SOLUTION TO THE CHAUTAUQUA PROBLEM

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Abstract. There exists an infinite ascending chain of finitely generated clones on a nine-element set for which the corresponding algebras are alternately finitely based and inherently nonfinitely based.

In 1981, at the Chautauqua resort near Boulder, Colorado, in the presence of a large walnut salad, G. McNulty posed the following problem (published in [5, p. 227]):

Problem. Let A be a finite set with at least three elements. In the lattice of clones over A, is there always an infinite ascending chain $C_0, C_1, \ldots$ of finitely generated clones such that $\langle A, C_i\rangle$ is finitely based when i is even and not finitely based when i is odd?

McNulty was motivated to pose this problem by the fact that a negative answer might lead to a positive solution to a long-standing finite basis problem of Tarski. In the meantime, R. McKenzie has given a negative solution to Tarski's problem [3] and so one might suspect that the Chautauqua Problem has a positive answer. Until recently, however, it was not known whether there exists *any* finite set A with the Chautauqua property. In this paper we prove

Theorem 1. *On a nine-element set A it is possible to define an infinite sequence of operations F_n $(n < \omega)$ so that if $\mathbf{A}_n = \langle A, \, (F_i : i \leq n)\rangle$, then for all $k < \omega$, $\mathbf{A}_{2k}$ is finitely based and every subdirectly irreducible member of $\mathbf{HSP}(\mathbf{A}_{2k})$ is in $\mathbf{HS}(\mathbf{A}_{2k})$, while $\mathbf{A}_{2k+1}$ is inherently nonfinitely based and $\mathbf{HSP}(\mathbf{A}_{2k+1})$ is residually large.*

This provides a strong affirmative answer to McNulty's question when restricted to nine-element sets. The theorem can be easily extended to any set with at least nine elements. McNulty subsequently observed that with more work, Theorem 1 can be extended to any set having at least three elements, fully solving the Chautauqua problem. (The latter result may eventually appear in [6].) Our proof of Theorem 1 combines several components of McKenzie's refutation of the RS conjecture and our finite basis theorem for residually finite varieties with a semilattice term operation.

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Following McKenzie [2, p. 3], for $n \geq 3$ we define the $n+1$-ary relation R_n on the set $\{a, b\}$ as follows:

$$R_n(x_0, x_1, \ldots, x_n) \quad \text{iff} \quad \begin{cases} & x_0 = a \text{ and } x_1 = \cdots = x_n = b, \\ \text{or} & x_0 = a \text{ and } x_i = b \text{ for exactly one } i \in \{1, \ldots, n\}, \\ \text{or} & x_0 = b \text{ and for all } i < j \leq n, \text{ if } x_j = b \text{ then } x_i = b. \end{cases}$$

If $f_0, \ldots, f_n \in \{a, b\}^{\mathbb{Z}}$ then $[\![R_n(f_0, \ldots, f_n)]\!]$ denotes the set

$$\{i \in \mathbb{Z} : R_n(f_0(i), \ldots, f_n(i)) \text{ is true}\}.$$

As a consequence of [2, Lemma 1.1] we have:

Fact 2. For each $n \geq 3$ there exist $f_0, \ldots, f_n \in \{a, b\}^{\mathbb{Z}}$ such that:

(1) For any $\sigma : \{0, \ldots, n\} \to \{0, \ldots, n\}$,

$$[\![R_n(f_{\sigma(0)}, \ldots, f_{\sigma(n)})]\!] = \mathbb{Z} \text{ iff } \sigma = \text{id}.$$

(2) For any $m \neq n$ $(m \geq 3)$ and any $\tau : \{0, \ldots, m\} \to \{0, \ldots, n\}$,

$$[\![R_m(f_{\tau(0)}, \ldots, f_{\tau(m)})]\!] \neq \mathbb{Z}.$$

Let $A = \{0, C, D, 1^a, 1^b, H^a, H^b, 2^a, 2^b\}$ where 1^a, H^b, etc. are formal names for distinct elements of A. For $\bar{x} \in \{a, b\}^{n+1}$ define $1^{\bar{x}} \in \{1^a, 1^b\}^{n+1}$ by $1^{\bar{x}} = (1^{x_0}, \ldots, 1^{x_n})$; define $H^{\bar{x}}$ and $2^{\bar{x}}$ similarly. Define the semilattice operation $\wedge$ on A by $x \wedge y = 0$ if $x \neq y$, and $x \wedge x = x$. For $n \geq 3$ let R_n^+ be the $n+1$-ary relation defined on A by

$$R_n^+ = \{1^{\bar{x}} : \bar{x} \in R_n\} \cup \{H^{\bar{x}} : \bar{x} \in R_n\} \cup \{2^{\bar{x}} : \bar{x} \in R_n\}.$$

