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Varieties Having Boolean Factor Congruences

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Every ring R with identity satisfies the following property: the factor ideals of R (i.e., those ideals I such that $I+J=R$ and $I \cap J=(0)$ for some ideal J) form a Boolean sublattice of the lattice of all ideals of R. The universal algebraic abstraction of this property is known as *Boolean factor congruences* (*BFC*) or as the *strict refinement property*; more examples of algebras having BFC are lattices, semilattices, and centerless groups. We take up the study of varieties all of whose members have BFC, and show that all known examples of such varieties have a first-order definable 4-ary relation witnessing BFC. We also show that if every member of a variety is centerless then the variety has BFC, but not vice versa; and that, for a certain class of varieties, BFC is equivalent to the absence of abelian algebras. © 1990 Academic Press, Inc.

INTRODUCTION

A *congruence* of an algebra **A** is the kernel $\{\langle a, b\rangle \in A^2 : f(a)=f(b)\}$ of a homomorphism f with domain **A**; it is a *factor congruence* of **A** if f is a projection onto a direct factor of **A**; and **A** has *Boolean factor congruences* (*BFC*) if the set of factor congruences of A forms a Boolean sublattice of the lattice of all congruences of **A**. Probably the best-known examples of algebras having BFC are rings with identity.

The property BFC grew out of the work of A. Tarski and others on the direct product decompositions of groupoids with identity [13, 19, 8]. B. Jónsson and Tarski proved that any groupoid with identity whose center subgroup (i.e., the largest abelian subgroup whose elements commute and associate with all elements of the groupoid) is finite has the so-called *refinement property* and hence has at most one representation as a direct product of directly indecomposable groupoids. They also noticed that a stronger property, called the *strict refinement property* (SRP), holds if the center is trivial, and they proved [19] that SRP is equivalent to the "factor sub-

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groupoids" forming a Boolean poset with respect to inclusion. Several years later C. C. Chang, Jónsson, and Tarski [5] generalized the definitions of SRP and BFC to the setting of arbitrary algebras (and even relational structures), proved that SRP and BFC are equivalent, and gave several broad classes of structures which have SRP. This line of research culminated in a paper of R. McKenzie [15], in which it was proved (among many other things) that a structure having a binary relation R has the SRP if R is thin (see Definition 5.1 below) and both $R \circ R^{\cup}$ and $R^{\cup} \circ R$ are connected.

In the same year in which McKenzie's paper appeared, S. Comer published an article [6] showing that the Pierce sheaf construction for rings with identity extends to any algeba having BFC. This construction proved to be especially important for monadic algebras [7] and, more generally, algebras belonging to any finitely generated discriminator variety [14, 21, 4]. However, the construction for arbitrary algebras having BFC was largely ignored (though see, e.g., [18]).

Very recently, D. Bigelow and S. Burris [2] began the study of *varieties* (i.e., equationally definable classes of algebras) all of whose members have BFC. They used the Pierce sheaf construction to show that the characterization of the finite B-separating groups by A. B. Apps [1] extends to any such variety (even though the variety of groups does not have BFC). Bigelow and Burris also showed that BFC is a Mal'cev property for varieties, but they are unable to find a corresponding Mal'cev condition.

Though not used in this paper, the notion of a *Mal'cev condition* as defined by G. Grätzer [11], W. Taylor [20], and W. D. Neumann [17] is central to our study. Let $\mathscr{P}$ be a property attributable to varieties. A *finite presentation* (*for varieties*) is a pair consisting of a finite set $\{f_1, ..., f_r\}$ of primitive operation symbols and a finite set $\{\sigma_1 \approx \tau_1, ..., \sigma_k \approx \tau_k\}$ of equational laws in variables and these symbols. A *strong Mal'cev condition for* $\mathscr{P}$ is a finite presentation Γ (as above) such that an arbitrary variety V satisfies $\mathscr{P}$ iff there are terms $t_1, ..., t_r$ in the language of V which, when substituted for $f_1, ..., f_r$, make the equations in Γ true in every member of V. $\mathscr{P}$ is also said to be *defined by* Γ. A *Mal'cev condition for* $\mathscr{P}$ is a countably infinite sequence $\Gamma_0, \Gamma_1, \cdots$ of finite presentations defining properties $\mathscr{P}_0, \mathscr{P}_1, ...,$ respectively, such that (i) $\mathscr{P}_i$ implies $\mathscr{P}_{i+1}$ for each $i \geqslant 0$, and (ii) $\mathscr{P}$ is equivalent to the disjunction of the $\mathscr{P}_i$'s. Finally, $\mathscr{P}$ is a *Mal'cev property* if there exists a Mal'cev condition for $\mathscr{P}$.

This paper grew out of our attempt to find a Mal'cev condition for BFC. Though we have not yet succeeded, we have found a simple definability property (*) which (i) implies BFC, and (ii) is true of every variety known to us which has BFC. In this paper we state the property (*) and prove that (*) implies BFC while each of the following properties of a variety V implies (*):

(1) V has the Fraser–Horn–Hu property (see Section 2—this includes all varieties of lattices and rings with identity).

(2) More generally, for every $\mathbf{A}, \mathbf{B} \in V$ and all congruences θ, θ' of $\mathbf{A} \times \mathbf{B}$,

$$(\pi_1(\theta) \times \Delta_B) \cap (\theta \circ \theta') \cap (\theta' \circ \theta) \subseteq \mathrm{trans}[\theta \cup \pi_1(\theta \cap \theta') \times \Delta_B].$$

(3) V has a positive-Horn-definable binary relation R which is thin and is such that both $R \circ R^{\cup}$ and $R^{\cup} \circ R$ are connected in every member of V.

We also show that (*) is equivalent to BFC for those varieties V such that either:

(4) V is congruence modular (i.e., the congruence lattice of each member of V satisfies Dedekind's modular law—this includes all varieties of groups, rings, lattices, and quasigroups); or

(5) V is a variety of Jónsson–Tarski algebras (see Section 4—this includes all varieties of groupoids with identity).

Moreover, for the varieties of type (4) or (5), BFC is equivalent to the absence of "abelian" algebras. A weaker (but general) result proved with the aid of (*) is that an arbitrary variety V has BFC if (6) every member of V is "centerless."

The property (2) has a natural Mal'cev condition, and we wondered[1] if (2) were equivalent to BFC. An appendix to this paper contains an example of a variety which satisfies (*) but neither (2) nor (6), hence settling the question negatively. The problem of whether (*) is equivalent to BFC remains open.

0. Basic Notions

Our terminology and notation follow that of Burris and Sankappanavar [3]. In particular, a *factor congruence* of an algebra $\mathbf{A}$ is a congruence $\theta \in \mathrm{Con}\,\mathbf{A}$ for which there exists a $\theta' \in \mathrm{Con}\,\mathbf{A}$ satisfying $\theta \circ \theta' = \nabla_A$ and $\theta \cap \theta' = \Delta_A$. If θ' is such a solution, then the pair θ, θ' is called a *pair of factor congruences.* There is a well-known correspondence between pairs of factor congruences of $\mathbf{A}$ and decompositions of $\mathbf{A}$ as a direct product of two algebras. $\mathbf{A}$ has *Boolean factor congruences* if its factor congruences form a Boolean sublattice of Con $\mathbf{A}$, and a class K of algebras has BFC if every member of K has BFC.

[1] Professor S. Burris posed this question at the Asilomar Conference on Algebras, Lattices, and Logic (July 1987).

DEFINITION 0.1. Suppose R_1, R_2, S are binary relations on the sets A_1, A_2, and $A_1 \times A_2$, respectively. Then $\pi_1(S)$, $\pi_2(S)$, and $R_1 \times R_2$ are the binary relations on A_1, A_2, and $A_1 \times A_2$, respectively defined by

$$\pi_i(S) = \{\langle a_i, b_i\rangle : \langle (a_1, a_2), (b_1, b_2)\rangle \in S\}$$

$$R_1 \times R_2 = \{\langle (a_1, a_2), (b_1, b_2)\rangle : \langle a_i, b_i\rangle \in R_i, i = 1, 2\}.$$

LEMMA 0.2. *For an algebra* **A**, *the following are equivalent*:

(1) **A** *has BFC.*

(2) *If* ϕ, ϕ' *and* θ, θ' *are pairs of factor congruences of* **A**, *then* $\phi \circ \theta = \theta \circ \phi$ *and* $(\theta \circ \phi) \cap \phi' \subseteq \theta$.

(3) *If* ϕ, ϕ' *and* θ, θ' *are pairs of factor congruences of* **A**, *then* $(\phi \circ \theta \circ \phi) \cap \phi' \subseteq \theta$.

(4) *If* $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ *and* θ *is a factor congruence of* $\mathbf{B} \times \mathbf{C}$, *then* $\pi_1(\theta) \times \Delta_C \subseteq \theta$.

