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Congruence lattices of powers of an algebra*

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In Memory of Evelyn Nelson

§0. Introduction

Let $\mathcal{P}$ be a property which can be attributed to 0, 1-sublattices of Eq (X), X arbitrary. (Eq (X) is the lattice of all equivalence relations on X.) For example, $\mathcal{P}$ could be modularity or permutability. It was proved in [3] that for every integer $k \geq 2$ there is an $n \geq 1$ such that for every algebra $\mathbf{A}$ of cardinality k, if Con $(\mathbf{A}^m)$ satisfies $\mathcal{P}$ for all $m \leq n$ then Con $(\mathbf{A}^m)$ satisfies $\mathcal{P}$ for all m. Let $n_{\mathcal{P}}(k)$ denote the least such n. The open problem is: given $\mathcal{P}$, to evaluate the function $n_{\mathcal{P}}$.

In this paper we establish an upper bound to $n_{\mathcal{P}}$ for each of the following properties $\mathcal{P}$: permutability, distributivity, modularity and the property of being skew-free. The upper bounds established here are much too large; for example, the bound for modularity is

$$n_{\text{Mod}}(k) \leq k^{4(k^{(k^4-k^3+k^2)}-1)}$$

when in fact, e.g., $n_{\text{Mod}}(2) = 2$. However, the method used to obtain these bounds may be of interest. The underlying idea is that the set $K = \{\mathbf{A}^m : m \geq 1\}$ contains algebras which to a certain extent act like free algebras; and the derivation of a Mal'cev condition for $\mathcal{P}$ (in varieties) can be mirrored in K.

§1. Permutability

In this section we modify Mal'cev's characterization [6] of congruence permutable varieties, so that it can apply to the class $\mathbf{P(A)}$ of all powers of a fixed algebra $\mathbf{A}$. This produces an upper bound to n_{Perm}.

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DEFINITION 1.1. Let K be a class of algebras, $\mathbf{F}$ an algebra, and $X \subseteq F$. $\mathbf{F}$ is a *pseudo-K-free algebra over X* if $\mathbf{F}$ has the universal mapping property for K over X; i.e. for every $\mathbf{A} \in K$ and every map $\alpha: X \to A$ there is a homomorphism $\beta: \mathbf{F} \to \mathbf{A}$ which extends α. In this case, X is called a set of *pseudo-K-free generators,* and $\mathbf{F}$ is said to be pseudo-K-free of *rank* $|X|$.

N.B. It is not required that X generate $\mathbf{F}$.

EXAMPLE 1.2. Let $\mathbf{A}$ be an algebra and $K = \mathbf{P}(\mathbf{A})$. Then for every cardinal κ, $\mathbf{A}^{(A^\kappa)}$ is pseudo-K-free of rank κ. Indeed, for each $\eta \in \kappa$, let $p_\eta : A^\kappa \to A$ be the ηth projection, and put $X = \{p_\eta : \eta \in \kappa\}$. Then $\mathbf{A}^{(A^\kappa)}$ is pseudo-K-free over X, since any map $\alpha: X \to A^I$ can be extended to a homomorphism $\beta: \mathbf{A}^{(A^\kappa)} \to \mathbf{A}^I$ by the formula

$$\beta(f)(i) = f(\langle \alpha(p_\eta)(i) \rangle_{\eta \in \kappa}).$$

PROPOSITION 1.3. *Let K be a class of algebras which contains a pseudo-K-free algebra $\mathbf{F}$ of rank 3. Then K is congruence permutable iff $\mathbf{F}$ is congruence permutable.*

