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Discriminating varieties

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Dedicated to Bjarni Jónsson on the occasion of his 70th birthday

Abstract. In this paper we determine those locally finite varieties that generate decidable discriminator varieties when augmented by a ternary discriminator term.

1. Introduction

The work of Burris, McKenzie and Valeriote on the decidability of varieties has pointed out the importance of understanding the structure of the decidable locally finite discriminator varieties. It is shown in [10] that if a locally finite variety $\mathscr{V}$ has a decidable theory then it must decompose as the varietal product of three special kinds of varieties: strongly Abelian; affine; and discriminator. This decomposition effectively reduces the overall problem of characterizing the decidable locally finite varieties down to the above three special cases.

In this paper we apply some recent results on discriminator varieties to arrive at a description of those locally finite varieties that generate decidable discriminator varieties when a ternary discriminator term is adjoined. Before we can state our result we will need to give some definitions and state some known results.

DEFINITION 1.1. A variety $\mathscr{V}$ in the language L is said to be **decidable** if Th($\mathscr{V}$), the set of all first-order sentences satisfied by all members of $\mathscr{V}$, is recursive (under a suitable encoding of the language). A variety is **hereditarily undecidable** if every subtheory of Th($\mathscr{V}$) is undecidable.

For an explanation of general methods of establishing decidability and undecidability please consult [4] or [10]. We will not make direct reference to these

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techniques in this paper although the main lemmas are derived (elsewhere) using them.

Discriminator varieties are very specialized varieties that can be regarded as generalizations of the variety of Boolean algebras. On any set A we can define the operation $t_A(x, y, z)$ as follows:

$$t_A(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}.$$

The operation t_A is called the **ternary discriminator** on A.

DEFINITION 1.2. A variety $\mathscr{V}$ is called a **discriminator variety** iff there exists a term $t(x, y, z)$ in the language of $\mathscr{V}$ such that $\mathscr{V}$ is $\mathsf{V}(\mathscr{K})$, the variety generated by $\mathscr{K}$, for some class $\mathscr{K}$ such that for all $\mathbf{A} \in \mathscr{K}$, $t^{\mathbf{A}} = t_A$ (i.e., the term t defines the discriminator on the universe of $\mathbf{A}$). Such a term t is called a **discriminator term** for $\mathscr{V}$.

Clearly any variety generated by a primal algebra, in particular the variety of Boolean algebras, is a discriminator variety.

An important theorem of S. Bulman-Fleming and H. Werner describes the structure of the members of a discriminator variety. The theorem and its proof may be found in [4]; however, we do not need it in this paper. All that we need is the following observation.

PROPOSITION 1.3. *Let $\mathscr{V}$ be a discriminator variety with discriminator term $t(x, y, z)$. Let $\mathscr{K}$ be the class of all $\mathbf{A} \in \mathscr{V}$ for which $t^{\mathbf{A}} = t_A$.* Clearly $\mathscr{V} = \mathsf{V}(\mathscr{K})$. *Moreover, $\mathscr{K}$ is a universal class (i.e., it is defined by a set of universal first order sentences).*

It follows from the above proposition that every discriminator variety is term equivalent to a variety of the form $\mathsf{V}(\mathscr{K}^t)$, where $\mathscr{K}$ is a universal class of algebras (in some language L), and $\mathscr{K}^t$ is the class obtained by adjoining a new ternary function symbol t to L and interpreting it as the discriminator operation in all members of $\mathscr{K}$. The variety $\mathsf{V}(\mathscr{K}^t)$ that we form in this way is said to have been obtained by **discriminating** the class $\mathscr{K}$. These comments suggest that in principle one can reach an understanding of the decidable locally finite discriminator varieties in terms of the universal classes that give rise to them upon discriminating. This is the approach that several authors have been following and the one that we adopt in this paper. Actually, we shall have little to say about universal classes in general. Our main contribution in this paper is a complete description of the locally finite *varieties* $\mathscr{V}$ such that $\mathsf{V}(\mathscr{V}^t)$ is decidable.