Proof. The equivalence of (1)–(3) was proved (for relational structures) by Chang, Jónsson, and Tarski in [5, Theorem 4.5], while (4) is just a restatement of (3). ▮

1. THE PROPERTY (*)

DEFINITION 1.1. Let $\mathscr{L}$ be a first-order language and K a class of $\mathscr{L}$-structures. The set of *K-factorable formulas* is the smallest set Γ of $\mathscr{L}$-formulas which contains the atomic formulas and is closed under &, $\exists$, $\forall$, and the following rule:

> Suppose $\alpha(\vec{x})$, $\beta(\vec{x}, \vec{y})$, $\gamma(\vec{x}, \vec{y}) \in \Gamma$. If $K \models \forall\vec{x}\, \exists\vec{y}\beta(\vec{x}, \vec{y})$ then $\forall\vec{y}[\beta(\vec{x}, \vec{y}) \to \gamma(\vec{x}, \vec{y})] \in \Gamma$. More generally, if $K \models \forall\vec{x}[\alpha(\vec{x}) \to \exists\vec{y}\beta(\vec{x}, \vec{y})]$, then $\alpha(\vec{x})\ \&\ \forall\vec{y}[\beta(\vec{x}, \vec{y}) \to \gamma(\vec{x}, \vec{y})] \in \Gamma$.

LEMMA 1.2. *K-factorable formulas are evaluated coordinatewise in direct products of members of K. That is, if* $\phi(x_1, \ldots, x_n)$ *is K-factorable,* $\mathbf{A}_i \in K$ $(i \in I)$ *and* $f_1, \ldots, f_n \in \prod_{i \in I} A_i$, *then*

$$\prod_{i \in I} \mathbf{A}_i \models \phi(f_1, \ldots, f_n) \quad \text{iff} \quad \mathbf{A}_i \models \phi(f_1(i), \ldots, f_n(i)) \quad \text{for all} \quad i \in I.$$

Proof. For each K-factorable formula $\phi(x_1, \ldots, x_n)$ let R_ϕ be a new n-ary relation symbol and let $\mathscr{L}^+$ be $\mathscr{L}$ together with all such new symbols R_ϕ. Let K^+ be the class of expansions of the members of K to $\mathscr{L}^+$ defined by the rule $R_\phi(\vec{x}) \leftrightarrow \phi(\vec{x})$. To prove the lemma we must show

$\mathbf{P}(K^+) \models R_\phi(\vec{x}) \leftrightarrow \phi(\vec{x})$ for each K-factorable formula ϕ; and to do this it suffices to show that there is a Horn sentence Θ_ϕ of type $\mathscr{L}^+$ such that:

(i) $K^+ \models \Theta_\phi$

(ii) $\models \Theta_\phi \to \forall\vec{x}[R_\phi(\vec{x}) \leftrightarrow \phi(\vec{x})]$.

We define appropriate sentences Θ_ϕ by recursion on ϕ.

If ϕ is atomic, then Θ_ϕ is $\forall\vec{x}[R_\phi(\vec{x}) \leftrightarrow \phi(\vec{x})]$.

$\Theta_{\phi\&\psi}$ is $\Theta_\phi \,\&\, \Theta_\psi \,\&\, \forall\vec{x}[R_{\phi\&\psi}(\vec{x}) \leftrightarrow [R_\phi(\vec{x}) \,\&\, R_\psi(\vec{x})]]$.

$\Theta_{\exists y\phi}$ and $\Theta_{\forall y\phi}$ are defined analogously.

Finally, suppose ϕ is $\alpha(\vec{x}) \,\&\, \forall\vec{y}[\beta(\vec{x},\vec{y}) \to \gamma(\vec{x},\vec{y})]$; then Θ_ϕ is the conjunction of Θ_α, Θ_β, Θ_γ, and the sentences

$$\forall\vec{x}(R_\phi \to R_\alpha)$$

$$\forall\vec{x}\vec{y}([R_\phi \,\&\, R_\beta] \to R_\gamma)$$

$$\forall\vec{x}\,\exists\vec{y}([R_\alpha \to R_\beta] \,\&\, [(R_\alpha \,\&\, R_\gamma) \to R_\phi]).$$

The claim then follows by induction on ϕ, the only nontrivial step being the verification that

$$K^+ \models \forall\vec{x}\exists\vec{y}([R_\alpha \to R_\beta] \,\&\, [(R_\alpha \,\&\, R_\gamma) \to R_\phi])$$

when ϕ is $\alpha \,\&\, \forall\vec{y}(\beta \to \gamma)$. If $\mathbf{A} \in K$ and $x_1, x_2, \ldots \in A$, argue by the following three cases for $\vec{x} - \mathbf{A} \models \neg\alpha(\vec{x})$, $\mathbf{A} \models \alpha(\vec{x}) \,\&\, \neg\phi(\vec{x})$, and $\mathbf{A} \models \phi(\vec{x})$—using the fact that $K \models \forall\vec{x}(\alpha \to \exists\vec{y}\beta)$ for the last case. ∎

COROLLARY 1.3. *Suppose* $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E} \in K$, $\alpha\colon \mathbf{B}\times\mathbf{C} \cong \mathbf{D}\times\mathbf{E}$ *and* $\phi(x_1, \ldots, x_n)$ *is K-factorable. Let* $\pi_1\colon \mathbf{D}\times\mathbf{E} \to \mathbf{D}$ *be the first projection. If* $\mathbf{B} \models \phi(b_1, \ldots, b_n)$ *and* $\mathbf{C} \models \phi(c_1, \ldots, c_n)$, *then*

$$\mathbf{D} \models \phi(\pi_1\alpha(b_1, c_1), \ldots, \pi_1\alpha(b_n, c_n)).$$

DEFINITION 1.4. A class K is said to *satisfy property* (*) if there is a K-factorable formula $\pi(x, y, z, w)$ such that:

(i) $K \models \pi(x, y, x, y)$

(ii) $K \models \pi(x, x, z, w)$

(iii) $K \models \pi(x, y, z, z) \to x \approx y$.

Such a formula *witnesses* (*) for K.

THEOREM 1.5. *Suppose* $\mathbf{A}$ *is an algebra,* $\mathbf{A}/\theta \in K$ *for every factor congruence* θ, *and* K *satisfies* (*). *Then* $\mathbf{A}$ *has BFC.*

Proof. Let $\pi(x, y, z, w)$ witness (*) for K. To show that $\mathbf{A}$ has BFC, it suffices by Lemma 0.2 (4) to show that if $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, $\langle \theta, \theta' \rangle$ is a pair of factor congruences of $\mathbf{B} \times \mathbf{C}$, $\langle (a, c), (b, d) \rangle \in \theta$ and $e \in C$, then $\langle (a, e), (b, e) \rangle \in \theta$. Put $\mathbf{D} = (\mathbf{B} \times \mathbf{C})/\theta$ and $\mathbf{E} = (\mathbf{B} \times \mathbf{C})/\theta'$, and let $\alpha\colon \mathbf{B} \times \mathbf{C} \cong \mathbf{D} \times \mathbf{E}$ be the canonical isomorphism $(u, v) \mapsto \langle (u, v)/\theta, (u, v)/\theta' \rangle$. Note that $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E} \in \mathbf{I}(K)$.

Since π witnesses (*) for $\mathbf{I}(K)$, we have $\mathbf{B} \models \pi(a, b, a, b)$ and $\mathbf{C} \models \pi(e, e, c, d)$ by conditions (i) and (ii) of Definition 1.4. Hence $\mathbf{D} \models \pi((a, e)/\theta, (b, e)/\theta, (a, c)/\theta, (b, d)/\theta)$ by Corollary 1.3. But $(a, c)/\theta = (b, d)/\theta$ by hypothesis, so $(a, e)/\theta = (b, e)/\theta$ by condition (iii) of Definition 1.3. ∎

COROLLARY 1.6. *If V is a variety and satisfies (*), then V has BFC.*

Theorem 1.5 remains true if $\mathbf{A}$ is an arbitrary first-order structure, provided that we replace "factor congruence" by "factor relation" as this is defined in [16, p. 307]. However, we are interested only in the algebraic case.

The next lemma provides a useful tool for proving that a given class K satisfies (*).

LEMMA 1.7. *For a class K of structures, the following are equivalent:*

(1) *K satisfies (*).*

(2) *There is a K-factorable formula $\pi(x, y, z, w)$ such that:*

 (i) $K \models \pi(x, y, x, y)$

 (ii) $K \models \exists x[\pi(x, x, z, w)\ \&\ \pi(x, x, z', w')]$

 (iii) $K \models \pi(x, y, z, z) \rightarrow x \approx y$.

(3) *There is a K-factorable formula $\phi(x, y, \vec{u})$ such that:*

 (i) $K \models \exists \vec{u} \phi(x, y, \vec{u})$

 (ii) $K \models \forall \vec{u} \phi(x, y, \vec{u}) \leftrightarrow x \approx y$.

(4) *There is a K-factorable formula $\theta(x, \vec{u})$ such that:*

 (i) $K \models \exists \vec{u}[\theta(x, \vec{u})\ \&\ \theta(y, \vec{u})]$

 (ii) $K \models \forall \vec{u}[\theta(x, \vec{u}) \leftrightarrow \theta(y, \vec{u})] \rightarrow x \approx y$.

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (4). Let $\theta(x, u_1, u_2, u_3)$ be $\pi(u_1, u_2, x, u_3)$. Clearly θ is K-factorable, and 2(ii) implies 4(i). Suppose $\forall \vec{u}[\theta(x, \vec{u}) \leftrightarrow \theta(y, \vec{u})]$. Since $\pi(x, y, x, y)$ by 2(i), it follows from the assumption that $\pi(x, y, y, y)$ and so $x = y$ by 2(iii).

