Decidable discriminator varieties with lattice stalks

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Abstract

We determine those universal classes of lattices which generate a decidable discriminator variety when augmented by a ternary discriminator term. They are the locally finite universal classes whose finite members are almost homogeneous.

1 Introduction

We are interested in determining which varieties have a decidable first-order theory. For locally finite varieties, R. McKenzie and M. Valeriote [7] have succeeded in reducing this problem to two special subproblems:

PROBLEM 1: *For which finite rings R with unit is the variety of unitary left R-modules decidable?*

PROBLEM 2: *Which locally finite discriminator varieties (in a finite language) are decidable?*

In this paper we continue our investigation [10, 8] of Problem 2.

The *ternary discriminator* on the set A is the function $t_A : A^3 \to A$ given by $t_A(x, y, z) = x$ if $x \neq y$, $t(x, x, z) = z$. A *discriminator variety* is a variety $\mathcal{V}$ for which there is a term $t(x, y, z)$ in the language of $\mathcal{V}$ which defines the ternary discriminator on the universe of every nontrivial subdirectly irreducible member of $\mathcal{V}$.

There is a canonical way to generate discriminator varieties. Let $\mathcal{K}$ be a universal (i.e., definable by a set of universal first-order sentences) class of algebras in the language L, let t be a ternary operation symbol not occurring in L, and let $\mathsf{L}(\mathsf{t})$ denote the language $\mathsf{L} \cup \{\mathsf{t}\}$. If $\mathbf{A} \in \mathcal{K}$ then $\mathbf{A}^{\mathsf{t}}$ denotes the $\mathsf{L}(\mathsf{t})$-algebra $\langle \mathbf{A}, t_A \rangle$, and $\mathcal{K}^{\mathsf{t}}$ denotes $\{\mathbf{A}^{\mathsf{t}} : \mathbf{A} \in \mathcal{K}\}$. Then the variety $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ generated by $\mathcal{K}^{\mathsf{t}}$ is a discriminator variety, and every discriminator variety is definitionally equivalent to one of that form.

In Section 4 we consider discriminator varieties of the form $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ where $\mathcal{K}$ is a universal class of lattices, and describe which classes produce decidable discriminator varieties. One characterization is the following. Let $\mathbf{Q}$ be the chain of rational numbers under their usual ordering; let $\mathbf{N}_\omega$ be the unique countably infinite lattice of height 3 having exactly one 4-element chain; let $\mathbf{M}_{\omega,2}$ be the unique countably infinite modular lattice of height 3 having exactly two coatoms (see Figure 1); and let $\mathbf{M}_{2,\omega}$ be the dual of $\mathbf{M}_{\omega,2}$. Then for any universal class $\mathcal{K}$ of lattices, $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is decidable if and only if $\mathcal{K}$ omits each of $\mathbf{Q}$, $\mathbf{N}_\omega$, $\mathbf{M}_{\omega,2}$, and $\mathbf{M}_{2,\omega}$.

The result we shall use to establish undecidability is the following special case of Lemma 4.1 from [10].

LEMMA 1.1 *Let $\mathbf{A}$ be an algebra in the language L, and $\mathbf{S}$ a subalgebra of $\mathbf{A}$. Suppose there exist first-order L-formulas $\mu(x)$, $\tau(x)$, and $\psi(z)$ such that, setting $M = \mu^{\mathbf{A}}|_S$, $T = M \cap \tau^{\mathbf{A}}|_S$, and $\mathrm{Aut}_M\mathbf{S} = \{\sigma \in \mathrm{Aut}\,\mathbf{S} : \sigma(M) = M\}$,*

- **(i)** *M is infinite while T is finite;*
- **(ii)** *$M = \bigcup\{\sigma(T) : \sigma \in \mathrm{Aut}_M\mathbf{S}\}$;*
- **(iii)** *$\psi^{\mathbf{A}} \neq \emptyset$ but $\psi^{\mathbf{A}}|_S = \emptyset$.*

Then the class $\mathsf{P}_{\mathrm{s}}(\{\mathbf{S}, \mathbf{A}\}^{\mathsf{t}})$ is hereditarily undecidable.

The reader may already see how Lemma 1.1 can be applied to each of the four forbidden lattices mentioned above (for $\mathbf{Q}$ one must choose $\mathbf{A}$ to be a proper sublattice of $\mathbf{Q}$).

Most of this paper is devoted to establishing that $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is decidable assuming that $\mathcal{K}$ omits $\mathbf{Q}$, $\mathbf{N}_\omega$, $\mathbf{M}_{\omega,2}$ and $\mathbf{M}_{2,\omega}$. The proof requires an improvement of existing technology for analyzing the first-order theory of discriminator varieties. Specifically, we isolate a version of homogeneity which is weaker than the one considered in [10], and we prove its relevance to the general Problem 2. This is the real contribution of this paper.

We adopt the following conventions. $\mathbf{A} \leq_{bp} \prod_{x\in X} \mathbf{A}_x$ means that $\mathbf{A}$ is a Boolean product of the family $(\mathbf{A}_x)_{x\in X}$ (see [3] or [4]). We refer to the

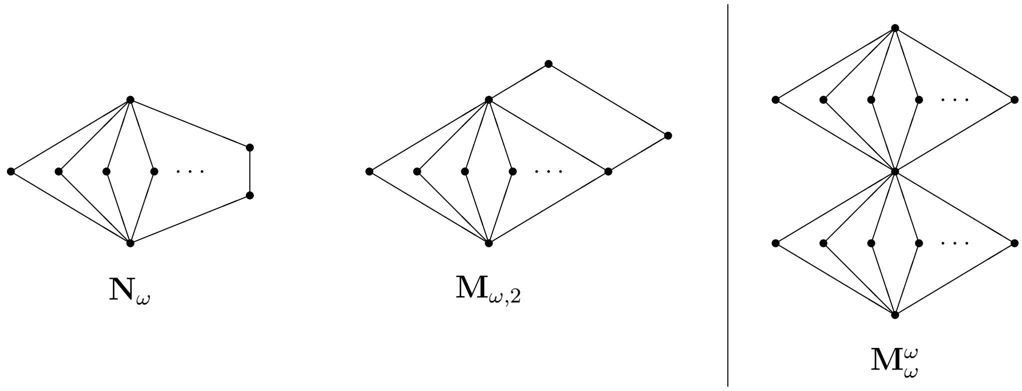

Figure 1:

members of $(\mathbf{A}_x)_{x\in X}$ as the *stalks* of $\mathbf{A}$. For a class $\mathcal{K}$, $\Gamma^a(\mathcal{K})$ denotes the class of all Boolean products with stalks from $\mathcal{K}$. When referring to a particular Boolean product of $(\mathbf{A}_x)_{x\in X}$ we shall assume that the Boolean topology on X is specified.

If $\mathbf{A} \leq_{bp} \prod_{x\in X} \mathbf{A}_x$ then A is a subuniverse of the $\mathsf{L}(\mathsf{t})$-algebra $\prod_{x\in X} \mathbf{A}_x^{\mathsf{t}}$. We denote the corresponding subalgebra of $\prod_{x\in X} \mathbf{A}_x^{\mathsf{t}}$ by $\mathbf{A}(\mathsf{t})$. Clearly $\mathbf{A}(\mathsf{t}) \in \mathsf{V}(\mathcal{K}^{\mathsf{t}})$. Conversely:

THEOREM 1.2 *(Bulman-Fleming, Werner [1]) Suppose $\mathcal{K}$ is a universal class of algebras.*

- **(i)** *If $\mathbf{B} \in \mathsf{V}(\mathcal{K}^{\mathsf{t}})$ and $|B| > 1$, then $\mathbf{B} \cong \mathbf{A}(\mathsf{t})$ for some $\mathbf{A} \in \Gamma^a(\mathcal{K})$.*
- **(ii)** *If moreover $\mathbf{B}$ is countable, then the Boolean product $\mathbf{A} \leq_{bp} \prod_{x\in X} \mathbf{A}_x$ referred to in (i) can be chosen so that the Boolean topology on X has only countably many clopen sets.*

If $\mathcal{K}$ is a class of algebras in the language L, then $\mathrm{Th}_\forall(\mathcal{K})$ denotes the set of all universal (first-order) L-sentences true in all members of $\mathcal{K}$. $\mathcal{K}_{fin}$ is the class of all finite members of $\mathcal{K}$. A lattice has height n if it contains an $(n+1)$-element chain but no $(n+2)$-element chain. We say that an algebra is n-generated if it is generated by a subset of cardinality at most n.

2 Homogeneity

The study of decidable discriminator varieties has recently become intertwined with the study of hereditarily homogeneous algebras. A locally finite algebra $\mathbf{A}$ is called *homogeneous* if every isomorphism between finite subalgebras of $\mathbf{A}$ can be extended to an automorphism of $\mathbf{A}$. If $\mathcal{K}$ is a locally finite universal class of algebras in a finite language, then we say that $\mathcal{K}$ is homogeneous provided each of its finite members is homogeneous. If $\mathcal{K}$ is homogeneous in this sense, then every countable member of $\mathcal{K}$ is *hereditarily homogeneous*, that is, each of its subalgebras is homogeneous.