The reader interested in some of the history of the problem will find the articles [2], [3] and [5] useful.

2. Tools

In this section we will state, without proof, the lemmas that are used to establish the main result of this paper. We will also describe the structure of the locally finite strongly Abelian varieties $\mathscr{S}$ that when discriminated produce decidable discriminator varieties, since this description will be needed later on. The first two lemmas were proved in [3].

LEMMA 2.1. *Let* $\mathbf{A}$ *be an* L*-algebra with an infinite subset* M. *Let* $\mathbf{S}$ *be the subalgebra of* $\mathbf{A}$ *generated by* M. *Suppose the following holds.*

(i) *In* $\mathbf{A}$ *there is a first order definable relation, Equiv*(x, y), *whose restriction to* S *defines an equivalence relation* $\equiv$ *such that*
   (a) *no 2 elements from* M *are related under* $\equiv$,
   (b) $\equiv$ *is invariant under automorphisms of* $\mathbf{S}$.
(ii) *any bijection between two finite subsets of* M *extends to an automorphism of* $\mathbf{S}$.
(iii) *for* $J \subseteq M$ *finite, the only elements from* M *that are* $\equiv$ *related to an element from the subalgebra of* $\mathbf{A}$ *generated by* J *belong to* J.
(iv) *there is a first order formula,* $\mu(x)$, *such that for* $a \in S$, $\mathbf{A} \models \mu(a)$ *if and only if* $a \equiv m$ *for some* $m \in M$.
(v) *there is a first order formula,* $\tau(x)$, *such that* $\mathbf{A} \models \tau(a)$ *for at least two elements from* M *and* $\mathbf{A} \models \neg\tau(b)$ *for* κ *many elements from* M, *where* κ *is some infinite cardinal.*
(vi) *the subsets of* A *defined by* μ *and* τ *are closed under* $\equiv$.

Then the class of all graphs of size at most κ *can be semantically embedded into the class* $\mathsf{P}_s(\mathbf{A}^t)$, *and hence this class is hereditarily undecidable.*

DEFINITION 2.2. A finite algebra is said to be **homogeneous** if every isomorphism between subalgebras extends to an automorphism of the algebra. For $\mathscr{K}$ a class of algebras, we way that $\mathscr{K}$ is **homogeneous** if every finite member of $\mathscr{K}$ is.

LEMMA 2.3. *Let* $\mathscr{K}$ *be a homogeneous, locally finite, finitely axiomatizable universal class of algebras with a finite language. Then the variety* $\mathsf{V}(\mathscr{K}^t)$ *is decidable.*

We now embark on a brief description of the locally finite strongly abelian varieties $\mathscr{V}$ such that $\mathsf{V}(\mathscr{V}^t)$ is decidable.

DEFINITION 2.4. An algebra is called **strongly Abelian** if for all terms $t(x, \bar{y})$, for all a, b, $\bar{c}$, $\bar{d}$ and $\bar{e}$,

$$t(a, \bar{c}) = t(b, \bar{d}) \rightarrow t(a, \bar{e}) = t(b, \bar{e}).$$

A variety is called **strongly Abelian** if each of its members is.

Strongly Abelian algebras can be viewed as generalizations of unary algebras. They were first introduced by Ralph McKenzie in [9]. The significance of these algebras, especially in the role they play in the classification of finite algebras and locally finite varieties has been demonstrated in [7, 9, 10].

The following construction produces strongly Abelian algebras and varieties that are not essentially unary. Let L be a language for k-sorted unary algebras. We construct a language L_k of (one-sorted) algebras as follows: for each sequence $f_1, \ldots, f_k$ of function symbols or unary projections of L, where the sort of the codomain of f_i is i, we include in L_k the k-ary function symbol $[f_1, \ldots, f_k]$. Note that if L is finite, then so is L_k.