(4) ⇒ (3). Let $\phi(x, y, u_1, u_2)$ be

$$\forall \vec{v}[[\theta(x, \vec{v})\ \&\ \theta(u_1, \vec{v})] \rightarrow \theta(y, \vec{v})]\ \&\ \forall \vec{v}[[\theta(y, \vec{v})\ \&\ \theta(u_2, \vec{v})] \rightarrow \theta(x, \vec{v})].$$

ϕ is K-factorable by 4(i). 3(i) is clearly true since we can choose $u_1 = y$ and $u_2 = x$, and $\forall u_1 u_2 \phi(x, x, u_1, u_2)$ is also clear. Suppose $\forall u_1 u_2 \phi(x, y, u_1, u_2)$; then in particular $\phi(x, y, x, y)$ and so $x = y$ by 4(ii).

(3) $\Rightarrow$ (1). Let $\pi(x, y, z, w)$ be $\forall \vec{u}[\phi(z, w, \vec{u}) \rightarrow \phi(x, y, \vec{u})]$. π is K-factorable by 3(i). Clearly condition (i) of Definition 1.4 is true, while (ii) and (iii) follow from 3(ii). ∎

2. The Fraser–Horn–Hu Property and a Generalization

In this section we show that each of the conditions (1), (2) stated at the beginning of this paper implies (*).

Definition 2.1. A variety V has the *Fraser–Horn–Hu property* if for all $\mathbf{A}_1, \mathbf{A}_2 \in V$ and $\theta \in \operatorname{Con} \mathbf{A}_1 \times \mathbf{A}_2$ there exist $\theta_i \in \operatorname{Con} \mathbf{A}_i$ such that $\theta = \theta_1 \times \theta_2$.

For example, every congruence distributive variety satisfies this property, as does the variety of rings with identity. It is immediate from Lemma 0.2(4) that every variety having the Fraser–Horn–Hu property also has BFC.

Theorem 2.2. *If a variety V has the Fraser–Horn–Hu property, then V satisfies* (*).

Proof. The usual Mal'cev condition for the Fraser–Horn–Hu property (see [9]) yields a principal congruence formula $\pi(x, y, z, w)$ which satisfies conditions (i) and (ii) of Definition 1.4. Any principal congruence formula is V-factorable (as it is $\exists$ & *atomic*) and satisfies condition (iii) of Definition 1.4. Hence π witnesses (*) for V. ∎

Definition 2.3. (i) A variety V *satisfies property* (I) if for all $\mathbf{A}, \mathbf{B} \in V$ and $\theta, \theta' \in \operatorname{Con} \mathbf{A} \times \mathbf{B}$,

$$(\pi_1(\theta) \times \Delta_B) \cap (\theta \circ \theta') \cap (\theta' \circ \theta) \subseteq \operatorname{trans}[\theta \cup \pi_1(\theta \cap \theta') \times \Delta_B],$$

where "trans" denotes the transitive closure.

(ii) *V satisfies property* (II) if for all $\mathbf{A}, \mathbf{B} \in V$ and $\theta, \theta' \in \operatorname{Con} \mathbf{A} \times \mathbf{B}$,

$$(\pi_1(\theta) \times \Delta_B) \cap (\theta \circ \theta') \cap (\theta' \circ \theta) \subseteq \operatorname{trans}[\theta \cup \pi_1(\theta') \times \Delta_B].$$

Clearly (I) implies (II). It is also easy to see (by Lemma 0.2(4)) that (I) implies BFC. Every variety with the Fraser–Horn–Hu property satisfies (I), as does the variety of semilattices.

Theorem 2.4. *If a variety V satisfies* (II), *then V satisfies* (*).

Proof. Let $\mathbf{A} = \mathbf{F}_V(x, y, z, w)$ and $\mathbf{B} = \mathbf{F}_V(a, b, c, d, e)$ be the V-free algebras on 4 and 5 generators, respectively. Let

$$\alpha_1 = \Theta_{\mathbf{A} \times \mathbf{B}}((x, a), (y, b))$$
$$\alpha_2 = \Theta_{\mathbf{A} \times \mathbf{B}}((x, e), (z, c))$$
$$\alpha_3 = \Theta_{\mathbf{A} \times \mathbf{B}}((y, e), (w, d))$$
$$\beta_1 = \Theta_{\mathbf{A} \times \mathbf{B}}((x, e), (w, d))$$
$$\beta_2 = \Theta_{\mathbf{A} \times \mathbf{B}}((y, e), (z, c))$$
$$\theta = \alpha_1 \vee \alpha_2 \vee \alpha_3$$
$$\theta' = \beta_1 \vee \beta_2.$$

Then $\langle (x, e), (y, e) \rangle \in (\pi_1(\theta) \times \Delta_B) \cap (\theta \circ \theta') \cap (\theta' \circ \theta)$, so by (II),

$$\langle (x, e), (y, e) \rangle \in \text{trans}[\theta \cup \pi_1(\theta') \times \Delta_B].$$

Thus there exist $n \geqslant 1$, $r_i \in A$, and $s_i, t_i \in B$ $(0 \leqslant i \leqslant 2n+1)$ such that:

$$(x, e) = (r_0, s_0), \qquad (y, e) = (r_{2n+1}, s_{2n+1})$$
$$\langle (r_{2i}, s_{2i}), (r_{2i+1}, s_{2i+1}) \rangle \in \theta \qquad (0 \leqslant i \leqslant n)$$
$$\langle (r_{2i-1}, t_{2i-1}), (r_{2i}, t_{2i}) \rangle \in \theta' \qquad (1 \leqslant i \leqslant n)$$
$$s_{2i-1} = s_{2i} \qquad (1 \leqslant i \leqslant n).$$

It follows from the definitions of θ and θ' that there are "3-generator congruence formulas" $\pi_i(x, y, u_1, v_1, u_2, v_2, u_3, v_3)$, $0 \leqslant i \leqslant n$, and "2-generator congruence formulas" $\pi'_i(x, y, u_1, v_1, u_2, v_2)$, $1 \leqslant i \leqslant n$, which witness the above claims of membership in θ and θ'; that is, each π_i and π'_i is of the form $\exists$ & *atomic* and:

$$\models \pi_i(x, y, u, u, v, v, w, w) \rightarrow x \approx y \qquad (0 \leqslant i \leqslant n) \qquad (1)$$

$$\models \pi'_i(x, y, u, u, v, v) \rightarrow x \approx y \qquad (1 \leqslant i \leqslant n) \qquad (2)$$

$$\mathbf{A} \times \mathbf{B} \models \pi_i((r_{2i}, s_{2i}), (r_{2i+1}, s_{2i+1}), (x, a), (y, b), (x, e), (z, c), (y, e), (w, d)) \qquad (0 \leqslant i \leqslant n) \qquad (3)$$

$$\mathbf{A} \times \mathbf{B} \models \pi'_i((r_{2i-1}, t_{2i-1}), (r_{2i}, t_{2i}), (x, e), (w, d), (y, e), (z, c)) \qquad (1 \leqslant i \leqslant n). \qquad (4)$$

Since each π_i and π'_i is evaluated coordinatewise in $\mathbf{A} \times \mathbf{B}$, and using the fact that $\mathbf{A}$ and $\mathbf{B}$ are V-free, it follows from (3) and (4) that:

$$V \models \forall xyzw\, \exists v_0 \cdots v_{2n+1}$$

$$\left[x \approx v_0 \,\&\, y \approx v_{2n+1} \,\&\, \mathop{\&}_{i=0}^{n} \pi_i(v_{2i}, v_{2i+1}, x, y, x, z, y, w) \right.$$

$$\left. \&\, \mathop{\&}_{i=1}^{n} \pi_i'(v_{2i-1}, v_{2i}, x, w, y, z) \right] \tag{5}$$

$$V \models \forall abcde\, \exists v_0 \cdots v_{2n+1}$$

$$\left[e \approx v_0 \,\&\, e \approx v_{2n+1} \,\&\, \mathop{\&}_{i=0}^{n} \pi_i(v_{2i}, v_{2i+1}, a, b, e, c, e, d) \right.$$

$$\left. \&\, \mathop{\&}_{i=1}^{n} v_{2i-1} \approx v_{2i} \right]. \tag{6}$$

Now let $\phi(x, y, u_1, u_2, u_3, u_4)$ be the formula

$$\exists v_0 \cdots v_{2n+1} \left[x \approx v_0 \,\&\, y \approx v_{2n+1} \,\&\, \mathop{\&}_{i=0}^{n} \pi_i(v_{2i}, v_{2i+1}, u_1, u_2, x, u_3, y, u_4) \right.$$

$$\left. \&\, \mathop{\&}_{i=1}^{n} v_{2i-1} \approx v_{2i} \right].$$

Clearly ϕ is V-factorable (as it is $\exists$ & *atomic*). We claim that ϕ satisfies the conditions of Lemma 1.7(3). First, suppose $\mathbf{A} \in V$ and $x, y \in A$. By (5) there exist $v_0, ..., v_{2n+1} \in A$ such that $x = v_0, y = v_{2n+1}$, and

$$\mathbf{A} \models \mathop{\&}_{i=0}^{n} \pi_i(v_{2i}, v_{2i+1}, x, y, x, y, y, x) \,\&\, \mathop{\&}_{i=1}^{n} \pi_i'(v_{2i-1}, v_{2i}, x, x, y, y).$$

The second conjunct together with (2) implies $v_{2i-1} = v_{2i}$ $(1 \leqslant i \leqslant n)$. Hence $\mathbf{A} \models \exists \vec{u} \phi(x, y, \vec{u})$, namely, $u_1 = u_4 = x$ and $u_2 = u_3 = y$.