Proof. First note that an algebra $\mathbf{A}$ is congruence permutable iff for all $a, b, c \in A$, $\langle a, c \rangle \in \Theta_{\mathbf{A}}(b, c) \circ \Theta_{\mathbf{A}}(a, b)$. Now suppose $\mathbf{F}$ is pseudo-K-free on $\{x, y, z\}$ and is congruence permutable. Then $\langle x, z \rangle \in \Theta_{\mathbf{F}}(y, z) \circ \Theta_{\mathbf{F}}(x, y)$, say

$$x \Theta_{\mathbf{F}}(y, z) p \Theta_{\mathbf{F}}(x, y) z. \tag{1}$$

Suppose $\mathbf{A} \in K$ and $a, b, c \in A$. Since $\mathbf{F}$ is pseudo-K-free on $\{x, y, z\}$ there is a homomorphism $\beta: \mathbf{F} \to \mathbf{A}$ such that $\beta(x) = a$, $\beta(y) = b$ and $\beta(z) = c$. It follows that

$$a \Theta_{\mathbf{A}}(b, c) \beta(p) \Theta_{\mathbf{A}}(a, b) c$$

since the homomorphism β preserves the principal congruence formulas which witness (1). So $\langle a, c \rangle \in \Theta_{\mathbf{A}}(b, c) \circ \Theta_{\mathbf{A}}(a, b)$ as required.

COROLLARY 1.4. *For any algebra $\mathbf{A}$, $\mathbf{P}(\mathbf{A})$ is congruence permutable iff $\mathbf{A}^{(A^3)}$ is congruence permutable.*

COROLLARY 1.5. $n_{\mathrm{Perm}}(k) \le k^3$.

It should be clear that one can obtain, in a similar manner, an upper bound to $n_{\mathscr{P}}$ for any propery $\mathscr{P}$ of the form

$$\forall\theta_1 \cdots \forall\theta_n[s(\vec{\theta}) \subseteq t(\vec{\theta})]$$

where $s(\vec{\theta})$, $t(\vec{\theta})$ are terms in $\cap$, $\vee$ and $\circ$, and $\vee$ does not occur in $s(\vec{\theta})$. (For the corresponding Mal'cev condition, see [8] or [9].) For example the property $\mathscr{P}$ of permutability combined with distributivity (characterized by Pixley in [7]) is equivalent to the universally quantified inclusion $\theta \cap (\psi_1 \circ \psi_2) \subseteq (\theta \cap \psi_2) \circ (\theta \cap \psi_1)$. Hence one can show that $n_{\mathrm{Perm+Dist}}(k) \leq k^3$. A second example is the weak distributive property WDist (m) defined by $\theta \cap (\psi_1 \circ^m \psi_2) \subseteq (\theta \cap \psi_1) \vee (\theta \cap \psi_2)$; the above method yields $n_{\mathrm{WDist}(m)}(k) \leq k^{m+1}$.

§2. Distributivity

This section contains a generalization of Jónsson's characterization [5] of congruence distributive varieties. The development is not as straightforward as it was for congruence permutability; this is because Jónsson's condition actually characterizes the property WDist(2):

$$\theta \cap (\psi_1 \circ \psi_2) \subseteq (\theta \cap \psi_1) \vee (\theta \cap \psi_2)$$

which is equivalent *in a variety* to congruence distributivity, but is strictly weaker than congruence distributivity in general.

NOTATION 2.1. Let $R, R_1, \ldots, R_n$ be subsets of A^2.

(i) $R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$; $R^1 = R$.
(ii) $\prod_{i=1}^n R_i = R_1 \circ R_2 \circ \cdots \circ R_n$.
(iii) trans (R) is the transitive closure of R.

DEFINITION 2.2. Let $\pi(x, y, u, v)$ be the principal congruence formula

$$\exists\vec{w}\left(x \approx t_1(z_1, \vec{w}) \;\&\; \left[\mathop{\&}_{i=1}^{m-1} t_i(z_i', \vec{w}) \approx t_{i+1}(z_{i+1}, \vec{w})\right] \;\&\; t_m(z_m', \vec{w}) \approx y\right).$$

The *length* of π is the positive integer m. The *type* of π is the function $\epsilon : \{0, \ldots, m-1\} \to \{1, -1\}$ defined by

$$\epsilon(i-1) = \begin{cases} 1 & \text{if } \langle z_i, z_i'\rangle = \langle u, v\rangle \\ -1 & \text{if } \langle z_i, z_i'\rangle = \langle v, u\rangle. \end{cases}$$

(The trivial principal congruence formula $\exists\vec{w}(x \approx y)$ has length 0 and no type.)