For $\mathbf{A}$ an L algebra with universes $A_1, \ldots, A_k$, we define an L_k algebra $\mathbf{A}[k]$ in the following way: the universe of $\mathbf{A}[k]$ is $A_1 \times \cdots \times A_k$ and for each $[f_1, \ldots, f_k]$ from L_k, define

$$[f_1, \ldots, f_k]^{\mathbf{A}[k]}(\vec{a}_1, \ldots, \vec{a}_k) = \langle f_1^{\mathbf{A}}(a_1^{i_1}), \ldots, f_k^{\mathbf{A}}(a_k^{i_k}) \rangle,$$

where $\vec{a}_i = \langle a_i^1, \ldots, a_i^k \rangle$ for all $i \leq k$ and for all $j \leq k$, the sort of the domain of $f_j^{\mathbf{A}}$ is i_j.

If $\mathcal{V}$ is a variety of k-sorted algebras of type L then it can be shown that the class $\mathcal{V}(k] = \{\mathbf{B}\colon \mathbf{B}$ is isomorphic to $\mathbf{A}[k]$ for some $\mathbf{A}$ from $\mathcal{V}\}$ is a strongly Abelian variety of L_k algebras.

The proof of the next theorem can be found in [3].

THEOREM 2.5. *Let $\mathcal{V}$ be a locally finite strongly Abelian variety of finite type. Then $\mathsf{V}(\mathcal{V}^t)$ is decidable if and only if $\mathcal{V}$ is term equivalent to a variety $\mathcal{W}[k]$, where $\mathcal{W}$ is a k-sorted unary variety such that every term of $\mathcal{W}$ is either constant or left invertible (i.e., for every term $t(x)$, either $\mathcal{W} \models t(x) = t(y)$ or there is some term $h(y)$ of the appropriate sort such that $\mathcal{W} \models h(t(x)) = x$).*

We would like to point out that every locally finite strongly Abelian variety of finite type is finitely axiomatizable [10, Theorem 0.17]; and if such a variety satisfies the conditions of the above theorem then it is homogeneous. (We leave the verification of this to the reader.) Thus Lemma 2.3 proves the sufficiency of these conditions. Their necessity can be established using Lemma 2.1; see [3]. We will use Lemma 2.3 in the section where we consider affine varieties. Unfortunately Lemma 2.1 turned out to be inadequate for the study of affine varieties. The following lemma, a slight simplification of a result proved by Willard in [12], is used in its place.

LEMMA 2.6. *Let* $\mathbf{A}$ *be an algebra of type* L, *and* $\mathbf{S}$ *a subalgebra of* $\mathbf{A}$. *Suppose there exist first-order* L*-formulas* $\mu(\bar{x})$, $\tau(\bar{x})$, *and* $\psi(\bar{z})$ *such that, setting* $M = \mu^{\mathbf{A}}|_S$, $T = M \cap \tau^{\mathbf{A}}|_S$, *and* $\mathrm{Aut}_M \mathbf{S} = \{\sigma \in \mathrm{Aut}\, \mathbf{S}\colon \sigma(M) = M\}$,

(i) M *is infinite while* T *is finite*;

(ii) $M = \bigcup \{\sigma(T)\colon \sigma \in \mathrm{Aut}_M \mathbf{S}\}$;

(iii) $\psi^{\mathbf{A}} \neq \varnothing$ *but* $\psi^{\mathbf{A}}|_S = \varnothing$.

Then the class $\mathsf{P}_s(\{\mathbf{S}, \mathbf{A}\}^t)$ *is hereditarily undecidable.*

3. Affine varieties

DEFINITION 3.1. A variety $\mathscr{V}$ is called **affine** if every algebra in $\mathscr{V}$ is polynomially equivalent to a unitary module over some ring.

It turns out that if $\mathscr{V}$ is affine then in fact there is a single ring $\mathbf{R} = \mathbf{R}(\mathscr{V})$ such that every algebra in $\mathscr{V}$ is polynomially equivalent to a unitary left $\mathbf{R}$-module and conversely, every unitary left $\mathbf{R}$-module is polynomially equivalent to some algebra from $\mathscr{V}$. It is important for us to note that in the case that $\mathscr{V}$ is a locally finite affine variety then the ring $\mathbf{R}(\mathscr{V})$ associated with $\mathscr{V}$ is finite. The details of this equivalence can be found in [2], Chapter 10 or in [6].