In this section we present several "finite perturbations" of the notion of $\mathcal{K}$ being homogeneous, each involving the existence of a finite (possibly empty) set $\mathcal{K}_0$ of finite members of $\mathcal{K}$ satisfying

(1) $\mathsf{S}(\mathcal{K}_0) \subseteq \mathrm{I}(\mathcal{K}_0)$.

(2) (If the language of $\mathcal{K}$ contains constant symbols): Every 0-generated member of $\mathcal{K}$ is in $\mathrm{I}(\mathcal{K}_0)$.

Definition 2.1 Suppose $\mathcal{K}$ is a locally finite universal class of algebras in a finite language, $\mathcal{K}_0$ is a finite set of finite members of $\mathcal{K}$ satisfying conditions (1) and (2) above, and $\mathbf{A} \in \mathcal{K}$. A *maximal $\mathcal{K}_0$-subuniverse of* $\mathbf{A}$ is a subuniverse A_0 such that (i) either $A_0 = \emptyset$ or $\mathbf{A}_0$ is isomorphic to some member of $\mathcal{K}_0$, and (ii) A_0 is maximal (among subuniverses of $\mathbf{A}$ ordered by inclusion) with respect to property (i).

Observe that if $\mathcal{K}_0$ satisfies conditions (1) and (2), then every algebra in $\mathcal{K}$ has at least one maximal $\mathcal{K}_0$-subuniverse; and if A_0 is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{A}$ and $\mathbf{B} \leq \mathbf{A}$, then $A_0 \cap B$ can be extended to a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{B}$ (in particular, if $A_0 \subseteq B$ then A_0 is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{B}$).

The perturbations of homogeneity that we have in mind are most easily stated in terms of "extension conditions," by which we mean the following. Suppose $\mathbf{A}$ is an algebra, A_0 is a subuniverse, and $\mathbf{B}$ and $\mathbf{B}'$ are subalgebras of $\mathbf{A}$ satisfying $B \cap A_0 = B' \cap A_0$. Then the *extension condition* $ext(\mathbf{A}, A_0, B, B')$ is the claim that every isomorphism $\sigma : \mathbf{B} \cong \mathbf{B}'$ satisfying $\sigma|_{B\cap A_0} = \mathrm{id}_{B\cap A_0}$ can be extended to an automorphism $\hat{\sigma}$ of $\mathbf{A}$ satisfying $\hat{\sigma}|_{A_0} = \mathrm{id}_{A_0}$. So for example, a locally finite algebra $\mathbf{A}$ is homogeneous if and only if $ext(\mathbf{A}, \mathrm{Sg}^{\mathbf{A}}(\emptyset), B, B')$ holds for all finite subalgebras $\mathbf{B}$ and $\mathbf{B}'$ of $\mathbf{A}$.

Definition 2.2 Let $\mathcal{K}$ be a locally finite universal class of algebras in a finite language, and $n \in \{0, 1, 2, 3\}$. $\mathcal{K}$ *satisfies* H_n if there exists a finite set $\mathcal{K}_0$ (called a *witnessing set*) of finite members of $\mathcal{K}$ satisfying the conditions (1) and (2) above and

- (if $n = 0$) for every $\mathbf{A} \in \mathcal{K}_{fin}$ there exists a maximal $\mathcal{K}_0$-subuniverse A_0 of $\mathbf{A}$ such that for all subalgebras $\mathbf{B}$, $\mathbf{B}'$ of $\mathbf{A}$ satisfying $A_0 \subseteq B$ and $A_0 \subseteq B'$, $ext(\mathbf{A}, A_0, B, B')$ holds.
- (if $n = 1$) for every $\mathbf{A} \in \mathcal{K}_{fin}$ and every maximal $\mathcal{K}_0$-subuniverse A_0 of $\mathbf{A}$, and for all subalgebras $\mathbf{B}$, $\mathbf{B}'$ of $\mathbf{A}$ satisfying $A_0 \subseteq B$ and $A_0 \subseteq B'$, $ext(\mathbf{A}, A_0, B, B')$ holds.
- (if $n = 2$) for every $\mathbf{A} \in \mathcal{K}_{fin}$ and every maximal $\mathcal{K}_0$-subuniverse A_0 of $\mathbf{A}$, and for all subalgebras $\mathbf{B}$, $\mathbf{B}'$ of $\mathbf{A}$ for which $B \cap A_0 = B' \cap A_0$, if $B \cap A_0$ is itself a maximal $\mathcal{K}_0$-subuniverse of both $\mathbf{B}$ and $\mathbf{B}'$ then $ext(\mathbf{A}, A_0, B, B')$ holds.
- (if $n = 3$) for every $\mathbf{A} \in \mathcal{K}_{fin}$ and every subuniverse A_0 of $\mathbf{A}$, if there exists $\mathbf{C} \in \mathcal{K}$ and a maximal $\mathcal{K}_0$-subuniverse C_0 of $\mathbf{C}$ such that $\mathbf{A} \leq \mathbf{C}$ and $A_0 = A \cap C_0$, then for all subalgebras $\mathbf{B}$, $\mathbf{B}'$ of $\mathbf{A}$ for which $B \cap A_0 = B' \cap A_0$, $ext(\mathbf{A}, A_0, B, B')$ holds.

For the sake of completeness, we shall also say that $\mathcal{K}$ satisfies H_∞ if $\mathcal{K}$ is homogeneous. H_∞ implies H_3 because if $\mathcal{K}$ is homogeneous then the set $\mathcal{K}_0$ consisting of one member from each isomorphism class of the 0-generated members of $\mathcal{K}$ witnesses H_3. And clearly $\mathrm{H}_3 \Rightarrow \mathrm{H}_2 \Rightarrow \mathrm{H}_1 \Rightarrow \mathrm{H}_0$ since H_n requires fewer extension conditions than does H_{n+1} with respect to a given set $\mathcal{K}_0$.

To indicate the extent of these conditions, we remark that (i) every universal class of the form $\mathsf{IS}(\mathcal{K}_0)$, where $\mathcal{K}_0$ is a finite set of finite algebras, satisfies H_3 ($\mathsf{S}(\mathcal{K}_0)$ is its witnessing set); (ii) H_0 is equivalent to the following condition: there exist locally finite universal classes $\mathcal{K}_1, \ldots, \mathcal{K}_r$ in the languages $\mathsf{L}_1, \ldots, \mathsf{L}_r$ respectively, where each L_i is an expansion of L by finitely many constant symbols, such that each $\mathcal{K}_i$ satisfies H_∞ and $\mathcal{K} = \bigcup_{i=1}^r \mathcal{K}_i|_{\mathsf{L}}$; (iii) for each $n < 3$, H_n is strictly weaker than H_{n+1} (this will be shown presently).

S. Burris, R. McKenzie, and M. Valeriote proved in [2] the following theorem: if $\mathcal{K}$ is a locally finite, finitely axiomatizable universal class of algebras in a finite language, and if $\mathcal{K}$ satisfies H_∞, then $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is decidable. In a recent paper [10] we showed that the last hypothesis can be weakened to H_3. (In [10, 8], H_3 was called *almost local homogeneity*.) In the present paper we shall further weaken the hypothesis to H_2. We conjecture that the decidability of $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is equivalent to some condition lying between H_1 and H_2, although at present we do not even know that H_0 is necessary for decidability.

An example of a class satisfying H_2 but not H_3 is the smallest universal class $\mathcal{K}$ of lattices containing the lattice $\mathbf{M}_\omega^\omega$ pictured in Figure 1. A set witnessing H_2 for $\mathcal{K}$ is the set consisting of a 5-element chain and its subchains. $\mathcal{K}$ does not satisfy H_3, for if $\mathcal{K}_0$ were a witnessing set for H_3 and C_0 were a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{M}_\omega^\omega$, then because C_0 is finite it would be possible to find a finite sublattice $\mathbf{A}$ of $\mathbf{M}_\omega^\omega$ and an atom x and a coatom y of $\mathbf{M}_\omega^\omega$ which belong to A but not to C_0. Put $A_0 = A \cap C_0$, $B = \{x\}$ and $B' = \{y\}$; then the extension condition $ext(\mathbf{A}, A_0, B, B')$ should hold, but doesn't. This example settles a question from [10] and one from [8].

An example of a class satisfying H_0 but not H_1 is the smallest universal class of lattices containing the lattice $\mathbf{N}_\omega$. A set witnessing H_0 is the set consisting of a 4-element chain and its subchains. It is shown in Section 4 that this class does not satisfy H_1.

