

Proof. Look at the entries of **a** and consider cases.

CASE 1. Every entry of **a** is 0. Then $\mathbf{a} \leq \mathbf{y}$ for all $\mathbf{y} \in F(G)^n$, so $g_{\mathbf{a}}$ can be chosen to be a projection.

CASE 2. Some entry of **a** is a loose vertex. Choose a loose vertex v of G and define

$$g_{\mathbf{a}}(\mathbf{x}) = \begin{cases} v & \text{if } \mathbf{a} \leq \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

CASE 3. No entry of **a** is a loose vertex, but some entry of **a** belongs to a bipartite component. Choose a bipartite component $\{v, v'\}$ of G and define

$$g_{\mathbf{a}}(\mathbf{x}) = \begin{cases} v & \text{if } \mathbf{a} \leq \mathbf{x} \\ v' & \text{if } \mathbf{a} \sqsubseteq \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

CASE 4. No entry of $\mathbf{a}$ is a loose vertex nor does any entry of $\mathbf{a}$ belong to a bipartite component, but some entry of $\mathbf{a}$ is a looped vertex. Choose a looped vertex v in G and define

$$g_{\mathbf{a}}(x) = \begin{cases} v & \text{if } \mathbf{a} \leq \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

□

THEOREM 6.7 *If* $\mathbf{F}(G)$ *is a dualizable finite flat graph algebra, then* $\mathbf{F}(G)$ *has enough total algebraic operations.*

Proof. Identical to the proof of Theorem 5.7. □

COROLLARY 6.8 *Every dualizable flat graph algebra is strongly dualizable.*

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