For $n \geq 3$ define $E_n : A^{n+2} \to A$ as follows: If $\bar{u} \notin R_n^+$ or $z \notin \{C, D\}$ then $E_n(\bar{u}, z) = 0$. If $\bar{x} \in R_n$ then

$$\begin{array}{lllllll} E_n(1^{\bar{x}}, C) & = & C & & & & \\ E_n(H^{\bar{x}}, C) & = & E_n(2^{\bar{x}}, D) & = & D & & \\ E_n(1^{\bar{x}}, D) & = & E_n(H^{\bar{x}}, D) & = & E_n(2^{\bar{x}}, C) & = & 0. \end{array}$$

Also for $n \geq 3$ define $S_n : A^{n+4} \to A$ by

$$S_n(\bar{u}, x, y, z) = \begin{cases} x & \text{if } R_n^+(\bar{u}) \text{ and } (x = y \text{ or } x = z) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{E} \subseteq \{E_n : 3 \leq n < \omega\}$ and $\mathcal{S} \subseteq \{S_n : 3 \leq n < \omega\}$, and put $\Gamma = \mathcal{E} \cup \mathcal{S}$ and $\mathbf{A}_\Gamma = \langle A, \wedge, \Gamma \rangle$.

Lemma 3. *Suppose that for every $n \geq 3$, if $E_n \in \Gamma$ then $S_n \in \Gamma$. Then*

(1) *Every subdirectly irreducible member of* $\mathbf{HSP}(\mathbf{A}_\Gamma)$ *is in* $\mathbf{HS}(\mathbf{A}_\Gamma)$.

(2) *If Γ is finite then $\mathbf{A}_\Gamma$ is finitely based.*

Lemma 4. *Suppose that there exists $n \geq 3$ such that $E_n \in \Gamma$ but $S_n \notin \Gamma$. Then*

(1) $\mathbf{HSP}(\mathbf{A}_\Gamma)$ *is residually large.*

(2) *If Γ is finite then $\mathbf{A}_\Gamma$ is inherently nonfinitely based.*

Proof of Lemma 3. We follow the by-now-familiar analysis of McKenzie of the finite subdirectly irreducible members of a variety generated by a finite height-1 semilattice-based algebra. Suppose $\mathbf{S}$ is a finite subdirectly irreducible member of $\mathbf{HSP}(\mathbf{A}_\Gamma)$. Among all triples $(I, \mathbf{B}, \theta)$ where I is a nonempty set, $\mathbf{B}$ is a subalgebra of $(\mathbf{A}_\Gamma)^I$, and θ is a congruence of $\mathbf{B}$ such that $\mathbf{S} \cong \mathbf{B}/\theta$, choose a triple with $|I|$ minimal. If $|I| = 1$ then $\mathbf{S} \in \mathbf{HS}(\mathbf{A}_\Gamma)$ as desired. Suppose $|I| > 1$. Define $B^{\text{top}} = \{f \in B : f(i) \neq 0 \text{ for all } i \in I\}$. Then (cf. [2, Lemmas 6.4–6.6] there exists $p \in B^{\text{top}}$ such that:

(1) $\{p\}$ is a θ-class. For all $f, g \in B$ we have $f \not\equiv g \pmod{\theta}$ iff there exists $\lambda \in \operatorname{Pol}_1 \mathbf{B}$ such that $\lambda(f) = p$ while $\lambda(g) \neq p$, or vice versa.
(2) For each $n \geq 3$ such that $S_n \in \Gamma$, there do not exist $f_0, \ldots, f_n \in B$ such that $[\![R_n^+(f_0, \ldots, f_n)]\!] = I$.