A class satisfying H_1 but not H_2 can be constructed as follows: in the language $\{f, b\}$, where f is unary and b is binary, let $\mathbf{A}$ be the algebra with universe $\{0, 1, 2, \ldots\} \cup \{a, b\}$ and operations specified by $f(n) = n$ for all $n < \omega$, $f(a) = b$, $f(b) = a$; and $b(x, y) = x$ unless $(x, y) = (0, a)$, while $b(0, a) = a$. Let $\mathcal{K}$ be the smallest universal class containing $\mathbf{A}$; it is locally finite and finitely axiomatizable. Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}$ whose universe is $\{a, b\}$, and let $\mathcal{K}_0 = \{\mathbf{C}\}$. Then $\mathcal{K}_0$ witnesses H_1 for $\mathcal{K}$. On the other hand, Lemma 1.1 can be used to show that $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is undecidable, so by the results in the next section, $\mathcal{K}$ does not satisfy H_2. This example refutes what we once thought was a plausible conjecture, namely, that $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is decidable if and only if $\mathcal{K}$ satisfies H_1.

3 Decidability

Until further notice, assume that $\mathcal{K}$ is a locally finite universal class of algebras in a finite language L containing no constant symbols, and that $\mathcal{K}_0$ is a finite set of finite members of $\mathcal{K}$ which satisfies $\mathsf{S}(\mathcal{K}_0) \subseteq \mathrm{I}(\mathcal{K}_0)$.

Because L has no constant symbols, it is possible to find a finite L-algebra $\mathbf{H}$ and a collection $\mathcal{C}$ of subuniverses of $\mathbf{H}$ satisfying

1. $\emptyset \in \mathcal{C}$.

2. $\mathrm{I}(\{\mathbf{C} \leq \mathbf{H} : C \in \mathcal{C} \text{ and } C \neq \emptyset\}) = \mathrm{I}(\mathcal{K}_0)$;

3. For every $C \in \mathcal{C}$ with $C \neq \emptyset$, every $\mathbf{A} \in \mathcal{K}_0$ and every embedding $\alpha : \mathbf{C} \hookrightarrow \mathbf{A}$ there exists a $D \in \mathcal{C}$ with $C \subseteq D$ and an isomorphism $\beta : \mathbf{A} \cong \mathbf{D}$ which satisfies and $\beta\alpha(c) = c$ for all $c \in C$.

Fix such a pair $(\mathbf{H}, \mathcal{C})$. The next definition is taken from [10].

Definition 3.1

(1) $\mathcal{K}/\mathbf{H}$ denotes the class of all $\mathbf{A} \in \mathcal{K}$ for which the set $A_0 := A \cap H$ is a common subuniverse of $\mathbf{A}$ and $\mathbf{H}$ and, if nonempty, $\mathbf{A}_0$ inherits the same operations from $\mathbf{A}$ as it does from $\mathbf{H}$.

(2) Suppose $\mathbf{A}, \mathbf{B} \in \mathcal{K}/\mathbf{H}$. An $\mathbf{H}$-*embedding from* $\mathbf{A}$ *to* $\mathbf{B}$ is an embedding $\sigma : \mathbf{A} \hookrightarrow \mathbf{B}$ which satisfies $\sigma(a) = a$ for all $a \in A \cap H$ and $\sigma(a) \notin H$ for all $a \in A \setminus H$. In this case write $\sigma : \mathbf{A} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{B}$. An $\mathbf{H}$-*isomorphism* is an isomorphism which is an $\mathbf{H}$-embedding in the above sense.

(3) If $\mathbf{A}, \mathbf{B} \in \mathcal{K}/\mathbf{H}$, write $\mathbf{A} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{B}$ to mean there exists an $\mathbf{H}$-embedding $\sigma : \mathbf{A} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{B}$, and $\mathbf{A} \overset{\mathbf{H}}{\cong} \mathbf{B}$ to mean there exists an $\mathbf{H}$-isomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Remark: If $\mathbf{A}$ and $\mathbf{B}$ are algebras in some expansion of L and if $\mathbf{A}|_{\mathsf{L}}, \mathbf{B}|_{\mathsf{L}} \in \mathcal{K}/\mathbf{H}$, then by $\mathbf{A} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{B}$ ($\mathbf{A} \overset{\mathbf{H}}{\cong} \mathbf{B}$) we mean the obvious thing, namely, there exists an embedding (isomorphism) from $\mathbf{A}$ to $\mathbf{B}$ which is an $\mathbf{H}$-embedding ($\mathbf{H}$-isomorphism) from $\mathbf{A}|_{\mathsf{L}}$ to $\mathbf{B}|_{\mathsf{L}}$.

Definition 3.2

(1) $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ is the class of those $\mathbf{A} \in \mathcal{K}/\mathbf{H}$ for which $A \cap H$ is both an element of $\mathcal{C}$ and a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{A}$.

(2) $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ is the class of all Boolean products $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$ which satisfy the following properties:

(i) $\mathbf{A}_x \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ for all $x \in X$.

(ii) For all $a \in A$ and $c \in \bigcup \mathcal{C}$, the set $a^{-1}(c)$ is clopen in X.

The next lemma follows from the proof of Lemma 3.4 in [10].

LEMMA 3.3 *With the above assumptions, for every countable member* $\mathbf{A}$ *of* $\Gamma^a(\mathcal{K})$ *there exists* $\mathbf{A}' \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ *having the same underlying Boolean space as* $\mathbf{A}$ *and satisfying* $\mathbf{A}(\mathsf{t}) \cong \mathbf{A}'(\mathsf{t})$*.* ▮

As in [2] and [10], the idea now is to show that if the stalks $\mathbf{A}_x$ of $\mathbf{A} \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ are "$\mathbf{H}$-homogeneous" in some specified sense, then one can achieve a satisfactory Feferman-Vaught-type analysis of $\mathbf{A}(\mathsf{t})$.

First, we impose one further requirement on the choice of $\mathbf{H}$ and $\mathcal{C}$, namely,

4. $\langle \mathcal{C}, \subseteq \rangle$ is a tree with $\emptyset$ being the root.

Next, we strengthen the notion of *quantifier-free* $\mathbf{H}$*-*n*-type* found in [10]. Let $\mathbf{A} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ with $A_0 = A \cap H$, and suppose $\bar{a} = \langle a_1, \ldots, a_n \rangle \in A^n$ ($n \geq 0$). Define $\mathrm{C}_{\mathbf{A}}(\bar{a})$ to be the $\subseteq$-least member C of $\mathcal{C}$ satisfying these properties:

1. $\mathrm{Sg}^{\mathbf{A}}(\{a_1, \ldots, a_n\} \cup C) \cap A_0 = C$.
2. (if $n > 0$) C is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{Sg}^{\mathbf{A}}(\{a_1, \ldots, a_n\} \cup C)$.

Clearly $C := A_0$ satisfies these properties, so $\mathrm{C}_{\mathbf{A}}(\bar{a})$ exists because $\mathcal{C}$ is a tree. With $\mathrm{C}_{\mathbf{A}}(\bar{a})$ so defined, $\mathrm{M}_{\mathbf{A}}(\bar{a})$ denotes the subuniverse of $\mathbf{A}$ generated by $\{a_1, \ldots, a_n\} \cup \mathrm{C}_{\mathbf{A}}(\bar{a})$. Thus if $n > 0$ then $\mathbf{M}_{\mathbf{A}}(\bar{a})$ is the smallest subalgebra of $\mathbf{A}$ which belongs to $\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$ and whose universe contains $\{a_1, \ldots, a_n\}$, while if $n = 0$ then $\mathrm{C}_{\mathbf{A}}() = \mathrm{M}_{\mathbf{A}}() = \emptyset$.

Definition 3.4 Fix an infinite sequence $\mathsf{b}_1, \mathsf{b}_2, \ldots$ of constant symbols not occurring in L. With $\mathbf{A}$ and $\bar{a}$ as above, the *quantifier-free* $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$*-*$n$*-type of* $\bar{a}$ *in* $\mathbf{A}$ is (some specification of) the $\mathbf{H}$-isomorphism type of the $\mathsf{L} \cup \{\mathsf{b}_1, \ldots, \mathsf{b}_n\}$-algebra $\langle \mathbf{M}_{\mathbf{A}}(\bar{a}); a_1, \ldots, a_n \rangle$ (assuming $n > 0$; if $n = 0$ then it is $\emptyset$). $\mathrm{typ}(\bar{a})$ will denote the quantifier-free $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$-$n$-type of $\bar{a}$ in $\mathbf{A}$.

For example, $\mathrm{typ}(\bar{a})$ can be taken to be what model theorists call the "quantifier-free n-type of $\bar{a}$ over the set $\mathrm{C}_{\mathbf{A}}(\bar{a})$." Here are some properties of quantifier-free $(\mathbf{H},\mathcal{C},\mathcal{K}_0)$-$n$-types.

LEMMA 3.5 *Suppose* $\mathbf{A},\mathbf{B} \in \mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0)$, $a_1,\ldots,a_n,c \in A$ *and* $b_1,\ldots,b_n$, $d \in B$.