In this section we will prove the following theorem.

THEOREM 3.2. *Let* $\mathscr{A}$ *be a locally finite affine variety in a finite language* L. *Let* $\mathbf{R}$ *be the ring* $\mathbf{R}(\mathscr{A})$ *associated with* $\mathscr{A}$. *Then* $\mathsf{V}(\mathscr{A}^t)$ *is decidable (not hereditarily undecidable) if and only if* $\mathbf{R}$ *is semi-simple.*

Before we prove this theorem we will need to prove some preliminary lemmas. The next result was pointed out to us by E. Kiss and W. Hodges. Any of the standard references on ring theory (e.g. [8]) will contain the necessary background.

LEMMA 3.3. *Let* $\mathbf{R}$ *be a finite ring with unit. Then* $\mathscr{M}_{\mathbf{R}}$, *the variety of all unitary left* $\mathbf{R}$*-modules, is homogeneous if and only if* $\mathbf{R}$ *is semi-simple.*

Proof. Suppose that $\mathbf{R}$ is semi-simple. Then using the well known structure theorem for finitely generated modules over a semi-simple artinian ring, one can easily prove that $\mathcal{M}_{\mathbf{R}}$ is homogeneous.

For the converse, suppose that $\mathcal{M}_{\mathbf{R}}$ is homogeneous. To prove that $\mathbf{R}$ is semi-simple, it will suffice to prove that every left ideal, I, of $\mathbf{R}$ is generated by an idempotent element e. Let I be a left ideal of $\mathbf{R}$ and consider the left $\mathbf{R}$-module $R \times I$. The two submodules $I \times \{0\}$ and $\{0\} \times I$ are isomorphic via the map γ that sends $(i, 0)$ to $(0, i)$. By the homogeneity of $R \times I$, it follows that γ extends to an automorphism Γ of $R \times I$. Let $\Gamma(1, 0) = (e, f)$. Then for $i \in I$,

$$(ie, if) = i\Gamma(1, 0) = \Gamma(i, 0) = (0, i),$$

and so $if = i$ for all $i \in I$. From this it follows that $ff = f$ (since $f \in I$) and I is generated by f. □

LEMMA 3.4. *Let $\mathcal{A}$ be an affine variety with ring $\mathbf{R}$. Then $\mathcal{M}_{\mathbf{R}}$ is homogeneous if and only if $\mathcal{A}$ is.*

Proof. Let $\mathcal{M}_{\mathbf{R}}$ be homogeneous and suppose that $\mathbf{A} \in \mathcal{A}$ is finite and $\mathbf{B}$ and $\mathbf{C}$ are two subalgebras of $\mathbf{A}$ that are isomorphic via the map γ. Choose a point 0 from B and let $0' = \gamma(0)$. Then, using the notation of Chapter 10 of [2], we have that the $\mathbf{R}$-modules $M(\mathbf{A}, 0)$ and $M(\mathbf{A}, 0')$ are isomorphic and that the sets B and C are submodules of $M(\mathbf{A}, 0)$ and $M(\mathbf{A}, 0')$, respectively. It also follows that as modules, $\mathbf{B}$ and $\mathbf{C}$ are isomorphic via the map γ.

Now, by the homogeneity of $\mathcal{M}_{\mathbf{R}}$, it follows that we can extend the map γ to an isomorphism Γ between $M(\mathbf{A}, 0)$ and $M(\mathbf{A}, 0')$. This map will also be an automorphism of the algebra $\mathbf{A}$, and since Γ extends γ, we are done.