We now claim that for every n-ary $F \in \Gamma$ and all $\bar{f} \in B^n$, $F(\bar{f}) \notin B^{\text{top}}$. Indeed, this follows immediately from the definitions of the operations, the hypothesis of the Lemma, and item 2 above. Therefore the only nonconstant unary polynomials of $\mathbf{B}$ which have p in their range are x and $x \wedge p$. Thus by item 1 above, the θ-classes are precisely $\{p\}$ and $B \setminus \{p\}$, and $\mathbf{B}/\theta$ is a 2-element meet semilattice in which all operations from Γ are identically equal to 0. But then $\mathbf{B}/\theta$ is isomorphic to the subalgebra of $\mathbf{A}_\Gamma$ having universe $\{0, C\}$, so again $\mathbf{S} \in \mathbf{HS}(\mathbf{A}_\Gamma)$. This proves the first item. The second item is a consequence of the first item, the fact that $\mathbf{A}_\Gamma$ has a semilattice term operation, and the finite basis theorem in [8]. $\square$

Proof of Lemma 4. Choose $n \geq 3$ such that $E_n \in \Gamma$ but $S_n \notin \Gamma$. Choose elements $f_0, \ldots, f_n \in \{a, b\}^{\mathbb{Z}}$ satisfying the statement of Fact 2. For any $f \in \{a, b\}^{\mathbb{Z}}$ and $g \in \{1, H, 2\}^{\mathbb{Z}}$ define $\langle g * f \rangle \in A^{\mathbb{Z}}$ by $\langle g * f \rangle(j) = g(j)^{f(j)}$. It will be convenient to represent the elements of $\{1, H, 2\}^{\mathbb{Z}}$ as follows: for any pair of disjoint subsets $X, Y \subseteq \mathbb{Z}$ define $g_{X,Y} : \mathbb{Z} \to \{1, H, 2\}$ by $g_{X,Y}(j) = 1$ for $j \in X$, $g_{X,Y}(j) = H$ for $j \in Y$, and $g_{X,Y}(j) = 2$ for $j \in \mathbb{Z} \setminus (X \cup Y)$. Thus if $f \in \{a, b\}^{\mathbb{Z}}$ then

$$\langle g_{X,Y} * f \rangle(j) = \begin{cases} 1^{f(j)} & \text{if } j \in X \\ H^{f(j)} & \text{if } j \in Y \\ 2^{f(j)} & \text{otherwise.} \end{cases}$$

Similarly, for $X \subseteq \mathbb{Z}$ define $h_X \in A^{\mathbb{Z}}$ by $h_X(j) = C$ if $j \in X$ and $h_X(j) = D$ otherwise. Note that if $X, Y \subseteq \mathbb{Z}$ are disjoint then computing in $(\mathbf{A}_\Gamma)^{\mathbb{Z}}$ we have

$$E_n(\langle g_{X,Y} * f_0 \rangle, \ldots, \langle g_{X,Y} * f_n \rangle, h_{X \cup Y}) = h_X. \tag{1}$$

To establish some kind of a converse, let

$$\begin{aligned}
\Lambda_0 &= \{\langle g * f_i\rangle : g \in \{1,H,2\}^{\mathbb{Z}} \text{ and } 0 \le i \le n\}\\
\Lambda_1 &= \{C,D\}^{\mathbb{Z}}\\
\Delta &= \{f \in A^{\mathbb{Z}} : 0 \in \operatorname{range}(f)\}\\
B &= \Lambda_0 \cup \Lambda_1 \cup \Delta.
\end{aligned}$$

Claim 5. *If $F \in \Gamma$ is k-ary, $r_0,\dots,r_{k-1} \in B$ and $F(r_0,\dots,r_{k-1}) \in A^{\mathbb{Z}} \setminus \Delta$, then $F = E_n$ and $(r_0,\dots,r_{k-1}) = (\langle g_{X,Y} * f_0\rangle,\dots,\langle g_{X,Y} * f_n\rangle, h_{X\cup Y})$ for some disjoint $X,Y \subseteq \mathbb{Z}$.*

Proof. Case 1: $F = E_m$ for some $m \ge 3$. Then $k = m+2$ and we must have $[\![R_m^+(r_0,\dots,r_m)]\!] = \mathbb{Z}$ and $r_{m+1} \in \{C,D\}^{\mathbb{Z}}$ by the definition of E_m. Hence $r_0,\dots,r_m \in \Lambda_0$ and $r_{m+1} \in \Lambda_1$, by the definition of B. Thus there exist $g_0,\dots,g_m \in \{1,H,2\}^{\mathbb{Z}}$ and $\sigma : \{0,\dots,m\} \to \{0,\dots,n\}$ such that $r_i = \langle g_i * f_{\sigma(i)}\rangle$ for $i \le m$. By the definition of R_m^+ we must have $g_0 = \cdots = g_m = g_{X,Y}$, say, and $[\![R_m(f_{\sigma(0)},\dots,f_{\sigma(m)})]\!] = \mathbb{Z}$. By Fact 2 this forces $m = n$ and $\sigma = \mathrm{id}$. Finally, the definition of E_n forces $r_{m+1} = h_{X\cup Y}$.