(i) *If* $\langle \mathbf{A}; a_1,\ldots,a_n\rangle \overset{\mathbf{H}}{\hookrightarrow} \langle \mathbf{B}; b_1,\ldots,b_n\rangle$ *then* $\mathrm{typ}(\bar{a}) = \mathrm{typ}(\bar{b})$.

(ii) *If* $\mathrm{typ}(\bar{a},c) = \mathrm{typ}(\bar{b},d)$ *then* $\mathrm{typ}(\bar{a}) = \mathrm{typ}(\bar{b})$.

Proof. (i) is left to the reader. (ii) is trivial if $n = 0$, so suppose $\mathrm{typ}(\bar{a},c) = \mathrm{typ}(\bar{b},d)$ with $n > 0$. By definition this means

$$\langle \mathbf{M}_{\mathbf{A}}(\bar{a},c); a_1,\ldots,a_n,c\rangle \overset{\mathbf{H}}{\cong} \langle \mathbf{M}_{\mathbf{B}}(\bar{b},d); b_1,\ldots,b_n,d\rangle. \tag{1}$$

Hence $\langle \mathbf{M}_{\mathbf{A}}(\bar{a}); a_1,\ldots,a_n\rangle \overset{\mathbf{H}}{\hookrightarrow} \langle \mathbf{A}; a_1,\ldots,a_n\rangle$ on the one hand, and

$$\begin{aligned}
\langle \mathbf{M}_{\mathbf{A}}(\bar{a}); a_1,\ldots,a_n\rangle &\overset{\mathbf{H}}{\hookrightarrow} \langle \mathbf{M}_{\mathbf{A}}(\bar{a},c); a_1,\ldots,a_n\rangle \\
&\overset{\mathbf{H}}{\cong} \langle \mathbf{M}_{\mathbf{B}}(\bar{b},d); b_1,\ldots,b_n\rangle \qquad \text{by equation (1)} \\
&\overset{\mathbf{H}}{\hookrightarrow} \langle \mathbf{B}; b_1,\ldots,b_n\rangle
\end{aligned}$$

on the other. So by (i), $\mathrm{typ}(\bar{a}) = \mathrm{typ}(\bar{b})$. ∎

Suppose $\mathbf{A} \in \mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0)$ and $\bar{a} \in A^n$. Then $\mathbf{M}_{\mathbf{A}}(\bar{a})$ is N-generated, where $N = n + |H|$. There are only finitely many N-generated algebras in $\mathcal{K}$ up to isomorphism (observation of Weispfennig [9]). Therefore for each $n \geq 0$ there are only finitely many quantifier-free $(\mathbf{H},\mathcal{C},\mathcal{K}_0)$-$n$-types realized in $\mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0)$. Let them be called $p_{n,1},\ldots,p_{n,\lambda(n)}$. The notation $p_{n+1,i} \vdash p_{n,j}$ shall mean the obvious thing, i.e., for all $\mathbf{A} \in \mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0)$ and $a_1,\ldots,a_n,c \in A$, if $\mathrm{typ}(\bar{a},c) = p_{n+1,i}$ then $\mathrm{typ}(\bar{a}) = p_{n,j}$. By Lemma 3.5(ii), for each $i = 1,\ldots,\lambda(n{+}1)$ there is a (unique) j such that $p_{n+1,i} \vdash p_{n,j}$. Similarly, if $\phi(x_1,\ldots,x_n)$ is a quantifier-free $\mathsf{L}(\mathsf{t})$-formula then $p_{n,i} \vdash \phi$ means $\mathbf{A}^{\mathsf{t}} \models \phi(\bar{a})$ whenever $\mathbf{A} \in \Gamma^a(\mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0))$, $\bar{a} \in A^n$, and $\mathrm{typ}(\bar{a}) = p_{n,i}$.

For each $n > 0$ fix a finite set $\mathcal{K}_n$ of finite members of $\mathcal{K}/(\mathbf{H},\mathcal{C},\mathcal{K}_0)$ with the following properties:

1. For each $n > 0$ and $i = 1,\ldots,\lambda(n)$ there exists an $\mathbf{A} \in \mathcal{K}_n$ and $\bar{a} \in A^n$ such that $\mathrm{typ}(\bar{a}) = p_{n,i}$ and $\mathrm{M}_{\mathbf{A}}(\bar{a}) = A$.

2. Conversely, for every $\mathbf{A} \in \mathcal{K}_n$ there exist $\bar{a} \in A^n$ such that $\mathrm{M}_{\mathbf{A}}(\bar{a}) = A$.

3. For all $\mathbf{A}, \mathbf{B} \in \mathcal{K}_n$, if $\mathbf{A} \overset{\mathbf{H}}{\cong} \mathbf{B}$ then $\mathbf{A} = \mathbf{B}$.

4. $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ for all $n > 0$.

Also let $\mathcal{K}_\omega = \bigcup_{n>0} \mathcal{K}_n$.

Now suppose $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$ is in $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$. For each $\mathbf{B} \in \mathcal{K}_\omega$ let $U_{\mathbf{B}} = \{x \in X : \mathbf{B} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{A}_x\}$. Because of item (ii) in the definition of $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$, $U_{\mathbf{B}}$ is an open subset of X. As in [10], $\vec{X}(\mathbf{A})$ denotes $\langle X, (U_{\mathbf{B}})_{\mathbf{B} \in \mathcal{K}_\omega} \rangle$ and $\vec{X}(\mathbf{A})^*$ denotes the dual Boolean algebra with distinguished family of ideals indexed by $\mathcal{K}_\omega$. Finally, if $\bar{a} \in A^n$ and $1 \leq i \leq \lambda(n)$ then by $[\![p_{n,i}(\bar{a})]\!]$ we mean the set of those $x \in X$ for which the quantifier-free $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$-$n$-type of $\langle a_1(x), \ldots, a_n(x) \rangle$ in $\mathbf{A}_x$ is $p_{n,i}$.

The next three lemmas provide the ingredients needed for the general argument from [10] to work.

LEMMA 3.6 *Suppose $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$ is in $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ and $\bar{a} \in A^n$ ($n \geq 0$). Then $([\![p_{n,i}(\bar{a})]\!])_{i=1}^{\lambda(n)}$ is a clopen partition of X.*

PROOF. It is enough to show that each $[\![p_{n,i}(\bar{a})]\!]$ is open. This follows from Lemma 3.5(i) and item (ii) in the definition of $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$. ▮

LEMMA 3.7 *Let $\mathcal{F} = \{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$ be a finite subset of $\mathcal{K}_\omega$. Suppose each $\mathbf{B}_i$ is n_i-generated, and let $N = n_1 + \cdots + n_k + |H| + 1$. Then for every $\mathbf{A} \leq_{bp} \prod_{x \in X} \mathbf{A}_x$ in $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ for which the Boolean algebra X^* of clopen sets of X is countable, there is a subalgebra $\mathbf{A}' \leq \mathbf{A}$ such that*

(i) $\mathbf{A}' \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ *(with the same topology on X).*

(ii) $U'_{\mathbf{B}} = U_{\mathbf{B}}$ *for each $\mathbf{B} \in \mathcal{F}$.*

(iii) $\mathbf{A}'_x$ *is N-generated for each $x \in X$.*

PROOF. This is a minor variation of Lemma 3.5 in [10]. To ensure that $\mathbf{A}'$ is in $\Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ it is enough to have $\mathbf{A}' \leq \mathbf{A}$, A' closed under patchwork, and $A_x \cap H \subseteq A'_x$ for each $x \in X$. The inclusion of $|H|$ in the formula for N is what allows us to meet this last demand. ▮

LEMMA 3.8 *Suppose, in addition to everything else assumed of $\mathcal{K}$ and $\mathcal{K}_0$, that $\mathcal{K}$ satisfies the homogeneity condition* H_2 *and that $\mathcal{K}_0$ is a witnessing set. Suppose further that $\mathbf{A} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$, that $\bar{a} \in A^n$, and that $p_{n+1,i}$ is a quantifier-free $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$-$(n{+}1)$-type satisfying $p_{n+1,i} \vdash \mathrm{typ}(\bar{a})$. Pick $\mathbf{B} \in \mathcal{K}_{n+1}$ and $\langle \bar{b}, d\rangle \in B^{n+1}$ satisfying $\mathrm{M}_{\mathbf{B}}(\bar{b}, d) = B$ and $\mathrm{typ}(\bar{b}, d) = p_{n+1,i}$. Then*

$$(\exists c \in A)(\mathrm{typ}(\bar{a}, c) = p_{n+1,i}) \quad \textit{if and only if} \quad \mathbf{B} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{A}.$$

Proof. ($\Rightarrow$) is obvious.
($\Leftarrow$) The claim is trivial if $n = 0$, so assume $n > 0$. Let $A_0 = A \cap H$ and suppose $\gamma : \mathbf{B} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{A}$. Also let $M = \mathrm{M}_{\mathbf{A}}(\bar{a})$; thus $\mathbf{M} \leq \mathbf{A}$, $\mathbf{M} \in \mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$, and there exists $\alpha : \mathbf{M} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{B}$ satisfying $\alpha(\bar{a}) = \bar{b}$. Finally let $M' = \gamma\alpha(M)$ and $D = \mathrm{Sg}^{\mathbf{A}}(M \cup \gamma(B) \cup A_0)$. We have:

1. $\mathbf{M}$, $\mathbf{M}'$ and $\mathbf{D}$ are finite members of $\mathcal{K}$ with $\mathbf{M} \leq \mathbf{D}$ and $\mathbf{M}' \leq \mathbf{D}$.