The following is essentially the easy half of Mal'cev's description of principal congruences.

LEMMA 2.3. *Suppose* $\mathbf{A}$ *is an algebra,* R *is a reflexive subuniverse of* $\mathbf{A}^2$, *and* π *is a principal congruence formula of length* $m \geq 1$ *and type* ϵ. *Then for all* $a, b, c, d \in A$, *if* $\langle c, d\rangle \in R$ *and* $\mathbf{A} \models \pi(a, b, c, d)$, *then* $\langle a, b\rangle \in \prod_{t=0}^{m-1} R^{\epsilon(t)}$.

The generalization of Jónsson's characterization begins here.

LEMMA 2.4. *Let* $\mathbf{A}$ *be an algebra and* $m \geq 1$. *Then* $(3)_m \Rightarrow (2) \Rightarrow (1)_m$ *where:*
$(1)_m$ *For all* $\theta, \psi_1, \psi_2, \ldots, \psi_{2m} \in \operatorname{Con} \mathbf{A}$, $\theta \cap \prod_{i=1}^{2m} \psi_i \subseteq \bigvee_{i=1}^{2m} (\theta \cap \psi_i)$.
(2) $\operatorname{Con} \mathbf{A}$ *is distributive.*
$(3)_m$ *for each* $\theta \in \operatorname{Con} \mathbf{A}$ *and all reflexive subuniverses* $R_1, R_2, \ldots, R_{2m}$ *of* $\mathbf{A}^2$,

$$\theta \cap \prod_{i=1}^{2m} R_i \subseteq \operatorname{trans}\left[\bigcup_{i=1}^{2m} \bigcup_{\epsilon \in \{1,-1\}^m} \left(\theta \cap \prod_{t=0}^{m-1} R_i^{\epsilon(t)}\right)\right].$$

Moreover, $(1)_m$ *is equivalent to*
$(1)'_m$ *For all* $a_1, \ldots, a_{2m+1} \in A$, $\langle a_1, a_{2m+1}\rangle \in \bigvee_{i=1}^{2m} [\Theta_{\mathbf{A}}(a_1, a_{2m+1}) \cap \Theta_{\mathbf{A}}(a_i, a_{i+1})]$.

Proof of $(3)_m \Rightarrow (2)$. Suppose $\mathbf{A}$ satisfies $(3)_m$. It suffices to show, by induction on $n \geq 1$, that for all $\theta, \psi_1, \ldots, \psi_{2^n \cdot m} \in \operatorname{Con} \mathbf{A}$,

$$\theta \cap \prod_{j=1}^{2^n \cdot m} \psi_j \subseteq \bigvee_{j=1}^{2^n \cdot m} (\theta \cap \psi_j).$$

If $n = 1$, set $R_i = \psi_i$ and apply $(3)_m$. If $n > 1$, set

$$R_i = \prod_{j=1}^{2^{n-1}} \psi_{2^{n-1}(i-1)+j}$$

and apply $(3)_m$ and the inductive hypothesis.

PROPOSITION 2.5. *Suppose* $m \geq 1$ *and* K *is a class of algebras containing a pseudo-K-free algebra* $\mathbf{F}$ *of rank* $2m + 1$. *Suppose further that* $\{x_1, \ldots, x_{2m+1}\} \subseteq F$ *is a set of* $2m + 1$ *pseudo-K-free generators, and that* $\mathbf{F}$ *satisfies* $(1)'_m$ *for the particular choice of elements* $a_i = x_i$ *"using principal congruence formulas of length* $\leq m$ *to witness the* $\Theta_{\mathbf{F}}(x_i, x_{i+1})$*'s"; i.e. there exist elements* $s_0, \ldots, s_r \in F$ *such that*

(i) $s_0 = x_1$, $s_r = x_{2m+1}$.
(ii) *For every* $j < r$, $\langle s_j, s_{j+1}\rangle \in \Theta_{\mathbf{F}}(x_1, x_{2m+1})$.
(iii) *For every* $j < r$ *there is an* $i(j) \in \{1, \ldots, 2m\}$ *and a principal congruence formula* π_j *of length* $\leq m$ *such that* $\mathbf{F} \models \pi_j(s_j, s_{j+1}, x_{i(j)}, x_{i(j)+1})$.