For the converse, suppose that $\mathcal{A}$ is homogeneous and let M be a finite left $\mathbf{R}$-module with isomorphic submodules $\mathbf{B}$ and $\mathbf{C}$. Let γ be an isomorphism between them. Again using the notation of [2], let $\mathbf{A} = \mathbf{A}(\mathbf{M}, \eta)$, where η is the zero map from the module $\mathbf{N}(\mathcal{A})$ into $\mathbf{M}$. Then by Theorem 10.1 of [2], we have that $\mathbf{A}$ belongs to $\mathcal{A}$. It also follows that B and C are subuniverses of $\mathbf{A}$ and that γ is an isomorphism between $\mathbf{B}$ and $\mathbf{C}$ when viewed as $\mathcal{A}$-algebras. By the homogeneity of $\mathcal{A}$, γ extends to an automorphism Γ of $\mathbf{A}$. It is easy to see that Γ is also an automorphism of $\mathbf{M}$ and so we have proved that $\mathcal{M}_{\mathbf{R}}$ is homogeneous. □

One also learns from Chapter 10 of [2] that every locally finite affine variety of finite type is finitely axiomatizable, and so we have the following result.

COROLLARY 3.5. *Let $\mathscr{A}$ be a locally finite affine variety of finite type such that its associated ring is semi-simple. Then* $\mathsf{V}(\mathscr{A}^t)$ *is decidable.*

To conclude this section we prove the following lemma. Theorem 3.2 will follow from this, and the previous corollary.

LEMMA 3.6. *Let $\mathscr{A}$ be a locally finite affine variety of finite type. If the ring $\mathbf{R} = \mathbf{R}(\mathscr{A})$ associated with $\mathscr{A}$ is not semi-simple, then* $\mathsf{V}(\mathscr{A}^t)$ *is hereditarily undecidable.*

Proof. It will suffice to prove this lemma in the case where the affine variety is in fact the variety $\mathscr{M}_{\mathbf{R}}$ of all unitary left **R**-modules, since, by Chapter 10 of [2], $\mathscr{M}_{\mathbf{R}}$ is interpretable into $\mathscr{A}$ and $\mathsf{V}(\mathscr{M}^t_{\mathbf{R}})$ is interpretable into $\mathsf{V}(\mathscr{A}^t)$.

So suppose that **R** is a finite ring that is not semi-simple. Let J be the Jacobson radical of **R**. Since **R** is not semi-simple, $J \neq \{0\}$ and so J contains some minimal, nontrivial left ideal I. We note that J annihilates the ideal I, so in particular, $I^2 = \{0\}$.

Let **A** be the left **R**-module $R \oplus \bigoplus_{i \in \omega} I$ and let **S** be the submodule $I \oplus \bigoplus_{i \in \omega} I$ of **A**. We will use Lemma 2.6 to prove that $\mathsf{V}(\mathbf{A}^t)$ is hereditarily undecidable.

Let $\langle a_0, a_1, \ldots, a_k \rangle$ be a listing of the elements of I, with $a_0 = 0$ and let $\mu(x_0, x_1, \ldots, x_k)$ be a quantifier free formula in the language of **R**-modules that describes the quantifier free type of the string $\langle a_0, a_1, \ldots, a_k \rangle$. Then $\langle b_0, b_1, \ldots, b_k \rangle \in M = \mu^{\mathbf{A}}|_S$ if and only if the b_i belong to S and the set $\{b_i : 0 \leq i \leq k\}$ forms a submodule of **A** isomorphic to I via the map that sends b_i to a_i for $0 \leq i \leq k$.

Let $\tau(\bar{x})$ be the formula

$$\mu(\bar{x}) \wedge \exists z \left(\bigwedge_{i=0}^{k} x_i = a_i \cdot z \right).$$

Then $\langle b_0, \ldots, b_k \rangle \in T = \tau^{\mathbf{A}}|_S$ if and only if $\{b_0, \ldots, b_k\} = I \oplus \bigoplus_{i \in \omega} \{0\}$ and the map that sends b_i to a_i is an isomorphism of **R**-modules.

By construction we have that M is infinite and T is a finite subset of M. Furthermore, given any $\bar{b}$ from T and $\bar{c}$ from M, there is an automorphism γ of **S** that maps $\bar{b}$ onto $\bar{c}$. This is because the ring $\mathbf{R}/J$ is semi-simple and we can regard **S** as an $\mathbf{R}/J$-module (since J annihilates S). The homogeneity of the class of $\mathbf{R}/J$-modules is enough to guarantee the existence of γ. Note also that since M is defined by a quantifier free formula then it is automatically preserved by γ.