Secondly, $V \models \forall \vec{u} \phi(x, x, \vec{u})$ by virtue of (6). Finally, suppose $\mathbf{A} \in V$, $x, y \in A$, and $\mathbf{A} \models \forall \vec{u} \phi(x, y, \vec{u})$. Then in particular $\mathbf{A} \models \phi(x, y, x, x, x, y)$, so there exist $v_0, ..., v_{2n+1} \in A$ such that $x = v_0, y = v_{2n+1}$, $v_{2i-1} = v_{2i}$ for $1 \leqslant i \leqslant n$, and

$$\mathbf{A} \models \mathop{\&}_{i=0}^{n} \pi_i(v_{2i}, v_{2i+1}, x, x, x, x, y, y).$$

This last fact together with (1) implies $v_{2i} = v_{2i+1}$ for $0 \leqslant i \leqslant n$. Hence $x = y$. ∎

3. Centerless Varieties

In this section we show that if every algebra in a variety V is centerless, then V satisfies (*). The first task is to describe a Mal'cev-like condition for the former property. For this purpose, fix an infinite sequence $\langle x_i \rangle_{i \geq 0}$ of variables.

Definition 3.1. Let $\mathscr{L}$ be a language for algebras and $A \neq \varnothing$. Then $TC(\mathscr{L}, A)$ is the set of all tuples

$$\langle t(x_0, ..., x_n), a, b, \vec{c}, \vec{d} \rangle,$$

where $n \geq 1$, t is an $\mathscr{L}$-term in the variables $\{x_0, ..., x_n\}$, a, $b \in A$, and $\vec{c}$, $\vec{d} \in A^n$.

Definition 3.2 (following [16]). Let $\mathbf{A}$ be an algebra and $\theta, \phi, \delta \in \operatorname{Con} \mathbf{A}$.

(i) *θ centralizes ϕ modulo δ* if for all $\langle t(x_0, ..., x_n), a, b, \vec{c}, \vec{d} \rangle$ in $TC(\mathscr{L}, A)$ such that $\langle a, b \rangle \in \theta$ and $\langle c_i, d_i \rangle \in \phi$ $(1 \leq i \leq n)$,

$$\langle t^{\mathbf{A}}(a, \vec{c}), t^{\mathbf{A}}(a, \vec{d}) \rangle \in \delta \quad \text{implies} \quad \langle t^{\mathbf{A}}(b, \vec{c}), t^{\mathbf{A}}(b, \vec{d}) \rangle \in \delta.$$

(ii) The *center* of $\mathbf{A}$, denoted $Z_{\mathbf{A}}$, is the greatest $\theta \in \operatorname{Con} \mathbf{A}$ such that θ centralizes ∇_A modulo Δ_A. $\mathbf{A}$ is *centerless* if $Z_{\mathbf{A}} = \Delta_A$.

(iii) The *(one-sided TC) commutator* of θ and ϕ, denoted $[\theta, \phi]$, is the least $\delta \in \operatorname{Con} \mathbf{A}$ such that θ centralizes ϕ modulo δ.

For groups, the above notions of center and commutator coincide with the usual ones (modulo the correspondence between congruences and normal subgroups).

There is a well-known explicit description of $Z_{\mathbf{A}}$ (see [16, Section 4.13] or [3, Section II.13]). The next definition and lemma show that $[\theta, \phi]$ also has an explicit description.

Definition 3.3. Let $\mathscr{L}$ be a language for algebras and $A \neq \varnothing$.

(i) Let $\mathscr{C}(A) = \{c_a : a \in A\}$ be a set of constant symbols indexed by A.

(ii) Let $\mathscr{F}_{TC}(\mathscr{L}, A) = \{f_\tau : \tau \in TC(\mathscr{L}, A)\}$ be a set of unary operation symbols indexed by $TC(\mathscr{L}, A)$.

(iii) Let $\Sigma_{TC}(\mathscr{L}, A)$ be the set of variable-free terms in the language $\mathscr{C}(A) \cup \mathscr{F}_{TC}(\mathscr{L}, A) \cup \{\circ\}$, where $\circ$ is a binary operation symbol.

(iv) Let $\mathbf{A}$ be an algebra of type $\mathscr{L}$. Then the maps l, $r: \Sigma_{TC}(\mathscr{L}, A) \to A$ are given recursively by

$$l_{c_a} = r_{c_a} = a.$$

$$l_{f_\tau(\sigma)} = t^{\mathbf{A}}(b, \vec{c}) \quad \text{and} \quad r_{f_\tau(\sigma)} = t^{\mathbf{A}}(b, \vec{d}) \quad \text{if} \quad \tau = \langle t, a, b, \vec{c}, \vec{d} \rangle.$$

$$l_{\sigma_1 \circ \sigma_2} = l_{\sigma_1} \quad \text{and} \quad r_{\sigma_1 \circ \sigma_2} = r_{\sigma_2}.$$

LEMMA 3.4. *Let* $\mathbf{A}$ *be an algebra of type* $\mathscr{L}$, $\theta, \phi \in \operatorname{Con} \mathbf{A}$, *and* $x, y \in A$. *Then* $\langle x, y \rangle \in [\theta, \phi]$ *iff there is a* $\sigma \in \Sigma_{TC}(\mathscr{L}, A)$ *such that*:

(1) *If* $\tau = \langle t(x_0, ..., x_n), a, b, \vec{c}, \vec{d} \rangle \in TC(\mathscr{L}, A)$ *and* $f_\tau(\sigma')$ *occurs as a subterm of* σ, *then*

(i) $\langle a, b \rangle \in \theta$

(ii) $\langle c_i, d_i \rangle \in \phi$ *for* $i = 1, ..., n$

(iii) $t^{\mathbf{A}}(a, \vec{c}) = l_{\sigma'}$ *and* $t^{\mathbf{A}}(a, \vec{d}) = r_{\sigma'}$.

(2) *If* $\sigma_1 \circ \sigma_2$ *occurs as a subterm of* σ, *then* $r_{\sigma_1} = l_{\sigma_2}$.

(3) $l_\sigma = x$ *and* $r_\sigma = y$.

Proof. Let S be the set of all $\langle x, y \rangle \in A^2$ for which there exists a $\sigma \in \Sigma_{TC}(\mathscr{L}, A)$ satisfying (1)–(3) above. It suffices to show that S is reflexive, symmetric, transitive, and is preserved by all unary polynomials of $\mathbf{A}$; that θ centralizes ϕ modulo S; and that if θ centralizes ϕ modulo δ, then $S \subseteq \delta$. We leave this tedious but doable exercise to the reader. ∎

Remark. It follows that the congruences $[\nabla_A, \nabla_A]$, $[\Theta_{\mathbf{A}}(x, y), \nabla_A]$, $[\nabla_A, \Theta_{\mathbf{A}}(x, y)]$ etc. can be described by certain $\exists$ & *atomic* formulas in the same way that $\Theta_{\mathbf{A}}(x, y)$ can be described by principal congruence formulas. We do not formalize this notion here.

LEMMA 3.5. *For an algebra* $\mathbf{A}$, *the following are equivalent*:

(1) *Every* $\mathbf{B} \in \mathbf{H}(\mathbf{A})$ *is centerless.*

(2) $[\theta, \nabla_A] = \theta$ *for all* $\theta \in \operatorname{Con} \mathbf{A}$.

(3) $\langle x, y \rangle \in [\Theta_{\mathbf{A}}(x, y), \nabla_A]$ *for all* $x, y \in A$.

Proof. Routine. (For help, see [16, Lemma 4.149 and the second half of Exercise 4.156, No. 11].) ∎

COROLLARY 3.6. *Let* V *be a variety of type* $\mathscr{L}$, $\mathbf{T}_{\mathscr{L}}(x, y)$ *the term algebra for* $\mathscr{L}$ *in the variables* $\{x, y\}$, *and* $\mathbf{F} = \mathbf{F}_V(x, y)$ *the* V*-free algebra on* $\{x, y\}$. *Then the following are equivalent*:

(1) *Every* $\mathbf{A} \in V$ *is centerless.*

(2) $\langle x, y\rangle \in [\Theta_{\mathbf{F}}(x, y), \nabla_F]$.

(3) *There is a* $\sigma \in \Sigma_{TC}(\mathscr{L}, T_{\mathscr{L}}(x, y))$ *such that*:

(a) *If* $\tau = \langle t(x_0, ..., x_n), p(x, y), q(x, y), \vec{r}(x, y), \vec{s}(x, y)\rangle \in TC(\mathscr{L}, T_{\mathscr{L}}(x, y))$ *and* $f_\tau(\sigma')$ *occurs as a subterm of* σ, *then*

(i) $V \models p(x, x) \approx q(x, x)$

(ii) $V \models t(p(x, y), \vec{r}(x, y)) \approx l_{\sigma'}(x, y)$ & $t(p(x, y), \vec{s}(x, y)) \approx r_{\sigma'}(x, y)$.

(b) *If* $\sigma_1 \circ \sigma_2$ *occurs as a subterm of* σ, *then* $V \models r_{\sigma_1}(x, y) \approx l_{\sigma_2}(x, y)$.

(c) $V \models l_\sigma(x, y) \approx x$ & $r_\sigma(x, y) \approx y$.