Then every $\mathbf{A} \in K$ satisfies $(3)_m$.

Proof. Suppose $\mathbf{A} \in K$, $\theta \in \operatorname{Con} \mathbf{A}$, $R_1, \ldots, R_{2m}$ are reflexive subuniverses of $\mathbf{A}^2$ and $\langle a_1, a_{2m+1}\rangle \in \theta \cap \prod_{i=1}^{2m} R_i$. Pick $a_2, \ldots, a_{2m} \in A$ so that $\langle a_i, a_{i+1}\rangle \in R_i$ for $1 \le i \le 2m$. Since $\mathbf{F}$ is pseudo-K-free over $\{x_1, \ldots, x_{2m+1}\}$ there is a homomorphism $\beta: \mathbf{F} \to \mathbf{A}$ which sends x_i to a_i for each i. Then

(i) $\beta(s_0) = a_1$, $\beta(s_r) = a_{2m+1}$.
(ii) For each $j < r$, $\langle \beta(s_j), \beta(s_{j+1})\rangle \in \theta$.
(iii) For each $j < r$, $\mathbf{A} \models \pi_j(\beta(s_j), \beta(s_{j+1}), a_{i(j)}, a_{i(j)+1})$.

For each j, assume (without loss of generality) that π_j has length m and let ϵ_j be its type. Then by Lemma 2.3,

$$\langle \beta(s_j), \beta(s_{j+1})\rangle \in \theta \cap \prod_{t=0}^{m-1} R_{i(j)}^{\epsilon_j(t)} \subseteq \bigcup_{i=1}^{2m} \bigcup_{\epsilon \in \{1,-1\}^m} \left(\theta \cap \prod_{t=0}^{m-1} R_i^{\epsilon(t)}\right)$$

for each $j < r$. Thus $\mathbf{A}$ satisfies $(3)_m$.

Remark. Suppose K is a variety; let $\mathbf{F} = \mathbf{F}_K(\bar{x}, \bar{y}, \bar{z})$ be the canonical K-free algebra on $\{\bar{x}, \bar{y}, \bar{z}\}$ obtained as a quotient of the term algebra $\mathbf{T}(x, y, z)$, and suppose $\mathbf{F}$ satisfies $(1)_1'$ for $a_1 = \bar{x}$, $a_2 = \bar{y}$, $a_3 = \bar{z}$. Pick $d_0, \ldots, d_n \in \mathbf{T}(x, y, z)$ such that

$$\bar{x} = \bar{d}_0[\Theta(\bar{x}, \bar{z}) \cap \Theta(\bar{x}, \bar{y})]\bar{d}_1[\Theta(\bar{x}, \bar{z}) \cap \Theta(\bar{y}, \bar{z})]\bar{d}_2 \cdots \bar{d}_n = \bar{z}.$$

Define $s_0, \ldots, s_{2n-2} \in \mathbf{T}(x, y, z)$ by $s_{2k+1} = d_{k+1}(x, y, z)$, $s_{4k} = d_{2k}(x, x, z)$ and $s_{4k+2} = d_{2k+1}(x, z, z)$. Then the elements $\bar{s}_0, \ldots, \bar{s}_{2n-2} \in F$ satisfy the conditions (i)–(iii) of Proposition 2.5 with $m = 1$.

Proposition 2.5 can be applied to any class K which has definable principal congruences, as the next corollary shows.