To complete the proof we must find a formula $\psi(z)$ such that $\mathbf{A} \models \psi(a)$ for some a, but $\mathbf{A} \models \neg\psi(b)$ for all $b \in S$. Setting $\psi(z)$ to be the formula

$$a_1 \cdot z \neq 0$$

works, since

$$\mathbf{A} \models \langle a_1, 0, 0, \ldots \rangle = a_1 \cdot \langle 1, 0, 0, \ldots \rangle$$

and

$$0 = a_1 \cdot \alpha$$

for all α from S.

4. The main result

DEFINITION 4.1. Let $\mathcal{V}$ be a variety with subvarieties $\mathcal{U}$ and $\mathcal{W}$. We say that $\mathcal{V}$ is the varietal product of $\mathcal{U}$ and $\mathcal{W}$ and write $\mathcal{V} = \mathcal{U} \otimes \mathcal{W}$ if $\mathcal{V}$ is the join of the varieties $\mathcal{U}$ and $\mathcal{W}$ and there is a term $b(x, y)$ of $\mathcal{V}$ such that $\mathcal{U} \models b(x, y) = x$ and $\mathcal{W} \models b(x, y) = y$.

An important consequence of the above definition is that every algebra in $\mathcal{V}$ is isomorphic to a direct product of an algebra from $\mathcal{U}$ with one from $\mathcal{W}$.

Putting together the results from the last two sections we have the following theorem.

THEOREM 4.2. *Let $\mathcal{V}$ be a locally finite variety in a finite language. Then the following are equivalent*:

(i) $\mathsf{V}(\mathcal{V}^t)$ *is decidable.*

(ii) $\mathsf{V}(\mathcal{V}^t)$ *is not hereditarily undecidable.*

(iii) $\mathcal{V}$ *is the varietal product of a strongly Abelian variety $\mathcal{S}$ and an affine variety $\mathcal{A}$ such that $\mathcal{S}$ satisfies the conditions of Theorem* 2.5 *and the ring associated with $\mathcal{A}$ is semi-simple.*

(iv) $\mathcal{V}$ *is a finitely axiomatizable, homogeneous variety.*

Proof. From Lemma 2.3 we know that if $\mathcal{V}$ is finitely axiomatizable and homogeneous then $\mathsf{V}(\mathcal{V}^t)$ is decidable. Clearly, if a variety is decidable then it is not hereditarily undecidable. Also, if (iii) holds then $\mathcal{V}$ is easily seen to be homogeneous since it is the varietal product of homogeneous varieties. The finite axiomatizability follows from Corollary 14.1 of [10], since in this case $\mathscr{V}$ is decidable and finitely generated. (One can also prove this directly, since each of the factors $\mathscr{S}$ and $\mathscr{A}$ is finitely axiomatizable.)

Finally, suppose that $\mathsf{V}(\mathscr{V}^{\iota})$ is not hereditarily undecidable. Then $\mathscr{V}$ is not hereditarily undecidable, as $\mathscr{V}$ is a reduct of a subclass of $\mathsf{V}(\mathscr{V}^{\iota})$. Then by Theorem 13.10 of [10], $\mathscr{V}$ is of the form $\mathscr{S} \otimes \mathscr{A} \otimes \mathscr{D}$ where $\mathscr{S}$ is strongly Abelian, $\mathscr{A}$ is affine and $\mathscr{D}$ is a discriminator variety. By a theorem of Burris in [1] the variety $\mathscr{D}$ must be trivial, since he proves that $\mathsf{V}(\mathscr{D}^{\iota})$ is hereditarily undecidable whenever $\mathscr{D}$ is a nontrivial discriminator variety. Theorem 2.5 and Theorems 3.2 can now be used to finish the proof. □

5. Other results

In closing, we would like to describe the few results known concerning the decidability of $\mathsf{V}(\mathscr{K}^{\iota})$ when $\mathscr{K}$ is not a variety, and illustrate them by considering their application to universal classes of lattices. The best result prior to 1989 was the theorem of H. Werner [5, 11], building on the work of S. Comer.