Proof. (1) ⇒ (2) and (2) ⇒ (3) follow from Lemmas 3.5 and 3.4, respectively, while (3) ⇒ (1) can be verified directly. ▮

Remark. Corollary 3.6(3) can be used to produce a Mal'cev condition for centerless varieties. In a similar manner one can use the results of Section 4 to obtain a Mal'cev condition for varieties containing no nontrivial abelian algebras ($[\nabla, \nabla] = \nabla$); likewise one can obtain Mal'cev conditions for: (1) varieties containing no contrivial abelian congruences ($[\theta, \theta] = \theta$), and (2) varieities satisfying $[\nabla, \theta] = \theta$. But that is not our goal here.

We now begin the proof that every centerless variety satisfies (*).

LEMMA 3.7. *Let K be a class of structures. Suppose there exist K-factorable formulas* $\theta_1(x, \vec{u}^1), ..., \theta_n(x, \vec{u}^n)$ *such that*:

(i) $K \models \exists \vec{u}^i[\theta_i(x, \vec{u}^i) \& \theta_i(y, \vec{u}^i)]$ *for each* $i = 1, ..., n$.

(ii) $K \models [\&_{i=1}^n \forall \vec{u}^i(\theta_i(x, \vec{u}^i) \leftrightarrow \theta_i(y, \vec{u}^i))] \rightarrow x \approx y$.

Then K satisfies (*).

Proof. Assume without loss of generality that the variables u_j^i are pairwise distinct and let $\theta(x, \vec{u}^1, ..., \vec{u}^n)$ be the formula

$$\mathop{\&}_{i=1}^{n} \theta_i(x, \vec{u}^i).$$

Then θ is K-factorable and satisfies the conditions of Lemma 1.7(4). ▮

THEOREM 3.8. *If V is a variety such that every* $\mathbf{A} \in V$ *is centerless, then V satisfies* (*).

Proof. Let $\mathscr{L}$ be the language of V and let $\sigma \in \Sigma_{TC}(\mathscr{L}, T_{\mathscr{L}}(x, y))$ be as in Corollary 3.6(3). Let

$$TC_\sigma = \{\tau \in TC(\mathscr{L}, T_{\mathscr{L}}(x, y)) : f_\tau \text{ occurs in } \sigma\}.$$

For each $\tau = \langle t(x_0, ..., x_n), p, q, \vec{r}, \vec{s} \rangle \in TC_\sigma$, let $\theta_{\tau,0}(x, u_0, ..., u_{2n})$ and $\theta_{\tau,1}(x, u_0, ..., u_{2n})$ be the formulas

$$t(p(x, u_0), u_1, ..., u_n) \approx t(p(x, u_0), u_{n+1}, ..., u_{2n})$$

and

$$t(q(x, u_0), u_1, ..., u_n) \approx t(q(x, u_0), u_{n+1}, ..., u_{2n}),$$

respectively.

We claim that the collection $\{\theta_{\tau,i} : \tau \in TC_\sigma, i = 0, 1\}$ satisfies the hypotheses of Lemma 3.7 (with $K = V$). Clearly $\models \theta_{\tau,i}(x, u_0, u_1, ..., u_n, u_1, ..., u_n)$ so condition (i) is easily met. Now suppose $\mathbf{A} \in V$, $a, b \in A$, and

$$\mathbf{A} \models \underset{\substack{\tau \in TC_\sigma \\ i=0,1}}{\&} \forall \vec{u}[\theta_{\tau,i}(a, \vec{u}) \leftrightarrow \theta_{\tau,i}(b, \vec{u})].$$

To show $a = b$ it suffices (by Corollary 3.6(3c)) to show $l^{\mathbf{A}}_{\sigma'}(a, b) = r^{\mathbf{A}}_{\sigma'}(a, b)$ for every subterm σ' of σ. This is done by induction on σ', the only nontrivial step being the case when $\sigma' = f_\tau(\sigma_1)$. Writing $\tau = \langle t, p, q, \vec{r}, \vec{s} \rangle$, one has

$$\begin{aligned} t^{\mathbf{A}}(p^{\mathbf{A}}(a, b), \vec{r}^{\mathbf{A}}(a, b)) &= l^{\mathbf{A}}_{\sigma_1}(a, b) && \text{(Corollary 3.6(3aii))} \\ &= r^{\mathbf{A}}_{\sigma_1}(a, b) && \text{(inductive hypothesis)} \\ &= t^{\mathbf{A}}(p^{\mathbf{A}}(a, b), \vec{s}^{\mathbf{A}}(a, b)) && \text{(3.6(3aii)).} \end{aligned}$$

Since $\mathbf{A} \models \forall \vec{u}[\theta_{\tau,0}(a, \vec{u}) \leftrightarrow \theta_{\tau,0}(b, \vec{u})]$ it follows that

$$t^{\mathbf{A}}(p^{\mathbf{A}}(b, b), \vec{r}^{\mathbf{A}}(a, b)) = t^{\mathbf{A}}(p^{\mathbf{A}}(b, b), \vec{s}^{\mathbf{A}}(a, b)).$$

By Corollary 3.6(3ai), $p^{\mathbf{A}}(b, b) = q^{\mathbf{A}}(b, b)$ and so

$$t^{\mathbf{A}}(q^{\mathbf{A}}(b, b), \vec{r}^{\mathbf{A}}(a, b)) = t^{\mathbf{A}}(q^{\mathbf{A}}(b, b), \vec{s}^{\mathbf{A}}(a, b)).$$

Since $\mathbf{A} \models \forall \vec{u}[\theta_{\tau,1}(a, \vec{u}) \leftrightarrow \theta_{\tau,1}(b, \vec{u})]$ it follows that

$$t^{\mathbf{A}}(q^{\mathbf{A}}(a, b), \vec{r}^{\mathbf{A}}(a, b)) = t^{\mathbf{A}}(q^{\mathbf{A}}(a, b), \vec{s}^{\mathbf{A}}(a, b)),$$

i.e., $l^{\mathbf{A}}_{\sigma'}(a, b) = r^{\mathbf{A}}_{\sigma'}(a, b)$ as required. ∎

4. Congruence Modular Varieties and Jónsson–Tarski Varieties

In congruence modular varieties and varieties of Jónsson–Tarski algebras, BFC is equivalent to (*), and also to the absence of abelian algebras. The consequence of BFC we use to establish the latter two conditions is contained in Corollary 4.4.

LEMMA 4.1. *If* $\mathbf{M}$ *is a nontrivial module over some ring, then* $\mathbf{M}^2$ *does not have BFC.*

Proof. Let $A_1 = M \times \{0\}$, $A_2 = \{0\} \times M$, and $A_3 = \Delta_M$. Each A_i is a submodule of $\mathbf{M}^2$, while $i \neq j$ implies $A_i + A_j = M^2$ and $A_i \cap A_j = \{\langle 0, 0 \rangle\}$. It follows (by the correspondence between submodules and congruences) that $\mathbf{M}^2$ does not have BFC. ∎

DEFINITION 4.2. Let **A**, **B** be algebras, not necessarily of the same type. **A** is a *polynomial reduct* of **B** if $A = B$ and every fundamental operation of **A** is a polynomial of **B**. **A** and **B** are *polynomially equivalent* if each is a polynomial reduct of the other.

LEMMA 4.3. *If* **A** *has BFC and* **A** *is a polynomial reduct of* **B**, *then* **B** *has BFC.*

Proof. Every factor congruence of **B** is also a factor congruence of **A**. The claim then follows from Lemma 0.2(3). ∎

COROLLARY 4.4. *If a variety V has BFC, then V contains no polynomial reducts of nontrivial modules.*

Proof. By Lemmas 4.1 and 4.3, and the fact that if **A** is a polynomial reduct of **B**, then $\mathbf{A}^2$ is a polynomial reduct of $\mathbf{B}^2$. ∎

DEFINITION 4.5. An algebra **A** is *abelian* if $Z_{\mathbf{A}} = \nabla_A$.

For example, a group is abelian iff it is commutative; a ring is abelian iff it is a zero ring; every module is abelian; no nontrivial lattice is abelian.

THEOREM 4.6. *Suppose V is a congruence modular variety. Then the following are equivalent*:

(1) *V has BFC.*

(2) *V satisfies* (*).

(3) *V has the Fraser–Horn–Hu property.*

(4) *V contains no nontrivial abelian algebras.*

Proof. (3) ⇒ (2) by Theorem 2.2.

(2) ⇒ (1) by Corollary 1.6.

(4) ⇔ (3) may be found in [10, Theorem 8.5].

(1)⇒(4) follows from Corollary 4.4 and the well-known theorem of C. Herrmann [12] that every abelian algebra in a congruence modular variety is polynomially equivalent to a module. ▮

DEFINITION 4.7. A *Jónsson–Tarski algebra* (or *algebra with zero*) is an algebra **A** whose type includes a binary operation symbol $+$ and a constant symbol 0 and which satisfies:

(i) $\mathbf{A} \models x+0 \approx x \;\&\; 0+x \approx x$.

(ii) $\{0\}$ is a subuniverse of **A**.