COROLLARY 2.6. *Suppose $m \ge 1$ and K is a class of algebras all of whose principal congruences are witnessed by principal congruence formulas of length $\le m$. If K contains a pseudo-K-free algebra $\mathbf{F}$ of rank $2m+1$, then K is congruence distributive iff $\mathbf{F}$ is congruence distributive.*

This yields an upper bound to the function n_{Dist}. The proof of Theorem 1 in [3] essentially showed the following: suppose $\mathbf{A}$ is finite and $\pi_1, \ldots, \pi_r$ are principal congruence formulas such that the maximal members (with respect to inclusion) of

$$\{\pi^{\mathbf{A}} \subseteq A^4 : \pi \text{ is a principal congruence formula}\}$$

are precisely $\pi_1^{\mathbf{A}}, \ldots, \pi_r^{\mathbf{A}}$; then the set $\{\pi_1, \ldots, \pi_r\}$ defines principal congruences in $\mathbf{P}(\mathbf{A})$. It is not hard to show that $\pi_1, \ldots, \pi_r$ can be chosen to have lengths bounded by some function of $|A|$.

LEMMA 2.7. *Suppose* $\mathbf{A}$ *is finite and* $k = |A|$. *Then for every principal congruence formula* π *there is a principal congruence formula* π_1 *of length no greater than*

$$m(k) = k^{(k^4-k^3+k^2)} - 1$$

such that $\pi^{\mathbf{A}} \subseteq \pi_1^{\mathbf{A}}$.

Proof. Suppose $\pi(x, y, u, v)$ has length $m \geq 1$; write π as $\exists w_1 \cdots \exists w_s \phi(x, y, u, v, \vec{w})$ where ϕ is

$$x \approx t_1(z_1, \vec{w}) \;\&\; \left[\mathop{\&}_{i=1}^{m-1} t_i(z_i', \vec{w}) \approx t_{i+1}(z_{i+1}, \vec{w})\right] \;\&\; t_m(z_m', \vec{w}) \approx y$$

and assume that there is no π_1 of shorter length than π satisfying $\pi^{\mathbf{A}} \subseteq \pi_1^{\mathbf{A}}$. Pick functions $w_1, \ldots, w_s : \pi^{\mathbf{A}} \to A$ such that for all $\langle a, b, c, d\rangle \in \pi^{\mathbf{A}}$,

$$\mathbf{A} \models \phi(a, b, c, d, \vec{w}(a, b, c, d)).$$

Let z_0', z_{m+1} be the variables x, y respectively and let $t_0(x, \vec{w})$ and $t_{m+1}(x, \vec{w})$ denote the term x. Define $f_0, \ldots, f_m : \pi^{\mathbf{A}} \to A$ by $f_0(a, b, c, d) = a$ and for $i > 0$,

$$f_i(a, b, c, d) = \begin{cases} t_i^{\mathbf{A}}(c, \vec{w}(a, b, c, d)) & \text{if } z_i' = u \\ t_i^{\mathbf{A}}(d, \vec{w}(a, b, c, d)) & \text{if } z_i' = v. \end{cases}$$

If $f_{i_0} = f_{i_1}$ for some $i_0 < i_1$, then the principal congruence formula π_1 given by

$$\exists \vec{w}\Bigg(\left[\mathop{\&}_{i=0}^{i_0-1} t_i(z_i', \vec{w}) \approx t_{i+1}(z_{i+1}, \vec{w})\right] \;\&\; t_{i_0}(z_{i_0}', \vec{w}) \approx t_{i_1+1}(z_{i_1+1}, \vec{w})$$
$$\;\&\; \left[\mathop{\&}_{i=i_1+1}^{m} t_i(z_i', \vec{w}) \approx t_{i+1}(z_{i+1}, \vec{w})\right]\Bigg)$$

has shorter length than π and satisfies $\pi^{\mathbf{A}} \subseteq \pi_1^{\mathbf{A}}$. Hence $f_0, \ldots, f_m$ are all distinct; so

$$m + 1 \leq |A^{\pi^{\mathbf{A}}}| \leq k^{(k^4-k^3+k^2)}.$$

COROLLARY 2.8. *Suppose* $\mathbf{A}$ *is finite of cardinality* k. *Then* $\mathbf{P}(\mathbf{A})$ *is congruence distributive iff* $\mathbf{A}^{(A^{2m(k)+1})}$ *is congruence distributive, where* $m(k) = k^{(k^4-k^3+k^2)} - 1$.

COROLLARY 2.9. $n_{\mathrm{Dist}}(k) \le k^{(2m(k)+1)} = k^{(2k^{(k^4-k^3+k^2)}-1)}$.