2. A_0 is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{D}$.

3. $M \cap A_0 = M' \cap A_0$ $(= \mathrm{C}_{\mathbf{A}}(\bar{a}))$ and the common intersection is a maximal $\mathcal{K}_0$-subuniverse of both $\mathbf{M}$ and $\mathbf{M}'$.

4. $\gamma\alpha : \mathbf{M} \cong \mathbf{M}'$ satisfies $\gamma\alpha|_{M \cap A_0} = \mathrm{id}_{M \cap A_0}$.

Because $\mathcal{K}_0$ witnesses H_2 for $\mathcal{K}$, $\gamma\alpha$ extends to an automorphism μ of $\mathbf{D}$ satisfying $\mu|_{A_0} = \mathrm{id}_{A_0}$. Let $\eta = \mu^{-1}\gamma$ and $c = \eta(d)$. Then $\eta : \mathbf{B} \overset{\mathbf{H}}{\hookrightarrow} \mathbf{A}$ and $\eta(\bar{b}) = \bar{a}$, so by Lemma 3.5(i) and our choice of $\mathbf{B}$, $\mathrm{typ}(\bar{a}, c) = \mathrm{typ}(\eta(\bar{b}), \eta(d)) = \mathrm{typ}(\bar{b}, d) = p_{n+1,i}$. ∎

Here is the principal result of this section.

THEOREM 3.9 *Suppose $\mathcal{K}$ is a locally finite universal class of algebras in a finite language. If $\mathrm{Th}_\forall(\mathcal{K})$ is decidable (for example, if $\mathcal{K}$ is finitely axiomatizable) and $\mathcal{K}$ satisfies* H_2*, then $\mathsf{V}(\mathcal{K}^{\mathrm{t}})$ is decidable.*

Proof. The proof is nearly identical to the proofs of Theorems 2.7 and 3.3 in [10], so we give only a sketch. Both the hypotheses and the conclusion are invariant under the transformation of $\mathcal{K}$ which replaces each constant by a constant unary operation, so we may assume that the language L of $\mathcal{K}$ contains no constant symbols. Let $\mathcal{K}_0$ witness the property H_2 for $\mathcal{K}$, and choose $\mathbf{H}$ and $\mathcal{C}$ as above. Because $\mathrm{Th}_\forall(\mathcal{K})$ is decidable, there exist effective representations of (1) the quantifier-free $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$-$n$-types realized in $\mathcal{K}$, and (2) the members of $\mathcal{K}_\omega$. As well, there are effective procedures which:

1. Given $n \geq 0$, list the quantifier-free $(\mathbf{H}, \mathcal{C}, \mathcal{K}_0)$-$n$-types $p_{n,1}, \dots, p_{n,\lambda(n)}$.
2. Given $n > 0$, list the members of $\mathcal{K}_n$.
3. Given $n > 0$ and $1 \leq i \leq \lambda(n)$, specify a member $\mathbf{B}$ of $\mathcal{K}_n$ for which there exists $\bar{b} \in B^n$ satisfying $\mathrm{typ}(\bar{b}) = p_{n,i}$ and $\mathrm{M}_{\mathbf{B}}(\bar{b}) = B$.
4. Given $n \geq 0$, $1 \leq i \leq \lambda(n+1)$, and $1 \leq j \leq n$, determine whether $p_{n+1,i} \vdash p_{n,j}$.
5. Given $n \geq 0$, $1 \leq i \leq \lambda(n)$, and a quantifier-free $\mathsf{L}(\mathsf{t})$-formula $\phi(x_1, \dots, x_n)$, determine whether $p_{n,i} \vdash \phi$.
6. Given $\mathbf{B} \in \mathcal{K}_\omega$, determine the least integer n such that $\mathbf{B}$ is n-generated.

These facts together with Lemmas 3.6 and 3.8 guarantee (as in [10]) the following Feferman-Vaught-like theorem: There is a recursive procedure which, given an arbitrary $\mathsf{L}(\mathsf{t})$-formula $\phi(x_1, \dots, x_n)$, produces a formula $\Phi(X_1, \dots, X_{\lambda(n)})$ in the first-order language for Boolean algebras with a family of distinguished ideals indexed by $\mathcal{K}_\omega$, such that for all $\mathbf{A} \in \Gamma^a(\mathcal{K}/(\mathbf{H}, \mathcal{C}, \mathcal{K}_0))$ and all $\bar{a} \in A^n$,

$$\mathbf{A}(\mathsf{t}) \models \phi(\bar{a}) \quad \text{if and only if} \quad \vec{X}(\mathbf{A})^* \models \Phi([\![p_{n,1}(\bar{a})]\!], \dots, [\![p_{n,\lambda(n)}(\bar{a})]\!]).$$

Lemma 3.7 then implies (again as in [10]) that there is an effectively computable function from the set of $\mathsf{L}(\mathsf{t})$-sentences to the set of positive integers, written $\phi \mapsto n(\phi)$, with the property that, for each ϕ, $\mathsf{V}(\mathcal{K}^{\mathsf{t}}) \models \phi$ if and only if $\mathsf{V}((\mathcal{K}_{n(\phi)})^{\mathsf{t}}) \models \phi$. Since $\mathcal{K}_{n(\phi)}$ is finite, $\mathsf{V}((\mathcal{K}_{n(\phi)})^{\mathsf{t}})$ is decidable (uniformly in the "input parameter" $\mathcal{K}_{n(\phi)}$) by the result of Burris and Werner [4]. Hence $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is decidable. ∎

4 Lattices

We shall use the following terminology. An M_λ is any lattice of height 2 having at least two atoms. If $\mathbf{L}$ is a lattice, then an M_λ *in* $\mathbf{L}$ is an M_λ which is a sublattice of $\mathbf{L}$. An M_λ is *free in* $\mathbf{L}$ if:

1. It is an interval sublattice $\mathbf{I}[a, b]$ of $\mathbf{L}$, and
2. Is such that each atom of $\mathbf{I}[a, b]$ is both meet-irreducible and join-irreducible in $\mathbf{L}$.

Definition 4.1 A *skeleton of* $\mathbf{L}$ is a subset $S \subseteq L$ satisfying:

(1) If $x \in L \setminus S$ then x is an atom of some free M_λ in $\mathbf{L}$.

(2) For each free M_λ in $\mathbf{L}$, exactly one of its atoms belongs to S.

Every lattice has a skeleton. Skeletons are sublattices, and any two skeletons of the same lattice are isomorphic. The following facts are easily proved from the definition: (i) if $\mathbf{L}$ is a lattice, S is a skeleton of $\mathbf{L}$, and a and b are elements of S such that $a \prec b$ in $\mathbf{S}$, then $a \prec b$ in $\mathbf{L}$; $\mathbf{L}$, then (ii) if $\mathbf{L}$ has a finite skeleton, then $\mathbf{L}$ is locally finite.

Here are two more obvious consequences. Let $\mathbf{L}$ be a lattice with a skeleton S, and let $(b_i)_{i<\eta}$ be the distinct elements of S which are atoms of free M_λs in $\mathbf{L}$. For each $i < \eta$ let X_i be the set of all atoms of the free M_λ containing b_i and let $\lambda_i = |X_i|$. Then: (iii) the isomorphism type of $\mathbf{L}$ is determined by the isomorphism type of the pointed skeleton $\langle \mathbf{S}; (b_i)_{i<\eta} \rangle$ and the sequence $(\lambda_i)_{i<\eta}$ of "dimensions"; (iv) for each family of bijections $(\sigma_i : X_i \longrightarrow X_i)_{i<\eta}$ satisfying $\sigma_i(b_i) = b_i$ there is an automorphism σ of $\mathbf{L}$ satisfying $\sigma|_S = \mathrm{id}_S$ and $\sigma|_{X_i} = \sigma_i$ for all $i < \eta$.

If $\mathcal{K}$ is a class of lattices then we say that $\mathcal{K}$ *has bounded skeletons* if there is a finite upper bound to the cardinality of the skeletons of members of $\mathcal{K}$.

LEMMA 4.2 *Suppose $\mathcal{K}$ is a universal class of lattices. $\mathcal{K}$ has bounded skeletons if and only if $\mathcal{K}$ omits each of* $\mathbf{Q}$, $\mathbf{N}_\omega$, $\mathbf{M}_{\omega,2}$, *and* $\mathbf{M}_{2,\omega}$.