Case 2: $F = S_m$ for some $m \ge 3$. Then $k = m+4$ and $[\![R_n^+(r_0,\dots,r_m)]\!] = \mathbb{Z}$ by the definition of S_m. By the analysis in Case 1 we get $m = n$, but we have assumed that $S_n \notin \Gamma$, so this case is vacuous. □

As a consequence of Claim 5 we see that B is the universe of a subalgebra $\mathbf{B}$ of $(\mathbf{A}_\Gamma)^{\mathbb{Z}}$ and $\theta := 0_B \cup \Delta^2$ is a congruence of $\mathbf{B}$. The quotient algebra $\mathbf{S} := \mathbf{B}/\theta$ is a height-1 meet semilattice in which all operations other than $\wedge$ and E_n are constantly equal to 0. Hence $\mathbf{S}$ is term-equivalent to its reduct $\langle S, \wedge, 0, E_n\rangle$. For the rest of this paper we will identify $\mathbf{S}$ with this reduct; we will also identify the universe of $\mathbf{S}$ with the set $\Lambda_0 \cup \Lambda_1 \cup \{0\}$. The nonzero values of E_n are completely specified by equation 1.

We shall construct two subalgebras of $\mathbf{S}$ which will prove the two items in the Lemma. First, for $i \le n$ define $2^{f_i}, 2_*^{f_i} \in \Lambda_0$ by $2^{f_i} = \langle g_{\emptyset,\emptyset} * f_i\rangle$ and $2_*^{f_i} = \langle g_{\emptyset,\{0\}} * f_i\rangle$. Let $\hat{D} = h_\emptyset$, the constant-D function, and let $\hat{D}_* = h_{\{0\}}$. Thus

$$\begin{array}{rcllllllll}
2^{f_i} &=& (\;\cdots & 2^{f_i(-1)} & 2^{f_i(0)} & 2^{f_i(1)} & \cdots & 2^{f_i(j)} & \cdots\;)\\
2_*^{f_i} &=& (\;\cdots & 2^{f_i(-1)} & H^{f_i(0)} & 2^{f_i(1)} & \cdots & 2^{f_i(j)} & \cdots\;)\\
\hat{D} &=& (\;\cdots & D & D & D & \cdots & D & \cdots\;)\\
\hat{D}_* &=& (\;\cdots & D & C & D & \cdots & D & \cdots\;)\\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Let $T_1 = \{2^{f_i} : i \le n\} \cup \{2_*^{f_i} : i \le n\} \cup \{\hat{D}, \hat{D}_*, 0\}$. Using equation 1 it can be seen that

$$E_n(2^{f_0},\dots,2^{f_n},\hat{D}) = E_n(2_*^{f_0},\dots,2_*^{f_n},\hat{D}_*) = \hat{D} \tag{2}$$

while $E_n(\bar{u}, z) = 0$ for all other inputs from T_1. Hence $\mathbf{T}_1$ is a subalgebra of $\mathbf{S}$. Now $\mathbf{T}_1$ is a height-1 meet-semilattice-based algebra in which 0 is an absorbing element for each fundamental operation, but some fundamental operation (namely E_n) does not commute with $\wedge$. Thus $\mathbf{HSP}(\mathbf{T}_1)$ is residually large by [7, Lemma 1.2]. Alternatively, one can appeal to McKenzie's 'bad genes' manuscript [4]: the gene of $\mathbf{T}_1$ is $G = (e, 0, \hat{D}, 0_{T_1})$ where $e(x) = x \wedge \hat{D}$, while the offending tolerance is 1_{T_1}. Note that $\mathrm{range}(e) = \{\hat{D}, 0\}$ and that $\mathbf{T}_1$ is subdirectly irreducible with monolith $\{0, \hat{D}\}^2 \cup 0_{T_1}$. Hence G is a gene. Since 0 is an absorbing element for every operation of $\mathbf{T}_1$, 1_{T_1} automatically rectangulates $\{0, \hat{D}\}^2$ modulo G. But equation 2 shows that 1_{T_1} fails to rectangulate itself modulo G; hence $\mathbf{HSP}(\mathbf{T}_1)$ is residually large by [4, Lemma 2.3].