§3. Modularity

In a similar manner, one can obtain an upper bound for the function n_{Mod} by imitating Day's characterization [4] of congruence modular varieties. We simply state the facts, without proof.

NOTATION 3.1. If $R, S \subseteq A^2$, then $R \circ^1 S = R$ and $R \circ^n S = R \circ (S \circ^{n-1} R)$ for $n > 1$.

LEMMA 3.2. *Let* $\mathbf{A}$ *be an algebra and* $m \ge 1$. *Then* $(6)_m \Rightarrow (5) \Rightarrow (4)_m$ *where*:

$(4)_m$ *For all* $\alpha, \beta, \gamma \in \mathrm{Con}\,\mathbf{A}$ *such that* $\alpha \ge \gamma$, $\alpha \cap (\beta \circ^{4m-1} \gamma) \subseteq (\alpha \cap \beta) \vee \gamma$.

(5) $\mathrm{Con}\,\mathbf{A}$ *is modular.*

$(6)_m$ *For all* $\alpha, \gamma \in \mathrm{Con}\,\mathbf{A}$ *such that* $\alpha \ge \gamma$, *and for every tolerance* Λ *of* $\mathbf{A}$, $\alpha \cap (\Lambda \circ^{4m-1} \gamma) \subseteq \mathrm{trans}\,(\alpha \cap \prod_{i=1}^m \Lambda) \vee \gamma$.

Moreover, $(4)_m$ *is equivalent to*

$(4)'_m$ *For all* $a_1, \ldots, a_{4m} \in A$, $\langle a_1, a_{4m}\rangle \in (\alpha \cap \beta) \vee \gamma$ *where*

$$\beta = \bigvee_{i=1}^{2m} \Theta_{\mathbf{A}}(a_{2i-1}, a_{2i})$$

$$\gamma = \bigvee_{i=1}^{2m-1} \Theta_{\mathbf{A}}(a_{2i}, a_{2i+1})$$

$$\alpha = \Theta_{\mathbf{A}}(a_1, a_{4m}) \vee \gamma.$$

PROPOSITION 3.3. *Suppose* $m \ge 1$ *and* K *is a class of algebras containing a pseudo-*K*-free algebra* $\mathbf{F}$ *of rank* $4m$. *Suppose further that* $\{x_1, \ldots, x_{4m}\} \subseteq F$ *is a set of* $4m$ *pseudo-*K*-free generators, and that* $\mathbf{F}$ *satisfies* $(4)'_m$ *for the particular choice of elements* $a_i = x_i$ *using principal congruence formulas of length* $\le m$ *to witness the* $\Theta_{\mathbf{F}}(x_{2i-1}, x_{2i})$*'s. Then every* $\mathbf{A} \in K$ *satisfies* $(6)'_m$.

COROLLARY 3.4. *Suppose* $\mathbf{A}$ *is finite of cardinality* k. *Then* $\mathbf{P}(\mathbf{A})$ *is congruence modular iff* $\mathbf{A}^{(A^{4m(k)})}$ *is congruence modular, where* $m(k)$ *is as in Lemma* 2.7.

COROLLARY 3.5. $n_{\mathrm{Mod}}(k) \le k^{4m(k)} = k^{4(k^{(k^4-k^3+k^2)}-1)}$.

§4. Skew-free

DEFINITION 4.1. Let $A_1, \ldots, A_n$ be nonempty sets, $n \geq 2$ and put $B = A_1 \times \cdots \times A_n$.

(i) Given $\theta_i \in \mathrm{Eq}(A_i)$ for $1 \leq i \leq n$, $\theta_1 \times \cdots \times \theta_n$ denotes

$$\{\langle \vec{a}, \vec{b} \rangle \in B^2 : \langle a_i, b_i \rangle \in \theta_i \quad \text{for each } i = 1, \ldots, n\}.$$

(ii) $\theta \in \mathrm{Eq}(B)$ is a *product equivalence relation* if $\theta = \theta_1 \times \cdots \times \theta_n$ for some $\theta_i \in \mathrm{Eq}(A_i)$.