THEOREM 5.1. *Suppose $\mathscr{K} = \mathbf{I}(\mathscr{K}_0)$ for some finite set $\mathscr{K}_0$ of finite algebras of finite type. Then $\mathsf{V}(\mathscr{K}^{\iota})$ is decidable.*

We can apply this theorem and Lemma 2.6 to prove the following.

PROPOSITION 5.2. *Let $\mathscr{K}$ be a universal class of distributive lattices. Then $\mathsf{V}(\mathscr{K}^{\iota})$ is decidable iff $\mathscr{K} = \mathbf{I}(\mathscr{K}_0)$ for some finite set $\mathscr{K}_0$ of finite lattices.*

Proof. The sufficiency of the condition follows from Theorem 5.1. To establish its necessity, suppose $\mathscr{K}$ is not of the specified form. Since $\mathscr{K}$ is locally finite and universal, $\mathscr{K}$ must contain finite lattices of arbitrarily large cardinality, and hence of arbitrarily large height (by distributivity). Thus $\mathscr{K}$ contains all finite chains, and hence all chains (by compactness).

Now let $\mathbf{A}$ be the chain $\{x \in \mathbf{Q} : x \leq 0 \text{ or } x \geq 1\}$ under the usual ordering of the rational naumbers, and let $S = A\backslash\{1\}$. Clearly 0 is definable in $\mathbf{A}$ by a formula $\tau(x)$ (since 0 is the unique element of $\mathbf{A}$ having an upper cover), while $\{\sigma(0) : \sigma \in \operatorname{Aut} \mathbf{S}\} = S$. Thus if we let $\mu(x)$ be $x = x$ and $\psi(z)$ be a formula asserting that z has a lower cover, then all the conditions of Lemma 2.6 are met. So $\mathsf{V}(\mathscr{K}^{\iota})$ is undecidable. □

This proof actually shows that if $\mathscr{K}$ is any universal class of lattices of unbounded height, then $\mathsf{V}(\mathscr{K}^{\iota})$ is hereditarily undecidable. Here is another application of Lemma 2.6.

PROPOSITION 5.3. *Let* LAT_3 *be the class of all lattices of height at most* 3. *Then* $\mathsf{V}(\text{LAT}_3^{\iota})$ *is hereditarily undecidable.*

Proof. Let $\mathbf{A}$ be the countably infinite lattice of height 3 having exactly one 4-element chain $0 \prec a \prec b \prec 1$, and let $S = A \setminus \{b\}$. Let $\mu(x)$, $\tau(x)$ and $\psi(z)$ be formulas asserting respectively that x has height 1, x belongs to a 4-element chain, and z has height 2 and belongs to a 4-element chain. Then μ, τ and ψ witness the hypotheses of Lemma 2.6. □

We remark that it is only a little harder to produce a modular lattice $\mathbf{A}$ in LAT_3 which witnesses the conditions of Lemma 2.6.

What about the class LAT_2 of all (modular) lattices of height at most 2? LAT_2 is not homogeneous (for example, the 3-element chain is rigid, yet all of its 1-element sublattices are isomorphic). But intuitively, at least, LAT_2 is nearly homogeneous. Willard [12] has formulated a notion of "almost local homogeneity" which is strong enough to ensure decidability, yet weak enough to encompass LAT_2. We reproduce the definitions and result here.

DEFINITION 5.4.

(1) Suppose $\mathbf{D}$ is a finite algebra and D_0 is a subuniverse. $\mathbf{D}$ is *homogeneous over* D_0 if for all subalgebras $\mathbf{B}$, $\mathbf{B}'$ of $\mathbf{D}$ satisfying $B \cap D_0 = B' \cap D_0$, every isomorphism $\sigma : \mathbf{B} \cong \mathbf{B}'$ satisfying $\sigma|_{B \cap D_0} = \text{id}_{B \cap B_0}$ can be extended to an automorphism $\hat{\sigma}$ of $\mathbf{D}$ satisfying $\hat{\sigma}|_{D_0} = \text{id}_{D_0}$.