PROOF. Each of the above four lattices is its own skeleton, so ($\Rightarrow$) follows. Conversely, assume that $\mathcal{K}$ is a universal class of lattices which omits each of the above four lattices.

Claim 1 *1. There is a finite upper bound to the height of members of $\mathcal{K}$.*

2. For sufficiently large n, the following is true: if $\mathbf{L} \in \mathcal{K}$, $a, b_1, \ldots, b_n, c$ are distinct elements of L with $a \prec b_i < c$ for all i, and $\{a, b_1, \ldots, b_n, c\}$ is an M_λ in $\mathbf{L}$, then $\mathbf{I}[a, c]$ is an M_λ in $\mathbf{L}$.

3. For sufficiently large n, if $\mathbf{L} \in \mathcal{K}$ and $\mathbf{I}[a,c]$ is an M_λ in $\mathbf{L}$ with at least n elements, then $\mathbf{I}[a,c]$ is a free M_λ in $\mathbf{L}$.

Proof. These are easy consequences of the compactness theorem. (1) is equivalent to the fact that $\mathcal{K}$ omits $\mathbf{Q}$. Because $\mathcal{K}$ omits $\mathbf{N}_\omega$, for sufficiently large n the hypotheses of (2) imply $b_i \prec c$ for all i. (If not, say $b_1 \not\prec c$, then because the height of $\mathbf{L}$ is bounded we could pick $x \in L$ such that $b_1 \prec x < c$. Then $b_i \not\leq x$ would imply $x \wedge b_i = a$ for $i \neq 1$, so $\{a, b_1, \ldots, b_n, x, c\}$ would be isomorphic to an arbitrarily large sublattice of $\mathbf{N}_\omega$ of height 3.) Now let $X = \{b \in L : a \prec b \prec c\}$. If $I[a,c] \neq X \cup \{a,c\}$, then because $\mathbf{L}$ has finite height we could pick a minimal element y of $I[a,c] \setminus (X \cup \{a,c\})$, which must be incomparable to each element of X. Hence $a \prec y$ (by minimality) and $\{a, b_1, \ldots, b_n, y, c\}$ is an M_λ. But then the previous argument implies $y \prec c$, so $y \in X$, a contradiction. Thus $I[a,c] = X \cup \{a,c\}$, so $\mathbf{I}[a,c]$ is an M_λ, which establishes (2). To show (3), suppose $\mathbf{L} \in \mathcal{K}$ has an interval $\mathbf{I}[a,c]$ which is an arbitrarily large (possibly infinite) M_λ but which is not free in $\mathbf{L}$. As the height of $\mathbf{L}$ is bounded, there is an atom b of $\mathbf{I}[a,c]$ and an element $x \in L \setminus I[a,c]$ such that either $x \prec b$ or $b \prec x$. If $b \prec x$ then then $I[a,c] \cup \{x, x \vee c\}$ is the universe of a sublattice of $\mathbf{L}$ which either contains $\mathbf{M}_{\omega,2}$ as a sublattice (if $I[a,c]$ is infinite) or is isomorphic to an arbitrarily large sublattice of $\mathbf{M}_{\omega,2}$ of height 3. Both alternatives contradict the fact that $\mathcal{K}$ omits $\mathbf{M}_{\omega,2}$. A similar argument using $\mathbf{M}_{2,\omega}$ works in case $x \prec b$. ∎

Claim 2 *There is a positive integer N with the following property: if S is a skeleton of some member of $\mathcal{K}$, then as a lattice $\mathbf{S}$ satisfies*

(*) *If $a, b_1, \ldots, b_n, c$ are distinct elements of S with $a \prec b_i < c$ for each i, and if $\{a, b_1, \ldots, b_n, c\}$ is an M_λ in $\mathbf{S}$, then $n \leq N$.*

Proof. Suppose instead that n can be arbitrarily large. Choose $\mathbf{L} \in \mathcal{K}$ such that $\mathbf{S} \leq \mathbf{L}$ and S is a skeleton of $\mathbf{L}$. By a remark following Definition 4.1, each b_i covers a in $\mathbf{L}$ as well as in $\mathbf{S}$. So by items (2) and (3) from the previous claim, the interval sublattice $\mathbf{I}[a,c]$ of $\mathbf{L}$ is a free M_λ in $\mathbf{L}$ (assuming n is sufficiently large). But then only one of its atoms should be in S, which is not the case. ∎

Until further notice, N will be a fixed positive integer satisfying the assertion in the previous claim.

Claim 3 *Let $\mathcal{C}$ be the class of all lattices $\mathbf{S} \in \mathcal{K}$ which satisfy the condition (*) given in Claim 2. There is a finite upper bound to the cardinality of the members of $\mathcal{C}$.*

PROOF. Note that $\mathcal{C}$ is closed under taking interval sublattices. We shall prove the claim by induction on the height of $\mathbf{S}$. If $\mathbf{S}$ has height 2, then $|S| \leq N + 2$ by condition (*). Suppose $h > 2$ and the claim is true for all members of $\mathcal{C}$ of height less than h. Let $\mathbf{S} \in \mathcal{C}$ be of height h; let 0 and 1 be its least and greatest elements respectively, and let A be its set of atoms. For each $a \in A$ the interval $\mathbf{I}[a, 1]$ is a member of $\mathcal{C}$ of height less than h, so the cardinality of $I[a, 1]$ is bounded. Thus assuming the claim to be false for the members of $\mathcal{C}$ of height h, it must be possible to choose $\mathbf{S}$ as above with $|A|$ arbitrarily large (perhaps infinite). Do so, and then pick a subset A_0 of A which is maximal with respect to the property that the join of any two distinct elements of A_0 is 1. By condition (*), $|A_0| \leq N$ and so $|A \setminus A_0|$ can be arranged to be sufficiently large.

By the maximality of A_0, for each $x \in A \setminus A_0$ there exists $a \in A_0$ such that $x \vee a < 1$. Because $|A_0|$ is bounded, a fixed $a_0 \in A_0$ can be chosen with the property that the set $X = \{x \in A \setminus A_0 : a_0 \vee x < 1\}$ is sufficiently large. For each $x \in X$ there is a coatom $c \in I[a_0, 1]$ satisfying $x \leq c$. Again because $I[a_0, 1]$ is bounded it is possible to choose a fixed coatom $c_0 \in I[a_0, 1]$ dominating a sufficiently large subset of X. But that contradicts the fact that $\mathbf{I}[0, c_0]$, being a member of $\mathcal{C}$ of height less than h, must have bounded cardinality. ▮

The previous two claims imply that $\mathcal{K}$ has bounded skeletons, proving ($\Leftarrow$) of Lemma 4.2. ▮

Here is the main result of the paper.

THEOREM 4.3 *For any universal class $\mathcal{K}$ of lattices, the following are equivalent:*

(1) $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ *is decidable.*

(2) $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ *is not hereditarily undecidable.*

(3) $\mathcal{K}$ *omits* $\mathbf{Q}$, $\mathbf{N}_\omega$, $\mathbf{M}_{\omega,2}$ *and* $\mathbf{M}_{2,\omega}$.

(4) $\mathcal{K}$ *has bounded skeletons.*

(5) *$\mathcal{K}$ is locally finite and satisfies* H_1.

(6) *$\mathcal{K}$ is locally finite, finitely axiomatizable, and satisfies* H_2.

Proof. (1) ⇒ (2) and (6) ⇒ (5) are trivial. (3) ⇔ (4) is a restatement of Lemma 4.2, while (6) ⇒ (1) follows from Theorem 3.9. We will show ¬(3) ⇒ ¬(2) & ¬(5) and then (4) ⇒ (6).

Suppose (3) fails; so $\mathcal{K}$ contains one of $\mathbf{Q}$, $\mathbf{N}_\omega$, $\mathbf{M}_{\omega,2}$ or $\mathbf{M}_{2,\omega}$.

Case 1. Assume $\mathbf{N}_\omega \in \mathcal{K}$. For future reference, name the elements of the unique 4-element chain in $\mathbf{N}_\omega$ by $0 \prec a \prec b \prec 1$, let the atoms of $\mathbf{N}_\omega$ other than a be a_i $(i < \omega)$, and put $M_\omega = N_\omega \setminus \{b\}$ and $M_\omega^- = M_\omega \setminus \{a\}$.

We first show that $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is hereditarily undecidable by verifying the hypotheses of Lemma 1.1. (This was already done in [8]; for the sake of completeness, we repeat the proof here.) Let $\mathbf{A} = \mathbf{N}_\omega$ and $\mathbf{S} = \mathbf{M}_\omega$. Note that the automorphism group of $\mathbf{M}_\omega$ acts transitively on its set of atoms. Let $\mu(x)$, $\tau(x)$ and $\psi(z)$ be formulas asserting respectively that x is an atom, x belongs to a 4-element chain, and z is a coatom belonging to a 4-element chain. Then μ, τ and ψ witness the hypotheses of Lemma 1.1.