LEMMA 4.8. *Suppose* **A** *is a nontrivial abelian Jónsson–Tarski algebra. Then there is a nontrivial* $\mathbf{B} \in \mathbf{H}(\mathbf{A}^2)$ *such that* **B** *is a polynomial reduct of a module.*

Proof. It can be shown (see e.g. [16, p. 298]) that the hypothesis implies that $+^{\mathbf{A}}$ is commutative, associative, cancellative, and commutes with every fundamental operation of **A**. Now let

$$\theta = \{\langle (a,b), (c,d)\rangle : a+d = b+c\}.$$

It follows from the above that $\theta \in \mathrm{Con}\mathbf{A}$, and clearly $\theta \neq \nabla_{A^2}$. Let $\mathbf{B} = \mathbf{A}^2/\theta$. Then $+^{\mathbf{B}}$ likewise is commutative, associative, and commutes with every fundamental operation of **B**. Moreover, $+^{\mathbf{B}}$ has inverses, namely,

$$-(a,b)/\theta = (b,a)/\theta.$$

Let R be the endomorphism ring of the abelian group $G = \langle B, +^{\mathbf{B}}, -, 0^{\mathbf{B}}\rangle$. It is an easy matter to show that **B** is a polynomial reduct of the canonical R-module on G. ▮

The next lemma gives a Mal'cev-like condition for varieties which contain no nontrivial abelian algebras. The proof is like the proofs of Lemma 3.4 and Corollary 3.6.

LEMMA 4.9: *Let V be a variety of type $\mathscr{L}$, $\mathbf{T}_{\mathscr{L}}(x,y)$ the term algebra for $\mathscr{L}$ in the variables $\{x,y\}$, and $\mathbf{F} = \mathbf{F}_V(x,y)$ the V-free algebra on $\{x,y\}$. Then the following are equivalent*:

(1) *V contains no nontrivial abelian algebras.*

(2) *$[\nabla_A, \nabla_A] = \nabla_A$ for every $\mathbf{A} \in V$.*

(3) $\langle x,y\rangle \in [\nabla_F, \nabla_F]$.

(4) *There is a $\sigma \in \Sigma_{TC}(\mathscr{L}, T_{\mathscr{L}}(x,y))$ satisfying all of the conditions of Corollary* 3.6(3) *except* (ai). ▮

THEOREM 4.10. *Suppose V is a variety of Jónsson–Tarski algebras. Then the following are equivalent*:

(1) *V has* BFC.

(2) *V satisfies* (*).

(3) *V contains no nontrivial abelian algebras.*

Proof. (2) ⇒ (1) by Corollary 1.6.

(1) ⇒ (3) by Corollary 4.4 and Lemma 4.8.

(3) ⇒ (2). We want to prove that V satisfies (*).

CLAIM 1. *It suffices to find a V-factorable formula $\rho(x, y)$ such that*

(i) $V \models \rho(x, x)$

(ii) $V \models \rho(0, x)$

(iii) $V \models \rho(x, 0) \to x \approx 0$.

Proof. Let $\pi(x, y, z, w)$ be

$$\forall u_1 u_2[z + u_1 \approx u_2 + w \to \exists u_3 u_4(x + u_3 \approx u_4 + y \,\&\, \rho(u_3, u_1) \,\&\, \rho(u_4, u_2))].$$

Then π is V-factorable and witnesses (*) for V.

Claim 2. *It suffices to find a V-factorable formula $\beta(x, \vec{u})$ such that*

(i) $V \models \exists \vec{u} \beta(x, \vec{u})$

(ii) $V \models \forall \vec{u} \beta(x, \vec{u}) \leftrightarrow x \approx 0$.

Proof. Let $\rho(x, y)$ be $\forall \vec{u}[\beta(y, \vec{u}) \to \beta(x, \vec{u})]$. Then ρ satisfies the conditions of Claim 1.

CLAIM 3. *It suffices to find V-factorable formulas $\beta_1(x, \vec{u}^1), ..., \beta_n(x, \vec{u}^n)$ such that*

(i) $V \models \exists \vec{u}^i \beta_i(x, \vec{u}^i) \qquad (1 \leqslant i \leqslant n)$

(ii) $V \models [\&_{i=1}^n \forall \vec{u}^i \beta_i(x, \vec{u}^i)] \leftrightarrow x \approx 0$.

Proof. Assume the variables u_j^i are distinct and let $\beta(x, \vec{u}^1, ..., \vec{u}^n)$ be

$$\mathop{\&}_{i=1}^{n} \beta_i(x, \vec{u}^i).$$

Then β satisfies the conditions of Claim 2.

CLAIM 4. *For each $(n+1)$-ary $\mathscr{L}$-term $t(x, \vec{y})$ there are V-factorable formulas $\gamma_i^t(x, x', \vec{y}, \vec{y}', \vec{u})$ $(1 \leqslant i \leqslant 6)$ such that*

(i) $V \models \&_{i=1}^6 \exists \vec{u} \gamma_i^t(x, x', \vec{y}, \vec{y}', \vec{u})$

(ii) $V \models \&_{i=1}^{6} \forall \vec{u} \gamma_i^t(0, 0, \vec{0}, \vec{0}, \vec{u})$

(iii) $V \models [\&_{i=1}^{6} \forall \vec{u} \gamma_i^t(x, x', \vec{y}, \vec{y}', \vec{u}) \ \& \ t(x, \vec{y}) \approx t(x, \vec{y}')] \rightarrow t(x', \vec{y}) \approx t(x', \vec{y}')$.

Proof. Let $\gamma_1^t, ..., \gamma_6^t$ be the following formulas respectively:

$$t(u_1, \vec{0}) + t(x', \vec{y}) \approx t(u_1 + x', \vec{y})$$

$$t(x, \vec{y}) + t(u_2, 0) \approx t(x + u_2, \vec{y})$$

$$t(x, \vec{y}') + t(u_3, 0) \approx t(x + u_3, \vec{y}')$$

$$t(u_4, \vec{0}) + t(x', \vec{y}') \approx t(u_4 + x', \vec{y}')$$

$$\exists v[v + t(x, \vec{0}) \approx u_5]$$

$$u_6 + [t(x, \vec{0}) + u_7] \approx [u_6 + t(x, \vec{0})] + u_7.$$

Condition (i) is true by virtue of the choice $u_i = 0$ for $i \neq 5$, $u_5 = t(x, \vec{0})$, and (ii) is true since $V \models t(0, \vec{0}) \approx 0$. Now suppose $\mathbf{A} \in V$, a, $a' \in A$, and $\vec{b}, \vec{b}' \in A^n$ are such that $t^{\mathbf{A}}(a, \vec{b}) = t^{\mathbf{A}}(a, \vec{b}')$ and

$$\mathbf{A} \models \mathop{\&}_{i=1}^{6} \forall \vec{u} \gamma_i^t(a, a', \vec{b}, \vec{b}', \vec{u}).$$

Since $\mathbf{A} \models \gamma_5^t(a, a', \vec{b}, \vec{b}', 0) \ \& \ \forall u_6 u_7 \gamma_6^t(a, a', \vec{b}, \vec{b}', u_6, u_7)$, the element $t^{\mathbf{A}}(a, \vec{0})$ has a left inverse (with respect to $+^{\mathbf{A}}$) and associates with every pair of elements of A. Thus to prove $t^{\mathbf{A}}(a', \vec{b}) = t^{\mathbf{A}}(a', \vec{b}')$ it suffices to show

$$t^{\mathbf{A}}(a, \vec{0}) + t^{\mathbf{A}}(a', \vec{b}) = t^{\mathbf{A}}(a, \vec{0}) + t^{\mathbf{A}}(a', \vec{b}'),$$

and this follows easily from $t^{\mathbf{A}}(a, \vec{b}) = t^{\mathbf{A}}(a, \vec{b}')$ and $\mathbf{A} \models \&_{i=1}^{4} \forall u_i \gamma_i^t(a, a', \vec{b}, \vec{b}', u_i)$ via the choice $u_1 = u_4 = a$, $u_2 = u_3 = a'$.

Now to finish the proof of the theorem, assume V contains no nontrivial abelian algebras. Let $\sigma \in \sum_{TC}(\mathscr{L}, T_{\mathscr{L}}(x, y))$ be as in Lemma 4.9, and let TC_σ be as in the proof of Theorem 3.8. For each $\tau = \langle t, p, q, \vec{r}, \vec{s} \rangle \in TC_\sigma$ and $i = 1, ..., 6$ let $\beta_{\tau, i}(x, \vec{u})$ be the formula

$$\gamma_i^t(p(x, 0), q(x, 0), \vec{r}(x, 0), \vec{s}(x, 0), \vec{u}).$$

Then the collection $\{\beta_{\tau, i}(x, \vec{u}) \colon \tau \in TC_\sigma, 1 \leqslant i \leqslant 6\}$ satisfies the conditions of Claim 3. ∎

5. Thin Binary Relations

Definition 5.1. Let R be a binary relation on a set A.

(1) R is *connected* if for all $a, b \in A$ there exist $n \geqslant 0$ and $c_0, ..., c_n \in A$ such that $c_0 = a$, $c_n = b$, and either $c_i R c_{i+1}$ or $c_{i+1} R c_i$ for each $i < n$.

(2) R is *thin* if $\{A, R\rangle$ satisfies the axiom

$$\forall u[(uRx \leftrightarrow uRy) \,\&\, (xRu \leftrightarrow yRu)] \rightarrow x \approx y.$$

(For example, every partial ordering of A is thin.)

(3) $R^{\cup} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$.

R. McKenzie proved that if R is a thin binary relation of A such that both $R \circ R^{\cup}$ and $R^{\cup} \circ R$ are connected, then $\langle A, R\rangle$ has Boolean factor relations ([15, Theorem 4.1]; the proof of a slightly weaker claim may also be found in [16, first corollary to Theorem 5.18]). It follows that if $\mathbf{A}$ is an algebra, $\rho(x, y)$ is an $\mathbf{H}(\mathbf{A})$-factorable formula, $\rho^{\mathbf{A}}$ is thin, and both $\rho^{\mathbf{A}} \circ (\rho^{\mathbf{A}})^{\cup}$ and $(\rho^{\mathbf{A}})^{\cup} \circ \rho^{\mathbf{A}}$ are connected, then $\mathbf{A}$ has BFC. In this section we show that if an entire variety has such a formula then the variety satisfies (*).