(2) If $\mathbf{A}$ is a locally finite algebra and A_0 is a finite subuniverse, then $\mathbf{A}$ is *locally homogeneous over* A_0 if every finite subalgebra $\mathbf{D}$ of $\mathbf{A}$ is homogeneous over $D \cap A_0$.

DEFINITION 5.5. Suppose $\mathscr{K}_0$ is a finite set of finite algebras and $\mathbf{A}$ is an algebra of the same type. A *maximal* $\mathscr{K}_0$*-subuniverse of* $\mathbf{A}$ is a subuniverse A_0 such that (i) either $A_0 = \varnothing$ or $\mathbf{A}_0$ is isomorphic to some member of $\mathscr{K}_0$, and (ii) A_0 is maximal (among subuniverses of $\mathbf{A}$ ordered by inclusion) with respect to property (i).

DEFINITION 5.6. Suppose $\mathscr{K}$ is a locally finite universal class of algebras of finite type. We shall say that $\mathscr{K}$ is *almost locally homogeneous* if there is a finite set

$\mathscr{K}_0$ of finite members of $\mathscr{K}$ satisfying:

(1) $\mathsf{S}(\mathscr{K}_0) \subseteq \mathbf{I}(\mathscr{K}_0)$;

(2) (If the type of $\mathscr{K}$ contains constant symbols): Every 0-generated member of $\mathscr{K}$ is in $\mathbf{I}(\mathscr{K}_0)$;

(3) If $\mathbf{A} \in \mathscr{K}$ and A_0 is a maximal $\mathscr{K}_0$-subuniverse of $\mathbf{A}$, then $\mathbf{A}$ is locally homogeneous over A_0.

LEMMA 5.7. *Suppose $\mathscr{K}$ is a finitely axiomatizable, locally finite, universal class of finite type. If $\mathscr{K}$ is almost locally homogeneous, the $\mathsf{V}(\mathscr{K}^t)$ is decidable.*

Let us return to the example of LAT_2. Let $\mathscr{K}_0$ be the set consisting of a 3-element chain and its subchains. Clearly if $\mathbf{L}$ is any lattice of height at most 1 and L_0 is a maximal $\mathscr{K}_0$-subuniverse of $\mathbf{L}$, then $L_0 = L$ and so $\mathbf{L}$ is automatically locally homogeneous over L_0. On the other hand, if $\mathbf{L}$ is a lattice of height 2, say $\mathbf{L} \cong M_\lambda$, and L_0 is a maximal $\mathscr{K}_0$-subuniverse of $\mathbf{L}$, then $\mathbf{L}_0$ is a 3-element chain consisting of the bottom and top elements of $\mathbf{L}$ and one atom a. Therefore, in verifying that $\mathbf{L}$ is locally homogeneous over L_0, one need only consider height-preserving isomorphisms between finite sublattices of $\mathbf{L}$, and these can always be extended as required. Hence LAT_2 is almost locally homogeneous; so by Lemma 5.7, $\mathsf{V}(\text{LAT}_2^t)$ is decidable.

Using Lemmas 2.6 and 5.7, Willard was able to characterize those locally finite universal classes $\mathscr{K}$ of unary algebras of finite type such that $\mathsf{V}(\mathscr{K}^t)$ is decidable. He found that for such classes $\mathscr{K}$, $\mathsf{V}(\mathscr{K}^t)$ is decidable iff $\mathscr{K}$ is almost locally homogeneous. Thus we pose the following two problems.

PROBLEM 1. Which locally finite universal classes $\mathscr{K}$ of lattices are almost locally homogeneous?

PROBLEM 2. Is it true that if $\mathscr{K}$ is a locally finite universal class of lattices for which $\mathsf{V}(\mathscr{K}^t)$ is decidable, then $\mathscr{K}$ is almost locally homogeneous?

Added in Proof: Problem 2 is solved (negatively) in [13].

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