Next suppose that $\mathcal{K}$ is locally finite and satisfies H_1; let $\mathcal{K}_0$ be a finite set of finite lattices witnessing H_1, and let L_0 be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{N}_\omega$. If $\mathbf{L}_0$ could be embedded in $\mathbf{M}_\omega^-$ (i.e., if $\{0, a, b, 1\} \not\subseteq L_0$), say $\mathbf{L}_0 \cong \mathbf{L}_1 \leq \mathbf{M}_\omega^-$, then L_1 must itself be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{N}_\omega$. Choose $a_i \notin L_1$. Then the isomorphism from $\mathbf{Sg}(L_1 \cup \{a_i\})$ to $\mathbf{Sg}(L_1 \cup \{a\})$ sending a_i to a and fixing all other elements cannot be extended to an automorphism of $\mathbf{N}_\omega$, violating H_1.

Hence every maximal $\mathcal{K}_0$-subuniverse of $\mathbf{N}_\omega$ must contain $\{0, a, b, 1\}$. Let L_0 be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{N}_\omega$ of greatest cardinality, and put $L_0' = L_0 \setminus \{b\}$. As before we can find a $\mathbf{L}_1 \leq \mathbf{M}_\omega^-$ such that $\mathbf{L}_1 \cong \mathbf{L}_0'$. Since $\mathbf{L}_0' \in \mathsf{IS}(\mathcal{K}_0) = \mathrm{I}(\mathcal{K}_0)$, we can extend L_1 to a maximal $\mathcal{K}_0$-subuniverse L_2 of $\mathbf{N}_\omega$. By the previous argument, $\{a, b\} \subseteq L_2 \setminus L_1$, so $|L_2| \geq |L_1| + 2 = |L_0| + 1$, contradicting our choice of L_0. This proves that $\mathcal{K}$ does not satisfy H_1.

Case 2. Assume $\mathbf{M}_{\omega,2} \in \mathcal{K}$. (A dual argument will take care of the case when $\mathbf{M}_{2,\omega} \in \mathcal{K}$.) Let b be the join-irreducible coatom of $\mathbf{M}_{\omega,2}$, let c be the other coatom, and let a be the unique lower cover of b.

To show that $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ is hereditarily undecidable, let $\mathbf{A} = \mathbf{M}_{\omega,2}$ and $\mathbf{S} = \mathbf{I}[0, c]$, and let $\mu(x)$, $\tau(x)$ and $\psi(z)$ be formulas asserting respectively that x

is an atom, x is meet-reducible, and z is a join-irreducible coatom. Then μ, τ and ψ witness the hypotheses of Lemma 1.1.

An argument like the one in Case 1 can be given to show that $\mathcal{K}$ does not satisfy H_1. One uses $I[0,c] \setminus \{a\}$ and $\{0,a,b,c,1\}$ in place of M_ω^- and $\{0,a,b,1\}$. The details are left to the reader.

Case 3. Assume $\mathbf{Q} \in \mathcal{K}$. We repeat the proof from [8] that $\mathcal{K}$ is hereditarily undecidable. Let $\mathbf{A} = \{x \in \mathsf{Q} : |x| \geq 1\}$ and $S = A \setminus \{1\}$. Note that $\mathbf{S} \cong \mathbf{Q}$ so the automorphism group of $\mathbf{S}$ is transitive. Let $\mu(x)$, $\tau(x)$ and $\psi(z)$ be formulas asserting respectively that $x = x$, x has an upper cover, and z has a lower cover. Then μ, τ and ψ witness the hypotheses of Lemma 1.1.

To show that $\mathcal{K}$ does not satisfy H_1, suppose instead that $\mathcal{K}$ is locally finite and $\mathcal{K}_0$ witnesses H_1 for $\mathcal{K}$, and let L_0 be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{A}$. Thus $\mathbf{L}_0$ is an n-element chain, and every n-element subchain of $\mathbf{A}$ is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{A}$. In particular, $L_1 = \{-n, -(n–1), \ldots, -2, -1\}$ is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{A}$. But the unique isomorphism from $L_1 \cup \{1\}$ to $L_1 \cup \{2\}$ does not extend to an automorphism of $\mathbf{A}$, contradicting H_1. This completes the proof that $\neg(3) \Rightarrow \neg(2)$ & $\neg(5)$.

(4) $\Rightarrow$ (6). Let N be an upper bound to the cardinality of skeletons of members of $\mathcal{K}$. Let $\mathcal{H}$ be the class of *all* lattices whose skeletons have at most N elements. $\mathcal{H}$ is a finitely axiomatizable locally finite universal class and $\mathcal{K} \subseteq \mathcal{H}$. Recall that if $\mathbf{L} \in \mathcal{H}$, S is a skeleton of $\mathbf{L}$, $b_1, \ldots, b_r$ are the distinct elements of S which are atoms of free M_λs in $\mathbf{L}$, and λ_i is the cardinal number of the set of atoms of the free M_λ in $\mathbf{L}$ containing b_i, then the pointed skeleton $\langle \mathbf{S}; b_1, \ldots, b_r \rangle$ together with the sequence of "dimensions" $(\lambda_1, \ldots, \lambda_r)$ determines $\mathbf{L}$ up to isomorphism. More significantly, if $\mathbf{L}'$ is another member of $\mathcal{H}$ with pointed skeleton $\langle \mathbf{S}'; b_1', \ldots, b_r' \rangle \cong \langle \mathbf{S}; b_1, \ldots, b_r \rangle$ and dimensions $\langle \lambda_1', \ldots, \lambda_r' \rangle$, then $\lambda_i \leq \lambda_i'$ for all i implies that $\mathbf{L}$ can be embedded in $\mathbf{L}'$.

Since there are up to isomorphism only finitely many pointed skeletons of members of $\mathcal{H}$, it follows (as in [10, Claim 5.6]) that the poset of isomorphism classes in $\mathcal{H}_{fin}$ under embeddability has no infinite antichains, and hence every universal subclass of $\mathcal{H}$ ($\mathcal{K}$ in particular) is finitely axiomatizable. Now we must show that $\mathcal{K}$ satisfies H_2. Let $\mathcal{K}'$ be the class of all $\mathbf{L} \in \mathcal{K}$ such that no free M_λ in $\mathbf{L}$ has more than N atoms. The next claim should be obvious.

Claim 4 $\mathcal{K}' = \mathrm{I}(\mathcal{K}_0)$ *for some finite set* $\mathcal{K}_0$ *of finite lattices.*

For the remainder of this section, let $\mathcal{K}_0$ be a set witnessing the previous claim.

Claim 5 $\mathsf{S}(\mathcal{K}_0) \subseteq \mathrm{I}(\mathcal{K}_0)$.

Proof. It must be shown that if $\mathbf{L}_1 \leq \mathbf{L} \in \mathcal{K}'$ and $\{a, b_1, \ldots, b_n, c\}$ is a free M_λ in $\mathbf{L}_1$ with $a < b_i < c$ (and $b_i \neq b_j$ for $i \neq j$) then $n \leq N$. If the interval in $\mathbf{L}$ from a to c is itself a free M_λ in $\mathbf{L}$, then $n \leq N$ by our choice of $\mathbf{L}$. Otherwise, we can show $n \leq N$ in the following way.

Choose a skeleton S of $\mathbf{L}$. Assume for simplicity that $S \cap \{b_1, \ldots, b_n\} = \{b_1, \ldots, b_k\}$ for some $k \leq n$. So if $k < i \leq n$ then b_i must be an atom of a free M_λ in $\mathbf{L}$, say $\mathbf{I}[u_i, v_i]$, and there must be an atom $d_i \neq b_i$ of $\mathbf{I}[u_i, v_i]$ which belongs to S. Working now in $\mathbf{L}$, it follows that $a \leq u_i \prec b_i, d_i \prec v_i \leq c$, and either the first or the last inequality is strict (since $\mathbf{I}[a, c]$ is not a free M_λ in $\mathbf{L}$). It can then be shown that $b_i \neq d_j$ for $1 \leq i \leq k < j \leq n$ and $d_i \neq d_j$ for $k < i < j \leq n$. So $b_1, \ldots, b_k, d_{k+1}, \ldots, d_n$ are distinct elements of S, which proves $n \leq |S| \leq N$. ∎

Claim 6 *If $\mathbf{L} \in \mathcal{K}$ and L_0 is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{L}$, then L_0 contains a skeleton of $\mathbf{L}$.*

Proof. Assume $\mathbf{L} \in \mathcal{K}$ and let L_0 be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{L}$. Choose a skeleton S of $\mathbf{L}$ with the property that if $\mathbf{I}[a, c]$ is a free M_λ of $\mathbf{L}$ at least one of whose atoms belongs to L_0, then the atom of $\mathbf{I}[a, c]$ contained in S also belongs to L_0. Let $L_1 = L_0 \cup S$. $\mathbf{L}_1$ is a sublattice of $\mathbf{L}$, so if we can show that $\mathbf{L}_1 \in \mathcal{K}'$ then it will follow from the maximality of L_0 that $S \subseteq L_0$.