Until further notice, R is a binary relation symbol, $\mathscr{R}$ is the class of all $\{R\}$-structures, and $K \subseteq \mathscr{R}$. The idea is to find an $\mathscr{R}$-factorable formula $\psi(x, y, z)$ which approximates the predicate $\forall u[(uRx \,\&\, uRy) \rightarrow uRz]$.

DEFINITION 5.2. For each $\{R\}$-formula ϕ, the *dual* of ϕ, denoted ϕ^{op}, is the formula obtained from ϕ by replacing each occurrence of a subformula of the form uRv by vRu.

LEMMA 5.3. *Suppose* $R^{\mathbf{A}}$ *is thin for every* $\mathbf{A} \in K$, *and there exist* $\mathscr{R}$-*factorable formulas* $\psi(x, y, z)$, $\psi^*(x, y, z)$ *such that*

(i) $K \models \psi(x, y, x) \,\&\, \psi(x, y, y)$

(ii) $K \models \psi(x, y, z) \rightarrow \forall u[(uRx \,\&\, uRy) \rightarrow uRz]$

(iii) $K \models \psi^*(x, y, x) \,\&\, \psi^*(x, y, y)$

(iv) $K \models \psi^*(x, y, z) \rightarrow \forall u[(xRu \,\&\, yRu) \rightarrow zRu]$.

Then K satisfies (*).

Proof. Let $\phi(x, y, u_1, u_2)$ be the $\mathscr{R}$-factorable (and hence K-factorable) formula

$$\psi(x, u_1, y) \,\&\, \psi^*(x, u_1, y) \,\&\, \psi(y, u_2, x) \,\&\, \psi^*(y, u_2, x).$$

Clearly $K \models \exists \vec{u}\phi(x, y, \vec{u}) \,\&\, \forall \vec{u}\phi(x, x, \vec{u})$ by hypotheses (i) and (iii). Now suppose $\mathbf{A} \in K$, $x, y \in A$, and $\mathbf{A} \models \forall \vec{u}\phi(x, y, \vec{u})$. Then in particular $\mathbf{A} \models \phi(x, y, x, y)$, so by hypotheses (ii) and (iv).

$$\mathbf{A} \models \forall u[uRx \leftrightarrow uRy] \,\&\, \forall u[xRu \leftrightarrow yRu].$$

By thinness, $x = y$. This shows that ϕ satisfies the conditions of Lemma 1.7(3); hence K satisfies (*). ∎

DEFINITION 5.4.

(1) For each $n \geqslant 1$, $C_n(x_0, x_1, ..., x_{2n})$ is the formula

$$\mathop{\&}_{i=0}^{n-1} [x_{2i+1} R x_{2i} \,\&\, x_{2i+1} R x_{2i+2}].$$

(2) $\psi_1(x, y, z)$ is the formula

$$\exists u C_1(x, u, y) \,\&\, \forall u[C_1(x, u, y) \rightarrow uRz].$$

For $n > 1$ $\psi_n(x, y, z)$ is recursively defined to be

$$\exists u_1 \cdots u_{2n-1} \, C_n(x, \vec{u}, y) \,\&\, \forall u_1 \cdots u_{2n-1}[C_n(x, \vec{u}, y)$$
$$\rightarrow \exists x'y'[u_1 Rx' \,\&\, u_{2n-1} Ry' \,\&\, \psi_{n-1}(x', y', z)]].$$

LEMMA 5.5.

(i) *Each* ψ_n is *$\mathscr{R}$-factorable.*

(ii) $\models \exists \vec{u} C_n(x, \vec{u}, y) \rightarrow [\psi_n(x, y, x) \,\&\, \psi_n(x, y, y)]$ *for each n.*

(iii) $\models [uRx \,\&\, uRy \,\&\, \psi_n(x, y, z)] \rightarrow uRz$ *for each n.*

Proof. (i) is clear.

(ii) This is certainly true for $n = 1$. Assume $n > 1$ and the claim is true for $n - 1$. If $\mathbf{A} \in \mathscr{R}$, $x, y, u_1, ..., u_{2n-1} \in A$ and $\mathbf{A} \models C_n(x, \vec{u}, y)$, then in particular $u_1 Rx$, $u_{2n-1} R u_{2n-2}$, and $\mathbf{A} \models \exists \vec{v} C_{n-1}(x, \vec{v}, u_{2n-2})$; hence $\mathbf{A} \models \psi_{n-1}(x, u_{2n-2}, x)$ by the inductive hypothesis. This shows that

$$\models \forall \vec{u}[C_n(x, \vec{u}, y) \rightarrow \exists x'y'[u_1 Rx' \,\&\, u_{2n-1} Ry' \,\&\, \psi_{n-1}(x', y', x)]]$$

and so $\models \exists \vec{u} C_n(x, \vec{u}, y) \rightarrow \psi_n(x, y, x)$. A similar argument shows $\models \exists \vec{u} C_n(x, \vec{u}, y) \rightarrow \psi_n(x, y, y)$.

(iii) This is trivial if $n = 1$. Assume $n > 1$ and the claim is true for $n - 1$. If $\mathbf{A} \in \mathscr{R}$, $x, y, z, u \in A$ and $\mathbf{A} \models uRx \,\&\, uRy \,\&\, \psi_n(x, y, z)$, define $u_{2i} = x$ $(1 \leqslant i < n)$ and $u_{2i+1} = u$ $(0 \leqslant i < n)$. Then $\mathbf{A} \models C_n(x, \vec{u}, y)$, so by hypothesis there exist $x', y' \in A$ such that

$$\mathbf{A} \models u_i Rx' \,\&\, u_{2n-1} Ry' \,\&\, \psi_{n-1}(x', y', z).$$

But $u_1 = u_{2n-1} = u$, so uRz by the inductive hypothesis. ∎

COROLLARY 5.6. *Suppose* $R^{\mathbf{A}}$ *is thin for every* $\mathbf{A} \in K$, *and*

$$K \models \forall xy[\exists \vec{u} C_m(x, \vec{u}, y) \,\&\, \exists \vec{v} C_n^{op}(x, \vec{v}, y)]$$

for some m, $n \geqslant 1$. *Then K satisfies* (*).

Proof. Let m, n be as above; then Lemma 5.5 and its dual imply that $\psi_m(x, y, z)$ and $\psi_n^{op}(x, y, z)$ satisfy the requirements of Lemma 5.3. ∎

COROLLARY 5.7. *Suppose K is closed under arbitrary direct products and, for every $\langle A, R\rangle \in K$, R is thin and both $R \circ R^{\cup}$ and $R^{\cup} \circ R$ are connected. Then K satisfies (*).*

Proof. Let $C_0(x, y)$ be the formula $x \approx y$. Then the connectedness hypotheses are equivalent to

$$K \models \forall xy\left(\bigvee\!\!\!\bigvee_{n\geq 0} \exists \vec{v} C_n^{op}(x, \vec{v}, y)\right) \& \forall xy\left(\bigvee\!\!\!\bigvee_{m\geq 0} \exists \vec{u} C_m(x, \vec{u}, y)\right).$$

It can be easily shown that if K' is any class of structures closed under direct products and $(\theta_i(\vec{x}))_{i\in I}$ is an indexed set of K'-factorable formulas such that

$$K' \models \forall \vec{x} \bigvee\!\!\!\bigvee_{i\in I} \theta_i(\vec{x})$$

then $K' \models \forall \vec{x}\theta_i(\vec{x})$ for some $i \in I$. Hence there exist m, $n \geq 0$ such that

$$K \models \forall xy \exists \vec{v} C_n^{op}(x, \vec{v}, y) \,\&\, \forall xy \exists \vec{u} C_m(x, \vec{u}, y).$$

If either $m = 0$ or $n = 0$, then K contains only trivial structures and the formula $x \approx x$ witnesses (*) for K. Otherwise the hypotheses of Corollary 5.6 are met. ▮

Of course, this last result can be applied to more general classes of structures.

COROLLARY 5.8. *Suppose K is a class of $\mathscr{L}$-structures closed under arbitrary direct products and $\rho(x, y)$ is a K-factorable formula such that, for every $\mathbf{A} \in K$, $\rho^{\mathbf{A}}$ is thin and both $\rho^{\mathbf{A}} \circ (\rho^{\mathbf{A}})^{\cup}$* and *$(\rho^{\mathbf{A}})^{\cup} \circ \rho^{\mathbf{A}}$ are connected. Then K satisfies (*).*

Proof. The preceding arguments work just as well with ρ replacing R and K replacing $\mathscr{R}$. ▮

APPENDIX

Let V be the variety of type $\{\cdot, 0\}$ defined by the axioms

$$x(yz) \approx (xy)z$$

$$xx \approx x$$

$$xyx \approx yx$$

$$0x \approx 0$$

$$xyz \approx yxz.$$

Define $x \leqslant y$ by $xy \approx x$. The first four axioms guarantee that $\leqslant$ is a partial order with least element 0 in every member of V. So V satisfies (*) by the results of Section 5. (Alternatively, the atomic formula $ux \approx u$ satisfies the conditions of Lemma 1.7(4).) We shall show that V does not satisfy property (II) defined in Section 2, and contains algebras with nontrivial center.