To prove the second item, for $k \in \mathbb{Z}$ let (k) denote $\{j \in \mathbb{Z} : j < k\}$ and for $i = 0, \ldots, n$ define $e_k^{f_i} = \langle g_{(k),\{k\}} * f_i \rangle$. Also define $v_k = h_{(k)}$ for each $k \in \mathbb{Z}$. Thus

$$\begin{array}{rcllllllllll} e_k^{f_i} & = & (& \cdots & 1^{f_i(j)} & \cdots & 1^{f_i(k-1)} & H^{f_i(k)} & 2^{f_i(k+1)} & \cdots & 2^{f_i(j')} & \cdots \;) \\ v_k & = & (& \cdots & C & \cdots & C & D & D & \cdots & D & \cdots \;) \\ v_{k+1} & = & (& \cdots & C & \cdots & C & C & D & \cdots & D & \cdots \;) \\ & & & & & & & \uparrow & & & & \\ & & & & & & & k & & & & \end{array}$$

Let $T_2 = \{e_k^{f_i} : i \leq n \text{ and } k \in \mathbb{Z}\} \cup \{v_k : k \in \mathbb{Z}\} \cup \{0\}$. Using equation 1 it can be seen that

$$E_n(e_k^{f_0}, \ldots, e_k^{f_n}, v_{k+1}) = v_k \text{ for each } k \in \mathbb{Z}, \tag{3}$$

while $E_n(\bar{u}, z) = 0$ for all other inputs from T_2. Hence $\mathbf{T}_2$ is a subalgebra of $\mathbf{S}$. There is an obvious automorphism σ of $\mathbf{T}_2$ defined by $\sigma(e_k^{f_i}) = e_{k-1}^{f_i}$, $\sigma(v_k) = v_{k-1}$, and $\sigma(0) = 0$. σ has 0 as a fixed point and has only finitely many other orbits, all infinite. Moreover, if X is the set of all $\bar{u} \in (T_2)^{n+2}$ such that $E_n(\bar{u}) \neq 0$, then σ is transitive on X. Finally, there is a nonconstant unary polynomial of $\mathbf{T}_2$ which maps v_1 to $\sigma(v_1)$ $(= v_0)$. It follows from [1, Theorem 1.1] that $\mathbf{T}_2$ is inherently nonfinitely based. Thus if Γ is finite then $\mathbf{A}_\Gamma$ is also inherently nonfinitely based. $\square$

To prove Theorem 1, let $F_0, F_1, F_2, F_3, F_4, \ldots$ be the sequence $\wedge, E_3, S_3, E_4, S_4, \ldots$ and apply Lemmas 3 and 4.

Note that if one is interested only in the alternations between residual largeness and residual finiteness, and does not require the algebras to be of finite type, then the above arguments prove more.

Theorem 6. *In the lattice of all clones on a nine-element set A there is a subposet order-isomorphic to $\langle Su(\omega), \subseteq \rangle$ with the following property: representing $Su(\omega)$ isomorphically as $Su(\omega) \times Su(\omega)$ and the subposet of clones as $\{\mathcal{C}_{I,J} : I, J \subseteq \omega\}$, and letting $\mathbf{A}_{I,J} = \langle A, \mathcal{C}_{I,J} \rangle$,*

(1) *If* $I \not\subseteq J$ *then* $\mathbf{HSP}(\mathbf{A}_{I,J})$ *is residually large.*
(2) *If* $I \subseteq J$ *then each subdirectly irreducible member of* $\mathbf{HSP}(\mathbf{A}_{I,J})$ *belongs to* $\mathbf{HS}(\mathbf{A}_{I,J})$.

Using the above theorem, one can show e.g. that for every countable limit ordinal λ there is an increasing chain $\{\mathcal{C}_\alpha : \alpha < \lambda\}$ of clones on the nine-element set A such that, with $\mathbf{A}_\alpha = \langle A, \mathcal{C}_\alpha \rangle$,

(1) If $\alpha = \omega \cdot \beta + n < \lambda$ with n an odd positive integer, then $\mathbf{HSP}(\mathbf{A}_\alpha)$ is residually large.
(2) If $\alpha = \omega \cdot \beta + n < \lambda$ with n an even nonnegative integer, then each subdirectly irreducible member of $\mathbf{HSP}(\mathbf{A}_\alpha)$ is in $\mathbf{HS}(\mathbf{A}_\alpha)$.

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