Suppose $\{a, b_1, \ldots, b_n, c\}$ is a free M_λ of $\mathbf{L}_1$, with $a \prec b_i \prec c$. Then $a \prec b_i \prec c$ in $\mathbf{L}$ as well, so if the interval in $\mathbf{L}$ from a to c is *not* a free M_λ in $\mathbf{L}$ then $\{b_1, \ldots, b_n\} \subseteq S$, which would imply $n \leq |S| \leq N$. On the other hand, if the interval in $\mathbf{L}$ from a to b *is* a free M_λ in $\mathbf{L}$ then $\{b_1, \ldots, b_n\} \subseteq L_0$ by our choice of S, so $\{a, b_1, \ldots, b_n, c\}$ is a free M_λ in $\mathbf{L}_0$ and hence $n \leq N$ as $\mathbf{L}_0 \in \mathcal{K}'$. This prove $\mathbf{L}_1 \in \mathcal{K}'$. ∎

We will now complete the proof of Theorem 4.3 by showing that $\mathcal{K}_0$ witnesses H_2 for $\mathcal{K}$. Let $\mathbf{L}$ be a finite member of $\mathcal{K}$, and let L_0 be a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{L}$. Suppose that $\mathbf{B}$ and $\mathbf{B}'$ are sublattices of $\mathbf{L}$ satisfying $B \cap L_0 = B' \cap L_0$ (=: B_0) and B_0 is a maximal $\mathcal{K}_0$-subuniverse of both $\mathbf{B}$

and $\mathbf{B}'$. Finally, suppose that σ is an isomorphism from $\mathbf{B}$ to $\mathbf{B}'$ satisfying $\sigma(x) = x$ for all $x \in B_0$.

By the previous claim, L_0 contains a skeleton of $\mathbf{L}$. Suppose $x \in B$ and $\sigma(x) \neq x$. Then $x \notin L_0$, so x is an atom of some free M_λ $\mathbf{I}[a, b]$ in $\mathbf{L}$. Because L_0 contains a skeleton of $\mathbf{L}$, L_0 contains at least one atom c of $\mathbf{I}[a, b]$. We claim that B_0 must also contain an atom c_x of $\mathbf{I}[a, b]$ For if this were false then we could put $B_1 = \mathrm{Sg}(B_0 \cup \{x\})$ and $L_1 = \mathrm{Sg}(B_0 \cup \{c\})$ and have $\mathbf{B}_0 < \mathbf{B}_1 \leq \mathbf{B}$ and $\mathbf{B}_1 \cong \mathbf{L}_1 \leq \mathbf{L}_0 \in \mathrm{I}(\mathcal{K}_0)$, contradicting the fact that B_0 is a maximal $\mathcal{K}_0$-subuniverse of $\mathbf{B}$. So $c_x \in B_0$ exists as claimed. Now since $\{x, c_x\} \subseteq B$ and since a and b belong to every skeleton of $\mathbf{L}$, it follows that $\{a, b\} \subseteq B \cap L_0$ $(= B_0)$ and hence $a = \sigma(a) < \sigma(x) < \sigma(b) = b$.

Thus if $x \in B$ and $\sigma(x) \neq x$, then x and $\sigma(x)$ are atoms in the same free M_λ of $\mathbf{L}$. It follows that σ can be extended to an automorphism $\hat{\sigma}$ of $\mathbf{L}$ satisfying $\hat{\sigma}|_{L_0} = \mathrm{id}_{L_0}$. This verifies $ext(\mathbf{L}, L_0, B, B')$ and proves that $\mathcal{K}_0$ witnesses H_2 for $\mathcal{K}$. ∎

5 Conclusion

Theorem 4.3 and its proof bear striking similarities to our analysis of unary algebras in [10]. (An algebra is *unary* if the arity of each of its fundamental operations is 1.) We can strengthen these similarities.

Let $\mathcal{K}$ be a locally finite universal class of unary algebras in a finite language and let $\mathbf{A} \in \mathcal{K}$. In [10] the following recipe for converting $\mathbf{A}$ to a vertex-colored poset was given. For $a \in A$ let $[a] = \{x \in A : \mathrm{Sg}^{\mathbf{A}}(x) = \mathrm{Sg}^{\mathbf{A}}(a)\}$ and let $[\mathbf{A}] = \{[a] : a \in A\}$. Defining $[a] \leq [b]$ to mean $\mathrm{Sg}^{\mathbf{A}}(a) \subseteq \mathrm{Sg}^{\mathbf{A}}(b)$ turns $[\mathbf{A}]$ into a poset (of finite height, since $\mathcal{K}$ is locally finite).

For $a \in A$ define $D_a = \{x \in A : [x] < [a]\}$. We say that $[a], [b] \in [\mathbf{A}]$ *have the same color* if $D_a = D_b$ and there exists an isomorphism $\sigma : \mathbf{Sg}^{\mathbf{A}}(a) \cong \mathbf{Sg}^{\mathbf{A}}(b)$ satisfying $\sigma|_{D_a} = \mathrm{id}_{D_a}$ and $\sigma([a]) = [b]$. A *color class in* $[\mathbf{A}]$ is the set of all $[a] \in [\mathbf{A}]$ of one fixed color.

Let us call a color class *free* if each of its elements is maximal (i.e., has no upper cover in the poset $[\mathbf{A}]$). Then by a *base for* $\mathbf{A}$ we mean a subset $\mathcal{B} \subseteq [\mathbf{A}]$ satisfying (1) if $[a] \in [\mathbf{A}] \setminus \mathcal{B}$ then $[a]$ is an element of a free color class in $[\mathbf{A}]$; (2) for each free color class in $[\mathbf{A}]$, exactly one of its elements is a member of $\mathcal{B}$. Bases coordinatize members of $\mathcal{K}$ in the same way that skeletons coordinatize lattices. More precisely, suppose $\mathbf{A} \in \mathcal{K}$, $\mathcal{B}$ is a base for $\mathbf{A}$, $B = \bigcup \mathcal{B}$, $([b_i])_{i<\eta}$ are the distinct elements of $\mathcal{B}$ belonging to free color classes of $[\mathbf{A}]$, and λ_i is the cardinality of the free color class containing $[b_i]$. Then $\mathbf{A}$ is determined up to isomorphism by the isomorphism type of $\langle \mathbf{B}; (b_i)_{i<\eta} \rangle$ and the sequence of "dimensions" $(\lambda_i)_{i<\eta}$.

By slightly extending the analysis in [10] we can prove the following theorem.

THEOREM 5.1 *Suppose $\mathcal{K}$ is a locally finite universal class of unary algebras in a finite language L. Let N be a positive integer bounding the maximum cardinality of the 1-generated members of $\mathcal{K}$. Then the following are equivalent:*

(1) $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ *is decidable.*

(2) $\mathsf{V}(\mathcal{K}^{\mathsf{t}})$ *is not hereditarily undecidable.*

(3) *$\mathcal{K}$ omits each member of a specific finite set $\mathcal{F}_{\mathsf{L},N}$ of countably infinite L-algebras. ($\mathcal{F}_{\mathsf{L},N}$ depends on L and N but not on $\mathcal{K}$.)*

(4) *$\mathcal{K}$ has bounded bases.*

(5) *$\mathcal{K}$ satisfies* H_1.

(6) *$\mathcal{K}$ is finitely axiomatizable and satisfies* H_3.

The similarity of Theorems 4.3 and 5.1 suggests to us that the characterization of decidable locally finite discriminator varieties is close at hand, and is intimately related to finite perturbations of homogeneity such as the ones described in Section 2. For further progress, we need a detailed understanding of the countable hereditarily homogeneous algebras belonging to arbitrary locally finite universal classes. This understanding will undoubtedly come from model theory. Such algebras are ω-categorical, and B. Hart and E. Hrushovski have shown (private communication) that they are ω-stable as well, so the structure theory of ω-categorical ω-stable structures can be applied.

In addition, several years ago A. H. Lachlan and others made a profound study of the countable stable homogeneous relational structures in a finite language (see e.g. [6, 5, **?**]). For a given language these structures fall into finitely many families in each of which the members are determined up to isomorphism by a finite sequence of dimensions. Motivated by these observations, we propose the following two problems for further study.

PROBLEM 1. Given a locally finite universal class $\mathcal{K}$ of algebras in a finite language and a positive integer r, describe the countable hereditarily homogeneous members of $\mathcal{K}$ of complete rank at most r.

PROBLEM 2. Given a locally finite universal class $\mathcal{K}$ of algebras in a finite language, describe the homogeneous universal subclasses of $\mathcal{K}$. Are they all finitely axiomatizable?

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