DEFINITION 1. For each finite set U, let $\mathbf{F}(U)$ be the algebra whose universe is

$$\{\langle U_1, u\rangle : U_1 \subseteq U, u \in U \backslash U_1\} \cup \{0\},$$

where 0 has the obvious interpretation and multiplication is given by

$$\langle U_1, u_1\rangle\langle U_2, u_2\rangle = \langle (U_1 \cup \{u_1\} \cup U_2) \backslash \{u_2\}, u_2\rangle,$$
$$0\langle U_1, u\rangle = \langle U_1, u\rangle\, 0 = 00 = 0.$$

LEMMA 2. *For each finite set U, $\mathbf{F}(U)$ is the V-free algebra on U via the identification of each $u \in U$ with $\langle \varnothing, u\rangle \in F(U)$. For each $\mathbf{A} \in V$ and choice of elements $a_u \in A$ $(u \in U)$ the unique homomorphism $\mathbf{F}(U) \to \mathbf{A}$ extending $u \mapsto a_u$ is given by*

$$\langle \{u_1, ..., u_n\}, u\rangle \mapsto a_{u_1} \cdots a_{u_n} a_u$$
$$0 \mapsto 0.$$

Proof. It is easy to check that $\mathbf{F}(U)$ is in V and is generated by U. To complete the proof it suffices to show that for each $\mathbf{A} \in V$ and $a_u \in A$ $(u \in U)$ the map $F(U) \to A$ given above is well-defined and is a homomorphism. To this end it suffices to show

(1) $V \models x_{\sigma(1)} \cdots x_{\sigma(n)} z \approx x_1 \cdots x_k z$ for every surjection $\sigma\colon \{1, ..., n\} \to \{1, ..., k\}$.

(2) $V \models x_i \approx 0 \to x_1 \cdots x_n \approx 0$ for each $i = 1, ..., n$.

The first claim is routine, and the second will be as soon as it is seen that $V \models x0 \approx 0$. Indeed, $x0 \approx 0x0 \approx 0$ by the third and fourth axioms, respectively. ∎

DEFINITION 3. Given $\mathbf{A} \in V$ and $a \in A$, define $f_a, g_a\colon A \to A$ by

$$f_a(x) = xa, \qquad g_a(x) = ax.$$

LEMMA 4. *Let* $\mathbf{A}, \mathbf{B} \in V$.

(1) *Every nonconstant unary polynomial of* $\mathbf{A}$ *is of the form* f_a *or* g_a *for some* $a \in A \backslash \{0\}$.

(2) *Every nonconstant unary polynomial of* $\mathbf{A}\times\mathbf{B}$ *is of one of the following forms*:

$$f_a\times f_b,\quad g_a\times g_b,\quad 0\times f_b,\quad 0\times g_b,\quad f_a\times 0,\quad g_a\times 0$$

for some $a\in A\setminus\{0\}$, $b\in B\setminus\{0\}$.

Proof. Suppose $t(x_0, ..., x_n)$ is an $(n+1)$-ary term which depends on all of its variables in $\mathbf{A}$ (or in $\mathbf{A}\times\mathbf{B}$). Then $t(\vec{x})$ corresponds in the canonical way to some

$$T_i=\langle\{x_0, ..., x_n\}\setminus\{x_i\}, x_i\rangle\in F(x_0, ..., x_n).$$

If $i=0$, then for every $\mathbf{C}\in V$ and $c_1, ..., c_n\in C$ the polynomial $t^{\mathbf{C}}(x,\vec{c})$ is identical to $g_d(x)$ where $d=c_1\cdots c_n$, while if $i\neq 0$ then $t^{\mathbf{C}}(x,\vec{c})=f_e(x)$ for $e=dc_i$. The lemma follows from these facts. ∎

DEFINITION 5. For each finite set U such that $0\notin U$, define $\sigma\colon F(U)\to U\cup\{0\}$ by

$$\sigma(0)=0,\qquad \sigma(\langle U_1, u\rangle)=u.$$

LEMMA 6. *Let* U, U' *be finite sets such that* $0\notin U\cup U'$; *let* $\mathbf{A}=\mathbf{F}(U)$ *and* $\mathbf{B}=\mathbf{F}(U')$, *and assume* $x, y\in U$ *and* $a, b\in U'$.

(1) *For all* $\langle p, q\rangle\in\Theta_{\mathbf{A}}(x, y)$, *either* $\sigma(p)=\sigma(q)$ *or* $\{\sigma(p),\sigma(q)\}=\{x, y\}$.

(2) *For all* $\langle (p, r), (q, s)\rangle\in\Theta_{\mathbf{A}\times\mathbf{B}}((x, a), (y, b))$, *either*:

(i) $(\sigma(p),\sigma(r))=(\sigma(q),\sigma(s))$, *or*

(ii) $\{(\sigma(p),\sigma(r)), (\sigma(q),\sigma(s))\}$ *is equal to one of the following sets*:

$$\{(x, a), (y, b)\},\qquad \{(x, 0), (y, 0)\},\qquad \{(0, a), (0, b)\}.$$

Proof of (2). Let S be the set of all $\langle (p, q), (r, s)\rangle\in(A\times B)^2$ which satisfy either of the two conditions above. It is easy to check that S is an equivalence relation on $A\times B$ which contains $\langle (x, a), (y, b)\rangle$. Thus it suffices to show that S is compatible with all nonconstant unary polynomials of $\mathbf{A}\times\mathbf{B}$. This will follow from Lemma 4 and the fact that $\sigma(pq)=\sigma(q)$ for any p, q in $A\setminus\{0\}$ (or in $B\setminus\{0\}$). ∎

Now let $\mathbf{A}$, $\mathbf{B}$, $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \theta, \theta'$ be as in the proof of Theorem 2.4, and let

$$\Sigma=\{(x, e), (w, e), (z, c), (y, c)\}.$$

LEMMA 7. *If* $(p, r), (q, s) \in A \times B$ *satisfy*

(i) $(\sigma(p), \sigma(r)) \in \sum$, *and*

(ii) $\langle (p, r), (q, s) \rangle \in \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup (\Theta_{\mathbf{A}}(x, w) \times \Delta_B) \cup (\Theta_{\mathbf{A}}(y, z) \times \Delta_B)$

then $(\sigma(q), \sigma(s)) \in \sum$.

Proof. This is immediate from Lemma 6. ▮

COROLLARY 8. $\langle (x, e), (y, e) \rangle \notin \text{trans}[\theta \cup \pi_1(\theta') \times \Delta_B]$; *hence* V *does not satisfy property* (II).

Proof. This follows from Lemma 7 and the fact that $\pi_1(\theta') \subseteq \Theta_{\mathbf{A}}(x, w) \vee \Theta_{\mathbf{A}}(y, z)$. ▮

To show that V contains algebras with nontrivial center, pick a set A and an element $0 \in A$, and define

$$xy = \begin{cases} 0 & \text{if } x = 0 \\ y & \text{otherwise.} \end{cases}$$

One can show that $\mathbf{A} = \langle A, \cdot, 0 \rangle \in V$. Define $\theta = \nabla_{A \setminus \{0\}} \cup \{\langle 0, 0 \rangle\}$.

LEMMA 9. $\theta \in \text{Con}\, \mathbf{A}$.

Proof. It suffices, by Lemma 4, to show that θ is preserved by f_a and g_a for every $a \in A \setminus \{0\}$. This is obvious. ▮

LEMMA 10. $Z_{\mathbf{A}} = \theta$.

Proof. It suffices to show that θ centralizes ∇_A modulo Δ_A, but $Z_{\mathbf{A}} \neq \nabla_A$. To prove the former claim, let $t(\vec{x})$ be an $(n+1)$-ary term which depends on all of its variables in $\mathbf{A}$, and let $\langle a, b \rangle \in \theta$ and $\vec{c}, \vec{d} \in A^n$ be such that $t^{\mathbf{A}}(a, \vec{c}) = t^{\mathbf{A}}(a, \vec{d})$. As in the proof of Lemma 4, pick $i \in \{0, ..., n\}$ such that $t(\vec{x})$ corresponds to $T_i = \langle \{x_0, ..., x_n\} \setminus \{x_i\}, x_i \rangle$ in $\mathbf{F}(x_0, ..., x_n)$. If $i = 0$ then there exist $u, v \in A$ such that $t^{\mathbf{A}}(x, \vec{c}) = ux$ and $t^{\mathbf{A}}(x, \vec{d}) = vx$; one can show $t^{\mathbf{A}}(b, \vec{c}) = t^{\mathbf{A}}(b, \vec{d})$ by breaking the argument into two cases according to whether or not $\langle u, v \rangle \in \theta$. If $i \neq 0$ then there exist $u, v \in A$ such that $t^{\mathbf{A}}(x, \vec{c}) = xu$ and $t^{\mathbf{A}}(x, \vec{d}) = xv$; in this case $t^{\mathbf{A}}(b, \vec{c}) = t^{\mathbf{A}}(b, \vec{d})$ follows by considering whether or not $a = 0$. Hence θ centralizes ∇_A modulo Δ_A.

If $\mathbf{C}$ is an abelian algebra in V, then for any $c \in C$, $00 = 0c$ implies $c0 = cc$, i.e., $0 = c$. Hence V contains no nontrivial abelian algebras, which proves $Z_{\mathbf{A}} \neq \nabla_A$. So $Z_{\mathbf{A}} = \theta$. ▮

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