(iii) $L \leq \mathrm{Eq}(B)$ is *skew-free* if every $\theta \in L$ is a product equivalence relation.

NOTATION 4.2. (i) Suppose $B = A_1 \times \cdots \times A_n$, $n \geq 2$. Then

$$\sigma(B) = \{\langle \vec{a}, \vec{b}, \vec{c}, \vec{d} \rangle \in B^4 : \text{for each } i = 1, \ldots, n,\ a_i = b_i \text{ or } \langle a_i, b_i \rangle = \langle c_i, d_i \rangle\}.$$

(ii) If $R \subseteq A^n$, then $p_1^R, \ldots, p_n^R$ denote the restrictions to R of the projections $p_1, \ldots, p_n : A^n \to A$.

The next lemma is an easy exercise (cf. [1], Lemma 1).

LEMMA 4.3. *Suppose* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *are algebras*, $n \geq 2$, *and* $\mathbf{B} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. *Then* Con $\mathbf{B}$ *is skew-free iff for every* $\langle \vec{a}, \vec{b}, \vec{c}, \vec{d} \rangle \in \sigma(B)$, $\langle \vec{a}, \vec{b} \rangle \in \Theta_{\mathbf{B}}(\vec{c}, \vec{d})$.

The following lemma shows that for finite $\mathbf{A}$, the set $K = \{\mathbf{A}^n : n \geq 2\}$ contains an algebra which is "pseudo-K-free with respect to σ."

LEMMA 4.4. *Suppose* $\mathbf{A}$ *is finite. Then* $\mathbf{A}^{\sigma(A)}$ *and* $p_1^{\sigma(A)}, p_2^{\sigma(A)}, p_3^{\sigma(A)}, p_4^{\sigma(A)} \in A^{\sigma(A)}$ *satisfy the following*:

(i) $\langle p_1^{\sigma(A)}, \ldots, p_4^{\sigma(A)} \rangle \in \sigma(A^{\sigma(A)})$.

(ii) *For every* $n \geq 2$ *and every* $\langle \vec{a}, \vec{b}, \vec{c}, \vec{d} \rangle \in \sigma(A^n)$, *there is a homomorphism* $\beta : \mathbf{A}^{\sigma(A)} \to \mathbf{A}^n$ *which sends* $p_1^{\sigma(A)}, \ldots, p_4^{\sigma(A)}$ *to* $\vec{a}, \ldots, \vec{d}$ *respectively.*

Proof. Define β by $\beta(f)_i = f(a_i, b_i, c_i, d_i)$.

COROLLARY 4.5. *Let* $\mathbf{A}$ *be finite. Then* Con $(\mathbf{A}^n)$ *is skew-free for every* $n \geq 2$ *iff* Con $(\mathbf{A}^{\sigma(A)})$ *is skew-free.*

Proof. Use Lemmas 4.3 and 4.4, and the fact that homomorphisms preserve principal congruence formulas.

COROLLARY 4.6. $n_{\mathrm{SkewF}}(k) \leq k^3 + k^2 - k$.

We close this paper with an upper bound to n_{Dist} which is better than the one established in §2. It is easy to see that if $\text{Con}(\mathbf{A}^n)$ is distributive then it is skew-free (cf. [2], Lemma IV.11.10). Not much harder is the fact that if $|A| = k$ and $\text{Con}(\mathbf{A}^n)$ is skew-free, then $\text{Con}(\mathbf{A}^n)$ is k-permutable. Hence $\text{Con}(\mathbf{A}^n)$ is distributive iff it satisfies $\theta \cap (\psi_1 \circ^k \psi_2) \subseteq (\theta \cap \psi_2) \circ^k (\theta \cap \psi_1)$ for all θ, ψ_1, ψ_2. It follows by the methods of §1 that this inclusion is true in $\text{Con}(\mathbf{A}^n)$ for every $n \geq 1$ iff it is true in $\text{Con}(\mathbf{A}^{(A^{k+1})})$. Hence $n_{\text{Dist}}(k) \leq k^{k+1}$.

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